Sets as extensions of properties

$$\{x \mid \underbrace{P}_{\text{a property}}(x)\}$$

always a set ?

↯ Russell's paradox (really a contradiction)

A safe way to build sets :

$$\{x \in \underbrace{S}_{\text{a set}} \mid \underbrace{P}_{\text{a property}}(x)\}$$

But this needs sufficiently big sets S

By fiat: $\mathbb{N}$ is a set

powerset $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is a set

Further constructions on sets

Let A, B be sets

Their product

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

ordered pair:

$(a,b) = (a',b')$ iff $a = a'$ and $b = b'$.

Can be realised as a set

$$(a,b) = \{ \{a\}, \{a,b\} \}$$

Their disjoint union (or disjoint sum)

$$A \oplus B = (\{0\} \times A) \cup (\{1\} \times B)$$

$$\underset{(0,a)}{\rotatebox{90}{$\in$}} \qquad\qquad \underset{(1,b)}{\rotatebox{90}{$\in$}}$$

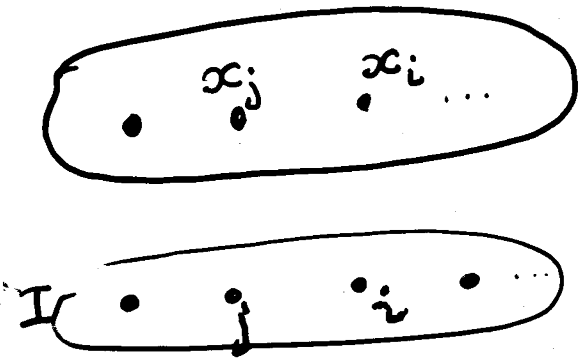
Indexed sets:

Let I be a set s.t.

x_i is an element for all $i \in I$

Then,

$\{x_i \mid i \in I\}$ is a set.



Let I be a set s.t.

A_i is a set for all $i \in I$.

The big union

$$\bigcup_{i \in I} A_i \quad \left[\text{or } \bigcup \{A_i \mid i \in I\}\right]$$

$$= \{x \mid \exists i \in I.\ x \in A_i\} \text{ is a set.}$$

The big intersection

$$\bigcap_{i \in I} A_i \quad \left[\text{or } \bigcap \{A_i \mid i \in I\}\right]$$

$$= \{x \mid \forall i \in I.\ x \in A_i\}.$$

[Needs I nonempty, or universe $\mathcal{U}$ so empty intersections are $\mathcal{U}$.]

Some Consequences

~> set of relations between sets X and Y:

$$\mathcal{P}(X \times Y).$$

~> set of partial functions from set X to set Y:

$$(X \rightharpoonup Y) = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a partial fn.} \}.$$

~> set of total functions from set X to set Y:

$$(X \to Y) = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a function} \}.$$

Higher Order Logic:

If $(x \in X \Rightarrow e \in Y)$, then

$$\lambda x \in X.\, e \in (X \to Y)$$

$$= \{ (x, e) \mid x \in X \}$$