Introduction to universal algebra

Defn: signature Σ =

- a set Σ of operation symbols
- an arity function $\mathrm{ar} : \Sigma \to \mathbb{N}$.

Defn: A Σ-algebra (or a model for Σ) A is:

- a set $|A|$ (the carrier),
- for each $f \in \Sigma$, a function $f^A : |A|^{\mathrm{ar}(f)} \to |A|$.

Defn: A Σ-algebra homomorphism $h : A \to B$

is a function $h : |A| \to |B|$ such that

$$h(f^A(a_1, \ldots, a_n)) = f^B(h(a_1), \ldots, h(a_n))$$

for each $f \in \Sigma$ and $a_1, \ldots, a_n \in |A|$.

Fact: Σ-algebras and their morphisms form a category Σ-$\mathbf{alg}$.

Terms, equations, theories

Defn: The set $T_\Sigma X$ of Σ-terms over X is defined by induction:

- if $x \in X$ then x is a term,
- if $f \in \Sigma$ and $t_1, \ldots, t_{\mathrm{ar}(f)}$ are terms then $f(t_1, \ldots, t_{\mathrm{ar}(f)})$ is a term.

Fact: for a Σ-algebra A,

every $g : X \to |A|$ extends uniquely to $g^\sharp : T_\Sigma X \to |A|$.

Defn: A Σ-equation is a pair of Σ-terms, written $s = t$.

A theory is a set of equations.

Defn: An equation $s = t$ holds in an algebra A

if for every $g : X \to |A|$ there is $g^\sharp(s) = g^\sharp(t)$.

Defn: An algebra A is a model (algebra) of a theory $\mathcal{T}$ if every equation in $\mathcal{T}$ holds in A.

Defn: $\mathcal{T}$-$\mathbf{alg}$ - the full subcategory of Σ-$\mathbf{alg}$ with models of $\mathcal{T}$ as objects

Generated and free algebras

Defn: An algebra A is generated by $g : X \to |A|$ if

$$\forall a \in |A|.\ \exists t \in T_\Sigma X.\ g^\sharp(t) = a$$

Defn: A $\mathcal{T}$-algebra A is free over X (with $g : X \to |A|$) if:

- A is generated by g, "no junk"
- no equations hold in A except those provable from $\mathcal{T}$. "no confusion "

Fact: For every $\mathcal{T}$ and X, a free $\mathcal{T}$-algebra over X exists.

Fact: If A is free over X (with $g : X \to |A|$) then for any $\mathcal{T}$-algebra B, every $h : X \to |B|$ extends uniquely to a homomorphism from A to B .

Free monoids

Defn. The free monoid over a set X is the set of finite sequences:

$$X^* = \bigcup_{n \in \mathbb{N}} X^n$$

with concatenation as multiplication and $\epsilon = \langle\rangle$ as unit.

Defn. The free commutative monoid over X

is the set of functions $f : X \to \mathbb{N}$

with pointwise addition as multiplication

and the constantly zero function as unit.

Similarly: free groups, free rings, free lattices etc.

Freedom vs. initiality

Roughly:

Monoid M is initial if
for every monoid N
there is a unique monoid morphism from M to N.

Monoid M is free over a set X if
every interpretation of X to a monoid N
extends to a unique monoid morphism from M to N.

For example, the monoid free over $\emptyset$ is initial.

Free objects

Consider a functor $G : \mathbf{D} \to \mathbf{C}$.

Defn. Given an object X in $\mathbf{C}$, a free object over X w.r.t. G is an object A in $\mathbf{D}$ with an arrow $\eta_X : X \to GA$ in $\mathbf{C}$ (the unit arrow) such that

for every B in $\mathbf{D}$ with an arrow $f : X \to GB$
there exists a unique arrow $f^\sharp : A \to B$ s.t. $Gf^\sharp \circ \eta_X = f$.

$$
\begin{tikzcd}
\mathbf{C} & & & \mathbf{D} \arrow[lll, "G"'] \\
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & GA \arrow[d, "Gf^\sharp"] & & A \arrow[d, dotted, "\exists!"', "f^\sharp"] \\
& GB & & B
\end{tikzcd}
$$