Functors

Functors = morphisms between categories

Defn. A functor $F : \mathbf{C} \to \mathbf{D}$ from $\mathbf{C}$ to $\mathbf{D}$ consists of:

- a function $F : |\mathbf{C}| \to |\mathbf{D}|$ (possibly large),
- for each $A, B \in |\mathbf{C}|$, a function

$$F : \hom_{\mathbf{C}}(A, B) \to \hom_{\mathbf{D}}(FA, FB), \qquad ●$$

s.t.

- $F(1_A) = 1_{FA}$ for $A \in |\mathbf{C}|$ (F preserves identities),
- $F(f \circ g) = Ff \circ Fg$ for $g : A \to B,\ f : B \to C$ in C
  (F preserves composition).

Defn. F is:

- full, if each ● is surjective,
- faithful, if each ● is injective.

Examples

- Identity functor $\mathrm{Id} : \mathbf{C} \to \mathbf{C}$, for any $\mathbf{C}$
- Constant functor $K_A : \mathbf{C} \to \mathbf{D}$ for any $A \in \mathbf{D}$:

  $K_A(B) = A, \quad K_A(f) = 1_A$ for any $B \in |\mathbf{C}|$, $f : B \to C$

- What is a functor between posets?

- Powerset functor: $\mathcal{P} : \mathbf{Sets} \to \mathbf{Sets}$
  -- $\mathcal{P}(A) = \{B \mid B \subseteq A\}$ for any set A
  -- $\mathcal{P}(f)(B) = \{f(b) \mid b \in B\}$ for any $f : A \to C$, $B \subseteq A$.

- Contravariant powerset functor: $\overleftarrow{\mathcal{P}} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$
  -- $\overleftarrow{\mathcal{P}}(A) = \{B \mid B \subseteq A\}$ for any set A
  -- $\overleftarrow{\mathcal{P}}(f)(D) = \{a \in A \mid f(a) \in D\}$ for any function $f : A \to B$ and $D \subseteq B$.

Examples ctd.

- Projection functors: $\mathbf{C} \xleftarrow{\pi_1} \mathbf{C} \times \mathbf{D} \xrightarrow{\pi_2} \mathbf{D}$
- The product functor: any *choice* in $\mathbf{C}$ of products $A \times B$ for every A and B, defines a functor:

$$\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

(action on arrows defined by pairing)

- Forgetful functors:

-- $U : \mathbf{Pos} \to \mathbf{Sets}$ $\qquad (A, \leq) \mapsto A$

-- $U : \mathbf{Mon} \to \mathbf{Sets}$ $\qquad (M, 1, \cdot) \mapsto M$ $\qquad$ etc.

They are faithful but usually not full.

- The free monoid functor: $F : \mathbf{Sets} \to \mathbf{Mon}$

-- on objects: $FX = X^* \ \left(= \bigcup_{n \in \mathbb{N}} X^n \right)$

-- on functions: $F(f)(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$