MPhil Advanced Computer Science
Topics in Logic and Complexity
Take Home Exam

There are ten questions below. Each is worth 20 marks. Answer as many questions as you wish. Your best five answers will be added to give a total mark out of 100.

1. In the lecture course, we saw that the problem of deciding, given a graph G and a sentence ϕ of first-order logic, whether $G \models \phi$ is PSPACE-complete. Here, you are to show that the same is true for Monadic Second-Order logic (MSO). That is, prove that the following decision problem is PSPACE-complete:

   Input: A graph G and a sentence ϕ of MSO.
   Decide: whether or not $G \models \phi$.

   To do this,

   (a) explain why the problem is decidable in polynomial space; and

   (b) prove that it is PSPACE-hard.

2. Recall that a Boolean formula is in *conjunctive normal form* if it is the conjunction of a collection of *clauses*, each of which is the disjunction of a set of *literals*. Each literal is either a propositional variable or the negation of a propositional variable. We say that a formula is in *3-CNF* if it is in conjunctive normal form and each clause contains exactly 3 literals. It is in *2-CNF* if it is in conjunctive normal form and each clause contains exactly 2 literals.

   The problem of deciding whether a given formula in 3-CNF is satisfiable is known to be NP-complete. Here, the aim is to show that the problem of deciding whether a given formula in 2-CNF is satisfiable is in NL.

   (a) Show that every clause containing 2 literals can be written as an implication in exactly two ways.

   For any formula ϕ in 2-CNF, define the directed graph G_ϕ to be the graph whose set of vertices is the set of all literals that occur in ϕ, and in which there is an edge from literal x to literal y if, and only if, the implication $(x \rightarrow y)$ is equivalent to one of the clauses in ϕ.

   (b) Show that ϕ is *unsatisfiable* if, and only if, there is a literal x such that there is a path in G_ϕ from x to $\neg x$ *and* a path from $\neg x$ to x.

   Recall that the problem of reachability in directed graphs is in NL and that NL = co-NL.

(c) Explain why from (a), (b) and the above statement, it follows that the problem of determining whether a formula in 2-CNF is satisfiable is in NL.

3. A *clique* in a graph $G = (V, E)$ is a set $X \subseteq V$ of vertices such that for any $u, v \in X$ if $u \neq v$ then (u, v) is an edge in E. The decision problem CLIQUE is the problem of deciding, given a graph G and a positive integer k whether or not G contains a clique with k or more elements. This problem is known to be NP-complete.

   We will represent this problem as a class of structures as follows. The vocabulary consists of two binary relations E and $<$ and one constant k. Consider structures $\mathcal{G} = (V, E, <, k)$ in this vocabulary where (V, E) is a graph, $<$ is a linear order on V and k is some element of V. We say that $\mathcal{G}$ is in CLIQUE if there is a set $X \subseteq V$ of vertices which forms a clique in the graph (V, E) and so that the number of elements in X is larger than the number of elements in $\{v \in V \mid v < k\}$, i.e. the number of elements before k in the linear order.

   (a) Give a sentence of existential second-order logic that defines the class of structures CLIQUE.

   (b) Prove that there is no sentence of first-order logic that defines this class of structures.

   Hint: for the second part, you will want to construct, for each p, a pair of structures $\mathbb{A}_p$ and $\mathbb{B}_p$ one of which is in CLIQUE and the other is not, but which are equivalent under $\equiv_p$. For this, you might find it useful to consider the proof, using Ehrenfeucht games, that evenness is not first-order definable on *linear orders*.

   (c) It is an open question whether or not CLIQUE as defined above is definable in LFP. What would be the consequences if it was definable? What would follow if you could prove it is not definable?

4. In this question, we will consider a different representation of the problem CLIQUE from Question 3 above. Let the vocabulary consist of two unary relations V and N, two binary relations E and $<$ and a constant k. Say that a structure $\mathbb{A} = (A, V, N, E, <, k)$ is in the class CLIQUE if: V and N are subsets of A; E is a binary relation on V; $<$ is a linear order on N; $k \in N$; and the graph (V, E) contains a clique with more elements than there are in the set $\{a \in N \mid a < k\}$.

   (a) Give a sentence of existential second-order logic that defines the class of structures CLIQUE in this sense.

   (b) Prove that there is no sentence of LFP that defines this class of structures.

Hint: for the second part, you will want to construct, for each k a pair of structures $\mathbb{A}_k$ and $\mathbb{B}_k$ one of which is in CLIQUE and the other is not, but which are equivalent under $\equiv^k$. For this, you might find it useful to consider the proof, using Ehrenfeucht games, that evenness is not definable in LFP on *unordered structures*.

(c) What can you conclude about whether CLIQUE is definable in PFP?

5. We say that a graph $G = (V, E)$ contains a *perfect matching* if there is a set $M \subseteq E$ of edges such that for every vertex $v \in V$, there is *exactly* one edge $e \in M$ such that e is incident on v.

   (a) Give a sentence of existential second-order logic that defines the class of graphs that contain a perfect matching.

   (b) Prove that there is no sentence of LFP that defines this class of structures.

   Hint: for the second part, recall the proof from the lecture that shows that Hamiltonicity is not definable in LFP.

6. Consider a vocabulary consisting of two unary relations P and O, one binary relation E and two constants s and t. We say that a structure $\mathbb{A} = (A, P, O, E, s, t)$ in this vocabulary is an *arena* if $P \cup O = A$ and $P \cap O = \emptyset$. That is, P and O partition the universe into two disjoint sets.

   An arena defines the following game played between a *player* and an *opponent*. The game involves a *token* that is initially placed on the element s. At each move, if the token is currently on an element of P it is *player* who plays and if it is on an element of O, it is *opponent* who plays. At each move, if the token is on an element a, the one who plays choses an element b such that $(a, b) \in E$ and moves the token from a to b. If the token reaches t at any point then *player* has won the game.

   We define GAME to be the class of arenas for which *player* has a strategy for winning the game. Note that in an arena $\mathbb{A} = (A, P, O, E, s, t)$, *player* has a strategy to win from an element a if *either* $a \in P$ and there is some move from a so that *player* still has a strategy to win after that move *or* $a \in O$ and for every move from a, *player* can win after that move.

   (a) Give a sentence of LFP that defines the class of structures GAME.

   We say that a collection $\mathcal{C}$ of decision problems is *closed under logarithmic space reductions* if whenever $A \in \mathcal{C}$ and $B \leq_L A$ (i.e. B is reducible to A by a logarithmic-space reduction) then $B \in \mathcal{C}$.

   The class of structures GAME defined above is known to be P-complete under logarithmic-space reductions.

   (b) Explain why this, together with (a) implies that the class of problems definable in LFP is *not* closed under logarithmic-space reductions.

7. If σ is a relational signature (i.e. it contains no function or constant symbols), and $\mathbb{A}$ and $\mathbb{B}$ are σ-structures, write $\mathbb{A}+\mathbb{B}$ for the structure whose universe is the disjoint union of the universes of $\mathbb{A}$ and $\mathbb{B}$ and where each relation symbol R of σ is interpreted by the corresponding union of its interpretations in $\mathbb{A}$ and $\mathbb{B}$. Similarly, write $n\mathbb{A}$ for the disjoint union of n copies of $\mathbb{A}$.

   (a) Show that, if $\mathbb{A} \equiv_q \mathbb{A}'$ and $\mathbb{B} \equiv_q \mathbb{B}'$, then $\mathbb{A}+\mathbb{B} \equiv_q \mathbb{A}'+\mathbb{B}'$.

   (b) Show that, for $n, m \geq q$, $n\mathbb{A} \equiv_q m\mathbb{A}$.

8. Suppose ϕ is formula of PFP, R is a relational variable, and $\mathcal{O}$ is the class of structures (in a vocabulary σ containing a binary relation symbol $<$) that interpret the symbol $<$ as a linear order.

   (a) Show there is a formula of PFP that is equivalent to $\exists R\phi$ on all structures in $\mathcal{O}$.

   (b) Use this fact to conclude that a class K of structures (not necessarily ordered) is definable by a sentence of the form $\exists R\phi$ (where ϕ is in PFP) if, and only if, $\{[\mathbb{A}]_< \mid \mathbb{A} \in K \text{ and } < \text{ is any order on } A\}$ is in PSPACE.

9. Consider a structure $\mathcal{E} = (A, E)$ where E is an equivalence relation on the set A, and let e_i denote the number of equivalence classes of E with exactly i elements. Define the k-index of $\mathcal{E}$ to be the k-tuple $(n_1, \ldots, n_k)$ where, for $i < k$, $n_i = \min(k, e_i)$ and $n_k = \min(k, \sum_{i \geq k} e_i,)$.

   (a) Show that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are two such structures with the same k-index, then $\mathcal{E}_1 \equiv^k \mathcal{E}_2$.

   It is known that if C is a class of structures where for each k there are only finitely many equivalence classes of structures with respect to $\equiv^k$, then every formula of LFP is equivalent, on C, to a formula of first-order logic. Use this and (a) to

   (b) prove that LFP is no more expressive than first-order logic on the class of finite equivalence relations.

10. We define the *3-chordal graph* on the set of vertices $\{0, \ldots, n-1\}$ to be the graph in which there is an edge from i to j if, and only if, $j = i + 1 \pmod n$ or $j = i + 2 \pmod n$.

    (a) Prove that the *3-chordal graph* on $\{0, \ldots, n-1\}$ is 3-colourable if, and only if, n is a multiple of 3.

    (b) Use this to show that 3-colourability is not definable in first-order logic.

    Hint: for the second part, you may wish to consider Hanf's theorem and the proof that connectivity is not first-order definable.