#### Examples

- For a monotonic function  $f: D \to C$  between posets, the free element over  $c \in C$  is the least element  $d \in D$  such that  $c \leq f(d)$ .
- $X^*$  is a free monoid over Xwrt. the forgetful functor  $G: Mon \to Sets$ .

Make unit arrows explicit!

- The path category of a graph G is free over G wrt. the forgetful functor  $U : Cat \rightarrow Graph$ .

**Exercise**: What is the free poset over a set X wrt. the forgetful functor  $G : \mathbf{Pos} \to \mathbf{Sets}$ ?

Exercise: What is the free object over (A,B) wrt. the diagonal functor  $\Delta:{\bf C}\to{\bf C}\times{\bf C}?$ 

### Facts about free objects

Fact: For a functor  $G : \mathbf{D} \to \mathbf{C}$ , free objects over  $X \in |\mathbf{C}|$ are initial objects in the comma category  $K_X \downarrow G$ where  $K_X : \mathbf{1} \to \mathbf{C}$  is the functor constant at X.

Corollary: Free objects, if they exist, are unique up to isomorphism.

Fact: If A is free over X wrt.  $G : \mathbf{D} \to \mathbf{C}$ then for each  $B \in |\mathbf{D}|$  there is a bijection  $(-)^{\sharp} : \hom_{\mathbf{C}}(X, GB) \cong \hom_{\mathbf{D}}(A, B)$ 

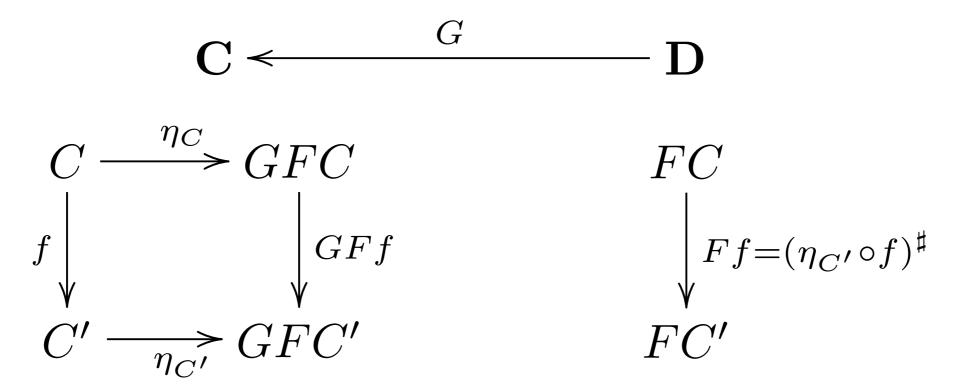
#### Free objects are functorial

Consider a functor  $G : \mathbf{D} \to \mathbf{C}$ . If every  $C \in |\mathbf{C}|$  has a free object  $FC \in |\mathbf{D}|$  wrt. Gthen the mapping  $C \mapsto FC$ 

$$f: C \to C' \quad \mapsto \quad (\eta_{C'} \circ f)^{\sharp}$$

defines a functor  $F : \mathbf{C} \to \mathbf{D}$ .

Further,  $\eta : \mathrm{Id}_C \to GF$  is a natural transformation.



# Left adjoints

Defn. A functor  $F : \mathbb{C} \to \mathbb{D}$  is left adjoint to  $G : \mathbb{D} \to \mathbb{C}$ with unit  $\eta : \mathrm{Id}_{\mathbb{C}} \to GF$  if for every  $C \in |\mathbb{C}|$ , FC with  $\eta_C$  is free over C wrt. G.

**Examples:** 

- the free monoid functor is left adjoint to  $G:\mathbf{Mon}\to\mathbf{Sets}$
- the path category functor is left adjoint to  $U:\mathbf{Cat}\to\mathbf{Graph}$
- "the" coproduct functor  $+: \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is left adjoint to the diagonal functor  $\Delta: \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ .

Exercise: What is a left adjoint to a monotonic function between posets?

## Facts about left adjoints

Theorem: Left adjoints to any fixed G, if they exist, are unique up to natural isomorphism.

**Theorem:** If F is left adjoint to G, then:

- F preserves colimits, - G preserves limits.

Theorem: Let D be locally small & complete (ie. have all limits). A functor  $G : \mathbf{D} \to \mathbf{C}$  has a left adjoint if and only if:

- G preserves limits,

- for every  $C \in |\mathbf{C}|$  there exists a set (ie not a proper class)  $\{f_i : C \to GD_i \mid i \in \mathcal{I}\}$  of arrows such that for each  $D \in \mathbf{D}$  and  $f : C \to GD$ , there exist  $i \in \mathcal{I}$  and  $g : D_i \to D$  such that  $f = Gg \circ f_i$ .