Even more examples

- $\mathbf{Pos}$: objects = posets, arrows = motononic functions

  ($f : (A, \leq_A) \to (B, \leq_B)$ s.t. $a \leq_A a' \implies f(a) \leq_B f(a')$)

- $\mathbf{Mon}$: objects = monoids, arrows = monoid homomorphisms

  $f : (M, \cdot_M, 1_M) \to (N, \cdot_N, 1_N)$ s.t.

  $$f(x \cdot_M y) = f(x) \cdot_N f(y) \qquad f(1_M) = 1_N$$

- $\mathbf{Grp}$: groups and group homomorphisms
- $\mathbf{Met}$: metric spaces and nonexpansive maps
- $\mathbf{Top}$: topological spaces and continuous maps
- $\mathbf{Mes}$: measurable spaces and measurable functions

...

Yet more examples

- $\mathbf{Rel}$: objects = sets, arrows = binary relations
  - -- relation composition:

  for $R \subseteq A \times B,\ S \subseteq B \times C$, define $S \circ R \subseteq A \times C$ by:

$$a(S \circ R)c \iff \exists b \in B.\ aRb,\ bSc$$

  -- identities = equality relations

- $\mathbf{Mat}$: objects = natural numbers,

  arrows in $\hom(m, n)$ = real-valued $m \times n$ matrices

  -- composition = matrix multiplication

  -- identities = unit matrices

Size issues

We could say "a category is a <u>set</u> of objects, <u>set</u> of arrows etc."

But we do not!

Problem: we want $\mathbf{Sets}$ to be a category,
but the set of all sets does not exist.

Look up
"set of all sets"

Formally, objects and arrows form classes.
(see *Mac Lane or Borceux for details*)

Like sets,
but can be larger

Defn. A category is small if its objects and arrows form sets.

Defn. A category is locally small if every $\hom(A, B)$ is a set.

Most categories that we will consider are locally small.

Subcategories

Defn. A subcategory of a category $\mathbf{C}$ is any category $\mathbf{D}$ s.t.

- $|\mathbf{D}| \subseteq |\mathbf{C}|$,
- for any $A, B \in |\mathbf{D}|$, $\hom_{\mathbf{D}}(A, B) \subseteq \hom_{\mathbf{C}}(A, B)$,
- composition on arrows in $\mathbf{D}$ coincides with that in $\mathbf{C}$,
- identities in $\mathbf{D}$ coincide with those in $\mathbf{C}$.

A subcategory is full if $\hom_{\mathbf{D}}(A, B) = \hom_{\mathbf{C}}(A, B)$
for all $A, B \in |\mathbf{D}|$.

Examples:

- $\mathbf{Sets}_{\mathrm{fin}}$ is a full subcategory of $\mathbf{Sets}$.
- $\mathbf{Sets}_{1\text{-}1}$ is a subcategory of $\mathbf{Sets}$, but it is not full.

Note: full subcategories are determined by their objects.

Duality principle

Given a category $\mathbf{C}$, the opposite (or dual) category $\mathbf{C}^{\mathrm{op}}$ is defined by:

- $|\mathbf{C}^{\mathrm{op}}| = |\mathbf{C}|$
- $\hom_{\mathbf{C}^{\mathrm{op}}}(A, B) = \hom_{\mathbf{C}}(B, A)$ for $A, B \in |\mathbf{C}|$
- composition: define $g \circ f$ in $\mathbf{C}^{\mathrm{op}}$ to be $f \circ g$ in $\mathbf{C}$.
- identities in $\mathbf{C}^{\mathrm{op}}$ are as in $\mathbf{C}$.

Fact. $(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}$

For every $\mathcal{K}$ (definition, theorem, ...), its dual (or co-$\mathcal{K}$) is obtained by reversing all arrows.

Fact. If a statement is true for all categories, then its dual is true for all categories.

Products of categories

Defn. The product $\mathbf{C} \times \mathbf{D}$ of categories $\mathbf{C}$ and $\mathbf{D}$ is defined by:

- $|\mathbf{C} \times \mathbf{D}| = |\mathbf{C}| \times |\mathbf{D}|$ (objects are pairs of objects)
- $\hom_{\mathbf{C}\times\mathbf{D}}((A,B),(C,D)) = \hom_{\mathbf{C}}(A,C) \times \hom_{\mathbf{D}}(B,D)$
  (arrows are pairs of arrows)
- composition and identities are defined pointwise:

$$(f,g) \circ (h,k) = (f \circ h, g \circ k)$$
$$1_{(A,B)} = (1_A, 1_B)$$

Exercises: What is the product of two monoids? Posets?
What is the opposite (dual) of a monoid? A poset?

Arrow categories

Defn. For a category $\mathbf{C}$, its arrow category $\mathbf{C}^{\rightarrow}$ is as follows:

- objects of $\mathbf{C}^{\rightarrow}$ are arrows of $\mathbf{C}$
- arrows from $f : A \rightarrow B$ to $g : C \rightarrow D$ are pairs of arrows $(h : A \rightarrow C,\ k : B \rightarrow D)$ such that:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \qquad\qquad k \circ f = g \circ h$$

- composition is pointwise, identities are pairs of identities.

Exercise. Check that composition is well-defined.

Slice categories

Defn. For an object A in $\mathbf{C}$, the slice category $\mathbf{C}/A$ is as follows:

- objects of $\mathbf{C}/A$ are arrows f in $\mathbf{C}$ with $\mathrm{cod}(f) = A$
- arrows from $f : B \to A$ to $g : C \to A$ are arrows $h : B \to C$ such that:

$$
\begin{tikzcd}
B \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
& A &
\end{tikzcd}
\qquad\qquad g \circ h = f
$$

- composition and identities are as in $\mathbf{C}$.

Exercise. Define the coslice category $A/\mathbf{C}$, where objects are arrows with $\mathrm{dom}(f) = A$.