Ch. 5 Inductive definitions

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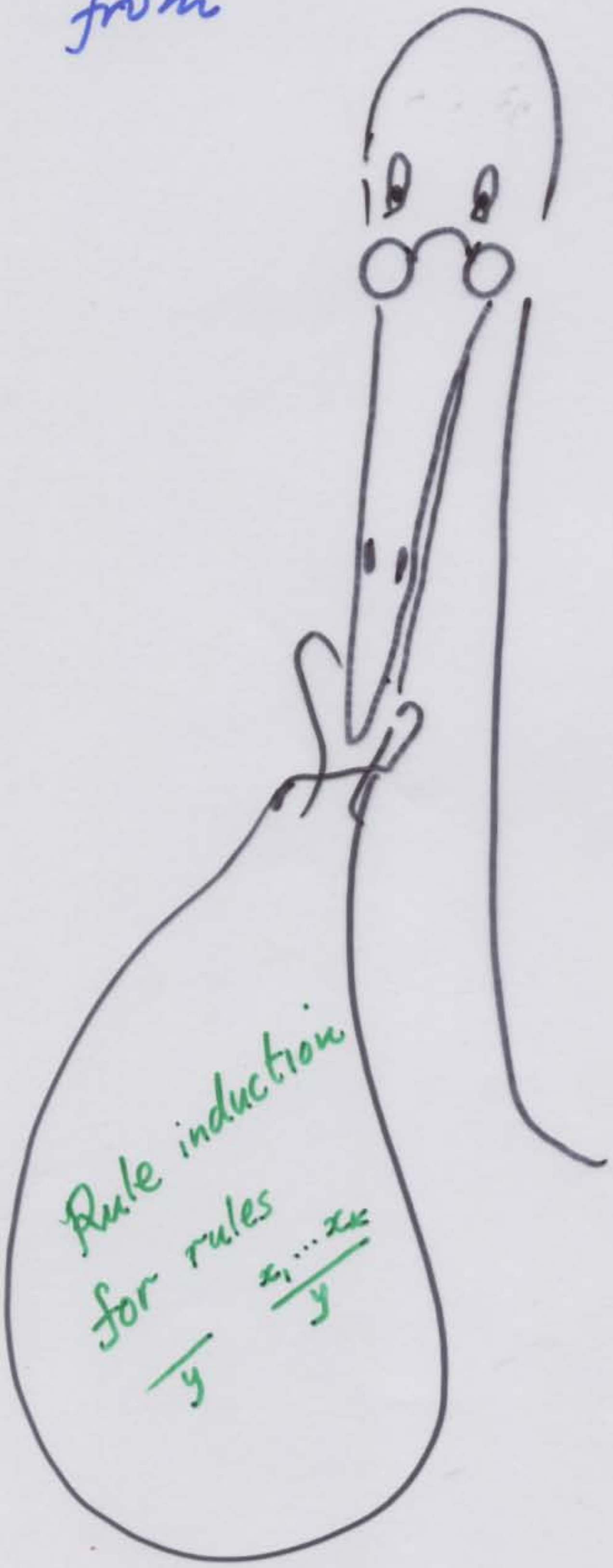

Boolean propositions from rules

$$A, B, \cdots ::= a, b, c, \cdots \mid true \mid false \mid A \wedge B \mid A \vee B \mid \neg A$$

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$$\frac{}{a}\; a \in Var \qquad \frac{}{true} \qquad \frac{}{false}$$

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$\neg a \wedge (b \vee \mathrm{true})$ a boolean propn.?

A derivation:

$$\dfrac{\dfrac{\dfrac{}{a}}{\neg a} \qquad \dfrac{\dfrac{}{b} \qquad \dfrac{}{\mathrm{true}}}{b \vee \mathrm{true}}}{\neg a \wedge (b \vee \mathrm{true})}$$

$\neg$  $\wedge$ ($b \vee \mathrm{true}$)

a boolean proposition
(assuming  $\notin \mathrm{Var}$) ?

The set of Boolean propositions is the

- set of elements for which there is a derivation [induction on derivations § 5.4]
- set built up by repeatedly applying the rules [least fixed points § 5.6]
- least set closed under the rules. [rule induction § 5.3]

Non-negative integers $\mathbb{N}_0$ from rules

- $0 \in \mathbb{N}_0$
- If $n \in \mathbb{N}_0$, then $n+1 \in \mathbb{N}_0$

$$\frac{}{0} \qquad\qquad \frac{n}{n+1}$$

$\mathbb{N}_0$ is the least set closed under the rules.

Alternative rules for $\mathbb{N}_0$

If $0, 1, \dots, n-1$ are in $\mathbb{N}_0$,

then $n \in \mathbb{N}_0$.

$$\frac{0,\ 1,\ \cdots,\ n-1}{n}$$

Strings Σ^*

Σ is a set of symbols, the alphabet

empty string $\quad \varepsilon \in \Sigma^*$

concatenation $\quad$ If $x \in \Sigma^*$ and $a \in \Sigma$,
then $ax \in \Sigma^*$

$$\frac{}{\varepsilon} \qquad\qquad \frac{x}{a\,x}\ a \in \Sigma$$

An instance of a rule:

$$\frac{x_1,\ x_2,\ \cdots,\ x_i,\ \cdots}{y}$$

Premise

Conclusion

a pair (X/y) where

$$X = \{x_1, x_2, \cdots, x_i, \cdots\}.$$

When X is finite, the rule is finitary.

NB. Can have $X = \emptyset$.

Rule induction:

$$\forall x \in I_R.\ P(x) \quad \text{if}$$

for all rules $(X/y) \in R$ s.t. $X \subseteq I_R$

$$\left(\forall x \in X.\ P(x) \right) \Rightarrow P(y).$$

R a set of rules

A set Q is R-closed iff

$$\forall (X/y) \in R.\quad X \subseteq Q \Rightarrow y \in Q$$

Define

$$I_R = \bigcap \{ Q \mid Q \text{ is } R\text{-closed} \}$$

need non-empty; is $\because$ R is a set.

Proposition 5.3

(i) I_R is R-closed

(ii) Q is R-closed $\Rightarrow$ $I_R \subseteq Q$.

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Define set inductively defined by R

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Transitive closure of a relation

Let $R \subseteq U \times U$.

Its transitive closure $R^+ \subseteq U \times U$ is given by:

$$\frac{}{(a,b)}\ (a,b) \in R \qquad \frac{(a,b) \quad (b,c)}{(a,c)}$$

$$R^+ = \{ (a,b) \in U \times U \mid \text{there is an } R\text{-chain from } a \text{ to } b. \}$$

$$a = a_1 \, R \, a_2 \, R \, a_3 \cdots R \, a_n = b$$

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$?\supseteq?$

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$R \; a_{n+1}$

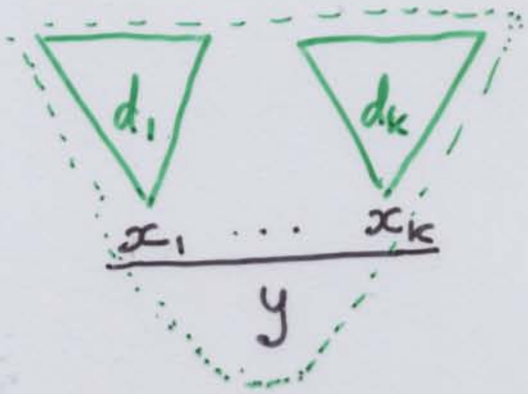
$R^* = R^+ \cup id_U$ reflexive, transitive closure.

Rule instances R

$$\frac{}{y} \qquad\qquad \frac{x_1 \quad \cdots \quad x_k}{y}$$

Rule instances R

R-derivations:



Rule instances R

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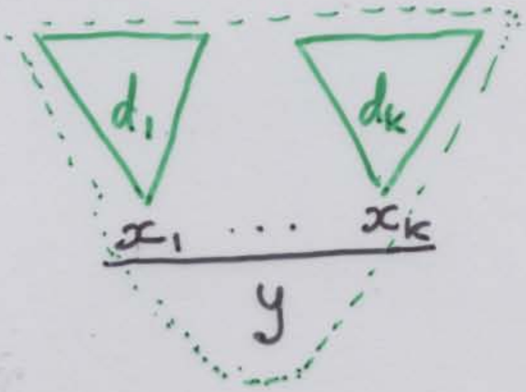

– rules for building derivations!

↳ Induction on derivations:

$P(d)$ for all R-derivations d
if for all rule instances $\{x_1, \dots, x_k\}/y$ in R
and R-derivations d_1 of $x_1, \dots, d_k$ of x_k

$$P(d_1) \,\&\, \cdots \,\&\, P(d_k) \Rightarrow P(\{d_1, \dots, d_k\}/y).$$

Important when property $P(d)$ depends on <u>whole</u> <u>derivation d.</u>

Theorem 5.15 $\quad I_R = \{ y \mid \exists \text{ R-derivation } d \text{ of } y \}.$

Alphabet $a, b, c, d, \dots \in \Sigma$.

The subset of Σ^* of 'words' is given by the rules:

(1) ab is a word;

(2) if ax is a word, then axx is a word;

(3) if $abbbx$ is a word, then ax is a word.

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Set of rules (rule instances):

$$R = \{(\emptyset/ab)\} \cup \{(\{ax\}/axx) \mid x \in \Sigma^*\} \cup \{(\{abbbx\}/ax) \mid x \in \Sigma^*\}.$$

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* The set of words is the least subset of Σ^* for which (1), (2) & (3), i.e. closed under the rules.