Facts about left adjoints

Theorem: Left adjoints to any fixed G, if they exist, are unique up to natural isomorphism.

Theorem: If F is left adjoint to G, then:

- F preserves colimits,
- G preserves limits.

Theorem: Let $\mathbf{D}$ be locally small & complete (ie. have all limits).
A functor $G : \mathbf{D} \to \mathbf{C}$ has a left adjoint if and only if:

- G preserves limits,
- for every $C \in |\mathbf{C}|$ there exists a set (ie not a proper class) $\{f_i : C \to GD_i \mid i \in \mathcal{I}\}$ of arrows such that for each $D \in \mathbf{D}$ and $f : C \to GD$, there exist $i \in \mathcal{I}$ and $g : D_i \to D$ such that $f = Gg \circ f_i$.

Cofree objects

Consider a functor $F : \mathbf{C} \to \mathbf{D}$.

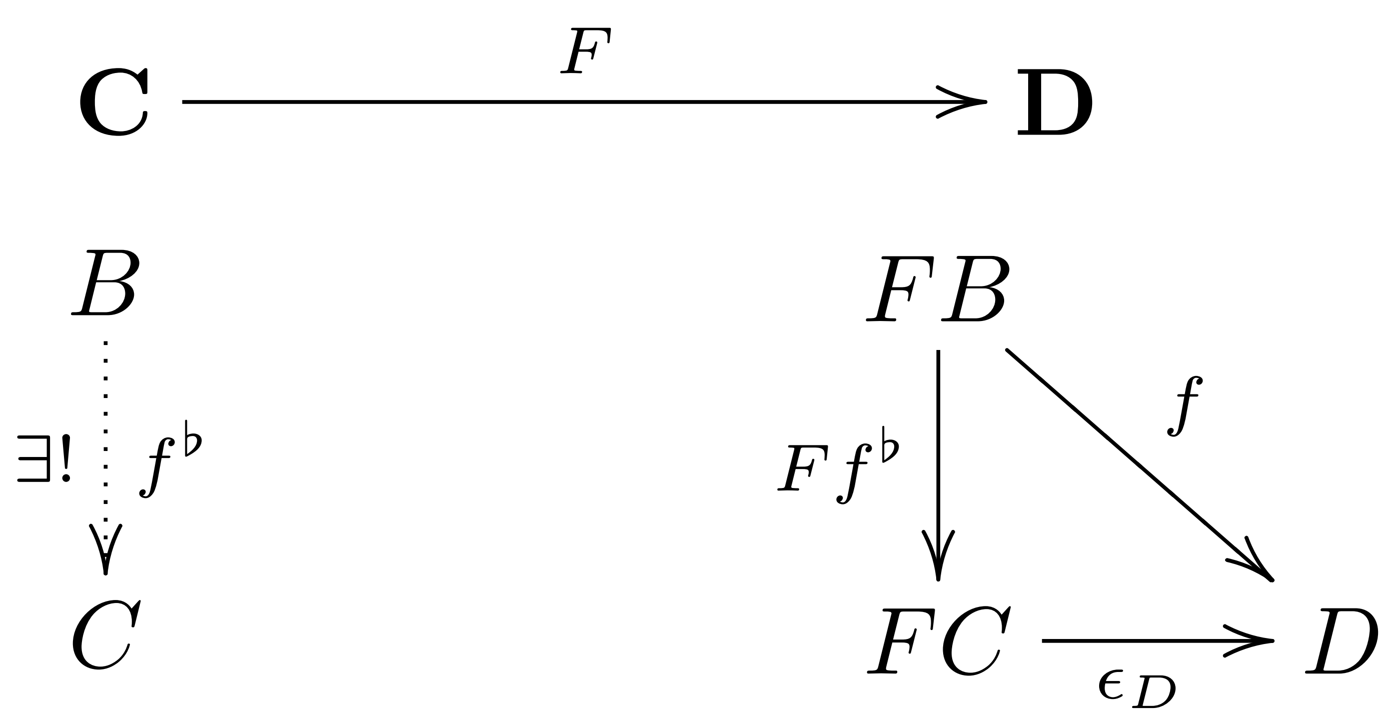
or: under

Defn. Given an object D in $\mathbf{D}$, a cofree object over D w.r.t. F is an object C in $\mathbf{C}$ with an arrow $\epsilon_D : FC \to D$ in $\mathbf{D}$

(the counit arrow)

such that

for every B in $\mathbf{C}$ with an arrow $f : FB \to D$
there exists a unique arrow $f^\flat : B \to C$ s.t. $\epsilon_D \circ F f^\flat = f$.



Examples

- For a monotonic function $f : C \to D$ between posets, the cofree element over $d \in D$ is the greatest element $c \in C$ such that $f(c) \leq d$.

Make count arrows explicit!

Exercise: What is the cofree object over (A, B) wrt. the diagonal functor $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$?

- Fix a set A. For a functor $A \times - : \mathbf{Sets} \to \mathbf{Sets}$, a cofree set over a set B is the set of functions B^A.

- A cofree set over a monoid $(M, \star, 1)$ wrt. the free monoid functor $(-)^* : \mathbf{Sets} \to \mathbf{Mon}$ is M.

Facts about cofree objects

Fact: For a functor $F : \mathbf{C} \to \mathbf{D}$, cofree objects over $D \in |\mathbf{D}|$ are final objects in the comma category $F \downarrow K_D$ where $K_D : \mathbf{1} \to \mathbf{D}$ is the functor constant at D.

Corollary: Cofree objects, if they exist, are unique up to isomorphism.

Fact: If C is cofree over D wrt. $F : \mathbf{C} \to \mathbf{D}$ then for each $B \in |\mathbf{C}|$ there is a bijection

$$(-)^\flat : \mathbf{C}(B, C) \cong \mathbf{D}(FB, D)$$

Cofree objects are functorial

Consider a functor $F : \mathbf{C} \to \mathbf{D}$.

If every $D \in |\mathbf{D}|$ has a cofree object $GD \in |\mathbf{C}|$ wrt. F

then the mapping

$$\begin{array}{rcl} D & \mapsto & GD \\ f : D \to D' & \mapsto & (f \circ \epsilon_D)^\flat \end{array}$$

defines a functor $G : \mathbf{D} \to \mathbf{C}$.

Further, $\epsilon : FG \to \mathrm{Id}_D$ is a natural transformation.

$$\mathbf{C} \xrightarrow{\quad F \quad} \mathbf{D}$$

$$\begin{array}{ccc} GD & \qquad & FGD & \xrightarrow{\epsilon_D} & D \\ \Big\downarrow{\scriptstyle Gf = (f \circ \epsilon_D)^\flat} & & \Big\downarrow{\scriptstyle FGf} & & \Big\downarrow{\scriptstyle f} \\ GD' & & FGD' & \xrightarrow[\epsilon_{D'}]{} & D' \end{array}$$

Right adjoints

Defn. A functor $G : \mathbf{D} \to \mathbf{C}$ is right adjoint to $F : \mathbf{C} \to \mathbf{D}$ with counit $\epsilon : FG \to \mathrm{Id}_{\mathbf{D}}$ if for every $D \in |\mathbf{D}|$, GD with ϵ_D is cofree over D wrt. F.

Examples:

- "the" product functor $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is right adjoint to the diagonal functor $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$.

- For any set A, a right adjoint to $A \times - : \mathbf{Sets} \to \mathbf{Sets}$ is denoted $(-)^A$ and defined by:

  B^A - the set of functions from A to B

  for $f : B \to C$, $\quad f^A = f \circ - \ : \ B^A \to C^A$

Defn: A category is cartesian closed if it has final objects, products and if each functor $A \times -$ has a right adjoint.