Lecture 2

Least Fixed Points

Thesis

All domains of computation are partial orders with a least element.

All computable functions are mononotic.

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

Monotonicity

- A function $f : D \to E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

Pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{\mathit{fix}(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f) \tag{lfp1}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \ \Rightarrow\ \mathit{fix}(f) \sqsubseteq d. \tag{lfp2}$$

Proof principle

Let D be a poset and let $f : D \to D$ be a monotone function with a least pre-fixed point $\mathit{fix}(f) \in D$.

For all $x \in D$, to prove that

$$\mathit{fix}(f) \sqsubseteq x$$

it is enough to establish that

$$f(x) \sqsubseteq x$$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

Thesis*

All domains of computation are complete partial orders with a least element.

All computable functions are continuous.

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \dots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \tag{lub1}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \tag{lub2}$$

A domain is a cpo that possesses a least element, $\bot$:

$$\forall d \in D \,.\, \bot \sqsubseteq d.$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $\mathit{dom}(f) \subseteq X$ and taking values in Y.

Partial order:

$$\begin{array}{lll} f \sqsubseteq g & \text{iff} & \mathit{dom}(f) \subseteq \mathit{dom}(g) \text{ and} \\ & & \forall x \in \mathit{dom}(f).\ f(x) = g(x) \\ & \text{iff} & \mathit{graph}(f) \subseteq \mathit{graph}(g) \end{array}$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \dots$ is the partial function f with $\mathit{dom}(f) = \bigcup_{n \geq 0} \mathit{dom}(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \mathit{dom}(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Least element $\bot$ is the totally undefined partial function.

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.
2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ in D,

   $$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

   for all $N \in \mathbb{N}$.
3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \dots \sqsubseteq e_n \sqsubseteq \dots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ $(m, n \geq 0)$ satisfies*

$$m \leq m' \;\&\; n \leq n' \;\Rightarrow\; d_{m,n} \sqsubseteq d_{m',n'}. \tag{†}$$

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{1,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{2,n} \;\sqsubseteq\; \dots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,1} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,3} \;\sqsubseteq\; \dots$$

Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right) .$$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that
$$f\Big(\bigsqcup_{n\geq 0} d_n\Big) = \bigsqcup_{n\geq 0} f(d_n) \quad \text{in } E.$$
- If D and E have least elements, then the function f is strict iff $f(\bot) = \bot$.

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by
$$\mathit{fix}(f) = \bigsqcup_{n\geq 0} f^n(\bot).$$
- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f\big(\mathit{fix}(f)\big) = \mathit{fix}(f)$, and hence is the least fixed point of f.

$\llbracket \mathbf{while}\ B\ \mathbf{do}\ C \rrbracket$

$$\begin{aligned}
&\llbracket \mathbf{while}\ B\ \mathbf{do}\ C \rrbracket \\
&\quad = \mathit{fix}(f_{\llbracket B \rrbracket, \llbracket C \rrbracket}) \\
&\quad = \textstyle\bigsqcup_{n\geq 0} {f_{\llbracket B \rrbracket, \llbracket C \rrbracket}}^{n}(\bot) \\
&\quad = \lambda s \in \mathit{State}. \\
&\qquad\quad \begin{cases}
\llbracket C \rrbracket^k(s) & \text{if } \forall\, 0 \leq i < k \in \mathbb{N}.\ \llbracket B \rrbracket(\llbracket C \rrbracket^i(s)) = \mathit{true} \\
 & \text{and } \llbracket B \rrbracket(\llbracket C \rrbracket^k(s)) = \mathit{false} \\
\uparrow & \text{if } \forall\, i \in \mathbb{N}.\ \llbracket B \rrbracket(\llbracket C \rrbracket^i(s)) = \mathit{true}
\end{cases}
\end{aligned}$$