Examples

- For a monotonic function $f : D \to C$ between posets, the free element over $c \in C$ is the least element $d \in D$ such that $c \leq f(d)$.

- X^* is a free monoid over X wrt. the forgetful functor $G : \mathbf{Mon} \to \mathbf{Sets}$.

Make unit arrows explicit!

- The path category of a graph G is free over G wrt. the forgetful functor $U : \mathbf{Cat} \to \mathbf{Graph}$.

Exercise: What is the free poset over a set X wrt. the forgetful functor $G : \mathbf{Pos} \to \mathbf{Sets}$?

Exercise: What is the free object over (A, B) wrt. the diagonal functor $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$?

Facts about free objects

Fact: For a functor $G : \mathbf{D} \to \mathbf{C}$, free objects over $X \in |\mathbf{C}|$ are initial objects in the comma category $K_X \downarrow G$ where $K_X : \mathbf{1} \to \mathbf{C}$ is the functor constant at X.

Corollary: Free objects, if they exist, are unique up to isomorphism.

Fact: If A is free over X wrt. $G : \mathbf{D} \to \mathbf{C}$ then for each $B \in |\mathbf{D}|$ there is a bijection

$$(-)^\sharp : \hom_{\mathbf{C}}(X, GB) \cong \hom_{\mathbf{D}}(A, B)$$

Free objects are functorial

Consider a functor $G : \mathbf{D} \to \mathbf{C}$.

If every $C \in |\mathbf{C}|$ has a free object $FC \in |\mathbf{D}|$ wrt. G

then the mapping

$$\begin{array}{ccc} C & \mapsto & FC \\ f : C \to C' & \mapsto & (\eta_{C'} \circ f)^\sharp \end{array}$$

defines a functor $F : \mathbf{C} \to \mathbf{D}$.

Further, $\eta : \mathrm{Id}_C \to GF$ is a natural transformation.

$$\begin{tikzcd} \mathbf{C} & & & \mathbf{D} \arrow[lll, "G"'] \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "f"'] & GFC \arrow[d, "GFf"] \\ C' \arrow[r, "\eta_{C'}"'] & GFC' \end{tikzcd} \qquad \begin{tikzcd} FC \arrow[d, "Ff = (\eta_{C'} \circ f)^\sharp"] \\ FC' \end{tikzcd}$$

Left adjoints

Defn. A functor $F : \mathbf{C} \to \mathbf{D}$ is left adjoint to $G : \mathbf{D} \to \mathbf{C}$
with unit $\eta : \mathrm{Id}_{\mathbf{C}} \to GF$ if for every $C \in |\mathbf{C}|$,
FC with η_C is free over C wrt. G.

Examples:

- the free monoid functor is left adjoint to $G : \mathbf{Mon} \to \mathbf{Sets}$
- the path category functor is left adjoint to $U : \mathbf{Cat} \to \mathbf{Graph}$
- "the" coproduct functor $+ : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$
  is left adjoint to the diagonal functor $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$.

Exercise: What is a left adjoint to a monotonic function between posets?

Facts about left adjoints

Theorem: Left adjoints to any fixed G, if they exist, are unique up to natural isomorphism.

Theorem: If F is left adjoint to G, then:

- F preserves colimits,
- G preserves limits.

Theorem: Let $\mathbf{D}$ be locally small & complete (ie. have all limits).
A functor $G : \mathbf{D} \to \mathbf{C}$ has a left adjoint if and only if:

- G preserves limits,
- for every $C \in |\mathbf{C}|$ there exists a set (ie not a proper class) $\{f_i : C \to GD_i \mid i \in \mathcal{I}\}$ of arrows such that for each $D \in \mathbf{D}$ and $f : C \to GD$, there exist $i \in \mathcal{I}$ and $g : D_i \to D$ such that $f = Gg \circ f_i$.