Continuous Mathematics

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Lecture notes

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Preface

These notes provide supplementary material for the *Continuous Mathematics* course as given in Michaelmas Term 2004. I am very grateful to Dr J. Daugman and Dr N.A. Dodgson for giving me access to their teaching materials for earlier variants of this course. I welcome feedback on the course — any errors in this material are my own.

Richard Gibbens
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1 Review of analysis

1.1 Limits, continuity and differentiability

1.1.1 The limit of a function

If we can make a real-valued function $f(x)$ as near as we like to a given number ℓ by making x sufficiently close to a number a then ℓ is said to be the *limit* of $f(x)$ as $x \to a$, written

$$\lim_{x \to a} f(x) = \ell\,. \tag{1}$$

The variable x is allowed to approach a from either direction. However, in many cases the limits will be different depending on whether x approaches a from the left or from the right. In such cases we write

$$\lim_{x \to a-} f(x) = \ell' \qquad \text{and} \qquad \lim_{x \to a+} f(x) = \ell'' \tag{2}$$

to denote the left and right limits, respectively. In all cases where the two separate limits are equal ($\ell' = \ell'' = \ell$, say) we write

$$\lim_{x \to a} f(x) = \ell\,. \tag{3}$$

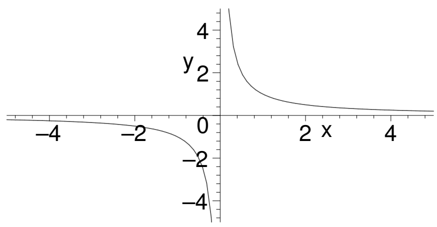

Figure 1: Infinite limits

In some cases a function may become arbitrarily large when x approaches some number a. When this occurs we write

$$\lim_{x \to a} f(x) = \infty\,. \tag{4}$$

In other words we allow the possibility that $\ell = \infty$. Similarly, we allow $\ell = -\infty$. For example, consider the case of $f(x) = \frac{1}{x}$ and $a = 0$, illustrated in Figure 1. In this case we have

$$\lim_{x \to 0-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 0+} f(x) = \infty\,. \tag{5}$$

Note that the notion of limit of $f(x)$ as $x \to a$ does not depend on the value of the function f at a, namely $f(a)$. Indeed, the function need not even be defined at $x = a$ for $\lim_{x \to a} f(x)$ to exist. Furthermore, if $f(x)$ is defined for $x = a$ the function value may be different from either its left or right limit at $x = a$. For example, consider the function, $f(x)$ defined as follows and illustrated in Figure 2

$$f(x) = \begin{cases} 1 & x \geq 0\,, \\ 0 & x < 0 \end{cases}\,. \tag{6}$$

More rigorously we can define the limit of a function $f(x)$ as $x \to a$ to be ℓ whenever for any number $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - \ell| < \epsilon \quad \text{when} \quad |x - a| < \delta\,. \tag{7}$$

Let $f(x)$ and $g(x)$ be two functions and a any number such that

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x) \tag{8}$$

both exist then the following three properties hold.

Theorem 1

$$\lim_{x\to a} \{f(x) + g(x)\} = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \tag{9}$$

Theorem 2

$$\lim_{x\to a} \{f(x)g(x)\} = \left\{\lim_{x\to a} f(x)\right\} \cdot \left\{\lim_{x\to a} g(x)\right\} \tag{10}$$

Theorem 3

$$\lim_{x\to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \tag{11}$$

whenever $\lim_{x\to a} g(x) \neq 0$.

1.1.2 Continuity

Having defined the notion of a limit of a function we can proceed to consider the notion of continuity. A real-valued function $f(x)$ is said to be *continuous at* $x = a$ if the following three conditions hold

(1) $\lim_{x\to a} f(x)$ exists,

(2) $f(x)$ is defined at $x = a$, and

(3) $\lim_{x\to a} f(x) = f(a)$.

A function that is not continuous at $x = a$ is *discontinuous at* a and a is then a *point of discontinuity* for the function f. Generally, if the graph of a function has a break in it at some value of x it is discontinuous at that point. For example, the functions illustrated in Figures 1 and 2 are both discontinuous at $x = 0$. The first function ($f(x) = 1/x$) becomes infinite at the point of discontinuity and is said to have an *infinite discontinuity at* $x = 0$; the second function remains finite at the discontinuity and is said to have a *finite discontinuity at* $x = 0$, accordingly. The function

$$f(x) = x - [x] \tag{12}$$

has an infinite number of finite discontinuities occuring whenever x is integer, as illustrated in Figure 3.

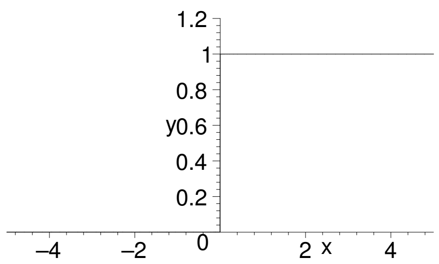

Figure 2: Limit with undefined function value

1.1.3 *Differentiability*

If, for a given value of x,

$$\frac{f(x+\delta x)-f(x)}{\delta x} \tag{13}$$

tends to a finite limit as $\delta x \to 0$ then this limit, denoted by $\frac{df}{dx}$ (or sometimes $f'(x)$), is called the *derivative* (or *differential coefficient*) of f for the value x. The function is then said to be *differentiable at* x. The function is said to be *differentiable* if the derivative exists at every value of x.

Consider the function $y = f(x)$ illustrated in Figure 4. Let P be a typical point on the graph with coordinates (x, y). Suppose that Q is some neighbouring point with coordinates $(x+\delta x, y+\delta y)$. The expression

$$\frac{f(x+\delta x)-f(x)}{\delta x} \tag{14}$$

is then the slope of the straight line joining points P and Q. If, as the point Q is allowed to approach the point P, the expression (14) approaches a limiting value, the curve has a tangent at P whose gradient is $f'(x)$.

The definition of the differential coefficient as a limit allows us not only to determine its value (when it exists) but also to derive the well known rules for differentiating the product and quotient of two functions, namely

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}\,, \tag{15}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}\,, \tag{16}$$

where f and g are two differentiable functions of x.

A necessary (but not sufficient) condition for a function f to be differentiable at some point x is that f is continuous at that point. For $(f(x+\delta x) - f(x))/\delta x$ can tend to a finite limit as $\delta x \to 0$ only if $f(x+\delta x) - f(x) \to 0$, that is, if f is continuous at x. Can you find a function that at some point is continous but is *not* differentiable?

1.2 *Power series and transcendental functions*

Before we consider power series and transcendental functions we must briefly review the convergence of infinite series.

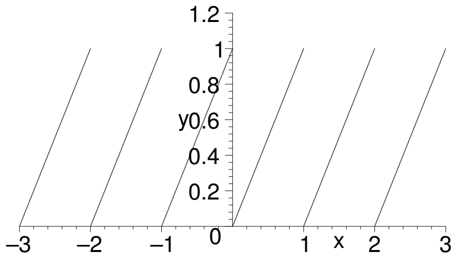

Figure 3: A function with an infinite number of (finite) discontinuities

1.2.1 Convergence of infinite series

If $a_1, a_2, a_3, \ldots$is a sequence of numbers then the sum of the first n numbers is called the nth *partial sum* and is represented by

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{r=1}^{n} a_r \, . \tag{17}$$

If the partial sums $S_1, S_2, \ldots$ converge to a finite limit, S, say, where

$$S = \lim_{n\to\infty} S_n \tag{18}$$

then S is defined as the sum of the infinite series

$$a_1 + a_2 + \ldots = \sum_{r=1}^{\infty} a_r \, , \tag{19}$$

and the infinite series is said to be *convergent.* Alternatively, if the sequence of partial sums tends to an infinite limit, or oscillates without tending to a limit the series is said to be *divergent.*

Example The geometric series

$$\sum_{r=0}^{\infty} ap^r = a(1 + p + p^2 + \ldots) \tag{20}$$

for some constant value a ($a \neq 0$) has partial sums ($p \neq 1$)

$$S_n = a\frac{1 - p^{n+1}}{1 - p} \, . \tag{21}$$

Hence, if $|p| < 1$

$$S = \lim_{n\to\infty} S_n = \frac{a}{1 - p} \tag{22}$$

and the series is convergent. However, the series is divergent when $|p| \geq 1$ since the partial sum S_n either increases without limit as $n \to \infty$ ($p \geq 1$), or oscillates either finitely ($p = -1$) or infinitely ($p < -1$).

There are a variety of tests for the convergence of series such as the *comparison test* and the *ratio test.* One such test that we shall need is *d'Alembert's ratio test* which takes the following form. If $\sum_{r=1}^{\infty} a_r$ is a series of positive or negative terms then the infinite series is convergent if

$$\lim_{r\to\infty} \left| \frac{a_{r+1}}{a_r} \right| = k < 1 \, , \tag{23}$$

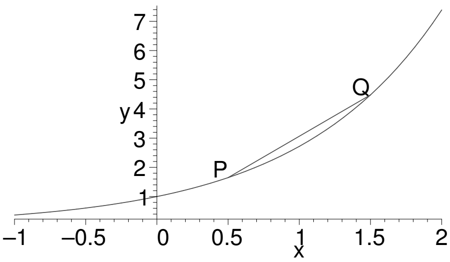

Figure 4: The slope of a function

and is divergent if

$$\lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| = k > 1\,. \tag{24}$$

Note that the test does not allow us to decide between convergence and divergence at the boundary case when $k = 1$.

1.2.2 *Power series*

Power series are an important type of infinite series given by

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \ldots \tag{25}$$

where $a_0, a_1, a_2, \ldots$ are constants. The values of x for which the power series converges can be found from d'Alembert's ratio test. Using this test, the series is convergent if

$$\lim_{r\to\infty}\left|\frac{a_{r+1}x^{r+1}}{a_r x^r}\right| = |x|\lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| = k < 1\,. \tag{26}$$

This condition is more conveniently express as $|x| < R$ where R is the *radius of converence* defined as

$$R = \lim_{r\to\infty}\left|\frac{a_r}{a_{r+1}}\right|\,, \tag{27}$$

provided the limit exists. That is to say the series is convergent provided x lies in the (open) interval

$$-R < x < R\,. \tag{28}$$

This interval is called the *interval (or range) of convergence.*

When $k = 1$, d'Alembert's ratio test gives us no information and consequently the series may converge or diverge if $|x| = R$. Again, from d'Alembert's ratio test, we know that the series diverges for any value of x outside the interval of convergence.

1.2.3 *Transcendental functions*

The following functions, members of a class of functions known as *transcendental functions*, are defined within their intervals of convergence by the following power series.

$$\sin x := x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{for all } x \tag{29}$$

$$\cos x := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{for all } x \tag{30}$$

$$\log_e(1+x) := x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for } -1 < x \leq 1 \tag{31}$$

$$e^x := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for all } x \tag{32}$$

$$\sinh x := \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{for all } x \tag{33}$$

$$\cosh x := \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad \text{for all } x \tag{34}$$

Power series possess a number of useful properties. The sum, difference or product of two power series with common intervals of convergence leads to a third power series which converges for the common interval of convergence of the original two series. If two power series converge for a common interval of convergence then one series may be subsituted into the other to give a third series which converges in that common interval. The transcendental functions make good examples to illustrate these properties. Consider, for example the series for $e^{e^{-x}}$. This may be obtained by subsituting $y = e^{-x}$ and using the power series for e^y. Hence,

$$e^{e^{-x}} = 1 + \left(1 - x + \frac{x^2}{2!} - \dots\right) + \frac{1}{2!}\left(1 - x + \frac{x^2}{2!} - \dots\right)^2 + \frac{1}{3!}\left(1 - x + \frac{x^2}{2!} - \dots\right)^3 + \cdots . \quad (35)$$

1.3 Taylor series

We now consider an important result which enables functions to be expanded in power series in x in a given interval.

Theorem 4 (Taylor's theorem) *If $f(x)$ is a continuous function of x with continuous derivatives $f'(x), f''(x), \dots$ up to and including $f^{(n)}(x)$ in a given interval $a \le x \le b$, and if $f^{(n+1)}(x)$ exists in $a < x < b$ then*

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + E_n(x) , \quad (36)$$

where

$$E_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \quad (37)$$

and $a < \xi < x$.

The term E_n is a remainder term and represents the error involved in approximating $f(x)$ by the *polynomial*

$$f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) . \quad (38)$$

Theorem 5 *If $\lim_{n\to\infty} E_n(x) = 0$ then $f(x)$ may be represented by the power series*

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots = \sum_{r=0}^{\infty} \frac{(x-a)^r}{r!}f^{(r)}(a) . \quad (39)$$

By changing x to $a + x$ we have the equivalent form

$$f(a+x) = f(a) + \frac{x}{1!}f'(a) + \frac{x^2}{2!}f''(a) + \cdots = \sum_{r=0}^{\infty} \frac{x^r}{r!}f^{(r)}(a) . \quad (40)$$

The Taylor series only represents the function $f(x)$ in the interval of convergence. The form of the coefficients in Taylor's series may be verified in the following manner. Let

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \cdots \quad (41)$$

where $A_0, A_1, A_2, \dots$ are constants. Differentiating term by term we have

$$f'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \cdots \quad (42)$$
$$f''(x) = 2A_2 + 3 \cdot 2A_3(x-a) + 4 \cdot 3A_4(x-a)^2 + \cdots \quad (43)$$
$$f'''(x) = 3!A_3 + 4!A_4(x-a) + \cdots , \quad (44)$$

and, in general,

$$f^{(n)}(x) = n!A_n + (n+1)!A_{n+1}(x-a) + \cdots . \quad (45)$$

Putting $x = a$ now gives

$$f(a) = A_0 \qquad f'(a) = A_1 \tag{46}$$
$$f''(a) = 2!A_2 \qquad f'''(a) = 3!A_3 \tag{47}$$
$$f^{(n)}(a) = n!A_n \tag{48}$$

which yields values for the constants $A_0, A_1, A_2, \ldots$ precisely as given by Taylor's theorem. A special case of the Taylor series is the *Maclaurin series* given by taking $a = 0$.

1.4 Complex variables

We are familiar with the real numbers and the associated notions of a limit, continuity and differentiation. The complex numbers are a useful generalization of real numbers. A *complex number*, denoted z, is defined as an ordered pair of real numbers (x, y) together with the *imaginary number* $i = \sqrt{-1}$ where

$$z = (x, y) = x + iy\,. \tag{49}$$

The (real) numbers x and y are called the *real* and *imaginary parts* of z, respectively. The main advantage of such a construction is that equations which have no solutions amongst real numbers, for example,

$$z^2 + 1 = 0 \tag{50}$$
$$z^2 - 2z + 2 = 0 \tag{51}$$

now have solutions in terms of complex numbers. We can readily see that $z = \pm i$ solves the first equation and $z = 1 \pm i$ solves the second.

We can see from the definition of the imaginary number i that powers of i may be simply expressed in terms of ± 1 and i itself. For example, we have that

$$i^2 = -1\,, \quad i^3 = i^2 i = -i\,, \quad i^4 = (i^2)^2 = 1\,, \tag{52}$$

and

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i\,, \quad i^{-2} = -1\,, \quad i^{-3} = \frac{1}{i^3} = \frac{1}{-i} = i\,. \tag{53}$$

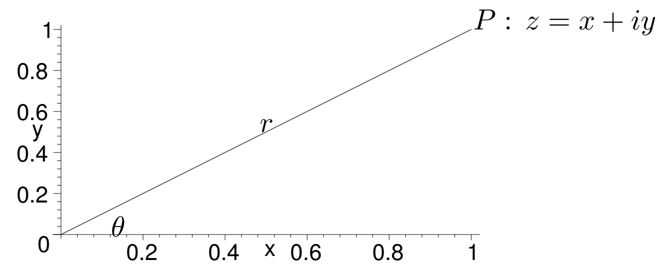

Figure 5: The Argand diagram

Given the definition of a complex number as an ordered pair of real numbers, there is a natural (1-1) correspondence between the infinite set of complex numbers and the points of a plane. The x-axis represents the real part of any complex number and the y-axis represents the imaginary part. For example, the point P with co-ordinates $(2, 3)$ uniquely represents the complex number $z = 2 + 3i$. We can refer to the x-axis as the *real axis* and the y-axis as the *imaginary axis* and the whole diagram is called the *Argand diagram*. We can also use polar co-ordinates r and θ to represent any non-zero complex number $z = x + iy \neq 0$ given by

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{54}$$

so that

$$z = x + iy = r(\cos\theta + i\sin\theta)\ . \tag{55}$$

The number r is called the *modulus* of z and is written $|z|$ (or sometimes $\mod z$). The angle θ is called the argument of z (denoted $\arg(z)$). Note that θ is not unique since the angles $\theta + 2k\pi$ (k any integer) are also arguments for z. We call the *principal value* of the argument (denoted $\mathrm{Arg}(z)$) that value of θ which satisfies the inequalities

$$-\pi < \theta \leq \pi\,. \tag{56}$$

Using the Argand diagram we can see that

$$|z| = r = \sqrt{x^2 + y^2}\,. \tag{57}$$

Note that we do *not* define a value for $\arg(0)$.

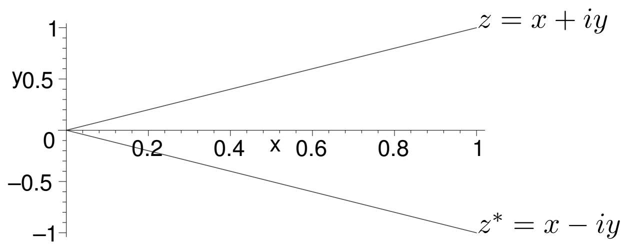

Figure 6: The complex conjugate

Associated with any complex number $z = x + iy$ is its *complex conjugate*, z^*, defined by

$$z^* = x - iy \tag{58}$$

and illustrated in Figure 6. Note that

$$|z| = |z^*| \tag{59}$$

and in the Argand diagram z^* is the mirror image of z in the real axis.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then the sum and difference of z_1 and z_2 are defined as

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{60}$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)\,. \tag{61}$$

It is interesting to see how this looks on an Argand diagram. If P represents z_1 and Q represents z_2 then the point R, forming the parallelogram $OPRQ$ represents $z_1 + z_2$. To see this, note that the point represented by R has real and imaginary parts given by $x_1 + x_2$ and $y_1 + y_2$, respectively.

The product of z_1 and z_2 is defined as

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\,. \tag{62}$$

The division of two complex numbers is defined in an analagous way so that

$$\frac{z_2}{z_1} = \frac{x_2 + iy_2}{x_1 + iy_1} = \frac{(x_2 + iy_2)(x_1 - iy_1)}{(x_1 + iy_1)(x_1 - iy_1)} = \frac{x_1x_2 + y_1y_2}{x_1^2 + y_1^2} + i\frac{x_1y_2 - y_1x_2}{x_1^2 + y_1^2} \tag{63}$$

provided $x_1^2 + y_1^2 = |z_1|^2 \neq 0$.

We can also use the power series definitions of the transcendental functions to define the following functions applied to complex argument z.

$$e^z := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \tag{64}$$

$$\sin z := z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \tag{65}$$

$$\cos z := 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots . \tag{66}$$

These series converge for $0 \leq |z| < \infty$. Using these series we see that

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{67}$$

for any real number θ. In summary, we have the following relationships

$$z = re^{i\theta} \tag{68}$$

where

$$r = |z| = \sqrt{zz^*} \tag{69}$$

$$\theta = \mathrm{Arg}(z) . \tag{70}$$

2 Fourier series

2.1 Introduction and general properties

We have seen that the Taylor and Maclaurin series allow us to represent certain functions as power series. Such functions must be continuous and infinitely differentiable within the interval of convergence of the power series. We will now consider how functions which may be neither differentiable nor continuous at certain points can be represented by a trigonometric series of the form

$$\frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx) \ , \tag{71}$$

where a_0, a_r and b_r are constants for $r = 1, 2, \ldots$. Since this trigonometric series is unchanged by replacing x by $x + 2k\pi$, where k is an integer, it must represent a periodic function in x of period 2π. Consequently, it is sufficient to consider any interval of length 2π and we choose the interval

$$-\pi < x \leq \pi \, . \tag{72}$$

Let $f(x)$ be an arbitrary function defined in this interval. Suppose the coefficients a_0, a_r, b_r are given by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx \, , \tag{73}$$

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos rx \, dx \, , \qquad r = 1, 2, 3, \ldots \ , \tag{74}$$

$$b_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin rx \, dx \, , \qquad r = 1, 2, 3, \ldots \tag{75}$$

then the resulting trigonometric series is called the *Fourier series* of $f(x)$ and the coefficients are the *Fourier coefficients*. The sum of a Fourier series is not necessarily equal to the function from which it is derived and the conditions under which the Fourier series of $f(x)$ converges to $f(x)$ in the sense that

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx) \tag{76}$$

depend on the particular function chosen. The following theorem states a set of sufficient conditions (known as the *Dirichlet's conditions*) which $f(x)$ must satisfy for (76) to be valid.

Theorem 6 *Suppose $f(x)$ is defined arbitrarily in the interval $-\pi < x \leq \pi$ and extended to other values of x by the periodicity condition $f(x + 2k\pi) = f(x)$ where k is an integer. Then, if in $-\pi < x \leq \pi$, $f(x)$ is continuous except for a finite number of points of finite discontinuities, and has only a finite number of maxima and minima, its Fourier series converges to $f(x)$ at all points in this interval where $f(x)$ is continuous. At a point of finite discontinuity, say $x = x_0$, the Fourier series converges to the value*

$$\frac{1}{2} \lim_{\delta \to 0} \{ f(x_0 + \delta) + f(x_0 - \delta) \} \ , \tag{77}$$

which is just the mean of the two limiting values of $f(x)$ as x approaches x_0 from the right and left-hand sides.

Functions satisfying the Dirichlet conditions (as stated in the theorem) are called *piecewise regular functions.*

2.1.1 *Derivation of Fourier coefficients*

If r and s are positive integers or zero then it follows by simple integration that

$$\int_{-\pi}^{\pi} \cos rx \cos sx \, dx = \begin{cases} 0 & \text{for } r \neq s \\ 2\pi & \text{for } r = s = 0 \\ \pi & \text{for } r = s > 0 \end{cases} \tag{78}$$

$$\int_{-\pi}^{\pi} \sin rx \sin sx \, dx = \begin{cases} 0 & \text{for } r \neq s \\ 0 & \text{for } r = s = 0 \\ \pi & \text{for } r = s > 0 \end{cases} \tag{79}$$

$$\int_{-\pi}^{\pi} \sin rx \cos sx \, dx = 0 \qquad \text{for all } r \text{ and } s \tag{80}$$

$$\int_{-\pi}^{\pi} \cos rx \, dx = \begin{cases} 0 & \text{for } r > 0 \\ 2\pi & \text{for } r = 0 \end{cases} \tag{81}$$

$$\int_{-\pi}^{\pi} \sin rx \, dx = 0 \qquad \text{for all } r \, . \tag{82}$$

Using these results and multiplying both sides of

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx) \tag{83}$$

by $\cos sx$ and integrating from $x = -\pi$ to π yields expressions for a_0 and a_r $(r = 1, 2, 3, \ldots)$. Similarly, multiplying both sides by $\sin sx$ and integrating from $x = -\pi$ to π yields expressions for b_r $(r = 1, 2, 3, \ldots)$.

Note that we can merge the expressions for a_0 and a_r into the single formula

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos rx \, dx \, , \tag{84}$$

as r takes values $r = 0, 1, 2, \ldots$.

2.1.2 *Even and odd functions*

If $f(x)$ is either an even or odd function of x in the interval $-\pi < x \leq \pi$ then the Fourier coefficients are simplified.

If $f(x)$ is an even function, that is $f(x) = f(-x)$, then

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \, dx \, , \tag{85}$$

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos rx \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos rx \, dx \qquad \text{for } r = 1, 2, 3 \, , \tag{86}$$

$$b_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin rx \, dx = 0 \qquad \text{for all } r \, . \tag{87}$$

Hence if $f(x)$ is an even function its Fourier series reduces to a series where all the sine terms vanish.

Alternatively, if $f(x)$ is an odd function, that is $f(x) = -f(-x)$, then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0\,, \tag{88}$$

$$a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx\,dx = 0 \qquad \text{for all } r\,, \tag{89}$$

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin rx\,dx\,, \qquad \text{for } r = 1, 2, 3, \ldots. \tag{90}$$

Hence if $f(x)$ is an odd function its Fourier series reduces to a series where all the cosine terms vanish.

2.1.3 Change of interval

Instead of expanding a function in the interval $-\pi < x \leq \pi$ let us instead work with the more general interval $-T < x \leq T$, where T is any given positive number. Suppose then that $f(x)$ is a piecewise regular function in $-T < x \leq T$ (that is, it satisifies Dirichlet's conditions) which is defined outside this interval by the periodicity condition $f(x + 2Tk) = f(x)$, where k is an integer. Thus, putting $z = \pi x/T$ we have that

$$f(x) = f\left(\frac{Tz}{\pi}\right) = F(z)\,, \tag{91}$$

where now $F(x)$ is a periodic function of z of period 2π. Hence in $-\pi < z \leq \pi$

$$F(z) = \frac{a_0}{2} + \sum_{r=1}^{\infty}\left(a_r \cos rz + b_r \sin rz\right)\,, \tag{92}$$

where

$$a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\cos rz\,dz\,, \quad (r = 0, 1, 2, \ldots)\,, \tag{93}$$

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\sin rz\,dz\,, \quad (r = 1, 2, 3, \ldots)\,. \tag{94}$$

Consequently, putting $z = \pi x/T$ in these expressions we have that

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty}\left(a_r \cos\frac{\pi xr}{T} + b_r \sin\frac{\pi xr}{T}\right) \tag{95}$$

where

$$a_r = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{\pi xr}{T}\,dx\,, \quad (r = 0, 1, 2, , \ldots)\,, \tag{96}$$

$$b_r = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{\pi xr}{T}\,dx\,, \quad (r = 1, 2, 3, \ldots)\,. \tag{97}$$

2.1.4 Compact complex representation

Finally, using the relation

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{98}$$

we may express $f(x)$ in the alternative form

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{irx} \tag{99}$$

where

$$c_r = \frac{1}{2}(a_r - ib_r) \qquad \text{and} \qquad c_{-r} = \frac{1}{2}(a_r + ib_r) \qquad \text{for } r > 0 \tag{100}$$

and

$$c_0 = \frac{1}{2}a_0 \,. \tag{101}$$

With these definitions

$$c_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-irx}\,dx \qquad \text{for } r = 0, \pm 1, \pm 2, \ldots . \tag{102}$$

Note that the coefficients, c_r, are complex numbers and that for each positive integer $r = 1, 2, 3, \ldots$ we have two coefficients: c_r and c_{-r}. Assuming $f(x)$ is a real-valued function, these two coefficients are complex conjugates

$$c_{-r} = c_r^* \qquad \text{for } r > 0 \,. \tag{103}$$

2.2 Examples

2.2.1 The square wave

Consider the square wave function given by

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \,, \\ 1 & 0 < x < \pi \,. \end{cases} \tag{104}$$

In this case $f(x)$ is an odd function and this implies that $a_r = 0$ for all $r \geq 0$. The terms b_r, $r = 1, 2, 3, \ldots$ are given by

$$b_k = \frac{2}{\pi}\int_0^{\pi} \sin rx\,dx \tag{105}$$

$$= \frac{2}{r\pi}(1 - \cos r\pi) \tag{106}$$

$$= \begin{cases} \frac{4}{r\pi} & r \text{ odd}\,, \\ 0 & r \text{ even}\,. \end{cases} \tag{107}$$

Hence the Fourier series expansion of the square wave function is

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) \,. \tag{108}$$

Figure 7 illustrates the way in which the leading terms of the Fourier series approximates the square wave. Notice how the Fourier series finds it hard not to *overshoot* near points of discontinuity in the function.

2.2.2 The sawtooth wave

Consider the Fourier series representing the sawtooth function $f(x) = x$ in the interval $-1 < x < 1$. Here $f(x)$ is an odd function so $a_r = 0$ for all $r \geq 0$ and we have a series of cosine terms alone. Using the change of interval expressions with the interval $-1 < x < 1$ we have for $r = 1, 2, \ldots$ that

$$b_r = \int_{-1}^{1} x \sin \pi r x\,dx \tag{109}$$

$$= -\frac{2}{\pi r}(-1)^r \,. \tag{110}$$

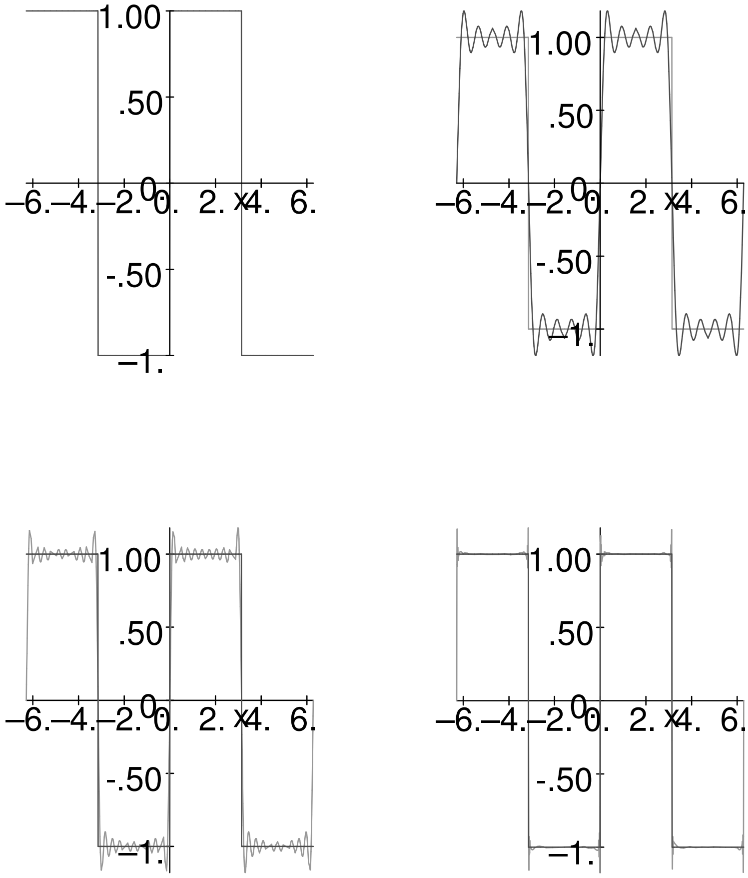

Figure 7: The square wave function and Fourier series of 5, 10 and 100 non-zero terms

Hence in the interval $-1 < x < 1$

$$f(x) = x = \frac{2}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \cdots\right). \tag{111}$$

At $x = \pm 1$, finite discontinuities occur. Hence at these points the series does not represent x but converges to the value

$$\frac{1}{2}\lim_{\delta\to 0}\{f(x_0+\delta) + f(x_0-\delta)\} = \frac{1}{2}\{1 + (-1)\} = 0\,. \tag{112}$$

Figure 8 shows the manner in which adding more terms improves the approximation to the sawtooth function.

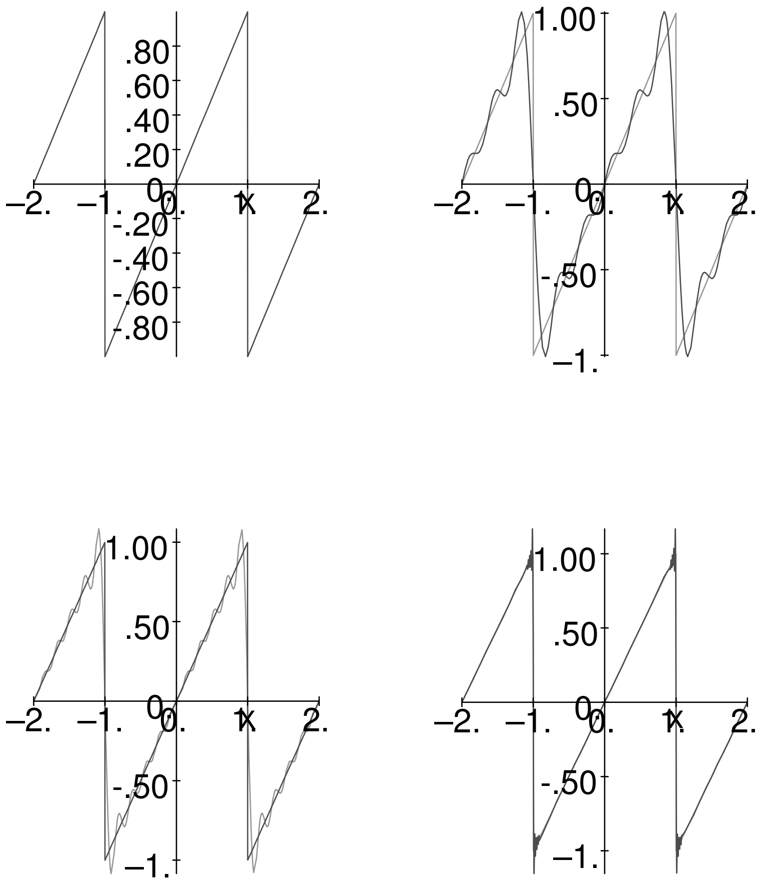

Figure 8: The sawtooth function and Fourier series of 5, 10 and 100 non-zero terms

3 Basis functions and decompositions

3.1 Expansions and basis functions

Taylor's series shows us ways to represent a function (for example, $\sin(x)$) in terms of a power series where each term is of the form $c_r x^r$. Fourier series, in a similar way, seek to represent certain functions by expressing them as a series of terms using the functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots \tag{113}$$

and we have seen that the resulting series may also be written more compactly as

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{irx} \, . \tag{114}$$

These are just two examples of a general approach where we seek to represent a function $f(x)$ in terms of linear combinations of certain other functions so that

$$f(x) = \sum_k c_k \Psi_k(x) \tag{115}$$

where the chosen functions $\Psi_k(x)$ are called the *expansion basis functions.* Thus for the Fourier series example the basis functions are the complex exponentials

$$\Psi_k(x) = e^{i\mu_k x} \tag{116}$$

where $k = 0, \pm 1, \pm 2, \ldots$ and the *frequency* of the kth basis function is $\mu_k = k$. Recall that we can change the length of the period from 2π and this will change the frequency used for the kth basis function. If the period is $2T$ then the frequency of the kth basis function becomes $\mu_k = \pi k/T$.

This approach proves to be very useful because it allows us to choose some universal set of functions and then represent many other functions in terms of just a set of numerical coefficients.

In the case of systems analysis a major benefit of doing this is that knowledge of how members of the chosen universal set of basis functions behave in the system gives us knowledge about how arbitrary input functions will be treated by the system.

3.2 Orthogonality, inner products and completeness

If the basis functions satisfy the rule that the integral of the conjugate product of any two distinct basis functions equals zero, that is,

$$\int_{-\infty}^{\infty} \Psi_k^*(x) \Psi_j(x) \, dx = 0 \quad (k \neq j) \tag{117}$$

then the set of basis functions is called *orthogonal.*

Such integrals are called *inner products*, often denoted using angle brackets

$$< \Psi_k(x), \, \Psi_j(x) > := \int_{-\infty}^{\infty} \Psi_k^*(x) \Psi_j(x) \, dx \, . \tag{118}$$

If, in addition to orthogonality, the set of basis functions has the property that the inner product of every basis function with itself is equal to one, that is,

$$< \Psi_j(x), \, \Psi_j(x) > = \int_{-\infty}^{\infty} \Psi_j^*(x) \Psi_j(x) \, dx = 1 \tag{119}$$

then the set of basis functions is said to be *orthonormal*.

The coefficients c_k can be conveniently determined by means of inner products of orthonormal basis functions with the given function $f(x)$. We find that, in general

$$c_k = < \Psi_k(x),\, f(x) > \; . \tag{120}$$

This operation resembles that of taking the projection of a vector in a given coordinate direction.

We say that a set of basis functions is *complete* when all functions of interest can be represented by an expansion of the form

$$\sum_k c_k \Psi(x)\, . \tag{121}$$

In other words, the space of basis functions spans the required set of functions. More rigorously, we can say that a set of basis functions is complete if no nontrivial function of interest, $f(x)$, is orthogonal to all the basis functions $\Psi_k(x)$. That is,

$$< f(x),\, \Psi_k(x) > = 0 \qquad \text{for all } k \tag{122}$$

implies that f is the trivial function $f(x) = 0$ for all x.

The set of basis functions of the form e^{ikx} $(k = 0, \pm 1, \pm 2, \ldots)$ is known to be complete for the space of piecewise regular functions of period 2π.

4 Representation of signals

4.1 Fourier transforms and their inverses

4.1.1 Introduction and general properties

We have seen that the Fourier series allows us to represent a periodic function within the limited range $-\pi < x \leq \pi$ (or, more generally, $-T < x \leq T$). We now look at the representation of *aperiodic* functions over the infinite range $-\infty < x < \infty$. Physically, this means resolving a single pulse or wave packet into sinusoidal waves.

Formally, the Fourier series representation can be extended to the infinite range so that

$$f(x) = \int_{-\infty}^{\infty} F(\mu) e^{i\mu x} \, d\mu \tag{123}$$

where $F(\mu)$ is given by the relation

$$F(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\mu x} \, dx \, . \tag{124}$$

The function $F(\mu)$ is known as the *Fourier transform* of $f(x)$. Equation (123) allows us to find the inverse function, $f(x)$, given a Fourier transform $F(\mu)$.

Several properties of Fourier transforms may be obtained directly from the definition.

Shift Shifting the original function $f(x)$ by some displacement α merely multiplies its Fourier transform by the factor $e^{-i\mu\alpha}$. Thus the Fourier transform of the shifted function $f(x-\alpha)$ is $F(\mu)e^{-i\mu\alpha}$.

Scale If the scale of the original function $f(x)$ changes (shrinks or expands) by a factor α, becoming $f(\alpha x)$, then the Fourier transform of the scaled function is $\frac{1}{|\alpha|}F(\mu/\alpha)$.

Differentiation Computing the derivative of a function corresponds to a multipication operation on the Fourier transform, specifically,

$$\left(\frac{d}{dx}\right)^m f(x) \quad \text{has Fourier transform} \quad (i\mu)^m F(\mu) \tag{125}$$

where m is the order of the derivative.

4.1.2 Convolution

Suppose the function $f(x)$ has Fourier transform $F(\mu)$ and the function $g(x)$ has Fourier transform $G(\mu)$. The *convolution* of $f(x)$ with $g(x)$, which is denoted $f * g$, combines these two functions to generate a third function $h(x)$ whose value at x is equal to the integral of the product of functions f and g after they undergo a relative shift by the amount x

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) \, d\alpha \, . \tag{126}$$

Thus the convolution is a way of combining two functions which in a sense uses one to blur the other, making all possible relative shifts between two functions when computing the integral of their product to obtain the corresponding output values.

Convolution is an extremely important operation in systems theory because it is one basis of describing how any linear system $h(t)$ acts on any input $s(t)$ to generate the corresponding output $r(t)$. The output is just given by the convolution of the input with the characteristic system response function, so that,

$$r(t) = h(t) * s(t)\,. \tag{127}$$

It may be shown that the Fourier transform, $H(\mu)$, of the convolution $h(x)$ is given by

$$H(\mu) = F(\mu)G(\mu) \tag{128}$$

where $F(\mu)$ and $G(\mu)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively.

This is a very useful results since it is much easier to multiply two functions $F(\mu)$ and $G(\mu)$ together than to convolve $f(x)$ and $g(x)$ together to obtain $h(x)$.

4.2 Wavelets

4.2.1 Brief introduction

Wavelets are a further method of representing functions which have received much interest in applied fields over the last several decades.

The approach fits into the general scheme of expansion using basis functions. Here we expand the functions $f(x)$ in terms of a doubly-infinite series

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} d_{jk}\Psi_{jk}(x) \tag{129}$$

where $\Psi_{jk}(x)$ are the basis functions.

The basis functions are all closely related and arise from *shifting* and *scaling* operations applied to a single function, $\Psi(x)$, known as the *mother wavelet*. The basis functions are given for integers j and k by

$$\Psi_{jk}(x) = \Psi(2^j x - k)\,. \tag{130}$$

A common example that we shall study here is the *Haar wavelet* whose mother function is both localised and oscillatory defined by

$$\Psi(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2}\,, \\ -1 & \text{if } \frac{1}{2} \le x < 1\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{131}$$

The Haar mother wavelet is illustrated in Figure 9.

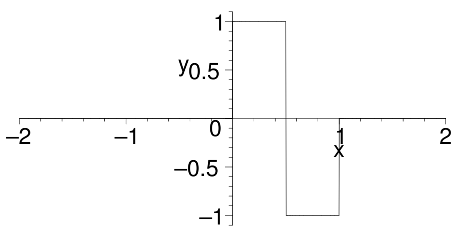

Figure 9: The Haar mother wavelet

The Haar mother wavelet oscillates and has a width (or scale) of one. The *dyadic dilates* of $\Psi(x)$, namely,

$$\ldots, \Psi(2^{-2}x), \Psi(2^{-1}x), \Psi(x), \Psi(2x), \Psi(2^2x), \ldots \tag{132}$$

have widths

$$\ldots, 2^2, 2^1, 1, 2^{-1}, 2^{-2}, \ldots \tag{133}$$

respectively. Since the dilate $\Psi(2^j x)$ has width 2^{-j}, its translates

$$\Psi(2^j - k) = \Psi(2^j(x - k2^{-j})), \qquad k = 0, \pm 1, \pm 2, \ldots \tag{134}$$

will cover the whole x-axis. The collection of coefficients d_{jk} are termed the *discrete wavelet transform* of the function $f(x)$.

How should we intrepret the values d_{jk}? Since the Haar basis function $\Psi(2^j x - k)$ vanishes except when

$$0 \leq 2^j x - k < 1\,, \quad \text{that is} \quad k2^{-j} \leq x < (k+1)2^{-j} \tag{135}$$

we see that d_{jk} gives us information about the bahaviour of f near the point $x = k2^{-j}$ measured on the scale of 2^{-j}. For example, the coefficients $d_{-10,k}$, $k = 0, \pm 1, \pm 2, \ldots$ correspond to variations of f that take place over intervals of length $2^{10} = 1024$ while the coefficients $d_{10,k}$ $k = 0, \pm 1, \pm 2, \ldots$ correspond to fluctuations of f over intervals of length 2^{-10}. These observations help explain how the discrete wavelet transform can be an exceptionally efficient scheme for representing functions.

4.2.2 Comparison between wavelet and Fourier analysis

Some of the practical motivations underlying the use of expansion basis functions such as Fourier analysis or wavelet analysis are

(1) improved understanding,

(2) denoising signals, and

(3) data compression.

By representation of signals or functions in other forms these tasks become easier. The approach taken with Fourier analysis represents signals in terms of trigonometric functions and as such is particularly suited to situations where the signal is relatively smooth and is not of limited extent.

Much naturally arising data has been found to be better represented using wavelets which are better able to cope with discontinuities and where the signal is of local extent. Generally, the efficiency of the representation depends on the types of signal involved. If your signal contains

(1) discontinuities (in both the signal and its derivatives), or

(2) varying frequency behaviour

then wavelets are likely to represent the signal more efficiently than is possible with Fourier analysis. One of the most useful features of wavelets is the ease with which a scientist can select the basis functions adapted for the given problem. The Haar mother wavelet is perhaps the simplest of a very wide class of possible wavelet systems used in practice today.

Many applied fields have started to make use of wavelets including astronomy, acoustics, signal and image processing, neurophysiology, music, magnetic resonance imaging, speach discrimination, optics, fractals, turbulence, earthquake prediction, radar, human vision, etc.