Lecture 1

Introduction

What is this course about?

- General area.

  Formal methods: Mathematical techniques for the specification, development, and verification of software and hardware systems.

- Specific area.

  Formal semantics: Mathematical theories for ascribing meanings to computer languages.

Why do we care?

- Rigour.

  ... specification of programming languages

  ... justification of program transformations

- Insight.

  ... generalisations of notions computability

  ... higher-order functions

  ... data structures

- Feedback into language design.

  ... continuations

  ... monads

- Reasoning principles.

  ... Scott induction

  ... Logical relations

  ... Co-induction

Styles of formal semantics

Operational.

Meanings for program phrases defined in terms of the *steps of computation* they can take during program execution.

Axiomatic.

Meanings for program phrases defined indirectly via the *axioms and rules* of some logic of program properties.

Denotational.

Concerned with giving *mathematical models* of programming languages. Meanings for program phrases defined abstractly as elements of some suitable mathematical structure.

Basic idea of denotational semantics

Syntax	$\xrightarrow{[\![-]\!]}$	Semantics
Recursive program	$\mapsto$	Partial recursive function
Boolean circuit	$\mapsto$	Boolean function
P	$\mapsto$	$[\![P]\!]$

Concerns:

- Abstract models (*i.e.* implementation/machine independent).
  $\rightsquigarrow$ Lectures 2, 3 and 4.
- Compositionality.
  $\rightsquigarrow$ Lectures 5 and 6.
- Relationship to computation (*e.g.* operational semantics).
  $\rightsquigarrow$ Lectures 7 and 8.

Characteristic features of a denotational semantics

- Each phrase (= part of a program), P, is given a denotation, $[\![P]\!]$ — a mathematical object representing the contribution of P to the meaning of *any* complete program in which it occurs.
- The denotation of a phrase is determined just by the denotations of its subphrases (one says that the semantics is compositional).

Basic example of denotational semantics (I)

IMP syntax

Arithmetic expressions

$$A ::= \underline{n} \mid L \mid A + A \mid \ldots$$

where n ranges over *integers* and L over a specified set of *locations* $\mathbb{L}$

Boolean expressions

$$B ::= \mathbf{true} \mid \mathbf{false} \mid A = A \mid \ldots \mid \neg B \mid \ldots$$

Commands

$$C ::= \mathbf{skip} \mid L := A \mid C; C \mid \mathbf{if}\ B\ \mathbf{then}\ C\ \mathbf{else}\ C \mid \mathbf{while}\ B\ \mathbf{do}\ C$$

Basic example of denotational semantics (II)

Semantic functions

$$
\begin{aligned}
\mathcal{A}: &\quad \mathbf{Aexp} \to (\mathit{State} \to \mathbb{Z}) \\
\mathcal{B}: &\quad \mathbf{Bexp} \to (\mathit{State} \to \mathbb{B}) \\
\mathcal{C}: &\quad \mathbf{Comm} \to (\mathit{State} \rightharpoonup \mathit{State})
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{Z} &= \{\ldots, -1, 0, 1, \ldots\} \\
\mathbb{B} &= \{\, \mathit{true}, \mathit{false} \,\} \\
\mathit{State} &= (\mathbb{L} \to \mathbb{Z})
\end{aligned}
$$

Basic example of denotational semantics (III)

Semantic function $\mathcal{A}$

$$\mathcal{A}[\![\underline{n}]\!] = \lambda s \in \mathit{State}.\, n$$

$$\mathcal{A}[\![L]\!] = \lambda s \in \mathit{State}.\, s(L)$$

$$\mathcal{A}[\![A_1 + A_2]\!] = \lambda s \in \mathit{State}.\, \mathcal{A}[\![A_1]\!](s) + \mathcal{A}[\![A_2]\!](s)$$

Basic example of denotational semantics (IV)

Semantic function $\mathcal{B}$

$$
\begin{aligned}
\mathcal{B}[\![\mathbf{true}]\!] &= \lambda s \in \mathit{State}.\, \mathit{true} \\
\mathcal{B}[\![\mathbf{false}]\!] &= \lambda s \in \mathit{State}.\, \mathit{false} \\
\mathcal{B}[\![A_1 = A_2]\!] &= \lambda s \in \mathit{State}.\, eq\big(\mathcal{A}[\![A_1]\!](s), \mathcal{A}[\![A_2]\!](s)\big)
\end{aligned}
$$

where $eq(a, a') = \begin{cases} \mathit{true} & \text{if } a = a' \\ \mathit{false} & \text{if } a \neq a' \end{cases}$

Basic example of denotational semantics (V)

Semantic function $\mathcal{C}$

$$[\![\mathbf{skip}]\!] = \lambda s \in \mathit{State}.\, s$$

NB: From now on the names of semantic functions are omitted!

A simple example of compositionality

Given partial functions $[\![C]\!], [\![C']\!] : \mathit{State} \rightharpoonup \mathit{State}$ and a function $[\![B]\!] : \mathit{State} \to \{\mathit{true}, \mathit{false}\}$, we can define

$$[\![\textbf{if } B \textbf{ then } C \textbf{ else } C']\!] = \lambda s \in \mathit{State}.\, \mathit{if}\big([\![B]\!](s), [\![C]\!](s), [\![C']\!](s)\big)$$

where

$$\mathit{if}(b, x, x') = \begin{cases} x & \text{if } b = \mathit{true} \\ x' & \text{if } b = \mathit{false} \end{cases}$$

Basic example of denotational semantics (VI)

Semantic function $\mathcal{C}$

$$[\![L := A]\!] \quad = \quad \lambda s \in \mathit{State}.\, \lambda \ell \in \mathbb{L}.\, \mathit{if}\big(\ell = L, [\![A]\!](s), s(\ell)\big)$$

Denotational semantics of sequential composition

Denotation of sequential composition $C; C'$ of two commands

$$[\![C; C']\!] = [\![C']\!] \circ [\![C]\!] = \lambda s \in \mathit{State}.\, [\![C']\!]\big([\![C]\!](s)\big)$$

given by composition of the partial functions from states to states $[\![C]\!], [\![C']\!] : \mathit{State} \rightharpoonup \mathit{State}$ which are the denotations of the commands.

Cf. operational semantics of sequential composition:

$$\frac{C, s \Downarrow s' \quad C', s' \Downarrow s''}{C; C', s \Downarrow s''}\,.$$

Fixed point property of $[\![\textbf{while } B \textbf{ do } C]\!]$

$$[\![\textbf{while } B \textbf{ do } C]\!] = f_{[\![B]\!],[\![C]\!]}([\![\textbf{while } B \textbf{ do } C]\!])$$

where, for each $b : \mathit{State} \to \{\mathit{true}, \mathit{false}\}$ and $c, w : \mathit{State} \rightharpoonup \mathit{State}$, we define

$$f_{b,c} : (\mathit{State} \rightharpoonup \mathit{State}) \to (\mathit{State} \rightharpoonup \mathit{State})$$

as

$$f_{b,c} = \lambda w \in (\mathit{State} \rightharpoonup \mathit{State}).\, \lambda s \in \mathit{State}.\, \mathit{if}\big(b(s), w(c(s)), s\big).$$

- Why does $w = f_{[\![B]\!],[\![C]\!]}(w)$ have a solution?
- What if it has several solutions—which one do we take to be $[\![\textbf{while } B \textbf{ do } C]\!]$?

Approximating $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$f_{[\![B]\!],[\![C]\!]}{}^{n}(\bot)$$

$$= \lambda s \in State. \begin{cases} [\![C]\!]^{k}(s) & \text{if } \exists\, 0 \le k < n.\ [\![B]\!]([\![C]\!]^{k}(s)) = false \\ & \text{and } \forall\, 0 \le i < k.\ [\![B]\!]([\![C]\!]^{i}(s)) = true \\ \uparrow & \text{if } \forall\, 0 \le i < n.\ [\![B]\!]([\![C]\!]^{i}(s)) = true \end{cases}$$

$$D \stackrel{\text{def}}{=} State \rightharpoonup State$$

- **Partial order $\sqsubseteq$ on D:**

  $w \sqsubseteq w'$ iff for all $s \in State$, if w is defined at s then so is w' and moreover $w(s) = w'(s)$.

  iff the graph of w is included in the graph of w'.

- **Least element $\bot \in D$ w.r.t. $\sqsubseteq$:**

  $\bot$ = totally undefined partial function

  = partial function with empty graph

  (satisfies $\bot \sqsubseteq w$, for all $w \in D$).