Discrete Mathematics:
Set theory for Computer Science

Supplementary material:
Ch 1 Mathematical argument & notation
Mathematical induction
Corrected version + additional exercises on my home page.
Please inform me of errors, confusing parts.

Mini seminars next term

Make sure you have a supervisor now!

Questions?
The mathematical statement

\[5 \text{ divides } 2^{3n+1} + 3^{n+1}\]

for all non-negative numbers \(n\)

says

\[5 \text{ divides } 2^{3.0+1} + 3^{0+1},\]
\[5 \text{ divides } 2^{3.1+1} + 3^{1+1},\]
\[5 \text{ divides } 2^{3.2+1} + 3^{2+1},\]
\[
\ldots\]
\[5 \text{ divides } 2^{3.185+1} + 3^{185+1},\]
\[
\ldots\]

Is it true?
falls
falls
falls
falls
'Domino theory' [McCarthy, Kennedy, Nixon, Kissinger, ...]

\[0 \quad 1 \quad 2 \quad \ldots \quad n \quad n+1 \quad \ldots\]

non-negative integers

\[ n \quad \Rightarrow \quad n+1 \]

\[0\]

\[\Rightarrow \quad \text{For all } n, \quad n\]
MATHEMATICAL INDUCTION

\[ P(\ ) \quad P(\ ) \quad P(\ ) \]

\[ P(\ ) \]

\[ P(\ ) \]
The principle of mathematical induction

To prove a property \( P(x) \) for all nonnegative integers \( x \), it suffices to prove

- the basis \( P(0) \)
- the induction step \( P(n) \Rightarrow P(n+1) \) for all nonnegative integers \( n \).

\[ P(x) \text{ is called the induction hypothesis} \]
Proposition. 5 divides $2^{3n+1} + 3^{n+1}$ for all nonnegative integers $n$.

Proof. By mathematical induction with induction hypothesis $D(n)$ that 5 divides $2^{3n+1} + 3^{n+1}$.

Basis: $2^{3*0+1} + 3^{0+1} = 2 + 3 = 5$ which is divisible by 5. Thus $D(0)$.

Induction step: Assume $D(n)$, where $n$ is a nonnegative integer.

$2^{3(n+1)+1} + 3^{(n+1)+1} = 2^{3n+4} + 3^{n+2} = 2^3(2^{3n+1} + 3^{n+1}) - 8.3^{n+1} + 3.3^{n+1} = 2^3(2^{3n+1} + 3^{n+1}) - 5.3^{n+1}$ by $D(n)$ this is divisible by 5.
"Avoiding the dots"

Definition by mathematical induction

To define a function $f(x)$ on all nonnegative integers $x$ it suffices to define

- $f(0)$ the function on 0
- $f(n+1)$ in terms of $f(n)$ for all nonnegative integers $n$.

Example. Factorial

$$n! = n \cdot (n-1) \cdots 2 \cdot 1$$

is defined by

$$0! = 1$$

$$(n+1)! = (n+1) \cdot n!$$

[The dots (ellipses) '...' can sometimes be a source of vagueness.]
Example.

W.r.t. \( x_0, x_1, \ldots, x_n, \ldots \)

the sum

\[
\sum_{i=0}^{n} x_i = x_0 + \ldots + x_n.
\]

Its definition by mathematical induction:

\[
\sum_{i=0}^{0} x_i = x_0,
\]

\[
\sum_{i=0}^{n+1} x_i = \left(\sum_{i=0}^{n} x_i\right) + x_{n+1}.
\]
\[ T(n+1) = T(n) + 1 + T(n) \]

So

\[ T(0) = 0 \]

\[ T(n+1) = 2 \cdot T(n) + 1 \]

Exercise: Prove \( T(n) = 2^n - 1 \) for all nonnegative integers \( n \).
Towers of Hanoi
Mathematical induction from basis integer $b$.

To prove a property $P(x)$ for all integers $x \geq b$ it suffices to prove

- the basis $P(b)$
- the induction step $P(n) \Rightarrow P(n+1)$ for all integers $n \geq b$.

Reduces to ordinary mathematical induction with $IH$ $P(x+b)$.

Exercise. Prove $n^2 > 2n$ for all integers $n \geq 3$. 
The Fibonacci numbers

\[ \text{fib}(0) = 0 \]
\[ \text{fib}(1) = 1 \]
\[ \text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2) \]
for \( n > 1 \).

Not quite a definition by mathematical induction!
Derivable from ordinary math.

Induction with IH:

\[ \forall k \ (0 \leq k < x), \ P(k) \]

i.e.

\[ P(0) \ & \ P(1) \ & \ ... \ & \ P(x-1) \]
Course-of-values induction

To prove a property $P(x)$ for all nonnegative integers $x$ it suffices to prove that

- for any nonnegative integer $n$, $P(n)$ follows from $P(0)$, $P(1)$, ..., and $P(n-1)$.
Golden ratio

\[ \frac{a+b}{a} = \frac{a}{b} \]

\[ \varphi = \frac{a}{b} \text{ def} \]

\[ \frac{\varphi}{\varphi + 1} = \varphi \]

\[ \varphi^2 = \varphi + 1 \]

\( \varphi \) is the positive soln. to

\[ x^2 = x + 1 \]

\[ \varphi = \frac{1 + \sqrt{5}}{2} \]

Other soln. \( \bar{\varphi} = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\varphi} \)

Nb. Both solns. satisfy

\[ x^n = x^{n-1} + x^{n-2} \]

for \( n > 1 \).
Proposition. \( \text{fib}(n) = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}} \) for all non-negative integers \( n \).

Proof. By course-of-values induction with induction hypothesis \( P(n) \):

\[
\text{fib}(n) = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}
\]

Case \( n = 0 \). By defn. \( \text{fib}(0) = 0 \). Also

\[
\frac{\varphi^0 - \overline{\varphi}^0}{\sqrt{5}} = 0.
\]

\( \therefore P(0) \)

Case \( n = 1 \). By defn. \( \text{fib}(1) = 1 \). Also

\[
\frac{\varphi^1 - \overline{\varphi}^1}{\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1.
\]

\( \therefore P(1) \)

Case \( n > 1 \). By defn. \( \text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2) \).

By I.H., \( \text{fib}(n) = \frac{\varphi^{n-1} - \overline{\varphi}^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \overline{\varphi}^{n-2}}{\sqrt{5}} \)

\[
= \frac{\varphi^{n-1} + \varphi^{n-2} - (\overline{\varphi}^{n-1} + \overline{\varphi}^{n-2})}{\sqrt{5}}
\]

\[
= \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}
\]

\( \therefore P(n) \) assuming \( P(0), P(1), \ldots, P(n-1) \).