Independent Set

Given a graph $G = (V, E)$, a subset $X \subseteq V$ of the vertices is said to be an *independent set*, if there are no edges (u, v) for $u, v \in X$.

The natural algorithmic problem is, given a graph, find the largest independent set.

To turn this *optimisation problem* into a *decision problem*, we define IND as:

> The set of pairs (G, K), where G is a graph, and K is an integer, such that G contains an independent set with K or more vertices.

IND is clearly in NP. We now show it is NP-complete.

Reduction

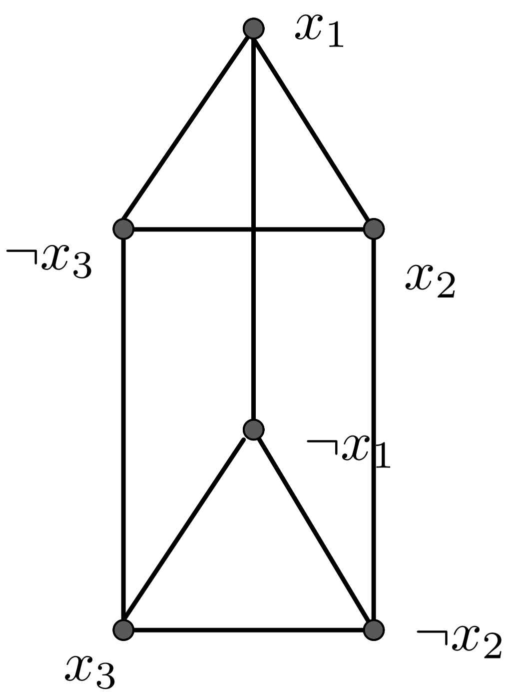
We can construct a reduction from 3SAT to IND.

A Boolean expression ϕ in 3CNF with m clauses is mapped by the reduction to the pair (G, m), where G is the graph obtained from ϕ as follows:

> G contains m triangles, one for each clause of ϕ, with each node representing one of the literals in the clause.
>
> Additionally, there is an edge between two nodes in different triangles if they represent literals where one is the negation of the other.

Example

$$(x_1 \lor x_2 \lor \neg x_3) \land (x_3 \lor \neg x_2 \lor \neg x_1)$$



Clique

Given a graph $G = (V, E)$, a subset $X \subseteq V$ of the vertices is called a *clique*, if for every $u, v \in X$, (u, v) is an edge.

As with IND, we can define a decision problem version:

CLIQUE is defined as:

> The set of pairs (G, K), where G is a graph, and K is an integer, such that G contains a clique with K or more vertices.

Clique 2

CLIQUE is in NP by the algorithm which *guesses* a clique and then verifies it.

CLIQUE is NP-complete, since

IND $\leq_P$ CLIQUE

by the reduction that maps the pair (G, K) to $(\bar{G}, K)$, where $\bar{G}$ is the complement graph of G.

k-Colourability

A graph $G = (V, E)$ is k-colourable, if there is a function

$$\chi : V \to \{1, \ldots, k\}$$

such that, for each $u, v \in V$, if $(u, v) \in E$,

$$\chi(u) \neq \chi(v)$$

This gives rise to a decision problem for each k.

2-colourability is in P.

For all $k > 2$, k-colourability is NP-complete.

3-Colourability

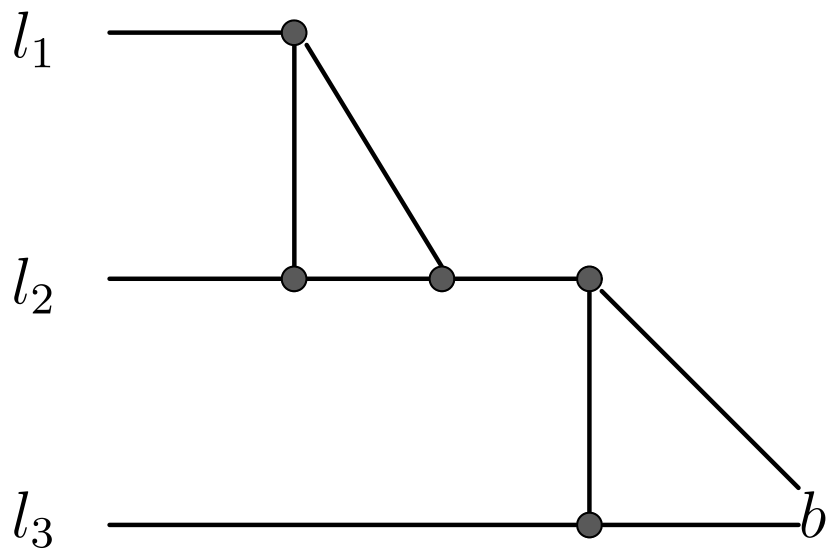
3-Colourability is in NP, as we can *guess* a colouring and verify it.

To show NP-completeness, we can construct a reduction from 3SAT to 3-Colourability.

For each variable x, have two vertices x, $\bar{x}$ which are connected in a triangle with the vertex a (common to all variables).

In addition, for each clause containing the literals l_1, l_2 and l_3 we have a gadget.

Gadget



With a further edge from a to b.