Complexity Theory
Lecture 4

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http://www.cl.cam.ac.uk/teaching/0910/Complexity/

Satisfiability

For Boolean expressions ϕ that contain variables, we can ask

> Is there an assignment of truth values to the variables which would make the formula evaluate to `true`?

The set of Boolean expressions for which this is true is the language SAT of *satisfiable* expressions.

This can be decided by a deterministic Turing machine in time $O(n^2 2^n)$.

An expression of length n can contain at most n variables.

For each of the 2^n possible truth assignments to these variables, we check whether it results in a Boolean expression that evaluates to `true`.

Is SAT $\in$ P?

Composites

Consider the decision problem (or *language*) Composite defined by:

$$\{x \mid x \text{ is not prime}\}$$

This is the complement of the language Prime.

Is Composite $\in$ P?

Clearly, the answer is yes if, and only if, Prime $\in$ P.

Hamiltonian Graphs

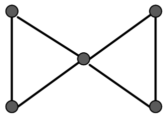
Given a graph $G = (V, E)$, a *Hamiltonian cycle* in G is a path in the graph, starting and ending at the same node, such that every node in V appears on the cycle *exactly once*.

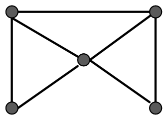
A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

The language HAM is the set of encodings of Hamiltonian graphs.

Is HAM $\in$ P?

Examples





The first of these graphs is not Hamiltonian, but the second one is.

Polynomial Verification

The problems Composite, SAT and HAM have something in common.

In each case, there is a *search space* of possible solutions.

> the factors of x; a truth assignment to the variables of ϕ; a list of the vertices of G.

The number of possible solutions is *exponential* in the length of the input.

Given a potential solution, it is *easy* to check whether or not it is a solution.

Verifiers

A verifier V for a language L is an algorithm such that

$$L = \{x \mid (x, c) \text{ is accepted by } V \text{ for some } c\}$$

If V runs in time polynomial in the length of x, then we say that

> L is *polynomially verifiable*.

Many natural examples arise, whenever we have to construct a solution to some design constraints or specifications.

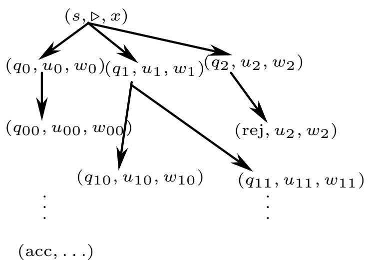
Nondeterministic Complexity Classes

We have already defined $\mathsf{TIME}(f(n))$ and $\mathsf{SPACE}(f(n))$.

$\mathsf{NTIME}(f(n))$ is defined as the class of those languages L which are accepted by a *nondeterministic* Turing machine M, such that for every $x \in L$, there is an accepting computation of M on x of length at most $f(n)$.

$$\mathsf{NP} = \bigcup_{k=1}^{\infty} \mathsf{NTIME}(n^k)$$

Nondeterminism



For a language in $\mathsf{NTIME}(f(n))$, the height of the tree is bounded by $f(n)$ when the input is of length n.

NP

A language L is polynomially verifiable if, and only if, it is in NP.

To prove this, suppose L is a language, which has a verifier V, which runs in time $p(n)$.

The following describes a *nondeterministic algorithm* that accepts L

1. input x of length n
2. nondeterministically guess c of length $\leq p(n)$
3. run V on (x, c)

NP

In the other direction, suppose M is a nondeterministic machine that accepts a language L in time n^k.

We define the *deterministic algorithm* V which on input (x, c) simulates M on input x.

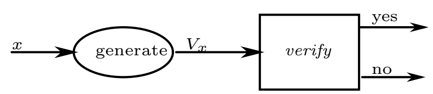
At the i^{th} nondeterministic choice point, V looks at the i^{th} character in c to decide which branch to follow.

If M accepts then V accepts, otherwise it rejects.

V is a polynomial verifier for L.

Generate and Test

We can think of nondeterministic algorithms in the generate-and test paradigm:



Where the *generate* component is nondeterministic and the *verify* component is deterministic.

Reductions

Given two languages $L_1 \subseteq \Sigma_1^\star$, and $L_2 \subseteq \Sigma_2^\star$,

A *reduction* of L_1 to L_2 is a *computable* function

$$f : \Sigma_1^\star \to \Sigma_2^\star$$

such that for every string $x \in \Sigma_1^\star$,

$$f(x) \in L_2 \text{ if, and only if, } x \in L_1$$