Ch. 4. Constructions on Sets

cf. constructions on types in prog. languages.
\[ R = \{ x \mid x \notin x^3 \} \]

Is \( R \cap R \neq R \)?

\[
\begin{align*}
R \cap R &= R \\
R \cup R &= R
\end{align*}
\]
\[ R = \{ x \mid x \not\in x \} \]

\[ R \in R \quad ? \]

\[ R \in R \Rightarrow R \not\in R \]

\[ R \not\in R \Rightarrow R \in R \]

\[ \not\exists \]

Sets as extensions of properties

\[ \{ x \mid P(x) \} \]

always a set?

\[ \mathcal{S} \] Russell's paradox (really a contradiction)

A safe way to build sets:

\[ \{ x \in S \mid P(x) \} \]

a set a property

But this needs sufficiently big sets \( S \)

By fiat: \( \mathbb{N} \) is a set

power set: \( \mathcal{P}(X) = \{ A \mid A \subseteq X \} \) is a set
Further constructions on sets

Let $A, B$ be sets

Their product

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

ordered pair:

$$(a, b) = (a', b') \iff a = a' \text{ and } b = b'.$$

Can be realised as a set

$$(a, b) = \{ \{ a \}, \{ a, b \} \}$$

Their disjoint union (or disjoint sum)

$$A \uplus B = (\{ 1 \} \times A) \cup (\{ 2 \} \times B)$$

$$\uplus_{(1, a)} \uplus_{(2, b)}$$
Indexed sets:

Let $I$ be a set s.t. $x_i$ is an element for all $i \in I$.

Then, $\{x_i \mid i \in I\}$ is a set.
Let $I$ be a set s.t.

$A_i$ is a set for all $i \in I$.

The big union

$$\bigcup_{i \in I} A_i \quad \text{[or } \bigcup \{A_i \mid i \in I\} \text{]}$$

$$= \{x \mid \exists i \in I. \ x \in A_i\} \text{ is a set.}$$

The big intersection

$$\bigcap_{i \in I} A_i \quad \text{[or } \bigcap \{A_i \mid i \in I\} \text{]}$$

$$= \{x \mid \forall i \in I. \ x \in A_i\}.$$ 

[Needs $I$ non-empty, or universe $U$ so empty intersections are $U$.]
Cantor's diagonal argument revisited

Let $X$ be a set. There is no bijection $\theta : X \rightarrow P(X)$.

Proof by contradiction. Suppose $\theta : X \rightarrow P(X)$ is a bijection.

Define $Y = \{ x \in X \mid x \notin \theta(x) \}$.

As $\theta$ is a bijection, there is $y \in X$ such that $\theta(y) = Y$. But ...
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As $\theta$ is a bijection, there is $y \in X$ such that $\theta(y) = Y$. But...

$y \in Y \Rightarrow y \in \theta(y) \Rightarrow y \notin Y$

$y \notin Y \Rightarrow y \notin \theta(y) = Y \Rightarrow y \in Y$

- contradiction! $\square$
Some Consequences

set of relations between sets $X$ and $Y$: $\mathcal{P}(X \times Y)$.

set of partial functions from set $X$ to set $Y$: $(X \rightarrow Y) = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a partial fn.} \}$.

set of total functions from set $X$ to set $Y$: $(X \rightarrow Y) = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a function} \}$. 
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$$P(X \times Y).$$

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$$(X \rightarrow \{T, F\}) \cong P(X)$$

i.e., powerset is definable from function space; a subset of $X$ corresponding to a characteristic function.
Higher Order Logic:

If \((x \in X \Rightarrow e \in Y)\), then

\[ \lambda x \in X. e \in (X \rightarrow Y) \]

\[ = \{ (x, e) \mid x \in X \} \]