Comma categories

Defn. Given two functors $\mathbf{C} \xrightarrow{F} \mathbf{E} \xleftarrow{G} \mathbf{D}$,
their comma category $F \downarrow G$ is defined as follows:

- objects are triples (C, f, D) for $C \in |\mathbf{C}|$, $D \in |\mathbf{D}|$, and $f : FC \to GD$ an arrow in $\mathbf{E}$
- arrows $(h, k) : (C, f, D) \to (C', f', D')$ are pairs $h : C \to C'$, $k : D \to D'$ such that $f' \circ Fh = Gk \circ f$.
- composition and identity defined componentwise.

Example. The category of graphs is a comma category:

$$\mathbf{Graph} = \mathrm{Id}_{\mathbf{Sets}} \downarrow \Delta \quad \text{for} \quad \Delta(X) = X \times X$$

Exercise: Show how arrow categories $\mathbf{C}^{\to}$ and slice categories $\mathbf{C}/A$, are comma categories.

Fact: If $\mathbf{C}$, $\mathbf{D}$ are complete and G preserves limits then $F \downarrow G$ is complete.

Multisorted sets

For a fixed set S,

- an S-sorted set is a family $A = (A_s)_{s \in S}$ of sets.
- an S-sorted function from A to B is a family

$$(f_s : A_s \to B_s)_{s \in S}$$

of functions.

S-sorted sets and functions form a category $\mathbf{Sets}^S$

Note: If S has n elements then

$$\mathbf{Sets}^S \cong \mathbf{Sets}^n = \underbrace{\mathbf{Sets} \times \cdots \times \mathbf{Sets}}_{n \text{ times}}$$

Isomorphism in $\mathbf{Cat}$

Variable sets

Defn. For a fixed poset $(I, \leq)$, an I-indexed set A consists of:

- a family of sets $(A_i)_{i \in I}$,
- a function $\alpha_{ij} : A_i \to A_j$ for $i \leq j$

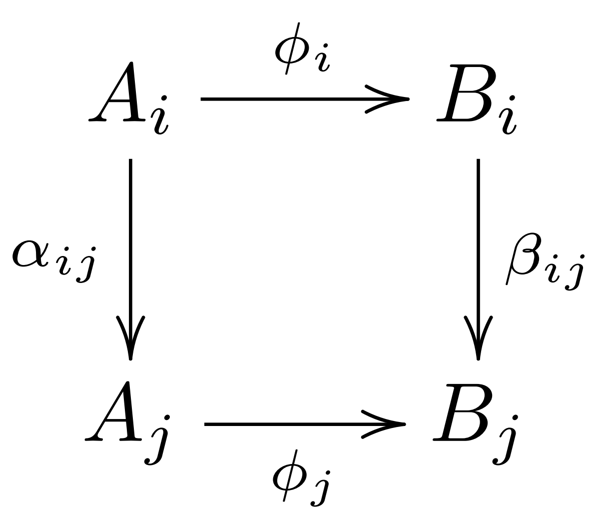
s.t.

- $\alpha_{ii} = 1_{A_i}$ for each i,
- $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ for $i \leq j \leq k$.

So it is just a functor $A : I \to \mathbf{Sets}$

Example: For $I = \mathbb{R}$, indexed sets are "sets varying through time".

Defn. An I-indexed function $\phi : A \to B$ is a family of functions $(\phi_i : A_i \to B_i)_{i \in I}$ such that:



for each $i \leq j$.

Natural transformations

Defn. For two functors $F, G : \mathbf{C} \to \mathbf{D}$,

a natural transformation $\phi : F \to G$ is a family

$$(\phi_C : FC \to GC)_{c \in |\mathbf{C}|}$$

of arrows in $\mathbf{D}$ indexed by objects in $\mathbf{C}$, such that

$$\begin{array}{ccc} FC & \xrightarrow{\phi_C} & GC \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FC' & \xrightarrow[\phi_{C'}]{} & GC' \end{array}$$

for each $f : C \to C'$ in $\mathbf{C}$.

The collection of all nat. transfs. from F to G denoted $Nat(F, G)$

Defn. ϕ is a natural isomorphism if every component ϕ_C is an isomorphism.

Examples

- identity transformation: $\mathrm{id_F} : F \to F$ for any $F : \mathbf{C} \to \mathbf{D}$

- singleton set: $\eta : \mathrm{Id}_{\mathbf{Sets}} \to \mathcal{P}$

$$\eta_X : X \to \mathcal{P}X \qquad \eta_X(x) = \{x\}$$

- Is there any transformation $\zeta : \mathcal{P} \to \mathrm{Id}_{\mathbf{Sets}}$?

$$\zeta_X : \mathcal{P}X \to X$$

No, e.g. the component at $\emptyset$ cannot exist...

How about nonempty powerset? $\zeta : \mathcal{P}^+ \to \mathrm{Id}_{\mathbf{Sets}}$?

No: take $X = \{\clubsuit, \spadesuit\}$, the naturality condition must fail for

$f : X \to X = \{\clubsuit \mapsto \spadesuit, \spadesuit \mapsto \clubsuit\}$

(NB. $(\mathcal{P}f)(X) = X$)