Shortest Path

- Generalize distance to weighted setting
- Digraph $G = (V,E)$ with weight function $w$: $E \rightarrow \mathbb{R}$ (assigning real values to edges)
- Weight of path $p = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is
  \[ w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}) \]
- Shortest path = a path of the minimum weight

Some slides of this lecture are taken from Alon Efrat's Introduction to algorithms.
Shortest-Path Problems

• Shortest-Path problems
  – Single-source (single-destination). Find a shortest path from a given source (vertex $s$) to each of the vertices.
  – Single-pair. Given two vertices, find a shortest path between them. Solution to single-source problem solves this problem efficiently, too.
  – All-pairs. Find shortest-paths for every pair of vertices. Dynamic programming algorithm.

Negative Weights and Cycles?

• Negative edges are OK, as long as there are no negative weight cycles (otherwise paths with arbitrary small “lengths” would be possible)
• Shortest-paths can have no cycles (otherwise we could improve them by removing cycles)
  – Any shortest-path in graph $G$ can be no longer than $n – 1$ edges, where $n$ is the number of vertices
Relaxation

• For each vertex \( v \) in the graph, we maintain \( v.d() \), the estimate of the shortest path from \( s \), initialized to \( \infty \) at the start
• Relaxing an edge \((u,v)\) means testing whether we can improve the shortest path to \( v \) found so far by going through \( u \)

\[
\text{Relax}(u,v,G) \\
\text{if } v.d() > u.d() + G.w(u,v) \text{ then} \\
v.setd(u.d() + G.w(u,v)) \\
v.setparent(u)
\]

Dijkstra's Algorithm

• Non-negative edge weights
• Greedy, similar to Prim's algorithm for MST
• Like breadth-first search (if all weights = 1, one can simply use BFS)
• Use \( Q \), a priority queue ADT keyed by \( v.d() \) (BFS used FIFO queue, here we use a PQ, which is re-organized whenever some \( d \) decreases)
• Basic idea
  – maintain a set \( S \) of solved vertices
  – at each step select "closest" vertex \( u \), add it to \( S \), and relax all edges from \( u \)
Dijkstra’s Pseudo Code

• Input: Graph G, start vertex s

\[\text{Dijkstra}(G, s)\]

01 for each vertex \(u \in G.V()\)
02 \(u.setd(\infty)\)
03 \(u.setparent(NIL)\)
04 \(s.setd(0)\)
05 \(S \leftarrow \emptyset\)  // Set S is used to explain the algorithm
06 \(Q.init(G.V())\)  // Q is a priority queue ADT
07 while not \(Q.isEmpty()\)
08 \(u \leftarrow Q.extractMin()\)
09 \(S \leftarrow S \cup \{u\}\)
10 for each \(v \in u.adjacent()\) do
11 \(\text{Relax}(u, v, G)\)
12 \(Q.modifyKey(v)\)

Dijkstra’s Example
Dijkstra’s Example (2)

Dijkstra(G,s)
01 for each vertex u ∈ G.V()
02 u.setd(∞)
03 u.setparent(NIL)
04 s.setd(0)
05 S ← ∅
06 Q.init(G.V())
07 while not Q.isEmpty()
08 u ← Q.extractMin()
09 S ← S U {u}
10 for each v ∈ u.adjacent() do
11 Relax(u, v, G)
12 Q.modifyKey(v)

Dijkstra’s Example (3)

Dijkstra(G,s)
01 for each vertex u ∈ G.V()
02 u.setd(∞)
03 u.setparent(NIL)
04 s.setd(0)
05 S ← ∅
06 Q.init(G.V())
07 while not Q.isEmpty()
08 u ← Q.extractMin()
09 S ← S U {u}
10 for each v ∈ u.adjacent() do
11 Relax(u, v, G)
12 Q.modifyKey(v)
**Dijkstra’s Running Time**

- Extract-Min executed \(|V|\) time
- Decrease-Key executed \(|E|\) time
- Time = \(|V| \cdot T_{\text{Extract-Min}} + |E| \cdot T_{\text{Decrease-Key}}\)
- \(T\) depends on different \(Q\) implementations

<table>
<thead>
<tr>
<th>(Q)</th>
<th>(T(\text{Extract-Min}))</th>
<th>(T(\text{Decrease-Key}))</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td>(\Theta(V))</td>
<td>(O(1))</td>
<td>(O(V^2))</td>
</tr>
<tr>
<td>binary heap</td>
<td>(\Theta(lg \ V))</td>
<td>(\Theta(lg \ V))</td>
<td>(O(E \ lg \ V))</td>
</tr>
<tr>
<td>Fibonacci heap</td>
<td>(\Theta(lg \ V))</td>
<td>(O(1)) (amort.)</td>
<td>(O(V \ lg \ V + E))</td>
</tr>
</tbody>
</table>

**Bellman-Ford Algorithm**

- Dijkstra’s doesn’t work when there are negative edges:
  - Intuition – we can not be greedy any more on the assumption that the lengths of paths will only increase in the future
- Bellman-Ford algorithm detects negative cycles (returns \textit{false}) or returns the shortest path-tree
Bellman-Ford Algorithm

Bellman-Ford\((G,s)\)
01 for each vertex \(u \in G.V()\)
02 \(u.setd(\infty)\)
03 \(u.setparent(\text{NIL})\)
04 \(s.setd(0)\)
05 for \(i \leftarrow 1 \text{ to } |G.V()-1| \) do
06 for each edge \((u,v) \in G.E()\) do
07 Relax\((u,v,G)\)
08 for each edge \((u,v) \in G.E()\) do
09 if \(v.d()>u.d()+G.w(u,v)\) then
10 return false
11 return true

Bellman-Ford Example

![Diagram](image-url)
Bellman-Ford Example

- Bellman-Ford running time:
  \[ (|V|-1)|E| + |E| = \Theta(VE) \]

RIP

- RIP = Routing Information Protocol
- Does not scale well, designed for small LANs
- Is a “distance vector protocol”
- Very simple, easy to configure, easy to implement
- Is most widely used routing protocol

Read the code!  http://www.quagga.net/
RIP History

- Developed at Xerox PARC in early 1980s
- Reimplemented in Berkeley UNIX
- 1988: Standardized in RFC 1058
- 1994: RIP-2, RFC 1723
  - Support CIDR addressing
  - Authentication
- 1997: RIPng for IPv6, RFC 2080

RIP Routing Table

<table>
<thead>
<tr>
<th>Destination</th>
<th>Next Hop</th>
<th>Metric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net A</td>
<td>Router 1</td>
<td>3</td>
</tr>
<tr>
<td>Net B</td>
<td>Direct</td>
<td>0</td>
</tr>
<tr>
<td>Net C, Host 3</td>
<td>Router 2</td>
<td>5</td>
</tr>
<tr>
<td>Default</td>
<td>Router 1</td>
<td>0</td>
</tr>
</tbody>
</table>

A destination is either a network, a host, or a “gateway of last resort”

The next hop is either a directly connected network or a directly connected router

Measures how many “hops away” is the destination
Basic RIP Protocol

Periodically exchange list of destinations and metrics with all neighboring routers

RIP routers exchange their entire “distance vector” every 30 seconds

Basic RIP Protocol (cont.)

Trust your neighbor...

Add
Dest. | Nxt Hop | Metric
A     | N       | m + c

Is Dest. A in my RIP Table?

Yes

Replace current entry with
Dest. | Nxt Hop | Metric
A     | N       | m + c

Is N my next hop for Dest. A and m + c is not the current metric?

Yes

Is m + c less than current metric for Dest. A?

NO
Counting to Infinity (and beyond!)

From RFC 1058

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3,B</td>
<td>2,D</td>
<td>3,B</td>
<td>1,∞</td>
</tr>
<tr>
<td>2</td>
<td>3,D</td>
<td>inf</td>
<td>3,B</td>
<td>1,∞</td>
</tr>
<tr>
<td>3</td>
<td>4,C</td>
<td>4,C</td>
<td>4,A</td>
<td>1,∞</td>
</tr>
<tr>
<td>4</td>
<td>5,C</td>
<td>5,C</td>
<td>5,A</td>
<td>1,∞</td>
</tr>
<tr>
<td>5</td>
<td>6,C</td>
<td>6,C</td>
<td>6,A</td>
<td>1,∞</td>
</tr>
<tr>
<td>6</td>
<td>7,C</td>
<td>7,C</td>
<td>6,D</td>
<td>1,∞</td>
</tr>
<tr>
<td>7</td>
<td>7,C</td>
<td>7,C</td>
<td>6,D</td>
<td>1,∞</td>
</tr>
</tbody>
</table>

OSPF

- OSPF = Open Shortest Path First
- Developed to address shortcomings of RIP
  - has rapid, loop-free convergence
  - does not count to infinity
- Link metrics between 0 and 65,535, no limit on path metric
- Is a “link state protocol”
- Has reputation for being complex
- Scales well
Each Router has a database representing the entire network that is constructed from the local knowledge at each router.

Building OSPF Routing Table

Compute locally using Link State Database!
That’s Easy!

Not so fast!

RIP RFC 1058 : 33 pages
OSPF RFC 2328 : 244 pages

Much of this complexity is related to the synchronization of the distributed, replicated link state database. Plus network modeling.

Hierarchical OSPF
Scalability: OSPF Areas

- LS database unique within an area
- Decentralize administration
- Reduce memory usage per router
- Reduce bandwidth used by flooding

Special OSPF protocol to exchange routes between areas. This is a “distance vector” protocol!

Link-state vs. vectoring

- Link state has faster convergence, but requires more memory, CPU, and message overhead
- Vectoring requires few resources, but convergence can be very slow. Counting to infinity can be a problem.
- Both protocols can induce transient forwarding loops during convergence
  - This is one of the issues addressed by Cisco’s EIGRP.