Continuous Mathematics

UNIVERSITY OF CAMBRIDGE

Computer Laboratory

Computer Science Tripos, Part IB & Part II (General)
Diploma in Computer Science

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Problem sheet

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1. Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ what are the real and imaginary parts of $z_3 = z_1 z_2$?

2. Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ what is the modulus, $|z_1|$, of z_1 and what is the modulus of $z_3 = z_1 z_2$?

3. Given $z_1 = x_1 + iy_1$ what is $\arg(z_1)$, the argument of z_1? Is it unique? What happens if $z_1 = 0$?

4. Given $z_1 = x_1 + iy_1$ with $x_1 \neq 0$ and $y_1 \neq 0$ show that

$$\text{Arg}(z_1) = \tan^{-1}(y_1/x_1) + \frac{\pi}{2}\text{sign}(y_1)(1 - \text{sign}(x_1))$$

where for $a \neq 0$

$$\text{sign}(a) = \begin{cases} +1 & a > 0 \\ -1 & a < 0 \end{cases}.$$

What happens if $x_1 = 0$ or $y_1 = 0$?

5. Suppose that $|z_1| = |z_2| = 1$. Using an Argand diagram, explain how computing their product $z_3 = z_1 z_2$ amounts to a rotation in the complex plane. Why is the multiplication of these complex variables reduced an addition? What is the value of $|z_3|$?

6. Given $z = \exp(2\pi i/5)$, what is the value of z^5? Explain your result using an Argand diagram.

7. Consider the complex exponential function $f(x) = \exp(2\pi i\omega x)$. What are the real and imaginary parts of $f(x)$ as functions of x?

8. For the imaginary number $i = \sqrt{-1}$, consider the quantity $\sqrt{i}$. Express $\sqrt{i}$ as a complex exponential. In what quadrant of the complex plane does it lie? What are the real and imaginary parts of $\sqrt{i}$? What is the modulus of $\sqrt{i}$?

9. Given $f(x) = \cos(1/x)$, does $\lim_{x\to 0} f(x)$ exist? What happens if instead $f(x) = x\cos(1/x)$?

10. Show that "continuity at $x = a$" does not imply "differentiable at $x = a$" by constructing a suitable counterexample.

11. Write down the Taylor's series approximation to the value of a function $f(b)$ given only the function and it's first three derivatives evaluated at $x = a$, namely, $f(a), f'(a), f''(a)$ and $f'''(a)$. You may assume that these derivatives exist and that f and each of its deriavtives is a continuous function.

12. Give an expression for computing $f(t)$ if we know only its projections $< f(t), \Psi_j(t) >$ onto this set of orthonormal basis functions $\{\Psi_j(t)\}$. Explain what is happening.

13. What will be the Fourier Transform of the m^{th} derivative of $f(x)$ with respect to x in terms of the Fourier Transform, $F(\mu)$, of $f(x)$: $\left(\frac{d}{dx}\right)^m f(x)$?

14. What happens to the Fourier Transform after shifting $f(x)$ by a distance α: $f(x - \alpha)$?

15. What happens to the Fourier Transform after dilating $f(x)$ by a factor a: $f(x/a)$?

16. What is the principal computational advantage of using orthogonal functions, over non-orthogonal ones, when representing a set of data as a linear combination of a universal set of basis functions?

    If $\Psi_k(x)$ belongs to a set of orthonormal basis functions, and $f(x)$ is a function or a set of data that we wish to represent in terms of these basis functions, what is the basic computational operation we need to perform involving $\Psi_k(x)$ and $f(x)$?

17. Any real-valued function $f(x)$ can be represented as the sum of one function $f_e(x)$ that has even symmetry (it is unchanged after being flipped around the origin $x = 0$) so that $f_e(x) = f_e(-x)$, plus one function $f_o(x)$ that has odd symmetry, so that $f_o(x) = -f_o(-x)$. Such a decomposition of any function $f(x)$ into $f_e(x) + f_o(x)$ is illustrated by

$$f_e(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$$

$$f_o(x) = \frac{1}{2}f(x) - \frac{1}{2}f(-x)\,.$$

Use this type of decomposition to explain why the Fourier transform of any real-valued function has *Hermitian symmetry:* its real-part has even symmetry, and its imaginary-part has odd symmetry.

Comment on how this redundancy can be exploited to simplify computation of Fourier transforms of real-valued, as opposed to complex-valued, data.

18. Newton's definition of a derivative in his formulation of The Calculus captures the notion of integer-order differentiation, *e.g.* the first or second derivative, etc. But in scientific computing we sometimes need a notion of fractional-order derivatives, as for example in fluid mechanics.

    Explain how "Fractional Differentiation" (derivatives of non-integer order) can be given precise quantitative meaning through Fourier analysis.

    Suppose that a continuous function $f(x)$ has Fourier Transform $F(\mu)$. Outline an algorithm (as a sequence of mathematical steps, not an actual program) for computing the 1.5^{th} derivative of some function $f(x)$

    $$\frac{d^{(1.5)}f(x)}{dx^{(1.5)}}$$

19. Given the definition of the Fourier transform and its inverse show that if α and A are non-zero constants then

    $$\widehat{F}(\mu) = A\int_{-\infty}^{\infty} f(x)e^{-i\alpha\mu x}\,dx$$

    implies that

    $$f(x) = \frac{|\alpha|}{2\pi A}\int_{-\infty}^{\infty} \widehat{F}(\mu)e^{i\alpha\mu x}\,d\mu$$

    In order to see what is going on start with the case $\alpha = 1$ and $A = 1/2\pi$.

20. Comment on the strengths and weakness of the Fourier analysis approach compared with an approach using wavelets.