# Hom functors

An arrow $f : A \to B$ in (locally small) $\mathbf{C}$ induces a function:

$$f \circ - : \mathrm{hom}(X, A) \to \mathrm{hom}(X, B)$$

for any object $X$.

- This gives a covariant hom-functor $\mathrm{Hom}(X, -) : \mathbf{C} \to \mathbf{Sets}$:

$$\mathrm{Hom}(X, A) = \mathrm{hom}_{\mathbf{C}}(X, A) \qquad \mathrm{Hom}(X, f) = f \circ -$$

Such functors are called representable.

- Also contravariant hom-functors $\mathrm{Hom}(-, X) = \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$:

$$\mathrm{Hom}(A, X) = \mathrm{hom}_{\mathbf{C}}(A, X) \qquad \mathrm{Hom}(f, X) = - \circ f$$

- Finally, the mixed-variant hom-functor:

$$\mathrm{Hom}(-, -) = \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$$

# Embeddings

Defn: A functor $F : \mathbf{C} \to \mathbf{D}$ is an embedding if it is:

- faithful,
- injective on objects.

Examples:

- if $\mathbf{C}$ is a subcategory of $\mathbf{D}$, then the inclusion is an embedding.
- the slice category $\mathbf{C}/A$ embeds into the arrow category $\mathbf{C}^{\to}$.
- $\mathbf{Sets}$ embeds into $\mathbf{Pos}$

Defn: A full embedding is an embedding that is full.

# Preservation by functors

Defn. $F : \mathbf{C} \to \mathbf{D}$ preserves monos (epis, isos etc.) if
for any $f : A \to B$ in $\mathbf{C}$, if $f$ is mono (epi, iso etc.)
then so is $F(f)$.

Fact. All functors preserve isos, sections and retractions.
(But not all preserve monos or epis!)

Defn. $F : \mathbf{C} \to \mathbf{D}$ preserves limits if it maps limiting cones
to limiting cones.

(Similarly, $F$ can preserve products, finite limits, colimits, etc.)

Theorem: Representable functors preserve limits.

Exercise: Show that the forgetful functor $U : \mathbf{Pos} \to \mathbf{Sets}$
is representable.

# Functor composition

For $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$, the composition:

$$G \circ F : \mathbf{C} \to \mathbf{E}$$

is defined by: $(G \circ F)(A) = G(F(A))$
$(G \circ F)(f) = G(F(f))$

**Categories form a category!**

$\mathbf{Cat}$ - the category of all small categories and functors.

(also $\mathbf{CAT}$ - the "superlarge category" of all categories)

Facts. The empty category is initial in $\mathbf{Cat}$,
the trivial one-object category is final,
product of categories is categorical product in $\mathbf{Cat}$.

Exercise: What are isomorphisms in $\mathbf{Cat}$?