Hamiltonian Graphs

Recall the definition of HAM—the language of Hamiltonian graphs.

Given a graph $G = (V, E)$, a *Hamiltonian cycle* in $G$ is a path in
the graph, starting and ending at the same node, such that every
node in $V$ appears on the cycle *exactly once*.

A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

The language HAM is the set of encodings of Hamiltonian graphs.
Hamiltonian Cycle

We can construct a reduction from 3SAT to HAM

Essentially, this involves coding up a Boolean expression as a graph, so that every satisfying truth assignment to the expression corresponds to a Hamiltonian circuit of the graph.

This reduction is much more intricate than the one for IND.
Travelling Salesman

Recall the travelling salesman problem

Given

- $V$ — a set of nodes.
- $c : V \times V \rightarrow \mathbb{N}$ — a cost matrix.

Find an ordering $v_1, \ldots, v_n$ of $V$ for which the total cost:

$$c(v_n, v_1) + \sum_{i=1}^{n-1} c(v_i, v_{i+1})$$

is the smallest possible.
Travelling Salesman

As with other optimisation problems, we can make a decision problem version of the Travelling Salesman problem.

The problem $\text{TSP}$ consists of the set of triples

$$(V, c : V \times V \rightarrow \mathbb{N}, t)$$

such that there is a tour of the set of vertices $V$, which under the cost matrix $c$, has cost $t$ or less.
Reduction

There is a simple reduction from HAM to TSP, mapping a graph \((V, E)\) to the triple \((V, c : V \times V \to \mathbb{N}, n)\), where

\[
c(u, v) = \begin{cases} 
1 & \text{if } (u, v) \in E \\
2 & \text{otherwise}
\end{cases}
\]

and \(n\) is the size of \(V\).
It is not just problems about formulas and graphs that turn out to be \( \text{NP} \)-complete.

Literally hundreds of naturally arising problems have been proved \( \text{NP} \)-complete, in areas involving network design, scheduling, optimisation, data storage and retrieval, artificial intelligence and many others.

Such problems arise naturally whenever we have to construct a solution within constraints, and the most effective way appears to be an exhaustive search of an exponential solution space.

We now examine three more \( \text{NP} \)-complete problems, whose significance lies in that they have been used to prove a large number of other problems \( \text{NP} \)-complete, through reductions.
The decision problem of 3D Matching is defined as:

Given three disjoint sets $X$, $Y$ and $Z$, and a set of triples $M \subseteq X \times Y \times Z$, does $M$ contain a matching?
I.e. is there a subset $M' \subseteq M$, such that each element of $X$, $Y$ and $Z$ appears in exactly one triple of $M'$?

We can show that 3DM is NP-complete by a reduction from 3SAT.
Reduction

If a Boolean expression $\phi$ in 3CNF has $n$ variables, and $m$ clauses, we construct for each variable $v$ the following gadget.

\begin{center}
\includegraphics{gadget.png}
\end{center}
In addition, for every clause $c$, we have two elements $x_c$ and $y_c$. If the literal $v$ occurs in $c$, we include the triple

$$(x_c, y_c, z_{vc})$$

in $M$.

Similarly, if $\neg v$ occurs in $c$, we include the triple

$$(x_c, y_c, \bar{z}_{vc})$$

in $M$.

Finally, we include extra dummy elements in $X$ and $Y$ to make the numbers match up.
Exact Set Covering

Two other well known problems are proved \textbf{NP}-complete by immediate reduction from \textbf{3DM}.

\textit{Exact Cover by 3-Sets} is defined by:

Given a set $U$ with $3n$ elements, and a collection $S = \{S_1, \ldots, S_m\}$ of three-element subsets of $U$, is there a sub collection containing exactly $n$ of these sets whose union is all of $U$?

The reduction from \textbf{3DM} simply takes $U = X \cup Y \cup Z$, and $S$ to be the collection of three-element subsets resulting from $M$. 

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Set Covering

More generally, we have the *Set Covering* problem:

Given a set $U$, a collection of $S = \{S_1, \ldots, S_m\}$ subsets of $U$ and an integer budget $B$, is there a collection of $B$ sets in $S$ whose union is $U$?