Transitive closure of a relation

Let $R \subseteq U \times U$.

Its transitive closure $R^+ \subseteq U \times U$ is given by:

$$\frac{}{(a,b)}\ (a,b) \in R \qquad\qquad \frac{(a,b) \quad (b,c)}{(a,c)}$$

$$R^+ = \{ (a,b) \in U \times U \mid \text{there is an } R\text{-chain from } a \text{ to } b. \}$$

$$a = a_1 \, R \, a_2 \, R \, a_3 \cdots R \, a_n = b$$

$R^* = R^+ \cup id_U$ reflexive, transitive closure.

Definition

A relation $\prec$ on a set A is well-founded iff there are no infinite descending chains

$$\cdots \prec a_n \prec \cdots \prec a_1 \prec a_0 .$$

Proposition 5.1 Let $\prec \subseteq A \times A$.

$\prec$ is well-founded iff every non-empty subset $Q \subseteq A$ has a minimal element m

i.e. $m \in Q$ and $\forall b \prec m.\ b \notin Q$.

Well-founded relations:

Examples

$m \prec n$ iff $m+1=n$ on $\mathbb{N}_0$ (or $\mathbb{N}$)

$m \prec n$ iff $m<n$ on $\mathbb{N}_0$ (or $\mathbb{N}$)

$A \prec B$ iff A is an immediate subproposition of B in Boolean propns.

Non-example

$\mathbb{Z}$ with $<$

For strings $u, u' \in \Sigma^*$

$u' < u$ iff $\mathrm{length}(u') < \mathrm{length}(u)$

determines a wfd relation on Σ^*.

Exercise 5.3

There is no $u \in \Sigma^*$ s.t. $au = ub$ for symbols $a \neq b$.

Proof. Assume there was. Then would be a $<$-minimal string u s.t.

$$au = ub.$$

But then $u = au'b$ & $u' < u$

$$\therefore \cancel{a}au'\cancel{b} = \cancel{a}u'b\cancel{b}$$

$$au' = u'b$$

contradiction!

The principle of well-founded induction

Let $\prec$ be wfd on A.

To prove $\forall a \in A.\ P(a)$

it suffices to prove that

for all $a \in A$

$$(\forall b \prec a.\ P(b)) \Rightarrow P(a).$$

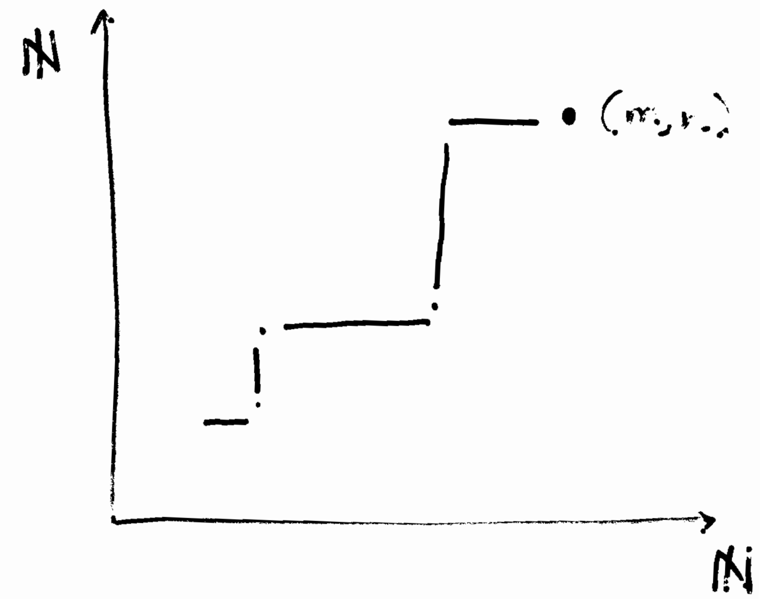
A well-founded relation on $\mathbb{N} \times \mathbb{N}$

$(m', n') < (m, n)$

iff

$(m', n') \neq (m, n)$ &

$m' \leq m$ & $n' \leq n$.



Proposition 5.9

(a) $hcf(m,n) = hcf(m, n-m)$ $\quad m<n$

(b) $hcf(m,n) = hcf(m-n, n)$ $\quad n<m$

(c) $hcf(m,m) = m$

Recall

- $hcf(m,n) \mid m$ & $hcf(m,n) \mid n$
- $k \mid m$ & $k \mid n \Rightarrow k \mid hcf(m,n)$

Euclid's algorithm for hcf

A reduction relation $\xrightarrow[E]{}$ on $\mathbb{N} \times \mathbb{N}$:

$$(m,n) \xrightarrow[E]{} (m, n-m) \quad \text{if} \quad m < n$$

$$(m,n) \xrightarrow[E]{} (m-n, n) \quad \text{if} \quad n < m$$

Theorem 5.10 For all $m, n \in \mathbb{N}$

$$\underbrace{(m,n) \xrightarrow[E]{}{}^{*} (hcf(m,n),\ hcf(m,n))}.$$

$$P(m,n) \overset{def}{\Leftrightarrow}$$

Prove $\forall m,n \in \mathbb{N}.\ P(m,n)$

by wfd induction w.r.t. $\prec \subseteq \mathbb{N} \times \mathbb{N}$

Building well-founded relations

Fundamental wfd relations
From inductive definitions

Transitive closure

If $\prec$ is wfd on A, then
$\prec^+$ is wfd on A.

Inverse image

Let $f : A \to B$.
If $\prec_B$ is wfd on B, then
$\prec_A$ is wfd on A, where

$$a' \prec_A a \iff_{def} f(a') \prec_B f(a).$$

Let $\prec_A$ be wfd on A, $\prec_B$ wfd on B.

Product

$\prec$ is wfd on $A \times B$ where

$$(a', b') \preceq (a, b) \Leftrightarrow_{def} a' \underset{A}{\preceq} a \ \& \ b' \preceq_B b.$$

$a' \prec_A a$ or $a' = a$

Lexicographic product

$\prec_{lex}$ is wfd on $A \times B$ where

$$(a', b') \prec_{lex} (a, b) \Leftrightarrow_{def}$$

$$a' \prec_A a \text{ or } (a' = a \ \& \ b' \prec_B b).$$

Defn. by well-founded induction
(Well-founded recursion)

Examples

- Defn. by mathl. induction:

$$f(0) = \sim\sim\sim$$
$$f(n+1) = \sim\sim\sim f(n) \sim\sim\sim$$

- Defn. by structural induction:

$$\mathrm{length}(a) = 1$$
$$\mathrm{length}(A \vee B) = \mathrm{length}(A) + \mathrm{length}(B) + 1$$
$$\vdots$$

- Defn. by wfd. induction on $<$ on $\mathbb{N}$

Fibonacci $\quad fib(1) = 1, \quad fib(2) = 1,$

$$fib(n) = fib(n-1) + fib(n-2) \text{ for } n > 2.$$

[cf. course-of-values induction]

Ackermann's function

$$ack : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$$

$$ack(0,n) = n+1$$

$$ack(m,0) = ack(m-1,1) \quad \text{if } m>0$$

$$ack(m,n) = ack(m-1, ack(m,n-1)) \quad \text{if } m,n>0$$

(i.e. $ack(m-1,k)$ where $k = ack(m,n-1)$)

- ack is defined because

$$(m, n-1) <_{lex} (m,n)$$

$$(m-1, k) <_{lex} (m,n)$$

$<_{lex}$ is lex. product of $<$ and $<$ on $\mathbb{N}_0$.

Well-founded recursion   P. 79

$\prec$ wfd on A

$$f(x) = \cdots f(x_1) \cdots f(x_2) \cdots\cdots \in B$$

where $\quad x_1 \prec x \;,\; x_2 \prec x \;,\; \ldots$

defines a unique function

$$f : A \longrightarrow B.$$

[NB. x etc. can be a pair, or tuple.]

Examinable material = what's been lectured = lecture plan P.2 = slides available from course web page.