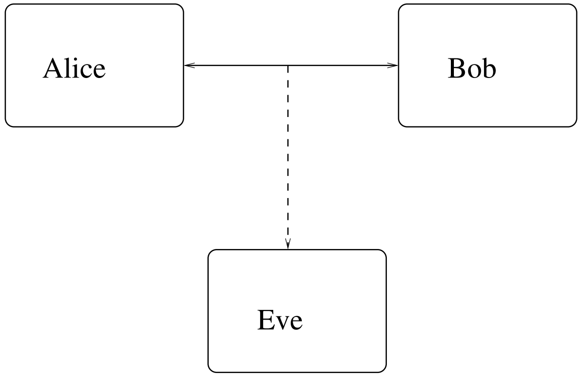
Cryptography



Alice wishes to communicate with Bob without Eve eavesdropping.

Private Key

In a private key system, there are two secret keys

e – the encryption key
d – the decryption key

and two functions D and E such that:

for any x,

$$D(E(x, e), d) = x$$

For instance, taking $d = e$ and both D and E as *exclusive or*, we have the *one time pad*:

$$(x \oplus e) \oplus e = x$$

One Time Pad

The one time pad is provably secure, in that the only way Eve can decode a message is by knowing the key.

If the original message x and the encrypted message y are known, then so is the key:

$$e = x \oplus y$$

Public Key

In public key cryptography, the encryption key e is public, and the decryption key d is private.

We still have,

for any x,

$$D(E(x, e), d) = x$$

If E is polynomial time computable (and it must be if communication is not to be painfully slow), then the function that takes $y = E(x, e)$ to x (without knowing d), must be in FNP.

Thus, public key cryptography is not *provably secure* in the way that the one time pad is. It relies on the existence of functions in $\mathsf{FNP} - \mathsf{FP}$.

One Way Functions

A function f is called a *one way function* if it satisfies the following conditions:

1. f is one-to-one.
2. for each x, $|x|^{1/k} \leq |f(x)| \leq |x|^k$ for some k.
3. $f \in \mathsf{FP}$.
4. $f^{-1} \notin \mathsf{FP}$.

We cannot hope to prove the existence of one-way functions without at the same time proving $\mathsf{P} \neq \mathsf{NP}$.

It is strongly believed that the RSA function:

$$f(x, e, p, q) = (x^e \bmod pq, pq, e)$$

is a one-way function.

UP

Though one cannot hope to prove that the RSA function is one-way without separating P and NP, we might hope to make it as secure as a proof of NP-completeness.

Definition
A nondeterministic machine is *unambiguous* if, for any input x, there is at most one accepting computation of the machine.

UP is the class of languages accepted by unambiguous machines in polynomial time.

UP

Equivalently, UP is the class of languages of the form

$$\{x \mid \exists y R(x, y)\}$$

Where R is polynomial time computable, polynomially balanced, *and* for each x, there is *at most one* y such that $R(x, y)$.

UP One-way Functions

We have

$$\mathsf{P} \subseteq \mathsf{UP} \subseteq \mathsf{NP}$$

It seems unlikely that there are any NP-complete problems in UP.

One-way functions exist *if, and only if,* $\mathsf{P} \neq \mathsf{UP}$.