Glynn Winskel

Discrete Mathematics:
Set theory for Computer Science

Supplementary material:

Ch 1 Mathematical argument & notation

Mathematical induction

Corrected version + additional exercises on my home page.

Please inform me of errors, confusing parts.

Mini seminars next term

Make sure you have a supervisor now!

Questions?

The mathematical statement

$$5 \text{ divides } 2^{3n+1} + 3^{n+1}$$

for all nonnegative numbers n

says

$$5 \text{ divides } 2^{3 \cdot 0+1} + 3^{0+1},$$

$$5 \text{ divides } 2^{3 \cdot 1+1} + 3^{1+1},$$

$$5 \text{ divides } 2^{3 \cdot 2+1} + 3^{2+1},$$

$$\dots$$

$$5 \text{ divides } 2^{3 \cdot 185+1} + 3^{185+1},$$

$$\dots$$

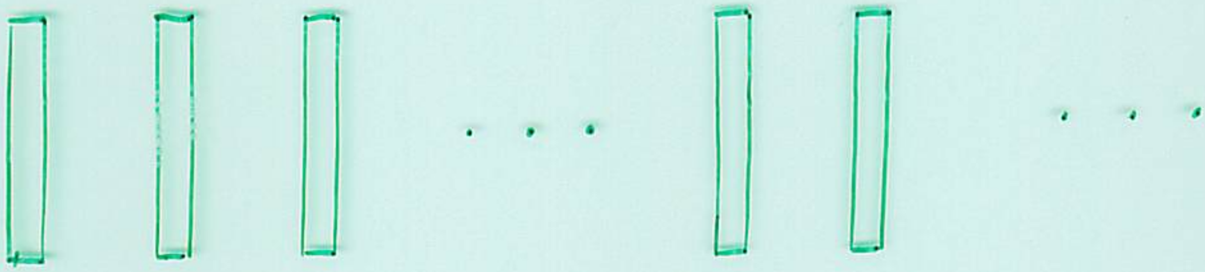
Is it true?

falls falls

falls

falls

'Domino theory' [McCarthy, Kennedy, Nixon, Kissinger, …]



$0 \quad 1 \quad 2 \quad \cdots \quad n \quad n+1 \quad \cdots$

nonnegative integers

$n \qquad \Rightarrow \qquad n+1$

0

$\Rightarrow$ For all n, n

MATHEMATICAL INDUCTION

P( ) P( )

P( )

P( )

The principle of mathematical induction

To prove a property $P(x)$ for all nonnegative integers x it suffices to prove

- the **basis** $P(0)$
- the **induction step** $P(n) \Rightarrow P(n+1)$ for all nonnegative integers n.

[$P(x)$ is called the **induction hypothesis**]

Proposition. 5 divides $2^{3n+1} + 3^{n+1}$ for all nonnegative integers n.

Proof. By mathematical induction with induction hypothesis $D(n)$ that

$$5 \text{ divides } 2^{3n+1} + 3^{n+1}.$$

Basis: $2^{3.0+1} + 3^{0+1} = 2+3 = 5$ which is divisible by 5. Thus $D(0)$

Induction step: Assume $D(n)$, where n is a nonnegative integer.

$$2^{3(n+1)+1} + 3^{(n+1)+1} = 2^{3n+4} + 3^{n+2} =$$

$$2^3\left(2^{3n+1} + 3^{n+1}\right) - 8.3^{n+1} + 3.3^{n+1}$$

$$= 2^3\left(2^{3n+1} + 3^{n+1}\right) - 5.3^{n+1}$$

by $D(n)$ this is divisible by 5.

"Avoiding the dots"

Definition by mathematical induction

To define a function $f(x)$ on all nonnegative integers x

it suffices to define

- $f(0)$ the function on 0
- $f(n+1)$ in terms of $f(n)$ for all nonnegative integers n.

Example. Factorial

$$n! = n.(n-1).\cdots.2.1$$

is defined by

$$0! = 1$$

$$(n+1)! = (n+1).\,n!$$

[The dots (ellipses) '...' can sometimes be a source of vagueness.]

Example.

W.r.t. $x_0, x_1, \cdots, x_n, \cdots$

the sum

$$\sum_{i=0}^{n} x_i = x_0 + \cdots + x_n .$$

Its definition by mathematical induction :

$$\sum_{i=0}^{0} x_i = x_0$$

$$\sum_{i=0}^{n+1} x_i = \left(\sum_{i=0}^{n} x_i\right) + x_{n+1}$$

$$T(n+1) = T(n) + 1 + T(n)$$

Correct notes P.16

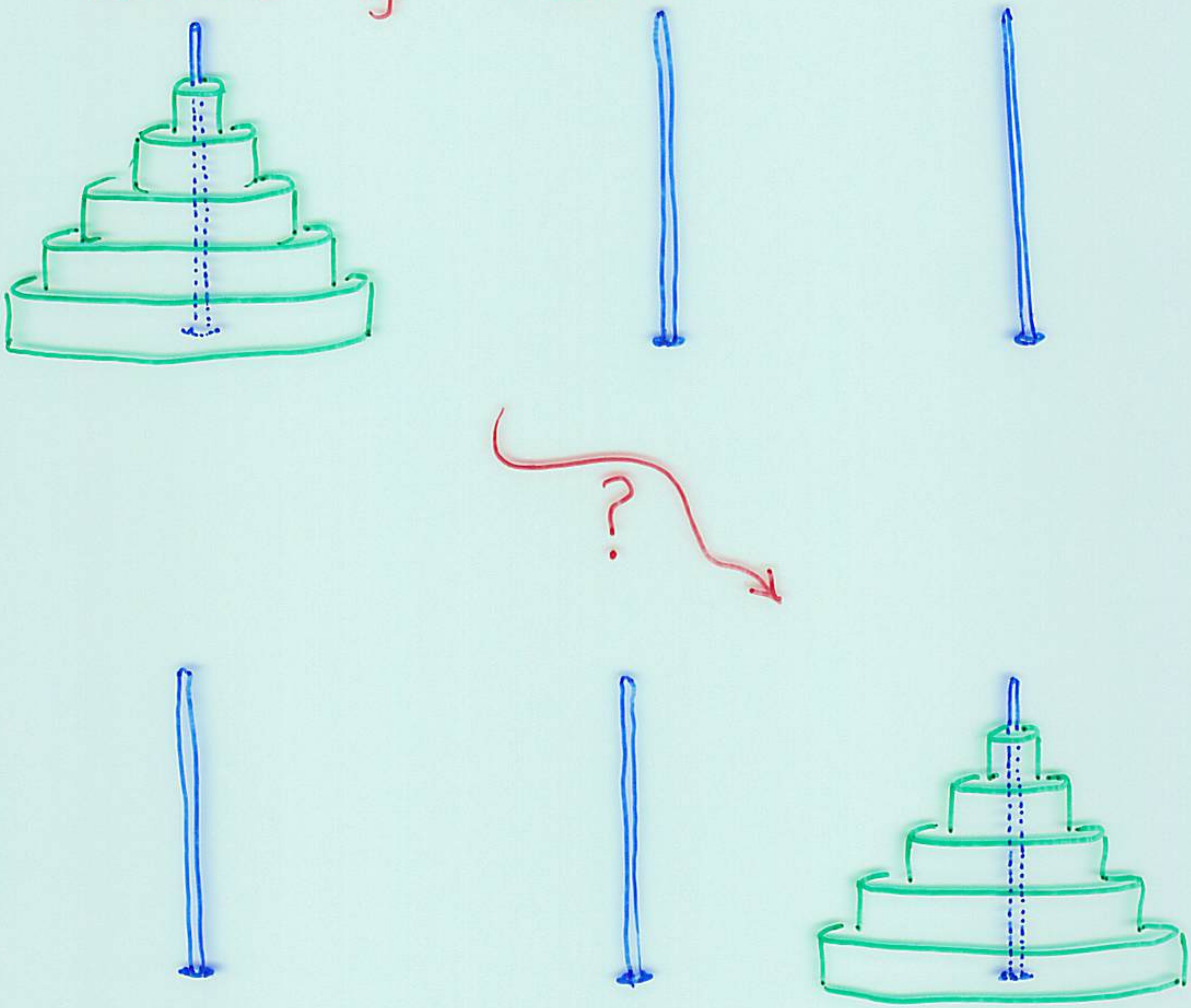
So

$$T(0) = 0$$

$$T(n+1) = 2.T(n) + 1$$

Exercise Prove $T(n) = 2^n - 1$ for all nonnegative integers n.

Towers of Hanoi



Mathematical induction from basis integer b.

To prove a property $P(x)$ for all integers $x \geqslant b$

it suffices to prove

- the basis $P(b)$
- the induction step $P(n) \Rightarrow P(n+1)$ for all integers $n \geqslant b$.

Reduces to ordinary mathematical induction with IH $P(x+b)$.

Exercise. Prove $n^2 > 2n$ for all integers $n \geqslant 3$.

The Fibonacci numbers

$$fib(0) = 0$$

$$fib(1) = 1$$

$$fib(n) = fib(n-1) + fib(n-2)$$

for $n > 1$.

Not quite a definition by mathematical induction!

Derivable from ordinary math. induction with IH

$$\forall k\,(0 \le k < x).\ P(k).$$

ie. $P(0)\ \&\ P(1)\ \&\ \cdots\ \&\ P(x-1)$

Course-of-values induction

To prove a property $P(x)$ for all nonnegative integers x it suffices to prove that

- for any nonnegative integer n $P(n)$ follows from

  $P(0)$, $P(1)$, $\ldots$ and $P(n-1)$.

Golden ratio

$$\frac{a+b}{a} = \frac{a}{b}$$

$$\varphi =_{def} \frac{a}{b}$$

$$\frac{\varphi+1}{\varphi} = \varphi$$

$$\varphi^2 = \varphi + 1$$

φ is the positive soln. to

$$x^2 = x + 1$$

$$\varphi = \frac{1+\sqrt{5}}{2}$$ Other soln. $$\bar{\varphi} = \frac{1-\sqrt{5}}{2} = \frac{-1}{\varphi}$$

Nb. Both solns. satisfy

$$x^n = x^{n-1} + x^{n-2}$$

for $n > 1$.

Proposition. $\mathrm{fib}(n) = \dfrac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$ for all nonnegative integers n.

Proof. By course-of-values induction with induction hypothesis $P(n)$:

$$\mathrm{fib}(n) = (\varphi^n - \bar{\varphi}^n)/\sqrt{5}$$

Case $n = 0$. By defn. $\mathrm{fib}(0) = 0$. Also $(\varphi^0 - \bar{\varphi}^0)/\sqrt{5} = 0$. $\therefore P(0)$

Case $n = 1$. By defn. $\mathrm{fib}(1) = 1$. Also $(\varphi^1 - \bar{\varphi}^1)/\sqrt{5} = \dfrac{2\sqrt{5}}{2\sqrt{5}} = 1$. $\therefore P(1)$

Case $n > 1$. By defn. $\mathrm{fib}(n) = \mathrm{fib}(n-1) + \mathrm{fib}(n-2)$.

By I.H., $\mathrm{fib}(n) = \dfrac{\varphi^{n-1} - \bar{\varphi}^{n-1}}{\sqrt{5}} + \dfrac{\varphi^{n-2} - \bar{\varphi}^{n-2}}{\sqrt{5}}$

$$= \frac{\varphi^{n-1} + \varphi^{n-2} - (\bar{\varphi}^{n-1} + \bar{\varphi}^{n-2})}{\sqrt{5}}$$

$$= \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}.$$

$\therefore P(n)$ assuming $P(0), \ldots P(n-1)$.

$\square$