Ch 3.

Relations & functions

- The extension of a property $P(x)$, $x \in \mathcal{U}$, is a set $\{x \in \mathcal{U} \mid P(x)\}$.

  What is the extension of a property

  $$P(x,y), \quad x \in \mathcal{U}, \; y \in V,$$

  or $$P(x,y,z), \quad x \in \mathcal{U}, \; y \in V, \; z \in W \;?$$

- In the 19c. it was recognised that functions? were often, but not always, associated with expressions e.g. x^2+3 for $x \in \mathbb{R}$.

Pairs & Products

$\{a, b\}$ unordered pair of a, b.

(a, b) ordered pair of a, b.

$(a, b) = (a', b') \Leftrightarrow a = a'$ & $b = b'$.

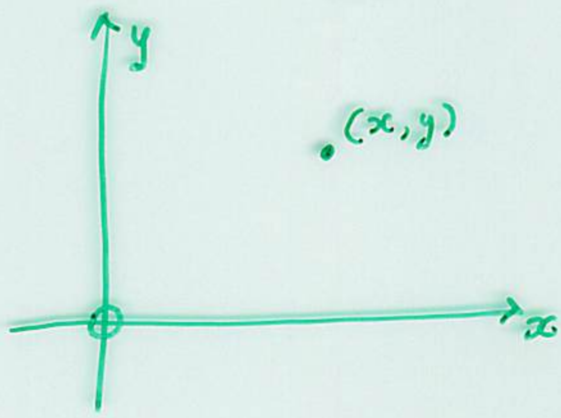
We could define $(a, b) =_{\text{def}} \{ \{a\}, \{a, b\} \}$.

The product of sets X and Y

$$X \times Y =_{\text{def}} \{ (a, b) \mid a \in X \text{ \& } b \in Y \}$$

Eg.

$\mathbb{R} \times \mathbb{R}$



Some laws:

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$$

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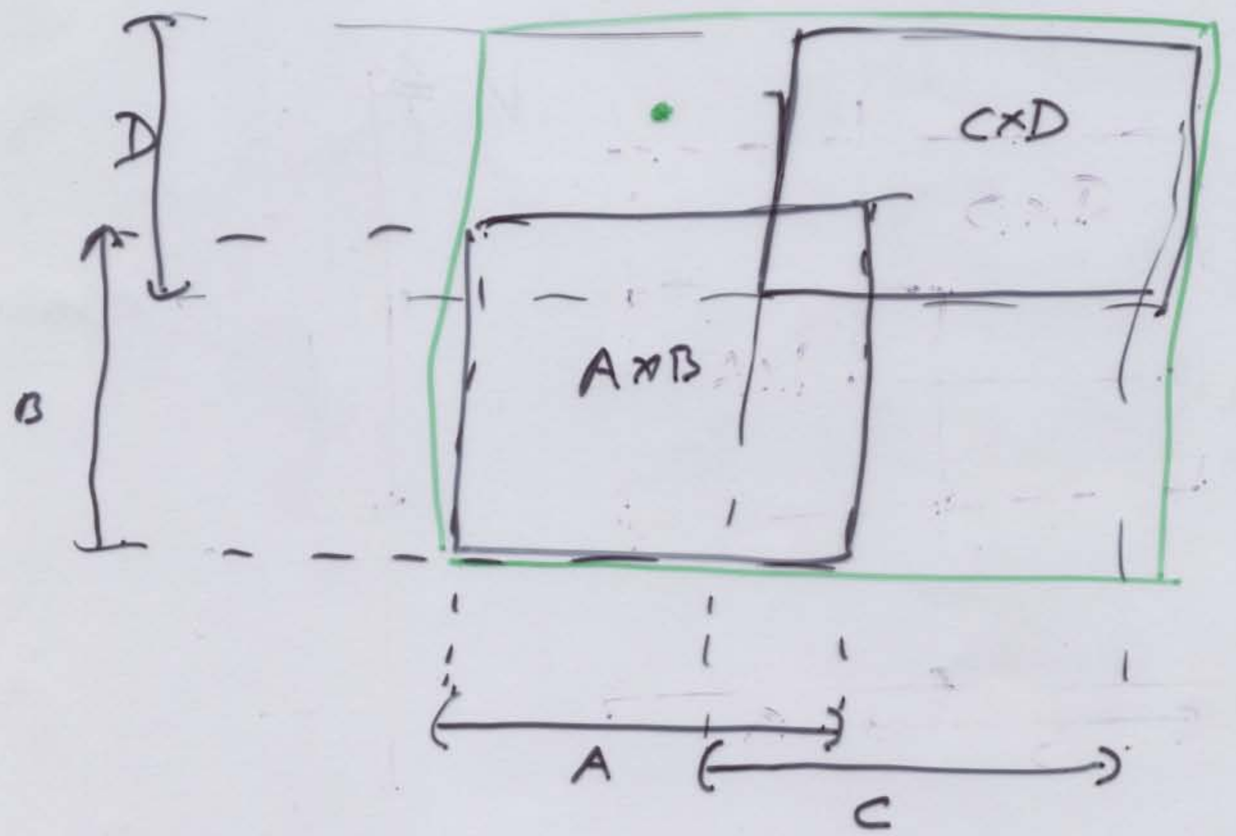
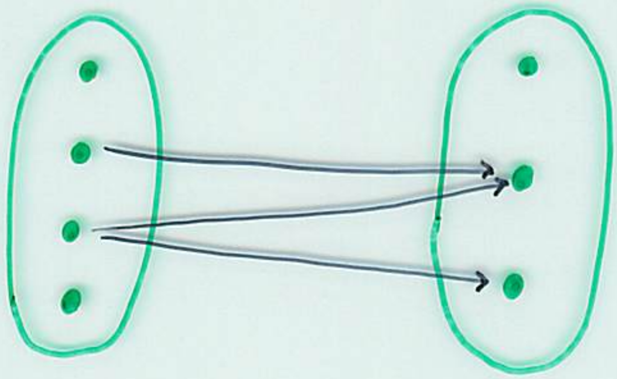


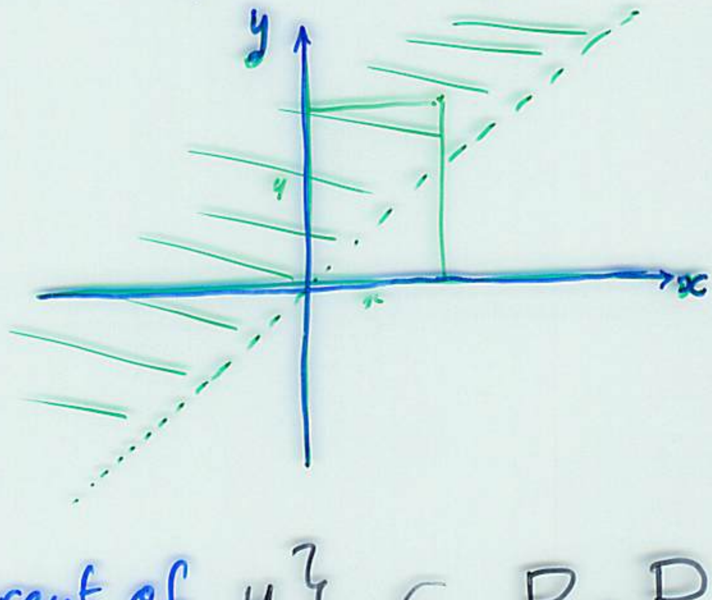
A binary relation between sets X, Y is a subset $R \subseteq X \times Y$



$(x, y) \in R$ often written $x \, R \, y$.

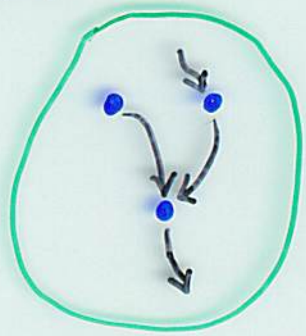
E.g.

- $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x < y\}$



- $\{(x,y) \mid x \text{ is parent of } y\} \subseteq P \times P$

P is set of people



A function from a set X to set Y

is a relation $f \subseteq X \times Y$

such that:

(1) $(x, y) \in f$ & $(x, y') \in f \Rightarrow y = y'$

for all $x \in X$, $y, y' \in Y$;

(2) $\forall x \in X \; \exists y \in Y. \; (x, y) \in f$

Write $f(x)$ for the unique y s.t. $(x, y) \in f$.

A partial function from X to Y is
a relation $f \subseteq X \times Y$ s.t. (1).

$$f : X \to Y$$

A function from a set X to set Y is a relation $f \subseteq X \times Y$ such that:

(1) $(x, y) \in f$ & $(x, y') \in f \Rightarrow y = y'$ for all $x \in X$, $y, y' \in Y$;

(2) $\forall x \in X \; \exists y \in Y. \; (x, y) \in f$

Write $f(x)$ for the unique y s.t. $(x, y) \in f$.

$$f : X \rightharpoonup Y$$

A partial function from X to Y is a relation $f \subseteq X \times Y$ s.t. (1).

Composing relations and functions.

$$R \subseteq X \times Y \qquad S \subseteq Y \times Z$$

Their composition:

$$S \circ R =_{def} \{ (x,z) \in X \times Z \mid \exists y \in Y.\ (x,y) \in R \ \&\ (y,z) \in S \}$$

Identity:

$$id_X \subseteq X \times X$$

$$id_X =_{def} \{ (x,x) \mid x \in X \}$$

Associativity:

$R \subseteq X \times Y,\ S \subseteq Y \times Z,\ T \subseteq Z \times W$

$$T \circ (S \circ R) = (T \circ S) \circ R$$

Composing relations and functions.

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Identity:

$$id_X \subseteq X \times X$$

$$id_X =_{def} \{ (x,x) \mid x \in X \}$$

Associativity:

$$R \subseteq X \times Y, \quad S \subseteq Y \times Z, \quad T \subseteq Z \times W$$

$$T \circ (S \circ R) = (T \circ S) \circ R$$

Composition of functions / partial fns is a function / partial function.

Special functions

Let $f : X \to Y$.

f is injective (1-1) iff
$\forall x, x' \in X.\ f(x) = f(x') \Rightarrow x = x'$

f is surjective (onto) iff
$\forall y \in Y\ \exists x \in X.\ y = f(x)$.

f is bijective (1-1 correspondence) iff
f is injective and surjective.

Proposition 3.9
$f : X \to Y$ is bijective iff it has an inverse
i.e. $g : Y \to X$ s.t. $g(f(x)) = x$ forall $x \in X$
and $f(g(y)) = y$ forall $y \in Y$.

Special functions

Let $f : X \to Y$.

f is injective (1-1) iff $\forall x, x' \in X.\ f(x) = f(x') \Rightarrow x = x'$

injective function = injection

f is surjective (onto) iff $\forall y \in Y\ \exists x \in X.\ y = f(x)$.

surjective function = surjection

f is bijective (1-1 correspondence) iff f is injective and surjective.

bijective fn. = bijection

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Direct and inverse image

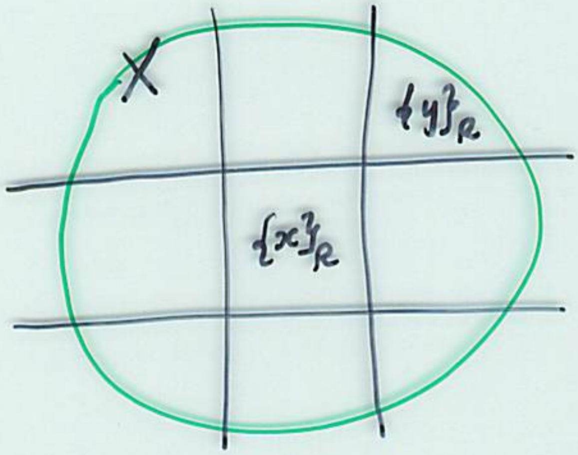
$$R \subseteq X \times Y$$

Let $A \subseteq X$. Its direct image under R

$$RA = \{ y \in Y \mid \exists x \in A.\ (x,y) \in R \}$$

Let $B \subseteq Y$. Its inverse image under R

$$R^{-1}B = \{ x \in X \mid \exists y \in B.\ (x,y) \in R \}$$



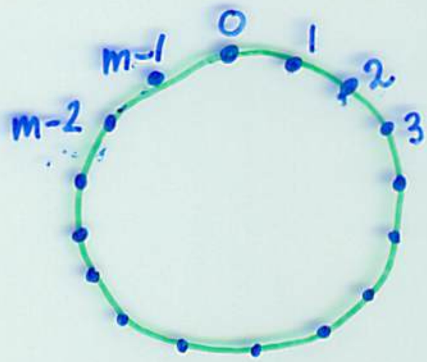
Partition:

- $x \in \{x\}_R$
- $\{x\}_R \cap \{y\}_R \neq \emptyset \Rightarrow \{x\}_R = \{y\}_R$

(1) $\{x\}_R \cap \{y\}_R \neq \emptyset \Rightarrow x R y$

(2) $x R y \Rightarrow \{x\}_R = \{y\}_R$

Examples of equivalence relations



- For $a, b \in \mathbb{Z}$, $m \in \mathbb{N}$

  $$a \equiv b \text{ mod } m$$

  iff m divides $a - b$
  i.e. a and b have same remainder when divided by m. [Ex. 3.18]

- 'Sameness' relations E.g. In this class
  'x and y have the same age'
  'x and y have the same college'.

- Equivalences on states of computation:
  bisimulation [Ex. 3.20], ...

Equivalence relations.

An equivalence relation on a set X is a relation

$$R \subseteq X \times X$$

which is

reflexive: $\forall x \in X.\ x R x$

symmetric: $\forall x, y \in X.\ x R y \Rightarrow y R x$

transitive: $\forall x, y, z \in X.\ x R y \,\&\, y R z \Rightarrow x R z$

Let $x \in X$. Its equivalence class

$$\{x\}_R =_{def} \{ y \in X \mid y R x \}$$

Theorem 3.13

$\{ \{x\}_R \mid x \in X \}$ is a partition of the set X.

Relations as structure – other examples

Directed graphs (X, R) where $R \subseteq X \times X$.

Partial orders $(P, \leq)$ where $\leq \subseteq P \times P$

s.t.

refl. $p \leq p$

tran. $p \leq q \;\&\; q \leq r \;\Rightarrow\; p \leq r$

antisym. $p \leq q \;\&\; q \leq p \;\Rightarrow\; p = q$

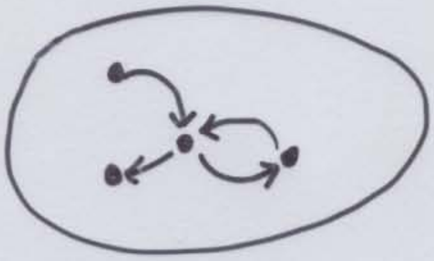
Cf. $\subseteq$ on sets

least upper bounds $\bigvee$ (cf. $\cup$)

greatest lower bounds $\bigwedge$ (cf. $\cap$)

Relations as structure – other examples

Directed graphs (X, R) where $R \subseteq X \times X$.



Partial orders $(P, \leq)$ where $\leq \subseteq P \times P$

s.t.

refl. $p \leq p$

tran. $p \leq q$ & $q \leq r \underset{\supseteq ?}{\Rightarrow} p \leq r$

antisym. $p \leq q$ & $q \leq p \underset{\subseteq ?}{\Rightarrow} p = q$

Cf. $\subseteq$ on sets

(1) least upper bounds $\vee$ zRx & zRy ∴ (cf. xRy $\cup$)

greatest lower bounds $\wedge$ xRz & zRy (cf. $\cap$)

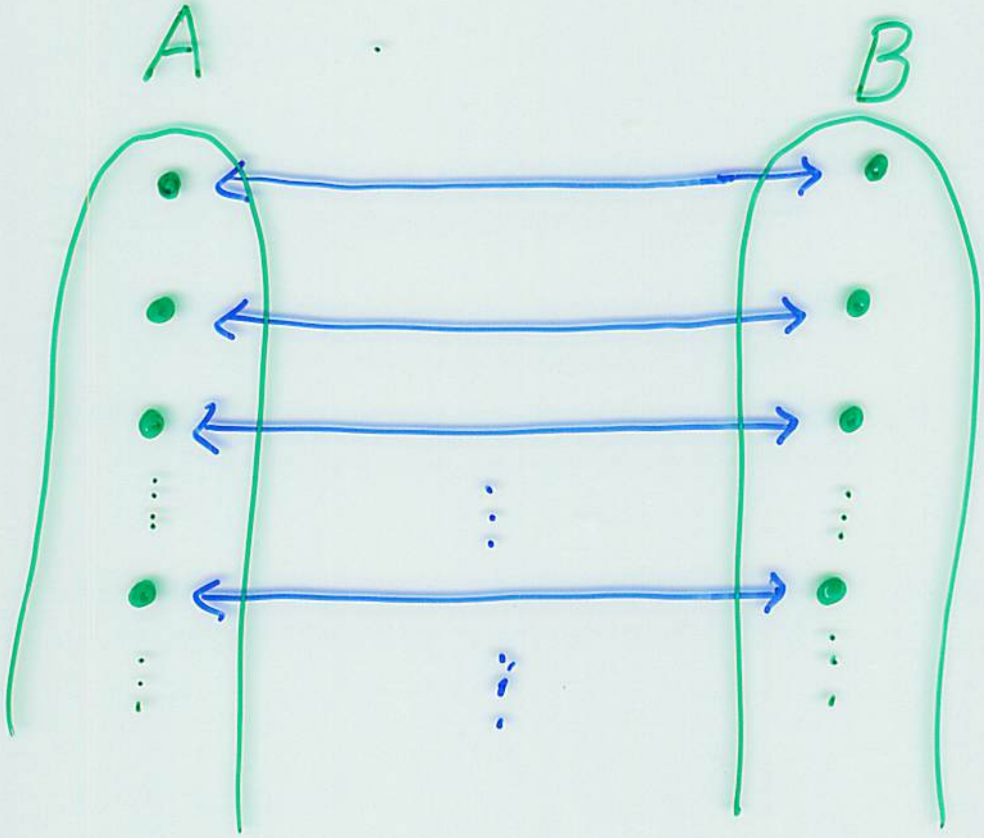
(2) Assume xRy.

Let $z \in \{x\}_R$ i.e. zRx. ∴ zRy ∴ $z \in \{y\}_R$.

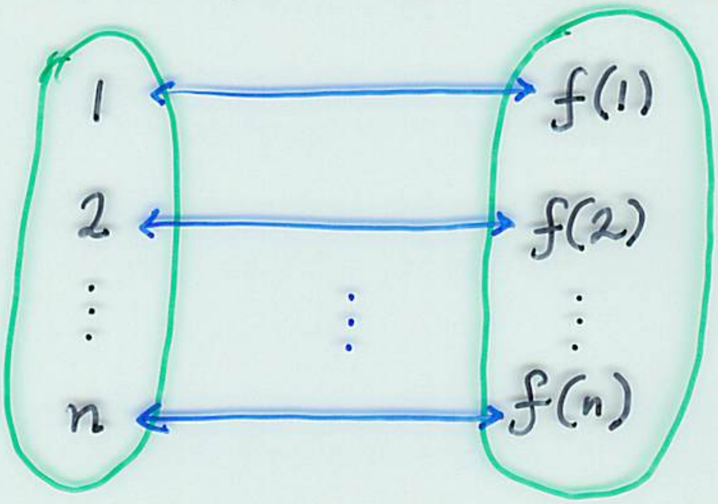
Let $w \in \{y\}_R$, i.e. wRy. Have yRx. ∴ wRx i.e. $w \in \{x\}_R$.

Size of sets
– countability.

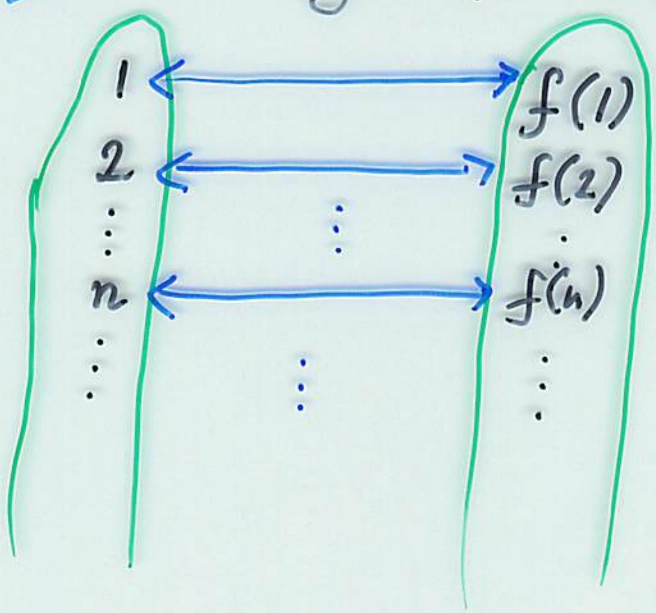
Two sets have the same size (or cardinality) iff there is a bijection between them :



A set A is *finite* iff there is a bijection $f : \{m \in \mathbb{N} \mid m \leq n\} \xrightarrow{\sim} A$ for some $n \in \mathbb{N}_0$



A set A is *countable* iff A is finite or there is a bijection $f : \mathbb{N} \xrightarrow{\sim} A$.



Lemma 3.25 Any subset A of $\mathbb{N}$ is countable.

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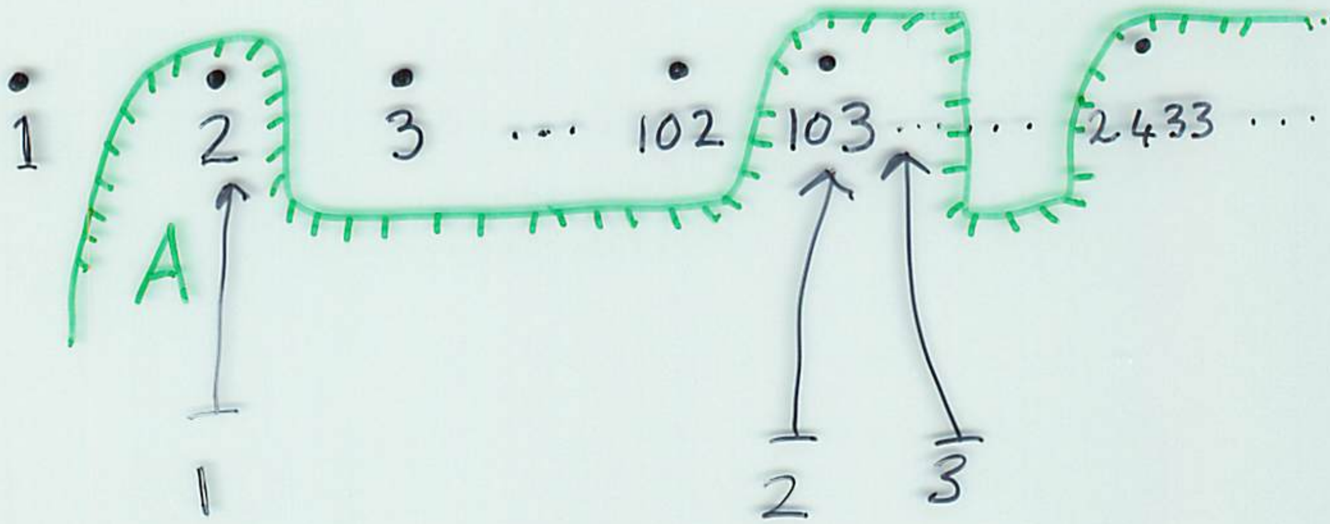
Proof idea:

Define $f : \mathbb{N} \longrightarrow A$ by mathl. ind.

$f(1)$ is least element of A if $A \neq \emptyset$;
undefined otherwise.

$f(n+1)$ is least element of A above $f(n)$ if $f(n)$ is defined & there is a member of A above $f(n)$;
undefined otherwise.



1
2
3
...
102
103
...
2433
...
A
1
2
3

The composition of

injections / surjections / bijections

is an

injection / surjection / bijection.

$$A \xrightarrow{f} B \xrightarrow{g} C$$

The composition of

injections / surjections / bijections

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$\rightarrowtail$ $\quad$ $\twoheadrightarrow$ $\quad$ $\xrightarrow{\simeq}$ (or $\cong$)

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Corollary 3.26

A set B is countable iff there is a bijection $g : A \xrightarrow{\sim} B$ where $A \subseteq \mathbb{N}$.

Lemma 3.27

A set B is countable iff there is an injection $f : B \rightarrowtail \mathbb{N}$.

Lemma 3.28 (VERY USEFUL!)

A set B is countable iff there is an injection $f : B \rightarrowtail A$ where A is countable.

In particular, a subset of a countable set is countable.

3.26 'only if' $\checkmark$

'if'

$$g \circ f : D \xrightarrow[\cong]{f} \underset{\displaystyle A}{\overset{\mathbb{N}}{\cup\!\mid}} \xrightarrow[\cong]{g} B$$

$$\| \\ \{1, 2, \ldots n, \ldots\}$$

3.27 'only if'

$$\underset{\displaystyle A}{\overset{\mathbb{N}}{\cup\!\mid}} \xrightarrow[\cong]{g} B$$

$$\underset{\displaystyle A}{\overset{\mathbb{N}}{\cup\!\mid}} \xleftarrow[\cong]{} B$$

$f = g^{-1}$

$f : B \rightarrowtail \mathbb{N}$ $\checkmark$

3.28 'only if' $\checkmark$

'if' $\quad f' \circ f : B \xrightarrow{f} A \underset{3.27}{\overset{f'}{\rightarrowtail}} \mathbb{N}$

ctble

Lemma 3.29 The set $\mathbb{N} \times \mathbb{N}$ is countable.

Corollary 3.30 The set $\mathbb{Q}^{+}$ is countable

Lemma 3.32 Suppose $A_1, A_2, \cdots, A_n, \cdots$ are countable sets. Their union

$$\bigcup_{n \in \mathbb{N}} A_n = \{ x \mid \exists n \in \mathbb{N}.\ x \in A_n \}$$ is countable.

Lemma 3.29 The set $\mathbb{N} \times \mathbb{N}$ is countable.

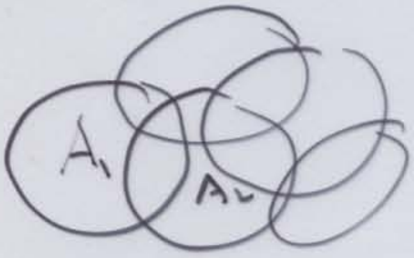
$f : \mathbb{N} \times \mathbb{N} \rightarrowtail \mathbb{N} \qquad f(m,n) = 2^m \times 3^n$

Corollary 3.30 The set $\mathbb{Q}^+$ is countable

$f : \mathbb{Q}^+ \rightarrowtail \mathbb{N} \times \mathbb{N} \qquad f(m/n) = (m,n)$

Lemma 3.32 Suppose $A_1, A_2, \cdots, A_n, \cdots$ are countable sets. Their union

$\bigcup_{n \in \mathbb{N}} A_n = \{ x \mid \exists n \in \mathbb{N}.\ x \in A_n \}$ is countable.



$A_n \xrightarrow{f_n} \mathbb{N}$

$h : \bigcup_{n \in \mathbb{N}} A_n \rightarrowtail \mathbb{N} \times \mathbb{N}$

$x \in \bigcup_{n \in \mathbb{N}} A_n$ least $x \in A_{n_x}$

$h(x) = (n_x, f_{n_x}(x))$

Assume $h(x) = h(y)$. $\therefore (n_x, f_{n_x}(x)) = (n_y, f_{n_y}(y))$

$\therefore \quad n_x = n_y = n$ say

$f_n(x) = f_n(y)$

$\therefore \quad x = y$ as f_n is injective.

Cor $\mathbb{Z} = \{0\} \cup \mathbb{N} \cup \{-n \mid n \in \mathbb{N}\}$ is countable.

Cor. 3.31.

A, B ctble

$\Rightarrow$ $A \times B$ ctble

Proof. $f_A : A \rightarrowtail \mathbb{N}$

$f_B : B \rightarrowtail \mathbb{N}$

? $f : A \times B \rightarrowtail \mathbb{N} \times \mathbb{N}$?

$f(a, b) = (f_A(a), f_B(b))$

Suppose $f(a, b) = f(a', b')$.

Then $(f_A(a), f_B(b)) = (f_A(a'), f_B(b'))$

$\therefore$ $f_A(a) = f_A(a')$ & $f_B(b) = f_B(b')$

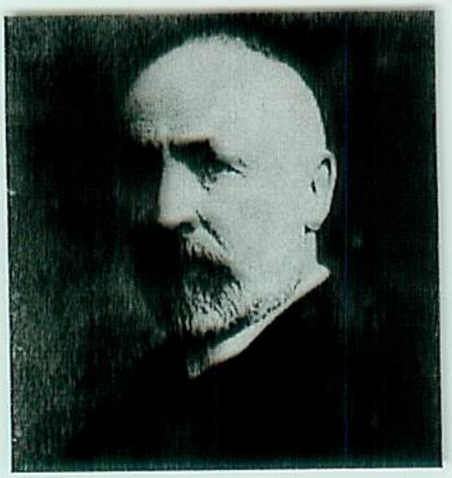
$\therefore$ $a = a'$ & $b = b'$

$\therefore$ $(a, b) = (a', b')$. $\square$

B ctble $\Leftarrow$

$B \overset{f}{\rightarrowtail} A$ ctble

unctble $\Rightarrow$ A unctble



Georg Cantor

Size of sets

'Diagonal argument'

Theorem 3.37 $\mathbb{R}$ is uncountable.

Theorem 3.37. $\mathbb{R}$ is uncountable.

Proof. By contradiction.

Assume $\mathbb{R}$ is countable.

Then $(0,1]$ is countable.

$$
\begin{array}{lllllll}
f(1) = & 0. & d_1^1 & d_2^1 & d_3^1 & \cdots & d_i^1 & \cdots \\
f(2) = & 0. & d_1^2 & d_2^2 & d_3^2 & \cdots & d_i^2 & \cdots \\
f(3) = & 0. & d_1^3 & d_2^3 & d_3^3 & \cdots & d_i^3 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \\
f(n) = & 0. & d_1^n & d_2^n & d_3^n & \cdots & d_i^n & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots &
\end{array}
$$

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\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
f(n) = & 0.\ d_1^n & d_2^n & d_3^n & \cdots & d_i^n & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots &
\end{array}
$$

$$r = 0.\ r_1\ r_2\ r_3\ \cdots\ r_i\ \cdots$$

$$r_i = \begin{cases} 1 & \text{if } d_i^i \neq 1 \\ 2 & \text{if } d_i^i = 1. \end{cases}$$

Via $(0,1] =_{def} \{ r \in \mathbb{R} \mid 0 < r \leq 1 \}$

is uncountable.

But $S =_{def} \{ s \in (0,1] \mid s \text{ can be expressed by a finite decimal} \}$

...

An algebraic number is a solution to a polynomial equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

where the coefficients $a_0, \ldots, a_n \in \mathbb{Z}$.

A transcendental is a real number which is not algebraic.

There are (uncountably many) transcendental numbers!

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There are (uncountably many) transcendental numbers!

$$\mathbb{Z} \cup \mathbb{Z}\times\mathbb{Z} \cup \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \cup \cdots$$

$$\cup \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{n} \cup \cdots$$

e, π

$$\underset{\text{uncttble}}{\mathbb{R}} = \underset{\text{cttble}}{\mathrm{Alg}} \cup \mathrm{Trans}$$

uncttble!