Polynomial Verification

The problems Composite, SAT and HAM have something in common.

In each case, there is a *search space* of possible solutions.

> the factors of x; a truth assignment to the variables of ϕ; a list of the vertices of G.

The number of possible solutions is *exponential* in the length of the input.

Given a potential solution, it is *easy* to check whether or not it is a solution.

Verifiers

A verifier V for a language L is an algorithm such that

$$L = \{x \mid (x, c) \text{ is accepted by } V \text{ for some } c\}$$

If V runs in time polynomial in the length of x, then we say that

> L is *polynomially verifiable*.

Many natural examples arise, whenever we have to construct a solution to some design constraints or specifications.

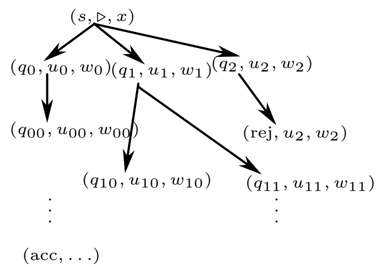
Nondeterministic Complexity Classes

We have already defined $\mathsf{TIME}(f(n))$ and $\mathsf{SPACE}(f(n))$.

$\mathsf{NTIME}(f(n))$ is defined as the class of those languages L which are accepted by a *nondeterministic* Turing machine M, such that for every $x \in L$, there is an accepting computation of M on x of length at most $f(n)$.

$$\mathsf{NP} = \bigcup_{k=1}^{\infty} \mathsf{NTIME}(n^k)$$

Nondeterminism



For a language in $\mathsf{NTIME}(f(n))$, the height of the tree is bounded by $f(n)$ when the input is of length n.

NP

A language L is polynomially verifiable if, and only if, it is in NP.

To prove this, suppose L is a language, which has a verifier V, which runs in time $p(n)$.

The following describes a *nondeterministic algorithm* that accepts L

1. input x of length n
2. nondeterministically guess c of length $\leq p(n)$
3. run V on (x, c)

NP

In the other direction, suppose M is a nondeterministic machine that accepts a language L in time n^k.

We define the *deterministic algorithm* V which on input (x, c) simulates M on input x.

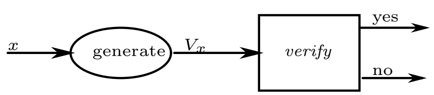
At the i^{th} nondeterministic choice point, V looks at the i^{th} character in c to decide which branch to follow.

If M accepts then V accepts, otherwise it rejects.

V is a polynomial verifier for L.

Generate and Test

We can think of nondeterministic algorithms in the generate-and test paradigm:



Where the *generate* component is nondeterministic and the *verify* component is deterministic.

Reductions

Given two languages $L_1 \subseteq \Sigma_1^\star$, and $L_2 \subseteq \Sigma_2^\star$,

A *reduction* of L_1 to L_2 is a *computable* function

$$f : \Sigma_1^\star \to \Sigma_2^\star$$

such that for every string $x \in \Sigma_1^\star$,

$$f(x) \in L_2 \text{ if, and only if, } x \in L_1$$

Resource Bounded Reductions

If f is computable by a polynomial time algorithm, we say that L_1 is *polynomial time reducible* to L_2.

$$L_1 \leq_P L_2$$

If f is also computable in $\mathsf{SPACE}(\log n)$, we write

$$L_1 \leq_L L_2$$

Reductions 2

If $L_1 \leq_P L_2$ we understand that L_1 is no more difficult to solve than L_2, at least as far as polynomial time computation is concerned.

That is to say,

If $L_1 \leq_P L_2$ and $L_2 \in \mathsf{P}$, then $L_1 \in \mathsf{P}$

We can get an algorithm to decide L_1 by first computing f, and then using the polynomial time algorithm for L_2.

Completeness

The usefulness of reductions is that they allow us to establish the *relative* complexity of problems, even when we cannot prove absolute lower bounds.

Cook (1972) first showed that there are problems in NP that are maximally difficult.

A language L is said to be NP-*hard* if for every language $A \in \mathsf{NP}$, $A \leq_P L$.

A language L is NP-*complete* if it is in NP and it is NP-hard.