# Knapsack

KNAPSACK is a problem which generalises many natural scheduling and optimisation problems, and through reductions has been used to show many such problems NP-complete.

In the problem, we are given $n$ items, each with a positive integer value $v_i$ and weight $w_i$.

We are also given a maximum total weight $W$, and a minimum total value $V$.

> Can we select a subset of the items whose total weight does not exceed $W$, and whose total value exceeds $V$?

# Reduction

The proof that KNAPSACK is NP-complete is by a reduction from the problem of Exact Cover by 3-Sets.

Given a set $U = \{1, \ldots, 3n\}$ and a collection of 3-element subsets of $U$, $S = \{S_1, \ldots, S_m\}$.

We map this to an instance of KNAPSACK with $m$ elements each corresponding to one of the $S_i$, and having weight and value

$$\Sigma_{j \in S_i} (m+1)^{j-1}$$

and set the target weight and value both to

$$\Sigma_{j=0}^{3n-1} (m+1)^j$$

# Scheduling

Some examples of the kinds of scheduling tasks that have been proved NP-complete include:

## Timetable Design

Given a set $H$ of *work periods*, a set $W$ of *workers* each with an associated subset of $H$ (available periods), a set $T$ of *tasks* and an assignment $r : W \times T \to \mathbb{N}$ of *required work*, is there a mapping $f : W \times T \times H \to \{0, 1\}$ which completes all tasks?

# Scheduling

## Sequencing with Deadlines

Given a set $T$ of *tasks* and for each task a *length* $l \in \mathbb{N}$, a release time $r \in \mathbb{N}$ and a deadline $d \in \mathbb{N}$, is there a work schedule which completes each task between its release time and its deadline?

## Job Scheduling

Given a set $T$ of *tasks*, a number $m \in \mathbb{N}$ of processors a length $l \in \mathbb{N}$ for each task, and an overall deadline $D \in \mathbb{N}$, is there a multi-processor schedule which completes all tasks by the deadline?

## Responses to NP-Completeness

*Confronted by an* NP*-complete problem, say constructing a timetable, what can one do?*

- It's a single instance, does asymptotic complexity matter?
- What's the critical size? Is scalability important?
- Are there guaranteed restrictions on the input? Will a special purpose algorithm suffice?
- Will an approximate solution suffice? Are performance guarantees required?
- Are there useful heuristics that can constrain a search? Ways of ordering choices to control backtracking?

# Validity

We define $\mathsf{VAL}$—the set of *valid* Boolean expressions—to be those Boolean expressions for which every assignment of truth values to variables yields an expression equivalent to `true`.

$$\phi \in \mathsf{VAL} \quad \Leftrightarrow \quad \neg\phi \notin \mathsf{SAT}$$

By an exhaustive search algorithm similar to the one for $\mathsf{SAT}$, $\mathsf{VAL}$ is in $\mathsf{TIME}(n^2 2^n)$.

Is $\mathsf{VAL} \in \mathsf{NP}$?

# Validity

$\overline{\mathsf{VAL}} = \{\phi \mid \phi \notin \mathsf{VAL}\}$—the *complement* of VAL is in NP.

Guess a a *falsifying* truth assignment and verify it.

Such an algorithm does not work for VAL.

In this case, we have to determine whether *every* truth assignment results in `true`—a requirement that does not sit as well with the definition of acceptance by a nondeterministic machine.

# Complementation

If we interchange accepting and rejecting states in a deterministic machine that accepts the language $L$, we get one that accepts $\overline{L}$.

If a language $L \in \mathsf{P}$, then also $\overline{L} \in \mathsf{P}$.

Complexity classes defined in terms of nondeterministic machine models are not necessarily closed under complementation of languages.

Define,

co-NP – the languages whose complements are in NP.

# Succinct Certificates

The complexity class NP can be characterised as the collection of languages of the form:

$$L = \{x \mid \exists y R(x, y)\}$$

Where $R$ is a relation on strings satisfying two key conditions

1. $R$ is decidable in polynomial time.
2. $R$ is *polynomially balanced*. That is, there is a polynomial $p$ such that if $R(x, y)$ and the length of $x$ is $n$, then the length of $y$ is no more than $p(n)$.

## Succinct Certificates

$y$ is a *certificate* for the membership of $x$ in $L$.

**Example:** If $L$ is $\mathsf{SAT}$, then for a satisfiable expression $x$, a certificate would be a satisfying truth assignment.