Ch. 4. Constructions on Sets

cf. constructions on types
in prog. languages.

$$R = \{x \mid x \notin x\}$$

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$$R \in R \Rightarrow R \notin R$$

$$R \notin R \Rightarrow R \in R \qquad \lightning\ !$$

Sets as extensions of properties

$$\{x \mid P(x)\}$$

a property

always a set ?

$\lightning$ Russell's paradox (really a contradiction)

A safe way to build sets :

$$\{x \in S \mid P(x)\}$$

a set   a property

But this needs sufficiently big sets S

By fiat: $\mathbb{N}$ is a set

powerset $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is a set

Further constructions on sets

Let A, B be sets

Their **product**

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

ordered pair:

$(a,b) = (a',b')$ iff $a = a'$ and $b = b'$.

Can be realised as a set

$$(a,b) = \{ \{a\}, \{a,b\} \}$$

Their **disjoint union** (or disjoint sum)

$$A \uplus B = (\{1\} \times A) \cup (\{2\} \times B)$$

$$\underset{(1,a)}{\Psi} \qquad \underset{(2,b)}{\Psi}$$

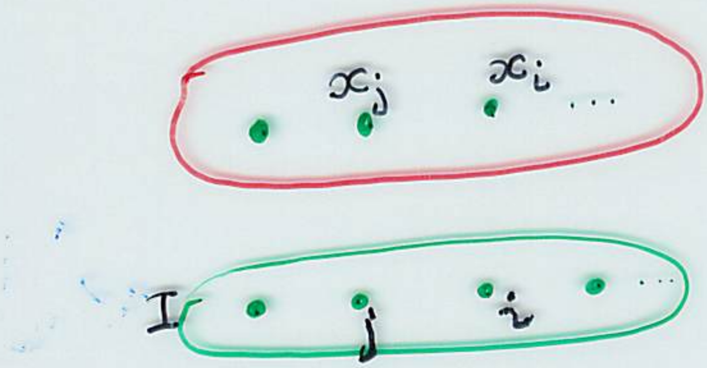
Indexed sets:

Let I be a set s.t.

x_i is an element for all $i \in I$

Then,

$\{x_i \mid i \in I\}$ is a set.



Let I be a set s.t.

A_i is a set for all $i \in I$.

The big union

$$\bigcup_{i \in I} A_i \quad [\text{or } \bigcup \{A_i \mid i \in I\}]$$

$$= \{x \mid \exists i \in I.\ x \in A_i\} \text{ is a set.}$$

The big intersection

$$\bigcap_{i \in I} A_i \quad [\text{or } \bigcap \{A_i \mid i \in I\}]$$

$\forall i.\ i \in \emptyset \Rightarrow x \in A_i$

$$= \{x \mid \forall i \in I.\ x \in A_i\}.$$

[Needs I nonempty, or universe $\mathcal{U}$ so empty intersections are $\mathcal{U}$.]

Cantor's diagonal argument revisited

Let X be a set. There is no bijection $\theta : X \to \mathcal{P}(X)$.

Proof By contradiction. Suppose

$$\theta : X \to \mathcal{P}(X)$$

is a bijection.

Define

$$Y = \{ x \in X \mid x \notin \theta(x) \}.$$

$$\begin{array}{c|ccccccc}
 & \cdots & x & \cdots & y & \cdots \\
\hline
\vdots & & \vdots & & \vdots & \\
\theta(x) & \cdots & \boxed{T} & \cdots & F & \cdots \\
\vdots & & \vdots & & \vdots & \\
\theta(y) & \cdots & F & \cdots & \boxed{F} & \cdots \\
\vdots & & \vdots & & \vdots &
\end{array}$$

As θ is a bijection, there is $y \in X$ such that $\theta(y) = Y$. But ...

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As θ is a bijection, there is $y \in X$ such that $\theta(y) = Y$. But ...

$$y \in Y \Rightarrow y \in \theta(y) \Rightarrow y \notin Y$$

$$y \notin Y \Rightarrow y \notin \theta(y) \Rightarrow y \in Y$$

– contradiction!

$\square$

Some Consequences

- set of relations between sets X and Y:

$$\mathcal{P}(X \times Y).$$

- set of partial functions from set X to set Y:

$$(X \rightharpoonup Y) = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a partial fn.} \}.$$

- set of total functions from set X to set Y:

$$(X \rightarrow Y) = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a function} \}.$$

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$$(X \to \{T, F\}) \cong \mathcal{P}(X)$$

i.e. powerset is definable from function space; a subset of X corresponding to a characteristic function.

Higher Order Logic:

If $(x \in X \Rightarrow e \in Y)$, then

$$\lambda x \in X.\, e \quad \in (X \longrightarrow Y)$$

$$= \{ (x, e) \mid x \in X \}$$