Resource pooling in congested networks: proportional fairness and product form

Neil Walton

Joint work with:
Frank Kelly and Laurent Massoulié

Statistical Laboratory, University of Cambridge.

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We review some recent results.

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First we consider an equivalence between single-path and multi-path routing...

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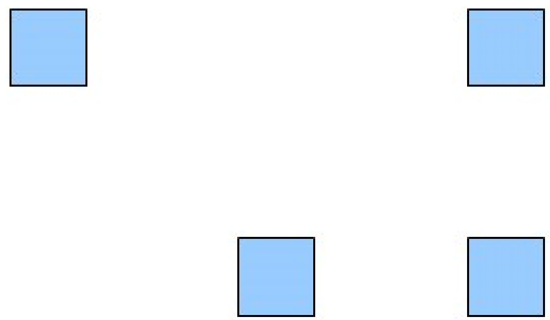

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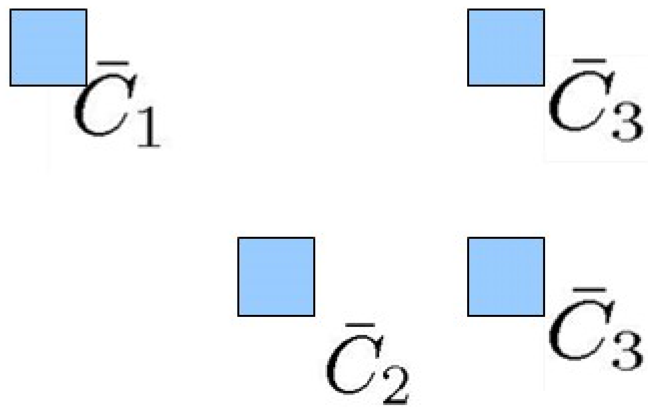

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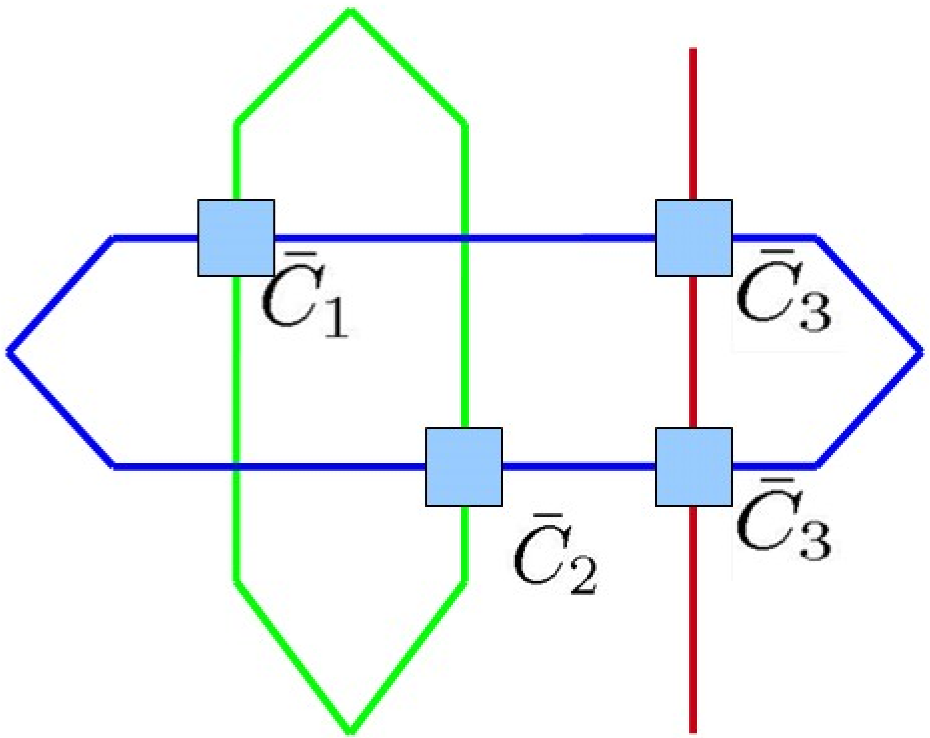

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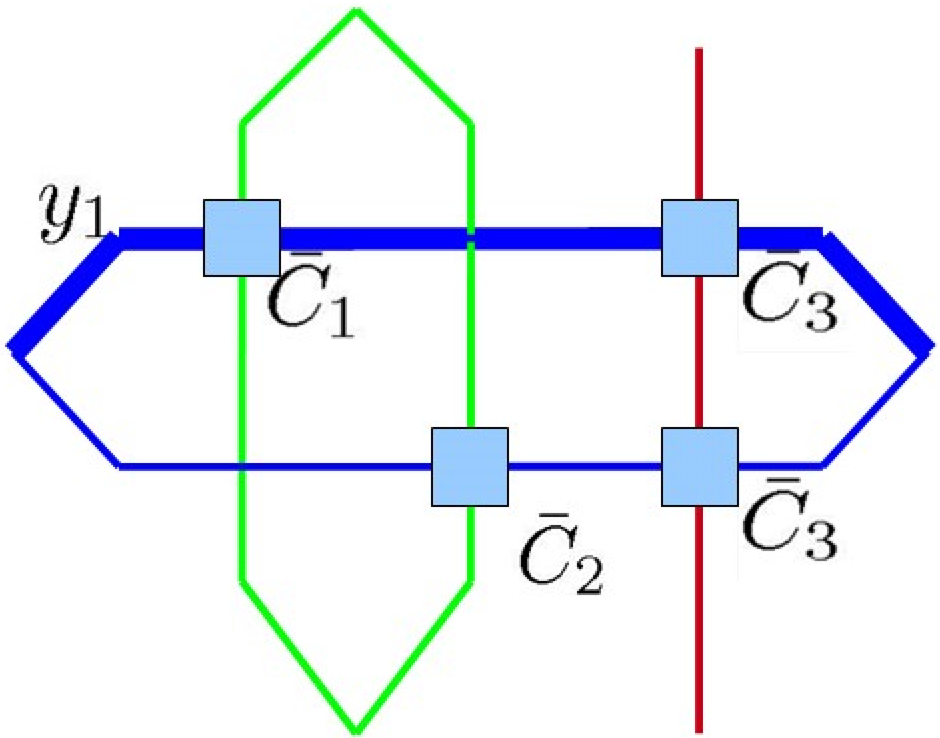

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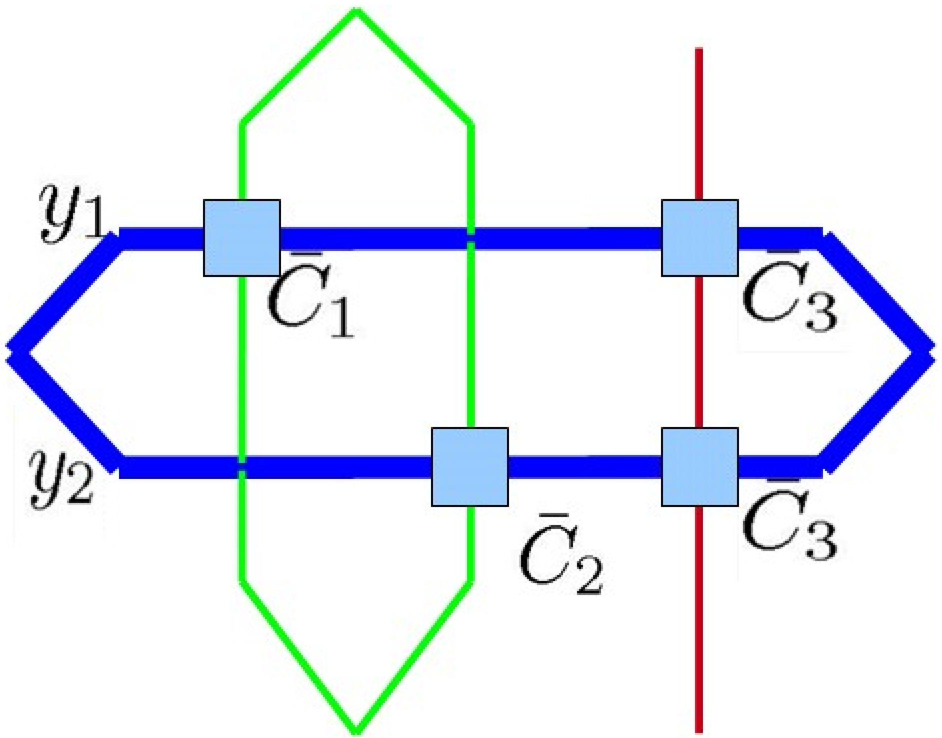

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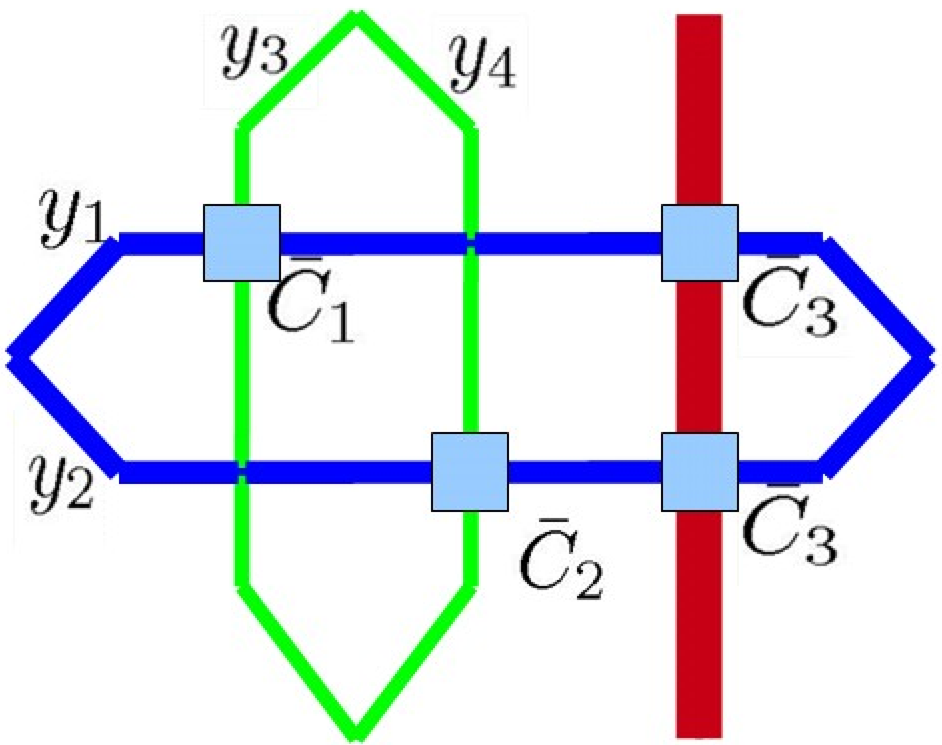

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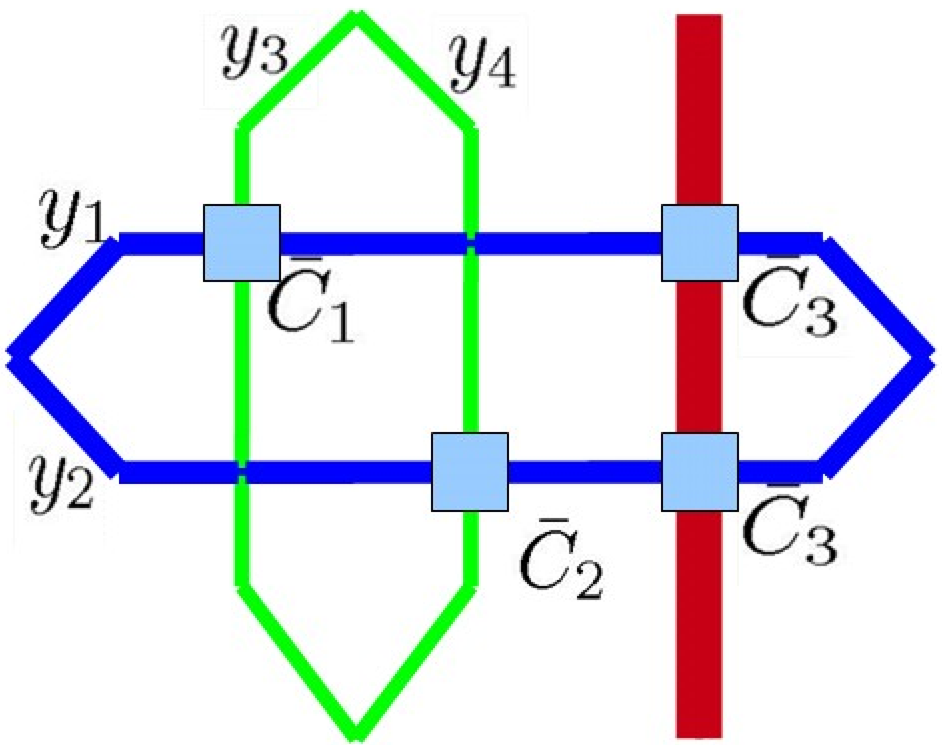

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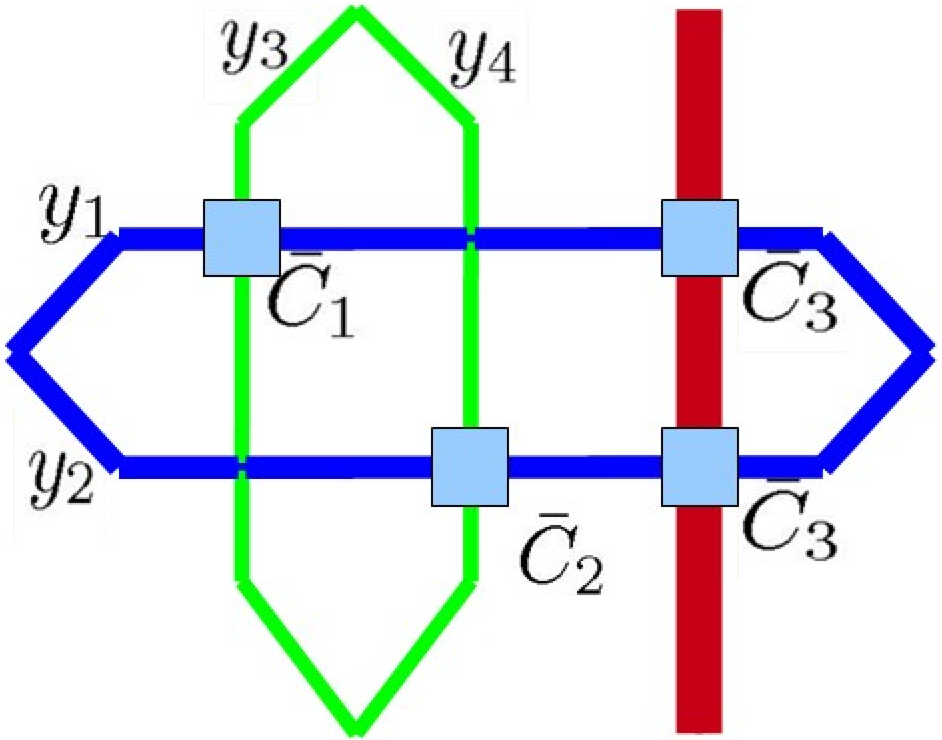

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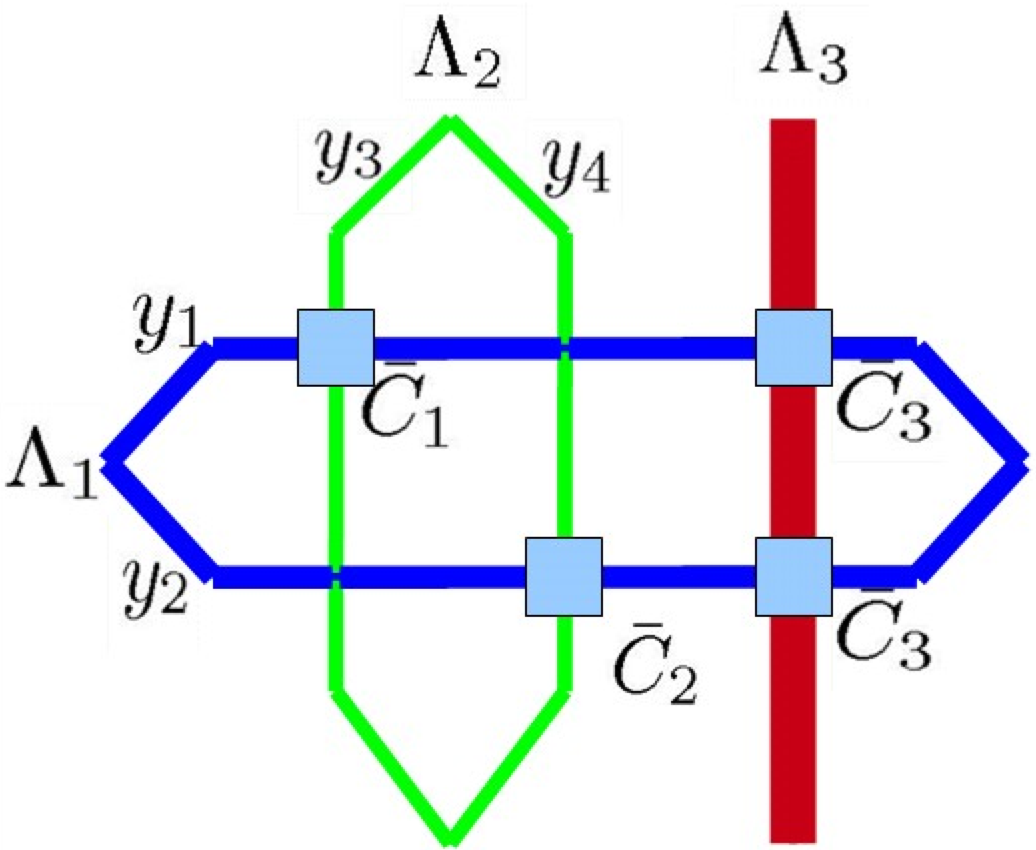

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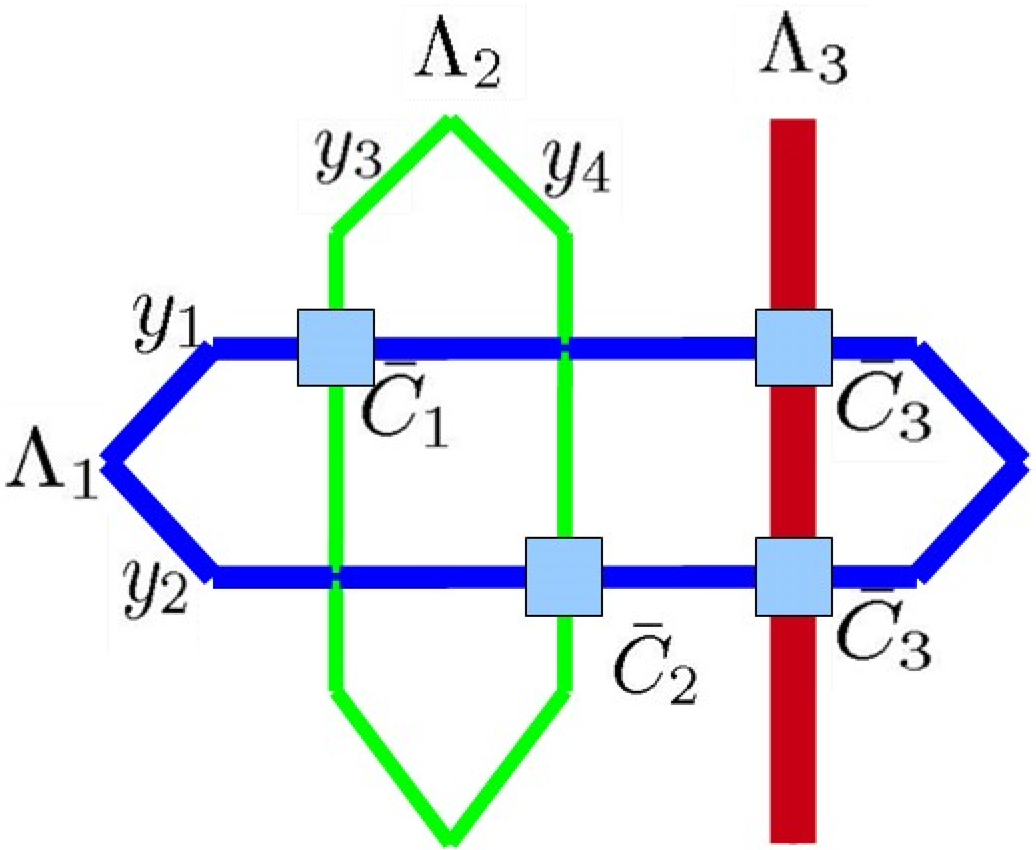

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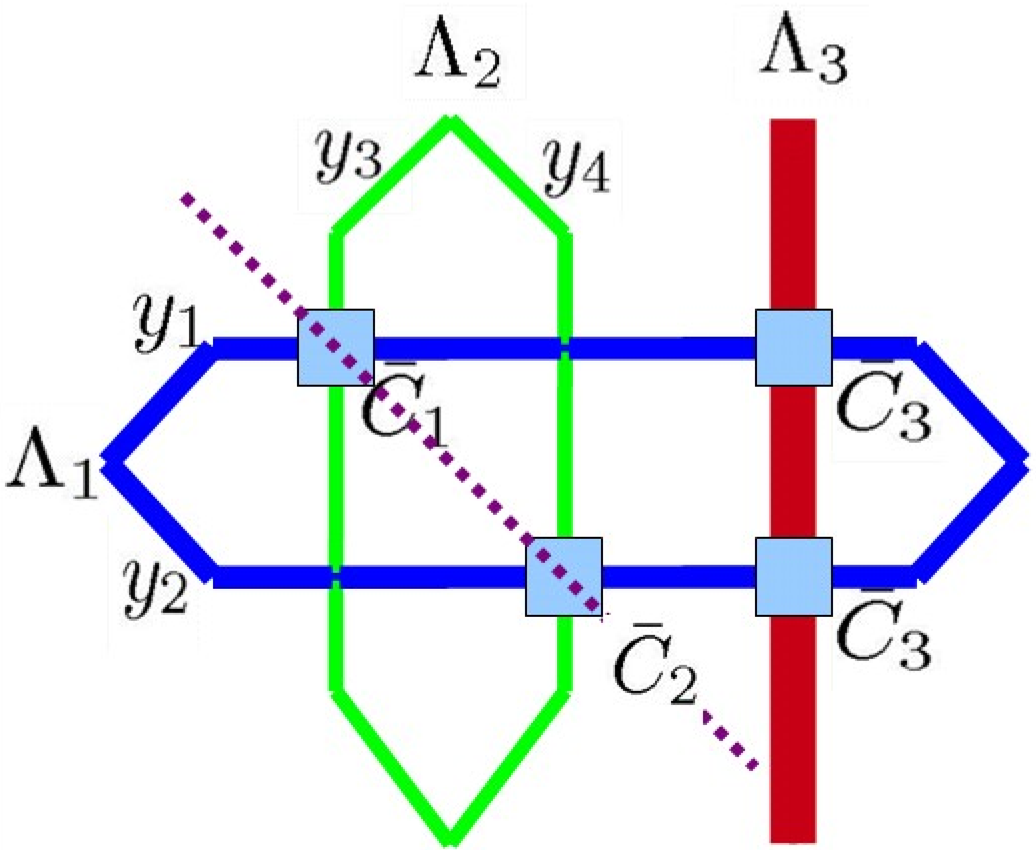

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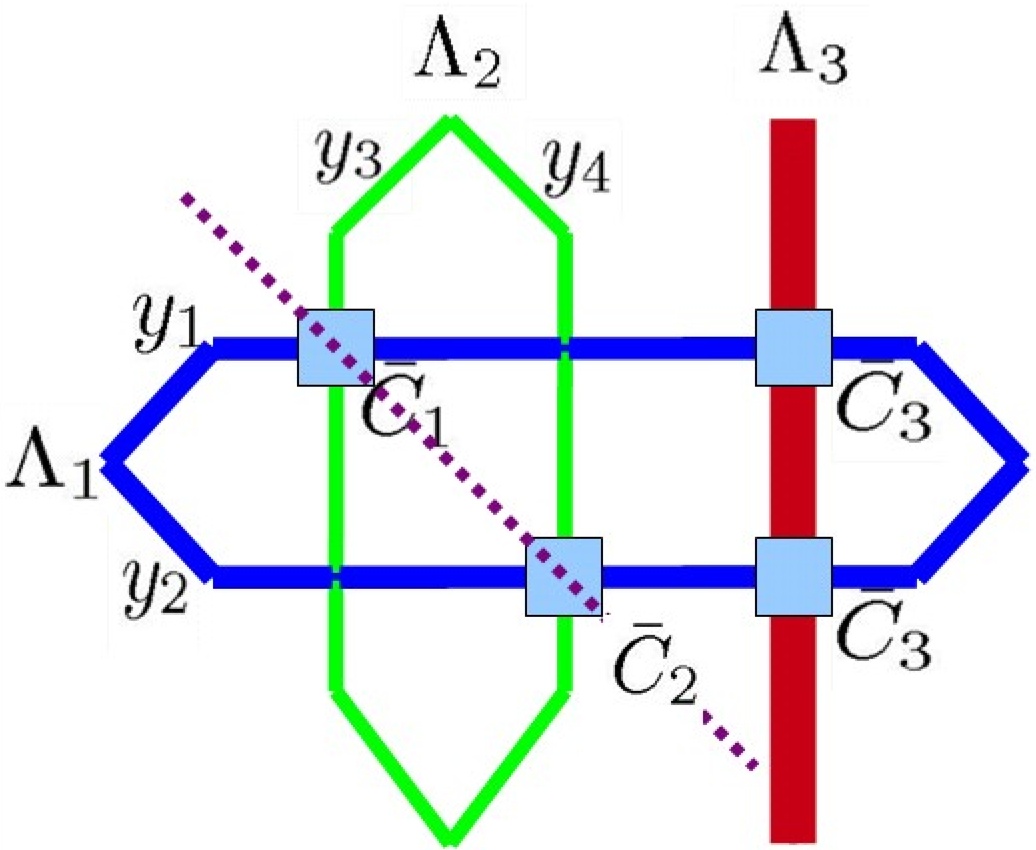

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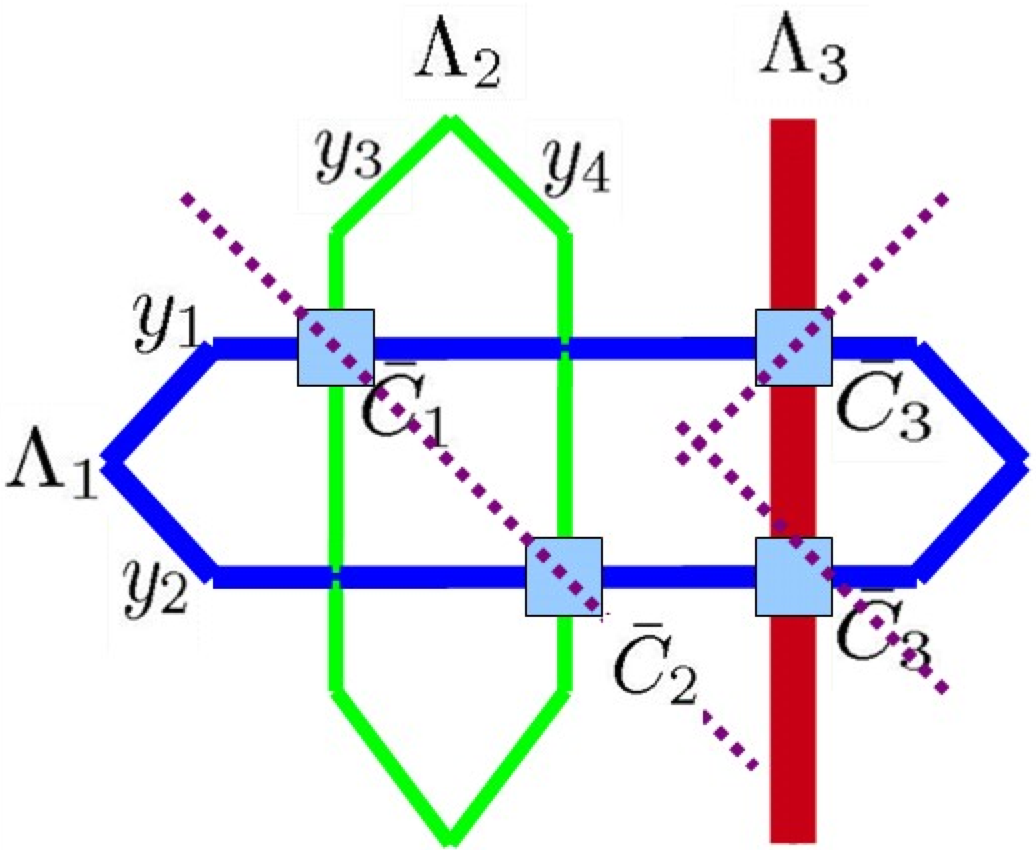

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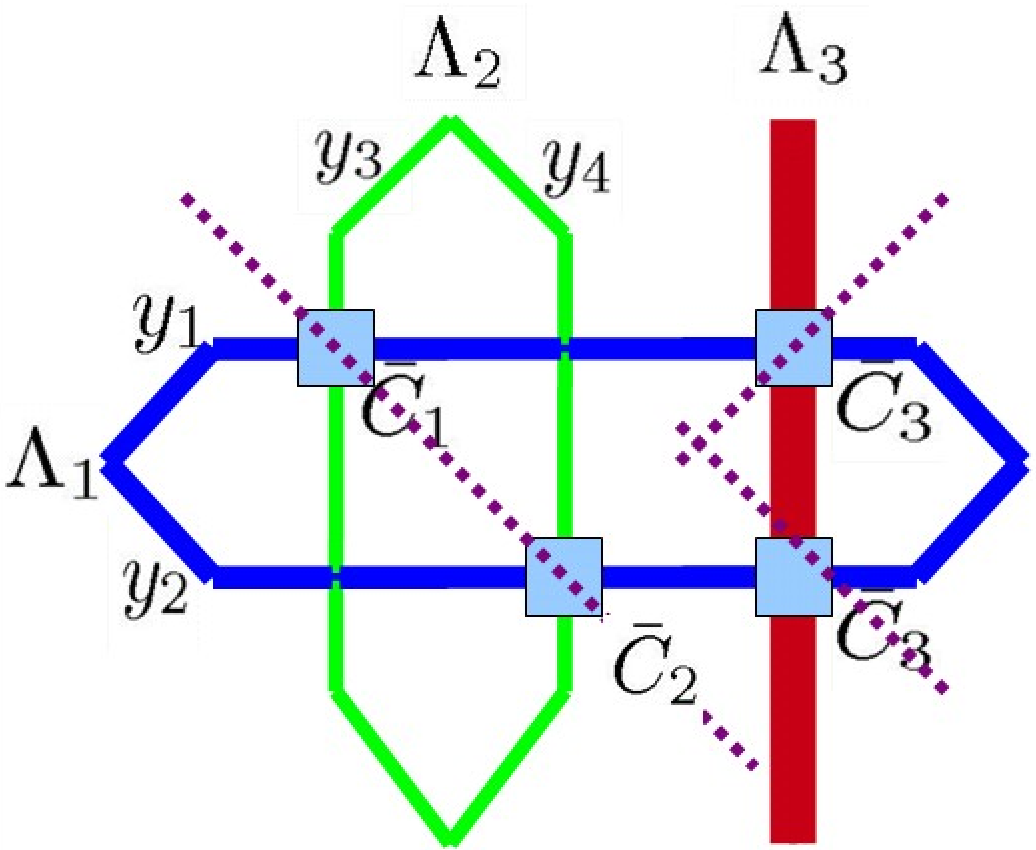

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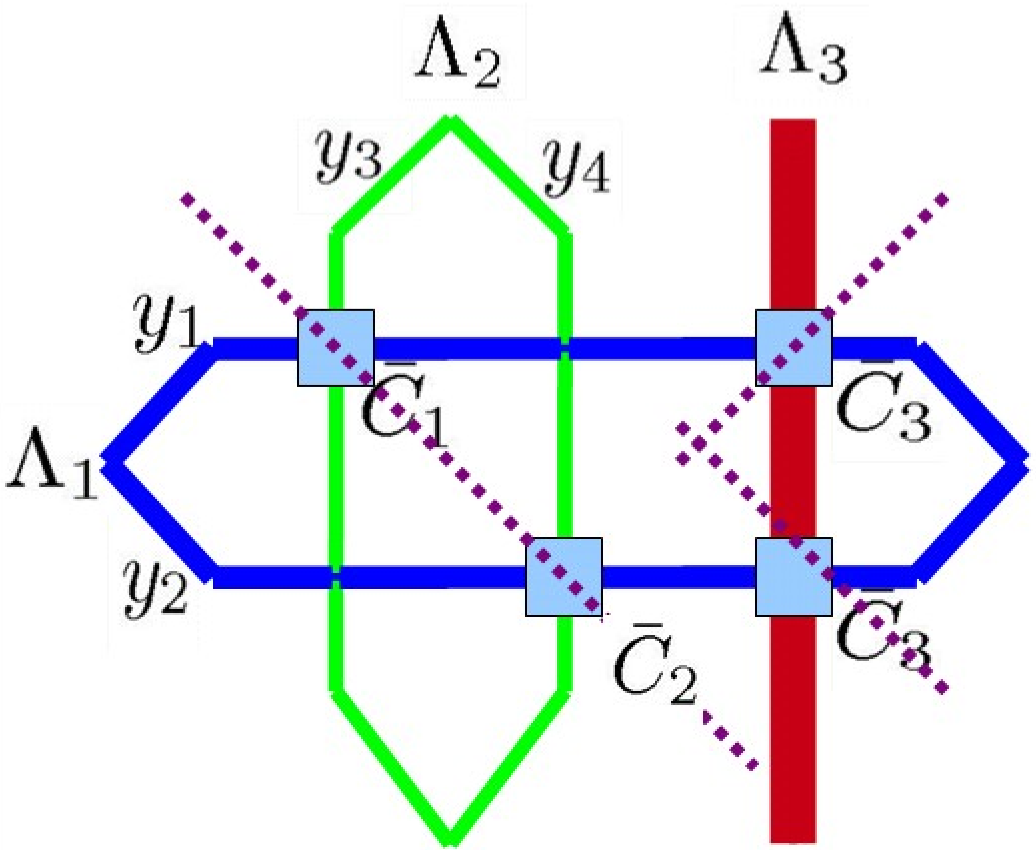

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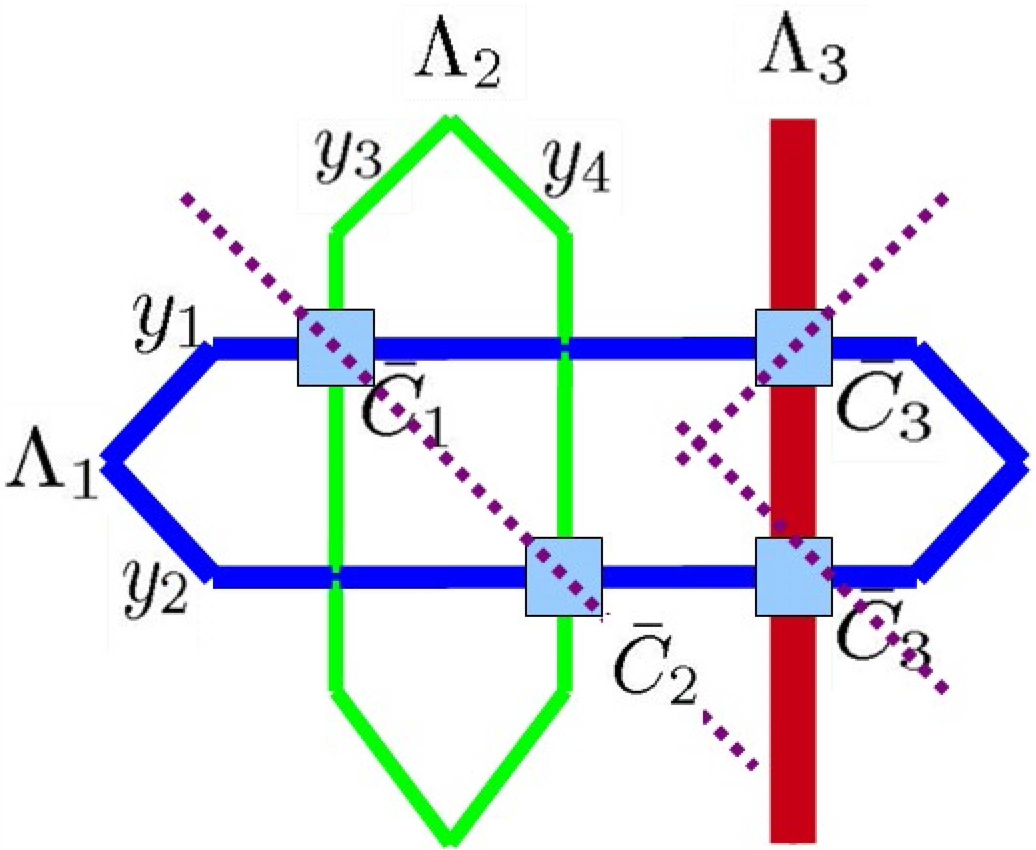

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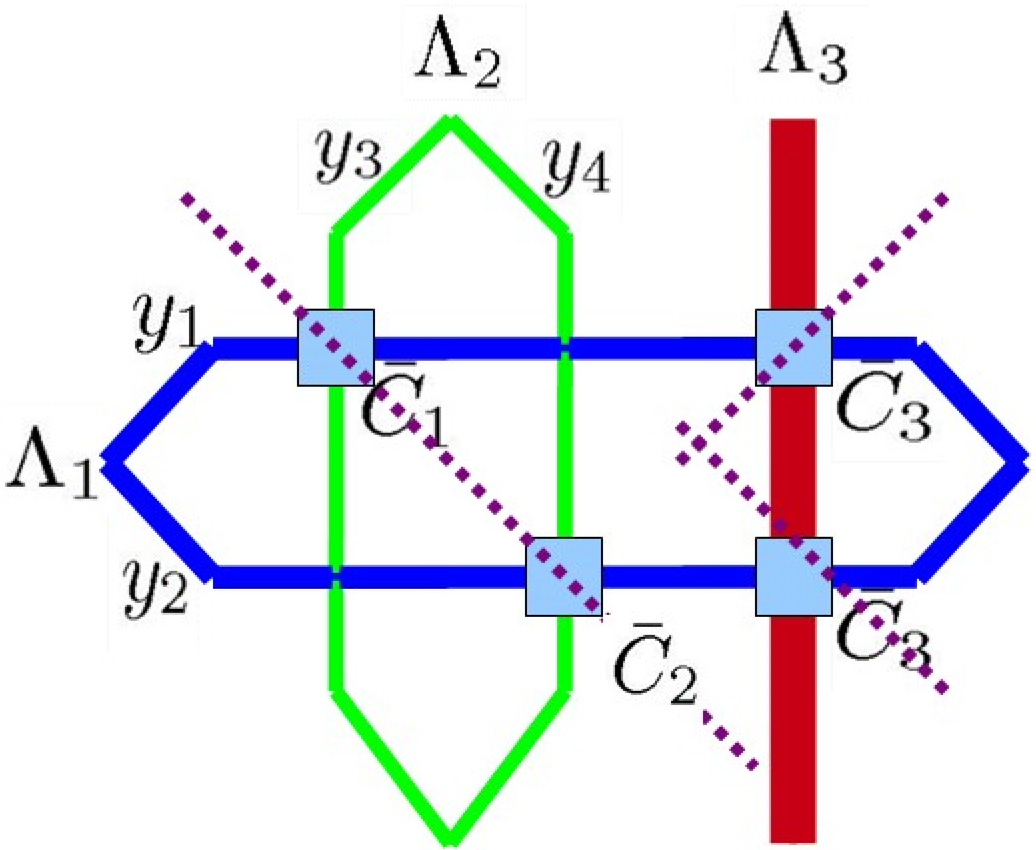

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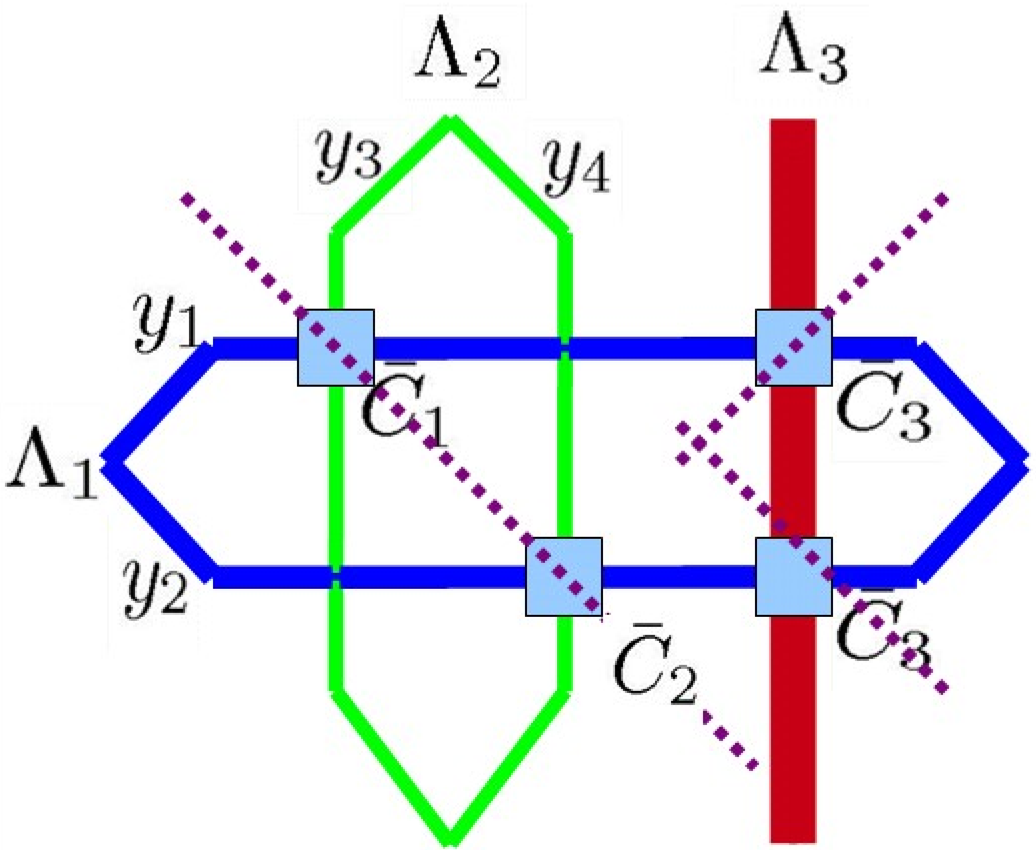

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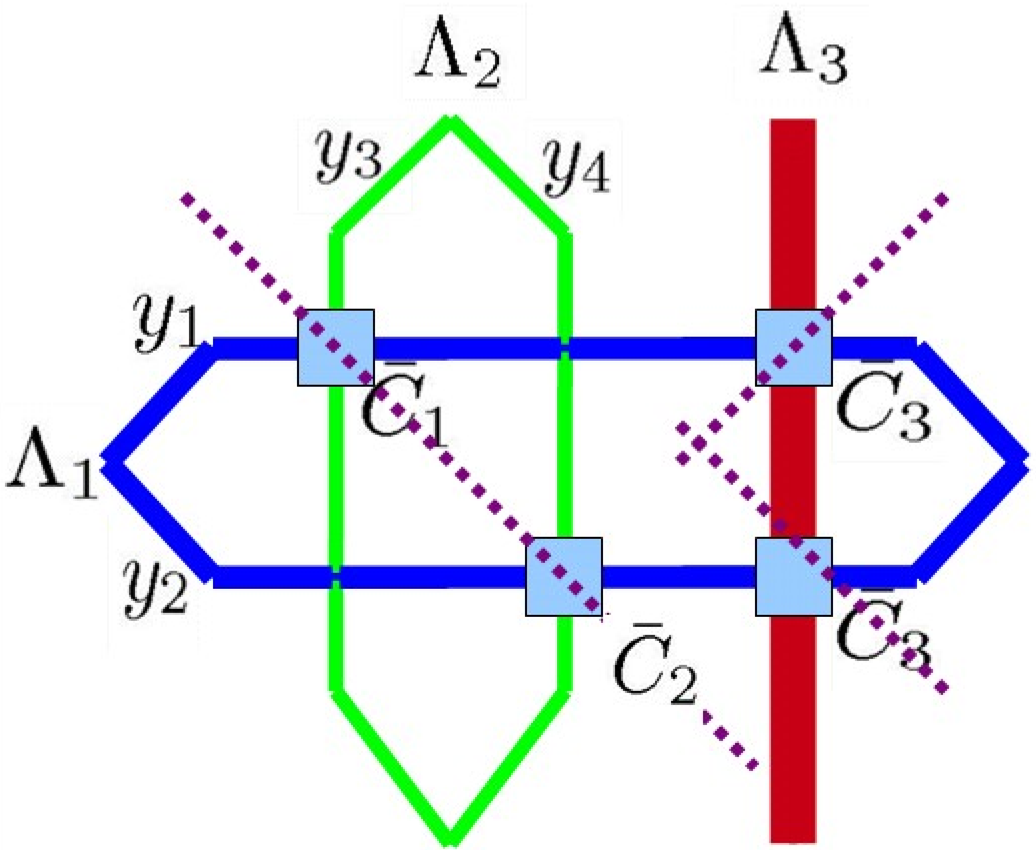

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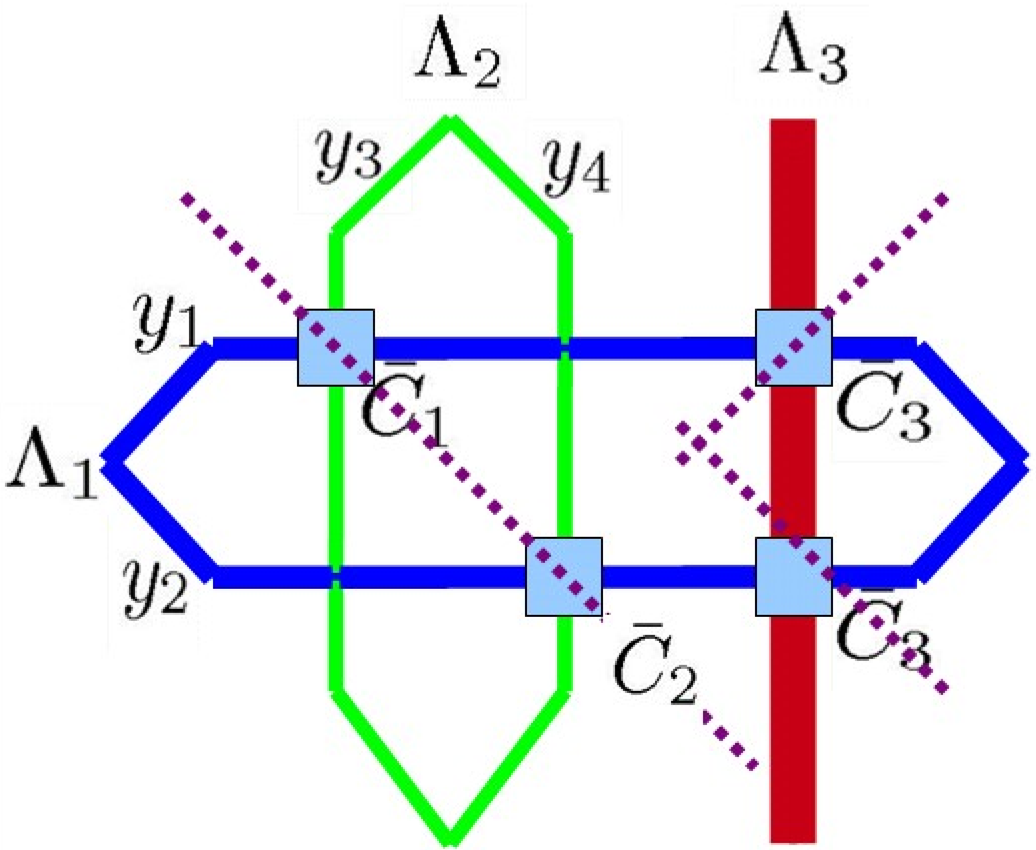
Set of Resource pools

A network (Kang, Kelly, Lee, Williams '09)



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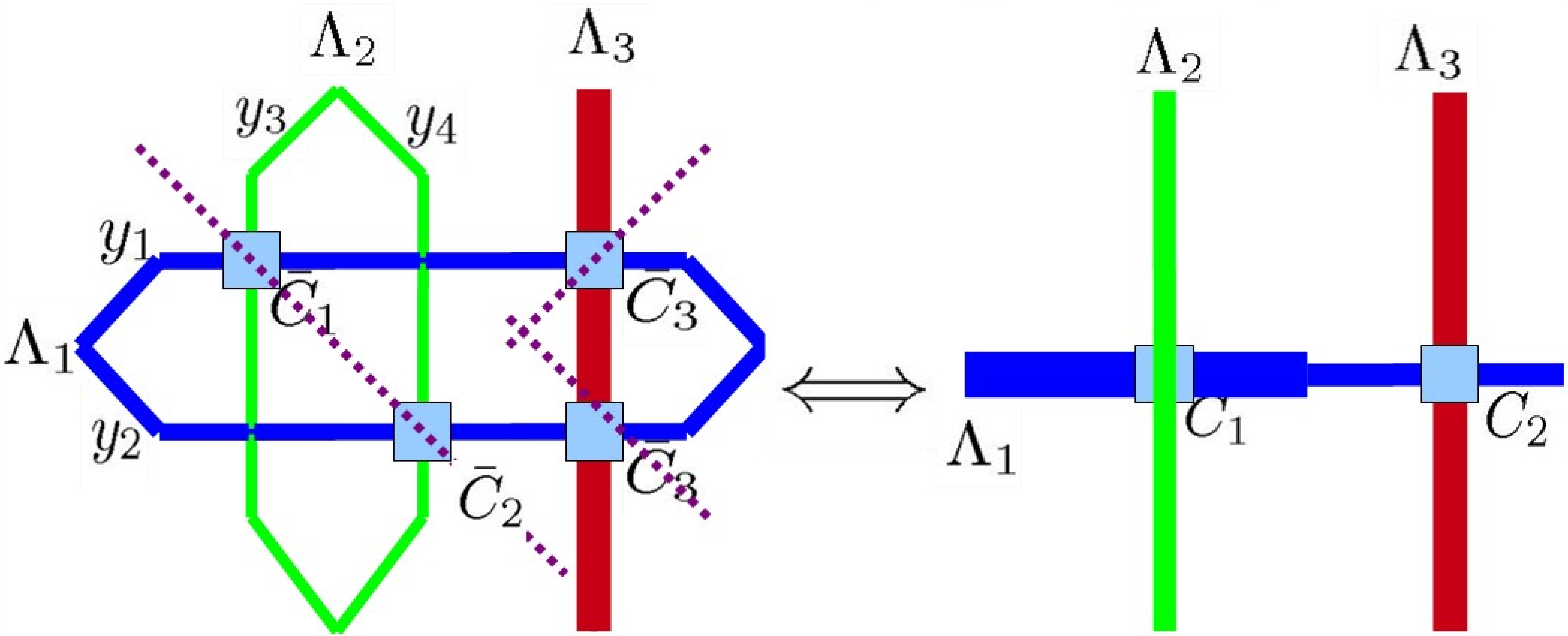

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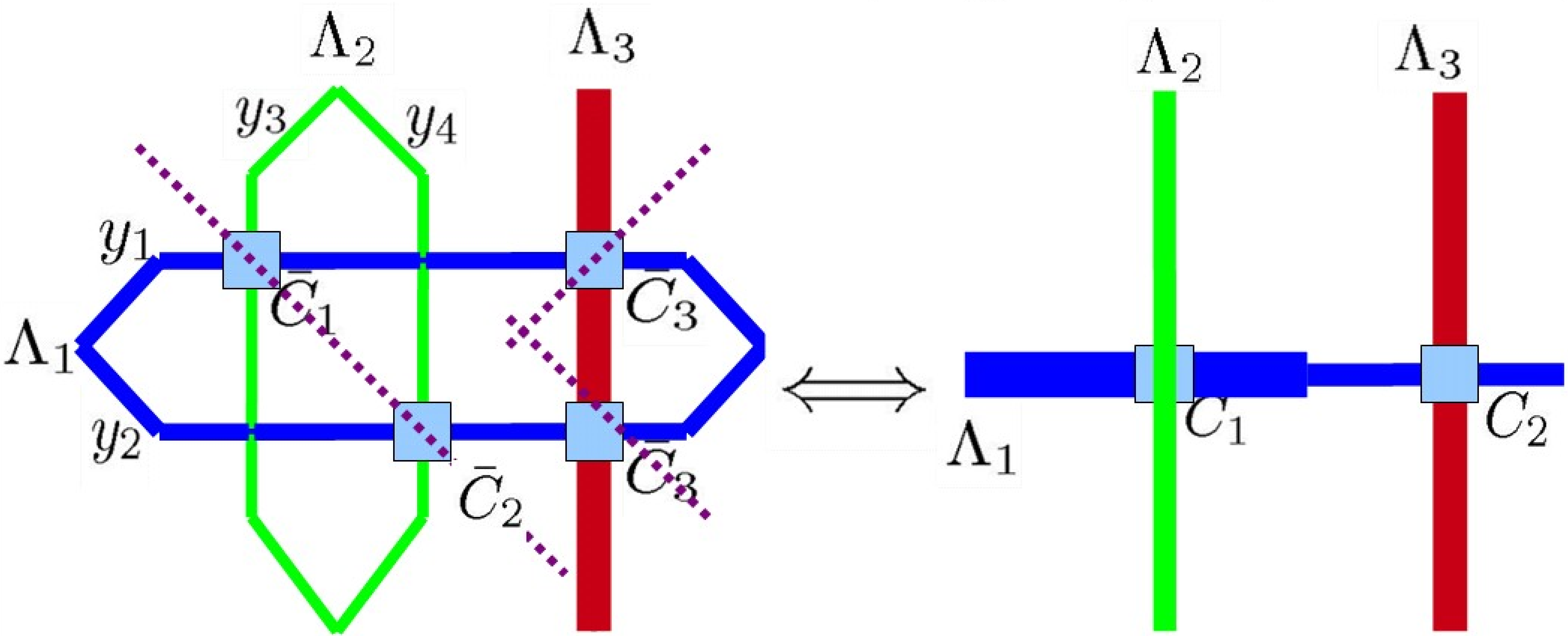

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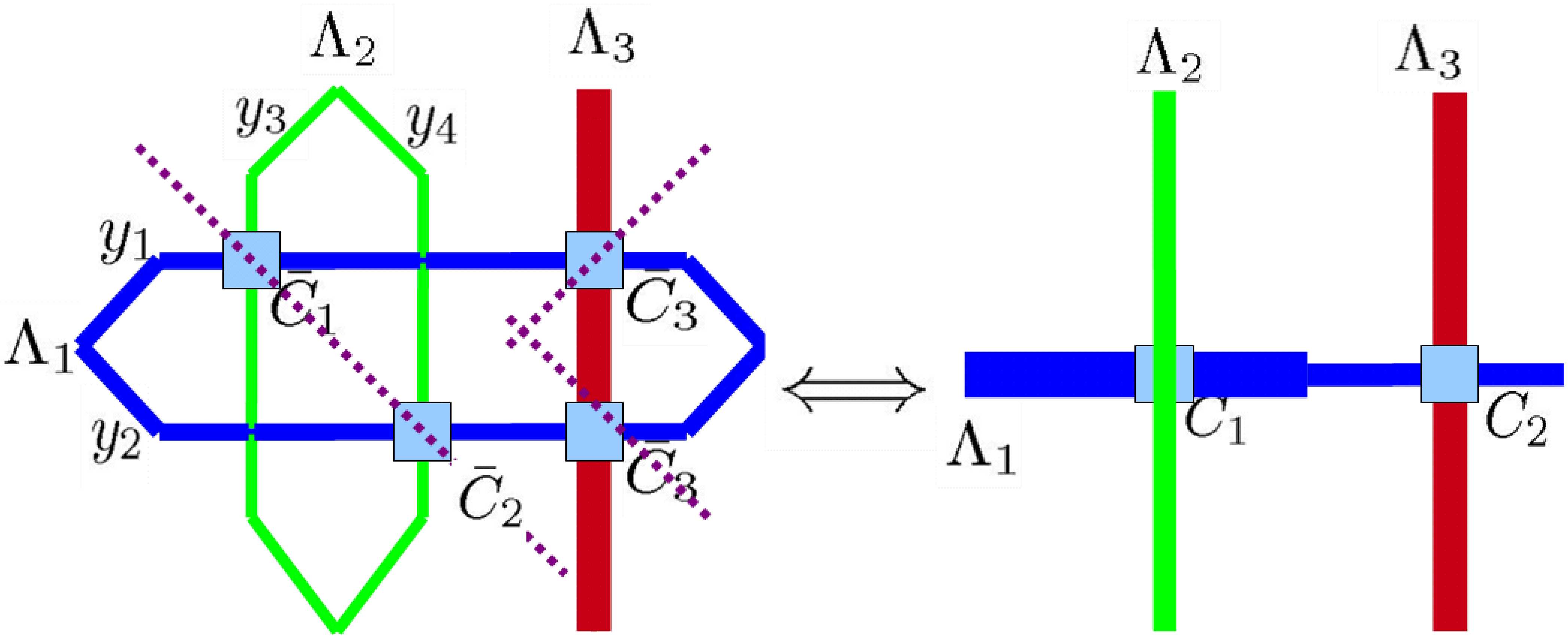

So multi-path routing is the same as single path routing when we pool resources

Proportional fairness

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$$\text{subject to} \quad \sum_i a_{ji}\Lambda_i \leq C_j, \quad j \in \mathcal{J}$$

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But

Proportional fairness does have some special properties…

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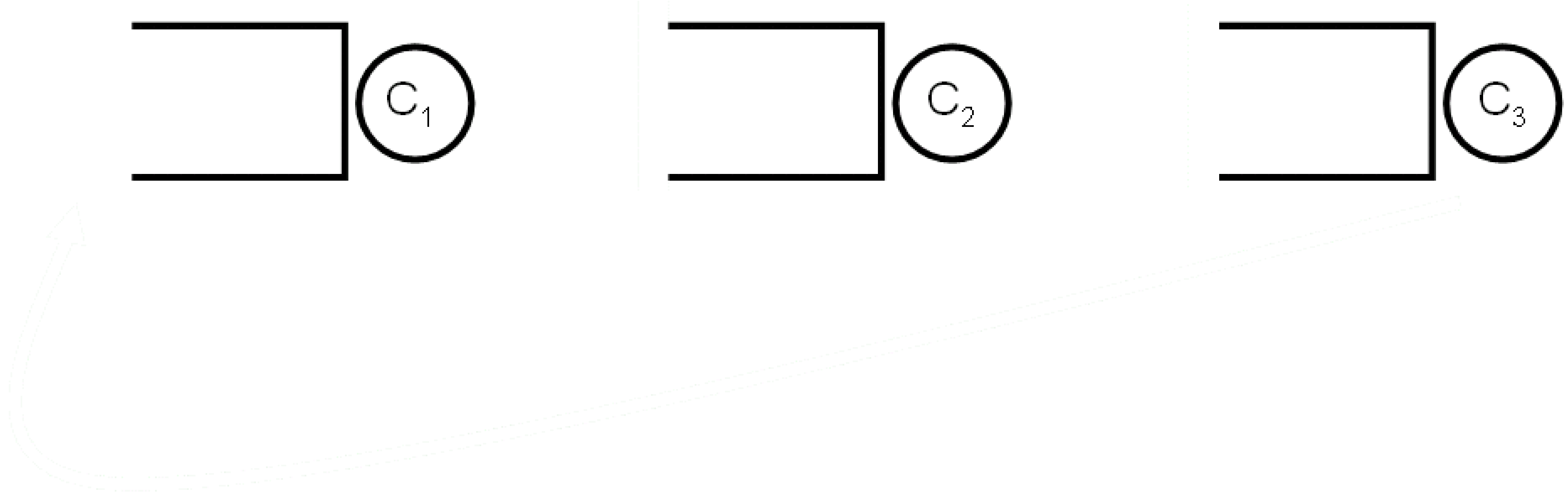
A Multi-class Network of Processor Sharing Queues



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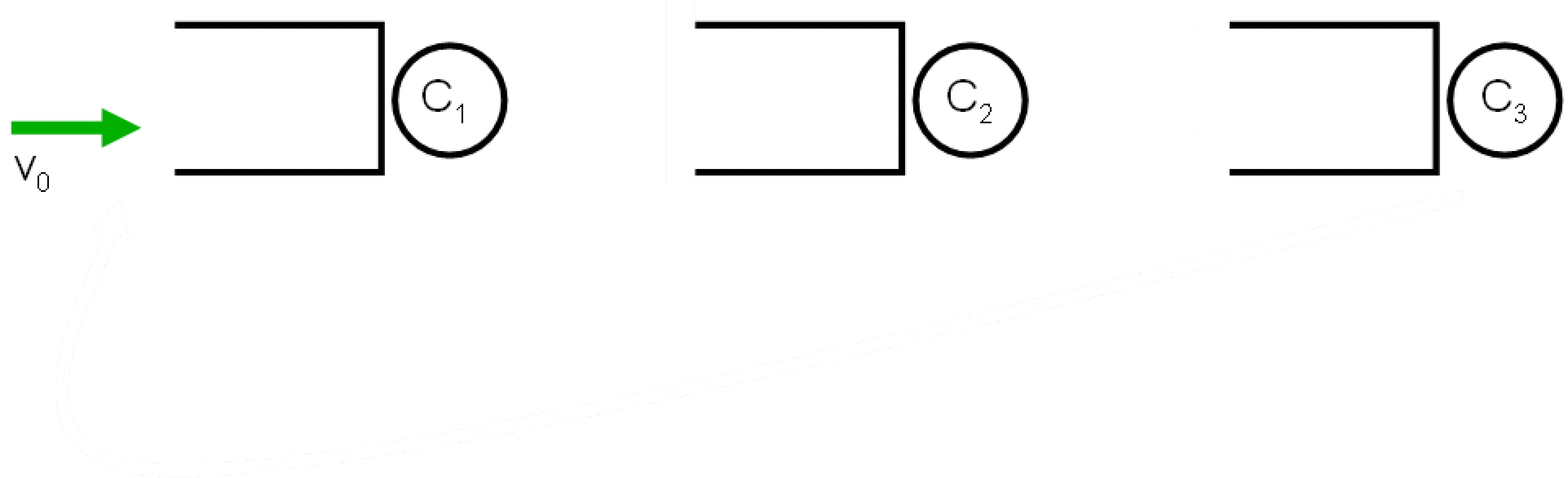

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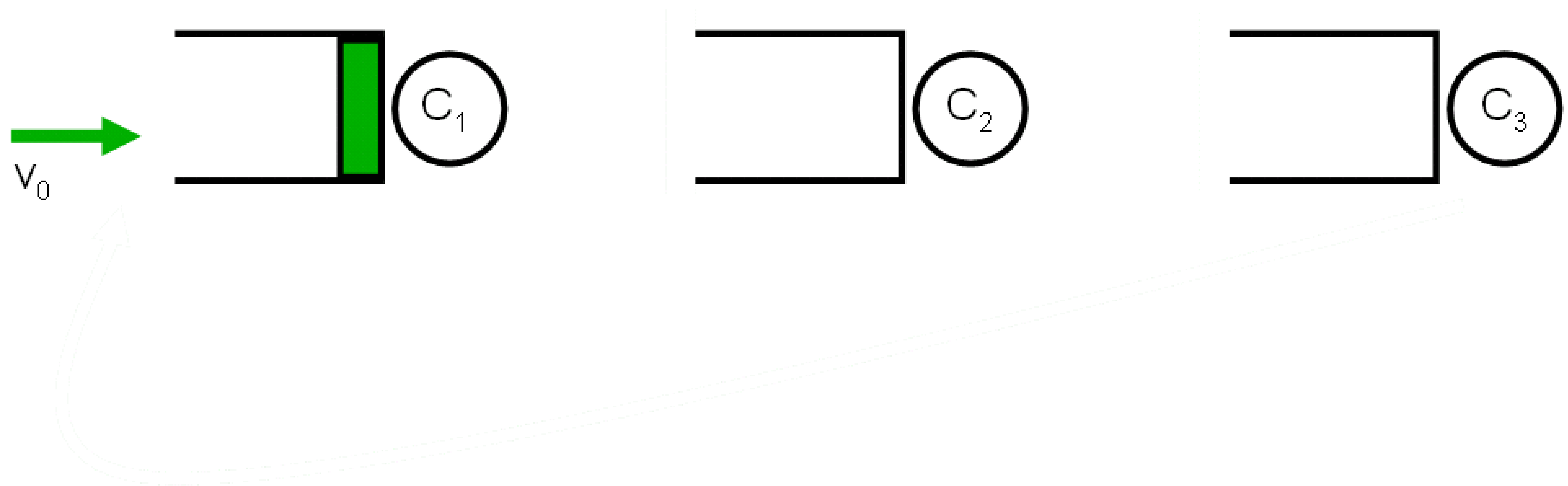

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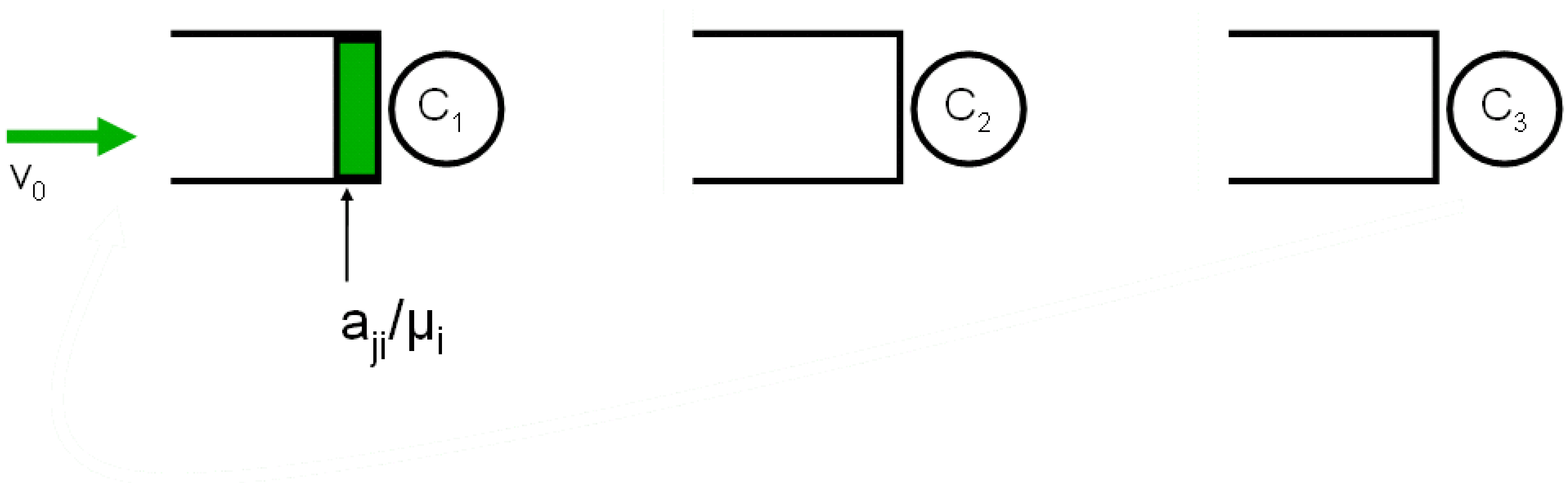

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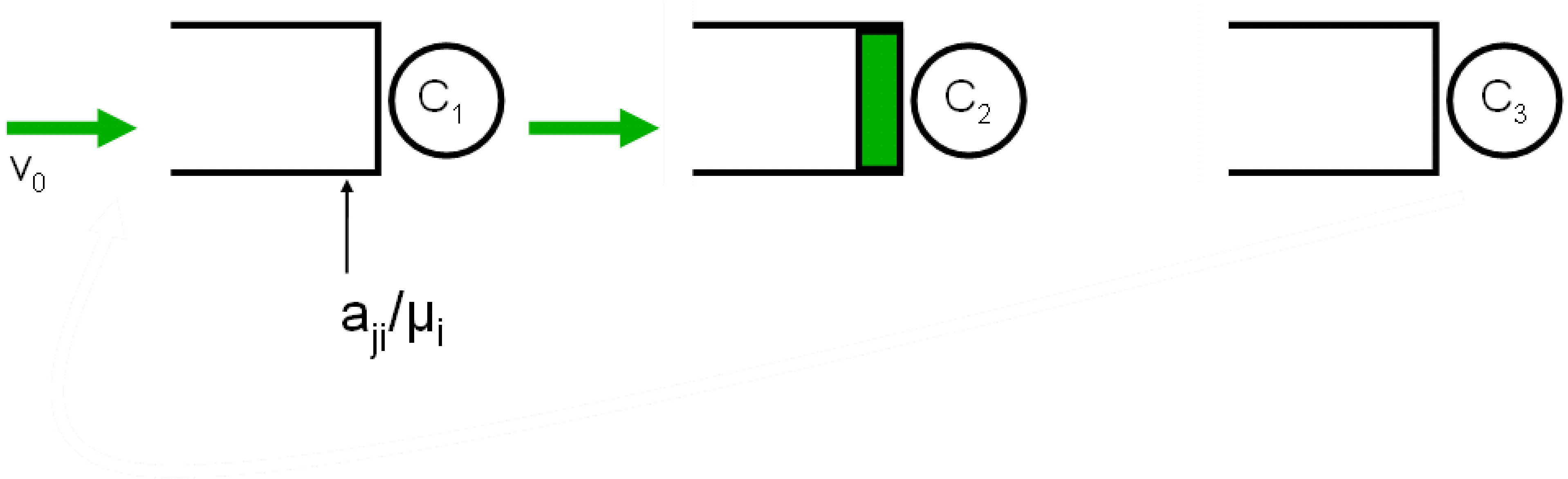

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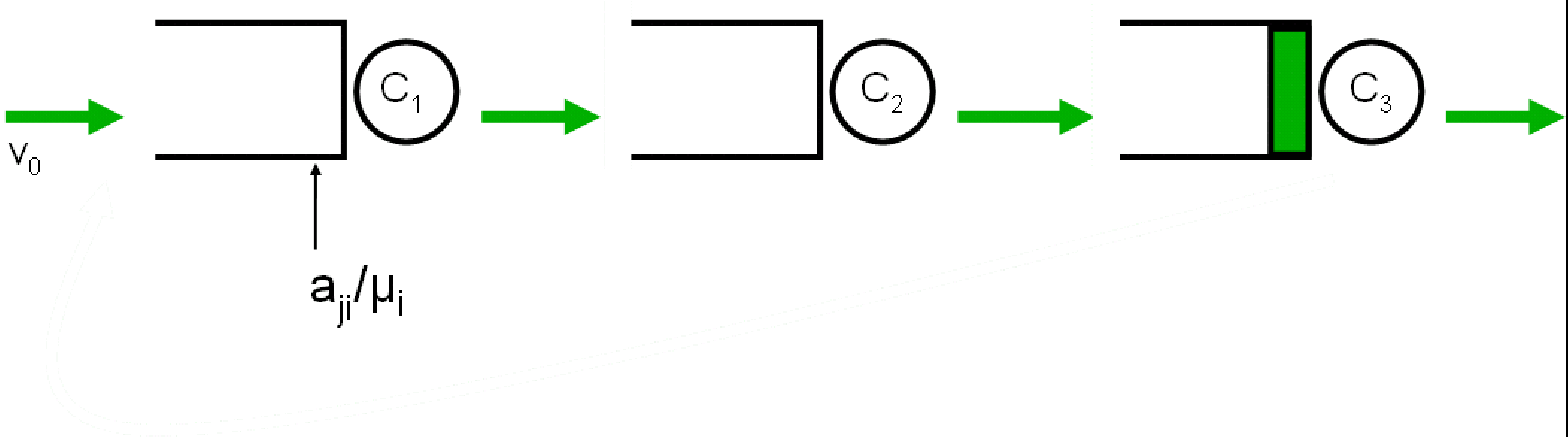

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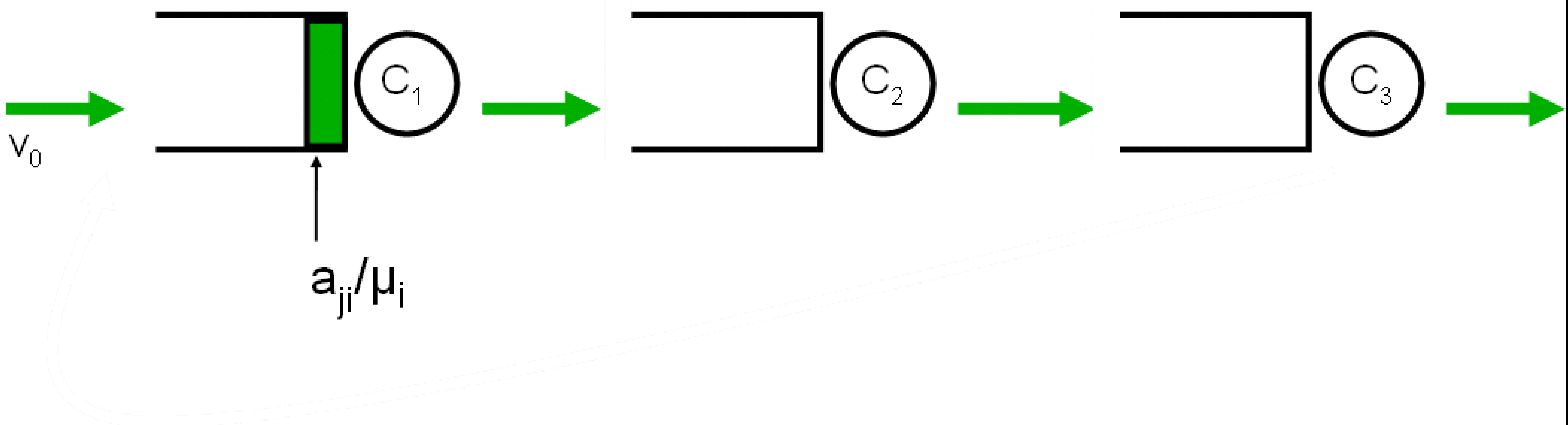

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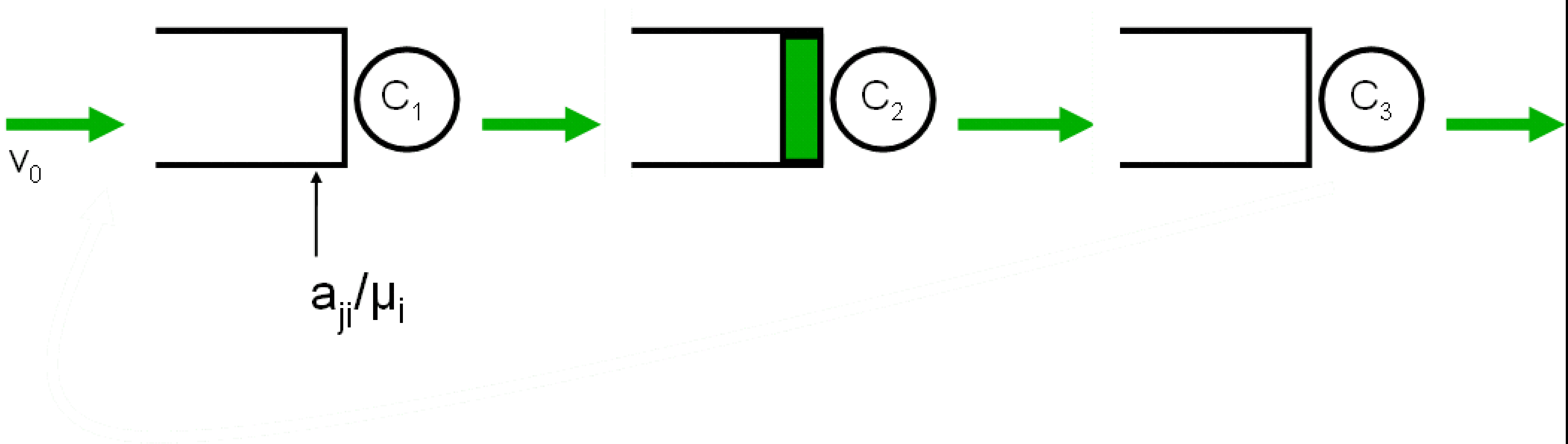

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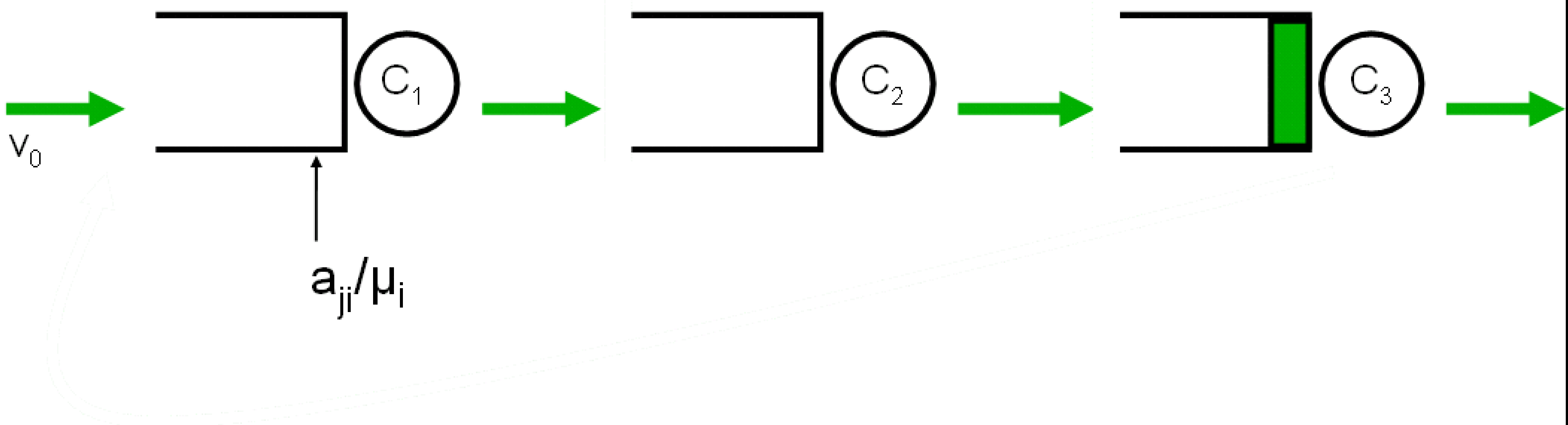

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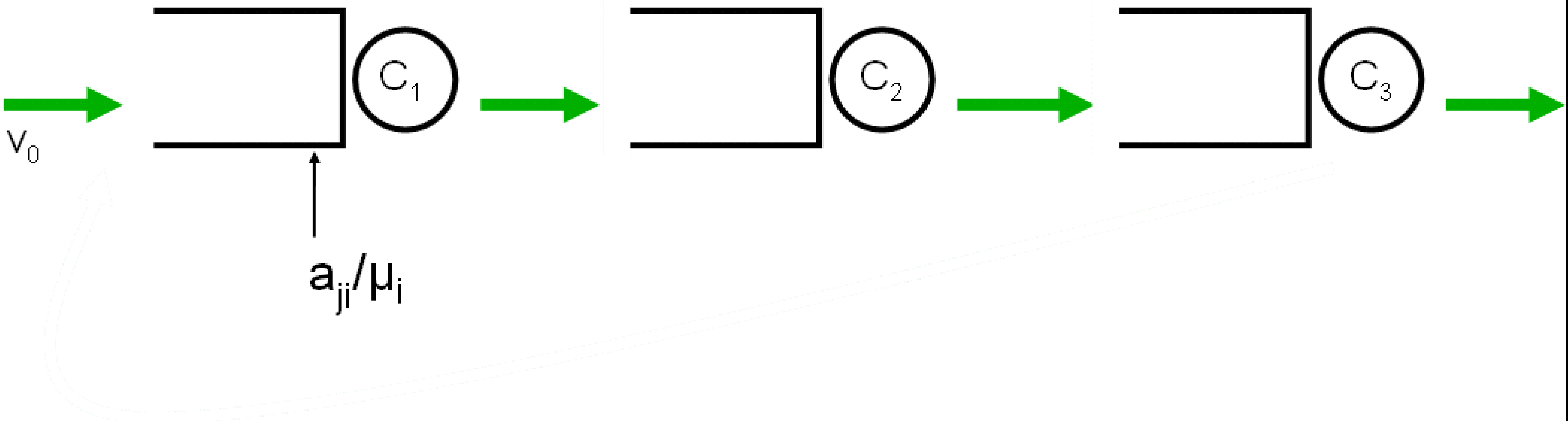

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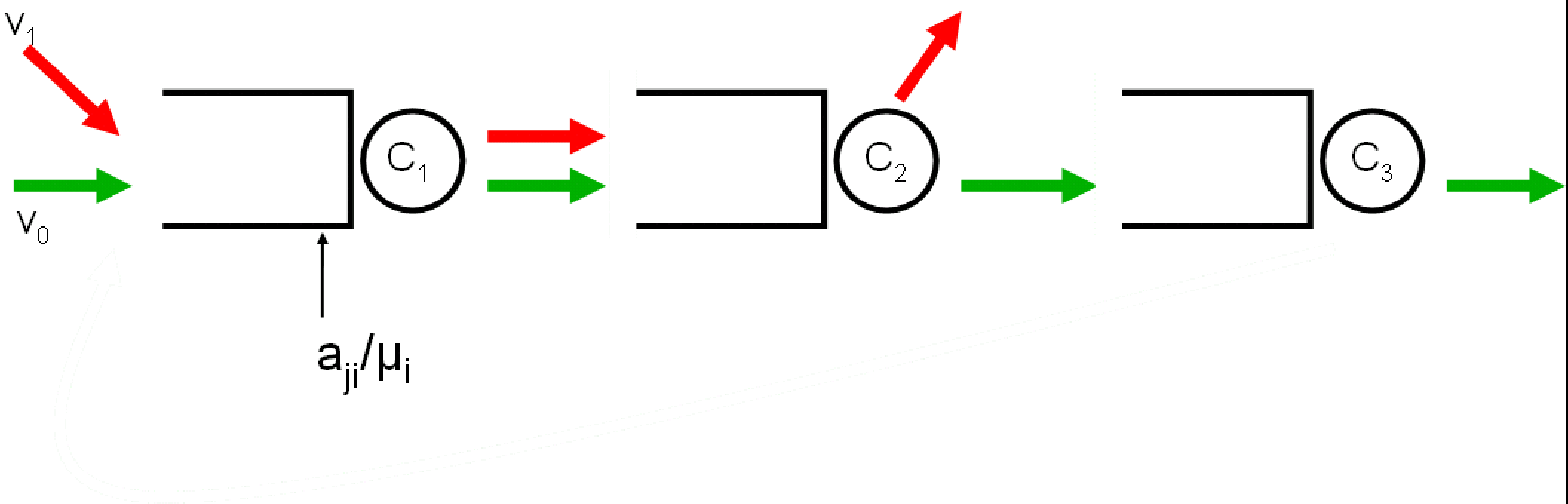

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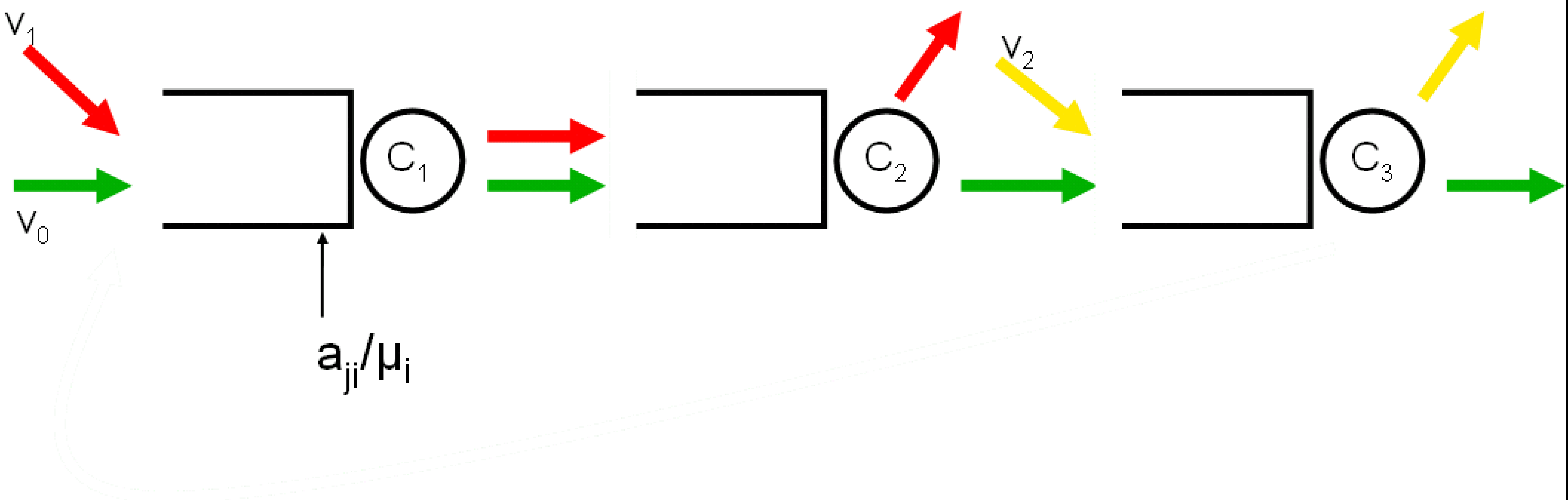

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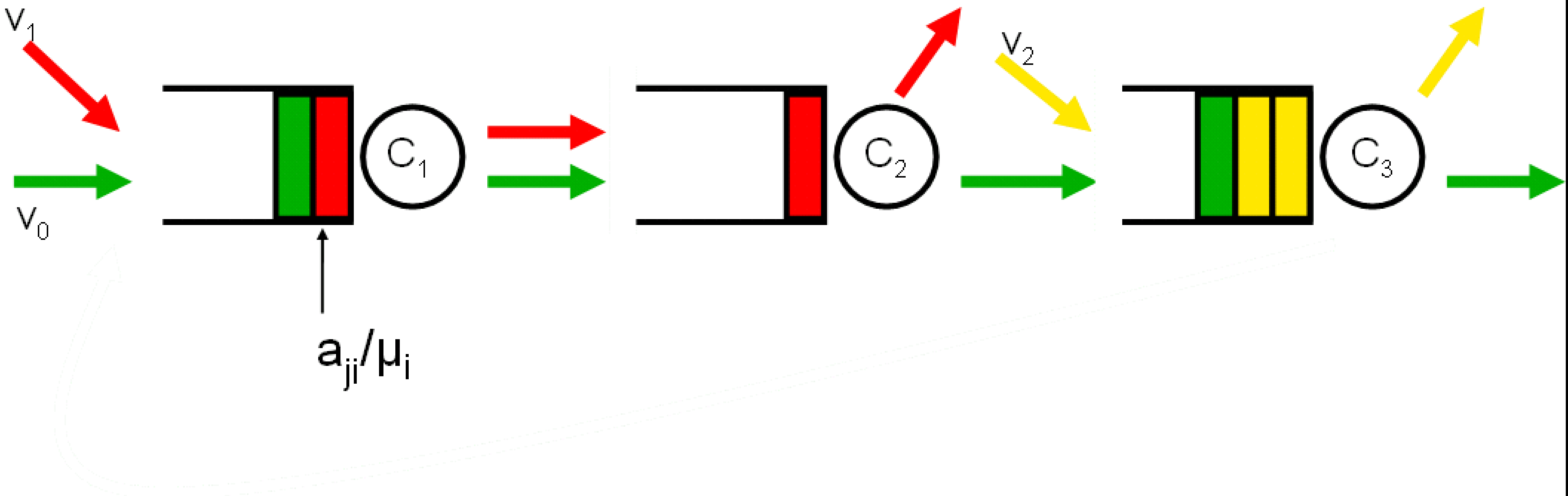

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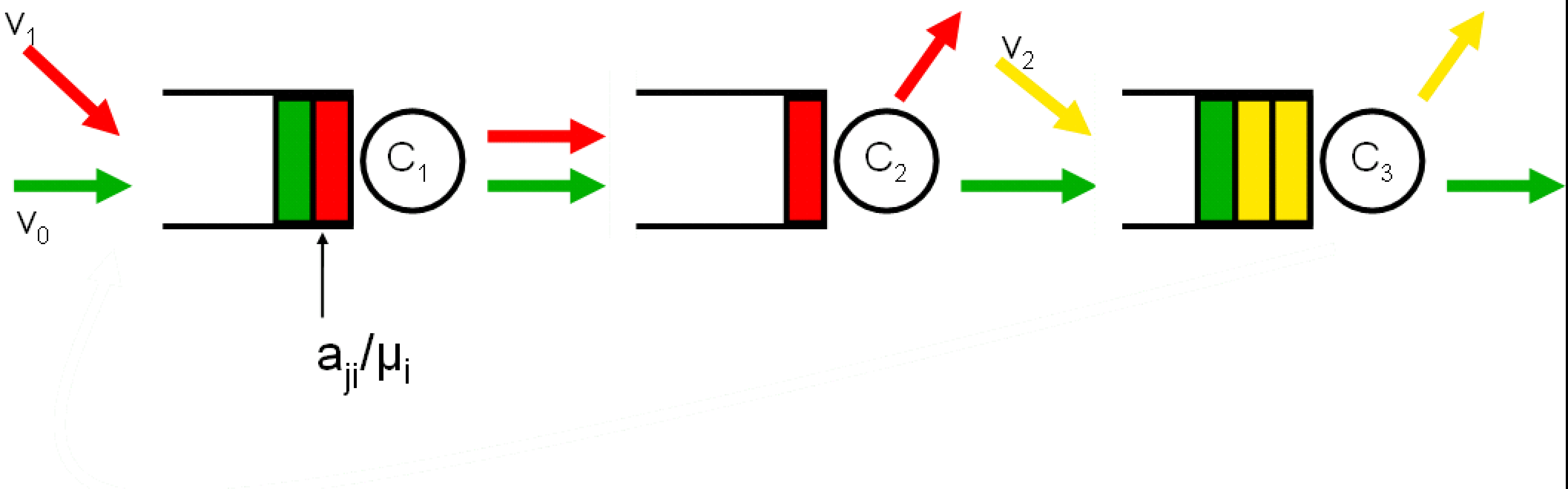

IMPORTANT POINT:

Queue sizes are independent Geometric Distributions

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This argument is due to: Schweitzer ’79, Kelly ’89, Roberts and Massoulie ’99

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#route *i* packets at queue *j*

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Since queues are stable therefore:

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These are the Kuhn-Tucker conditions for proportionally fair optimization!

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Idea shadow prices q_j are like queue sizes and so are independent.

Proportional fairness considers flows and resources, not packets and queues
So we must use a different model: (Roberts and Massoulie '98)

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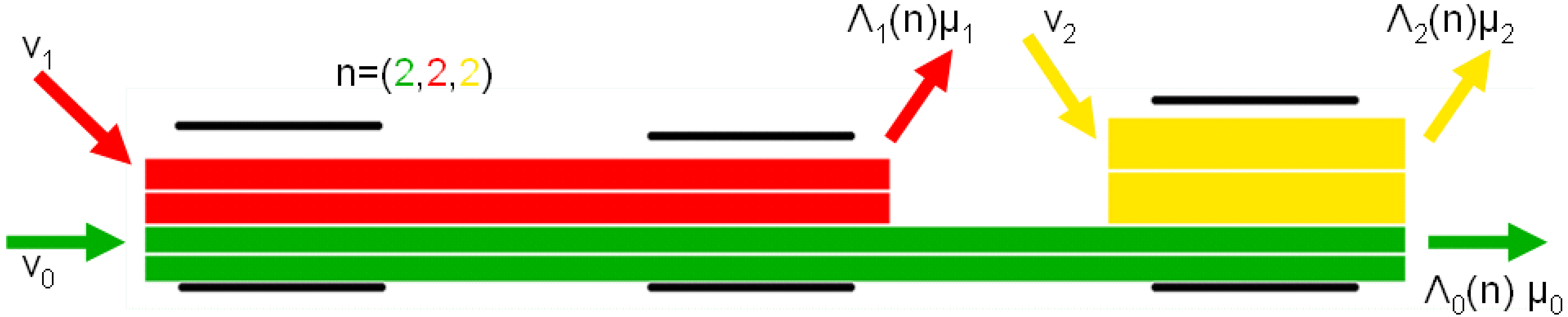

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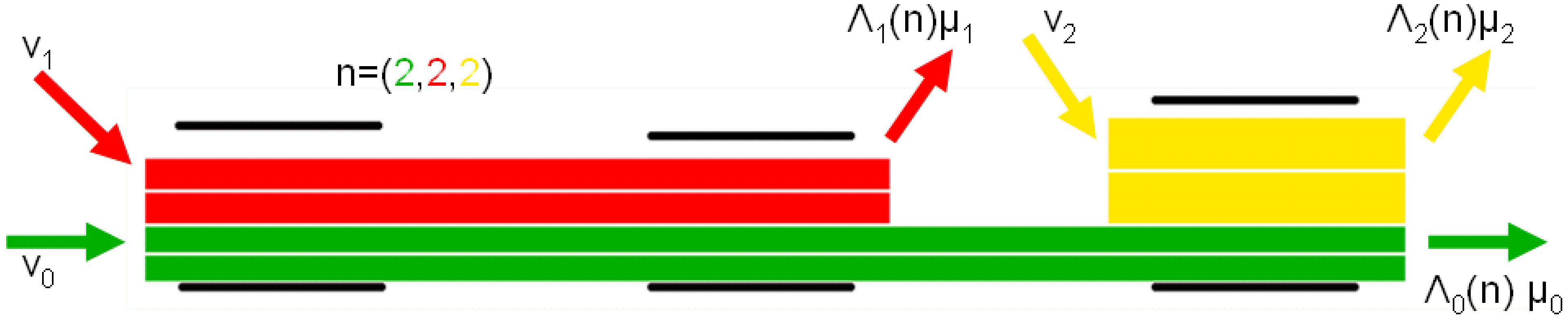

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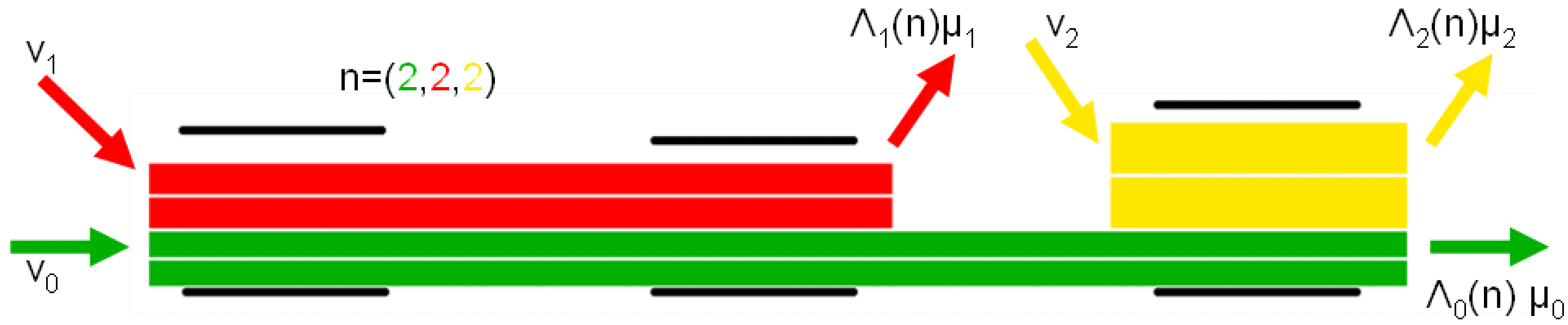

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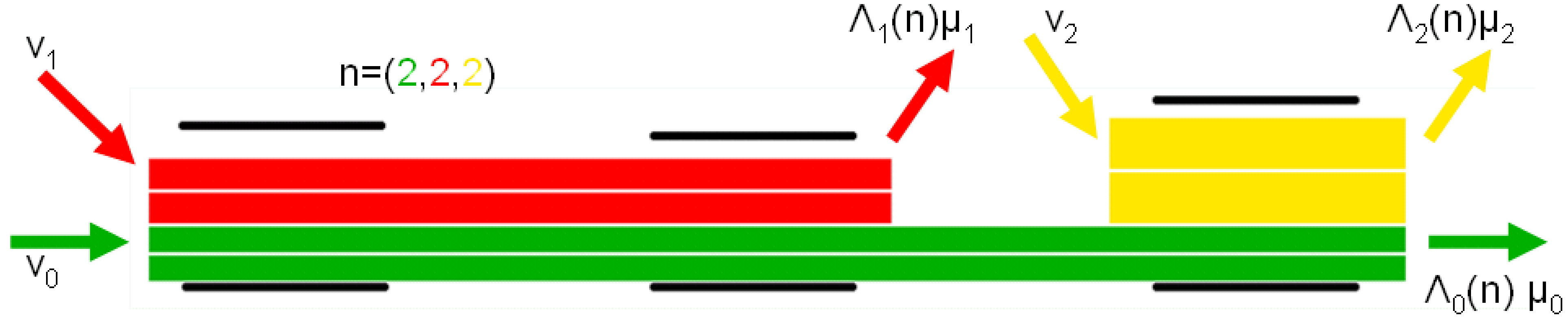

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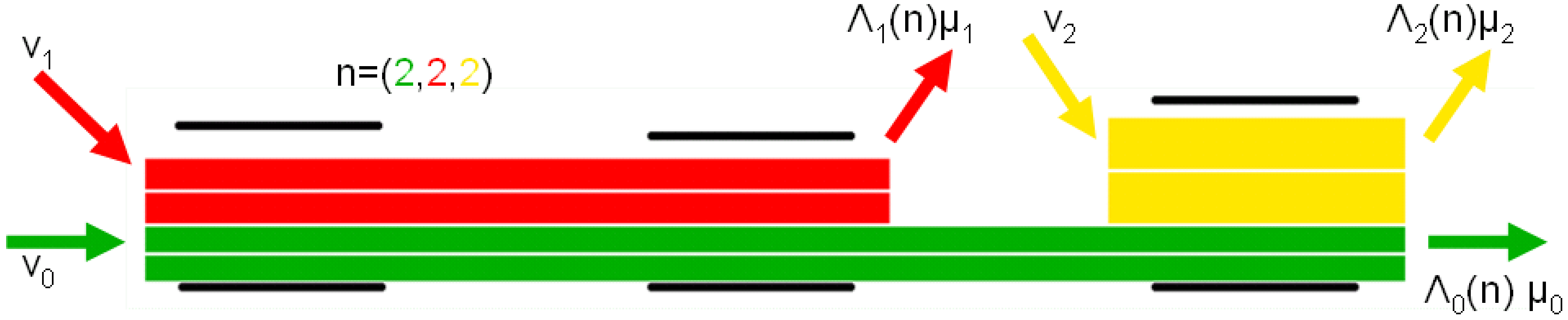

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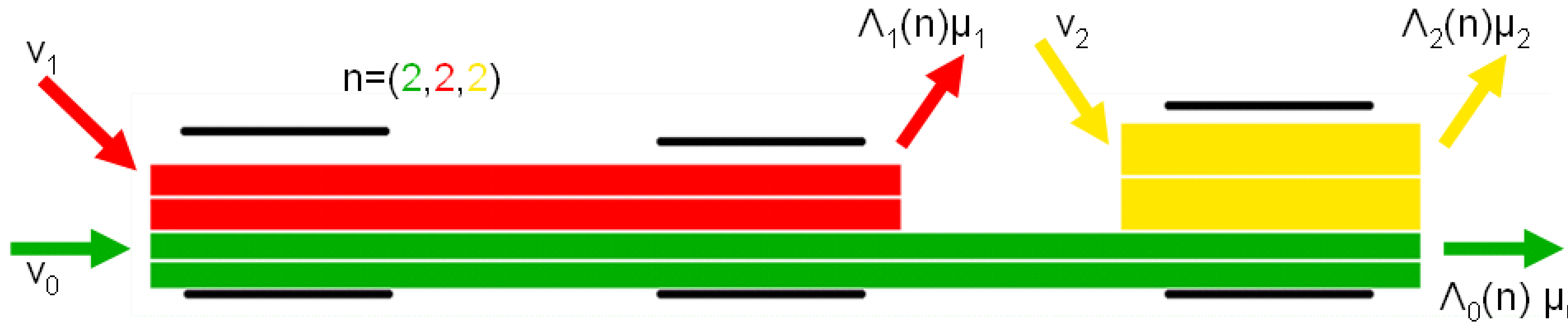

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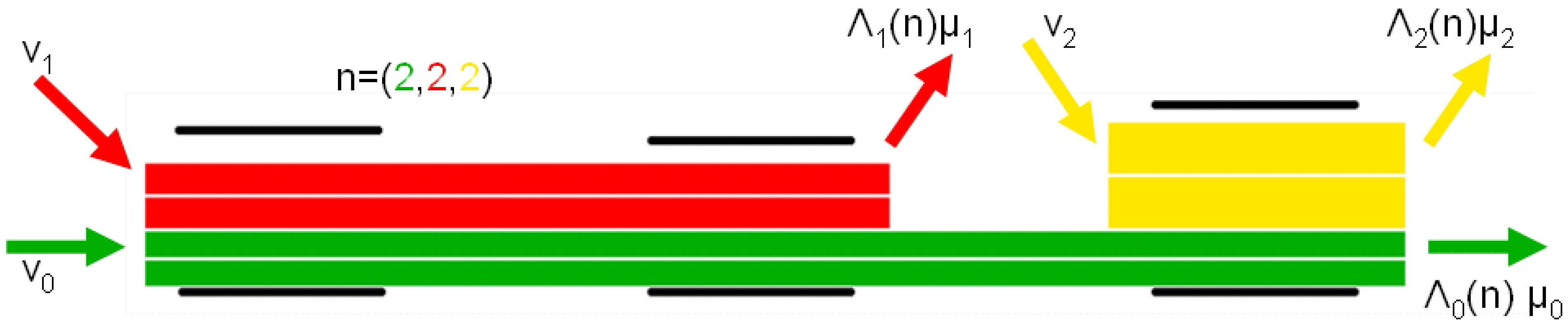

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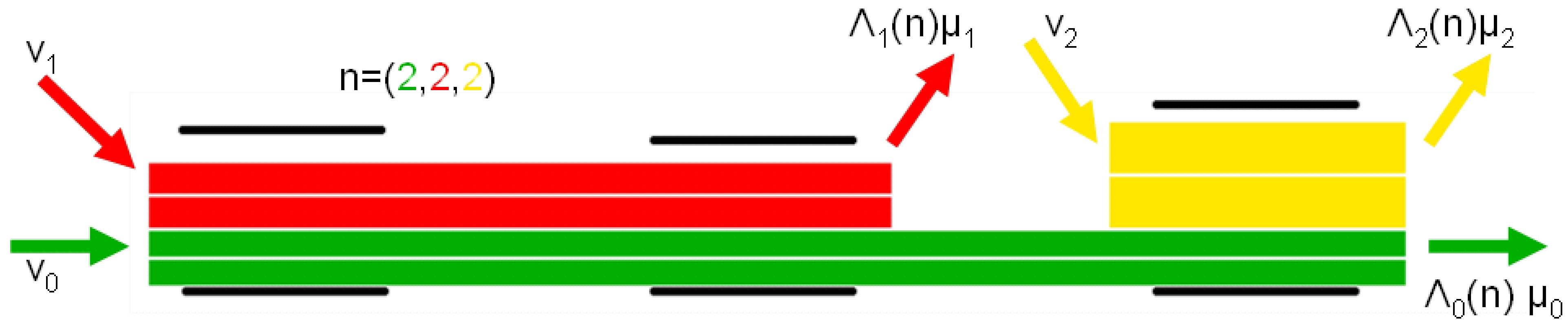

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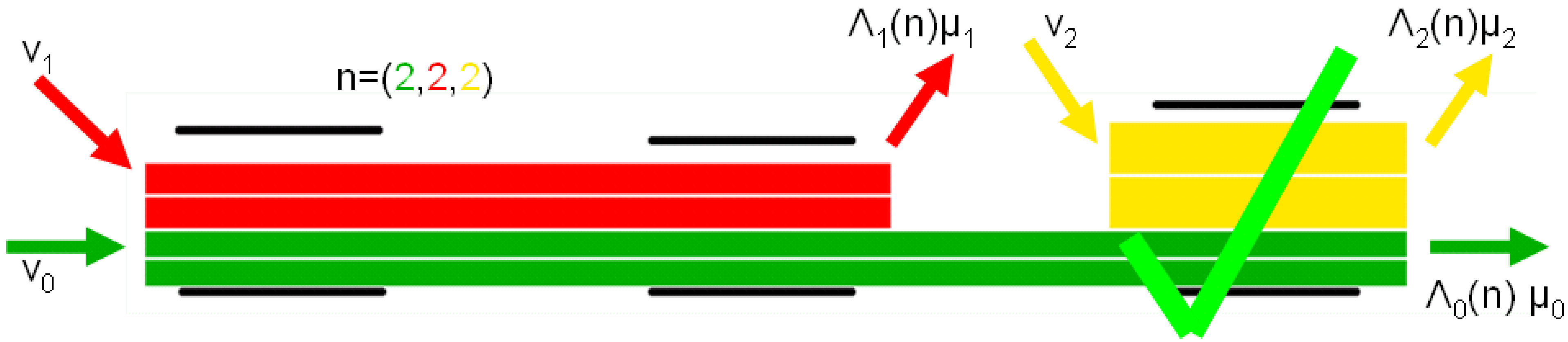

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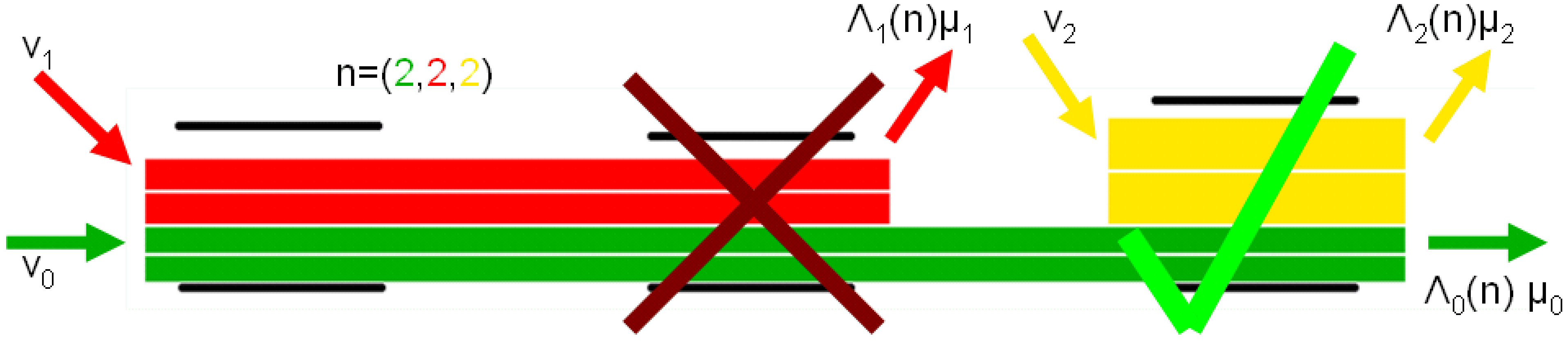

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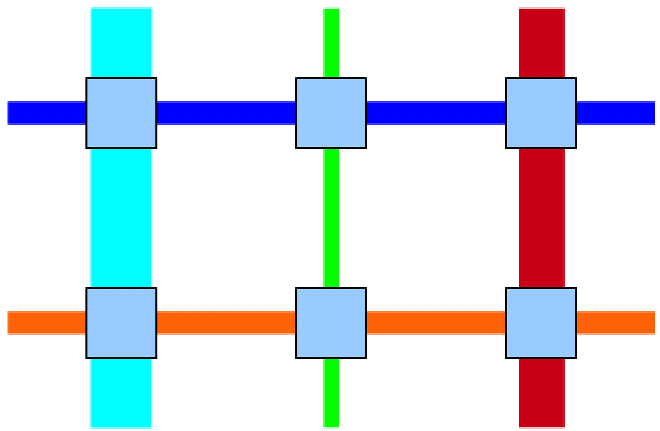

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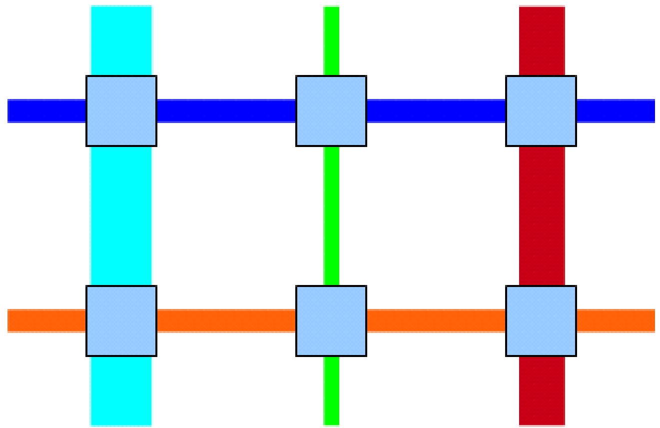

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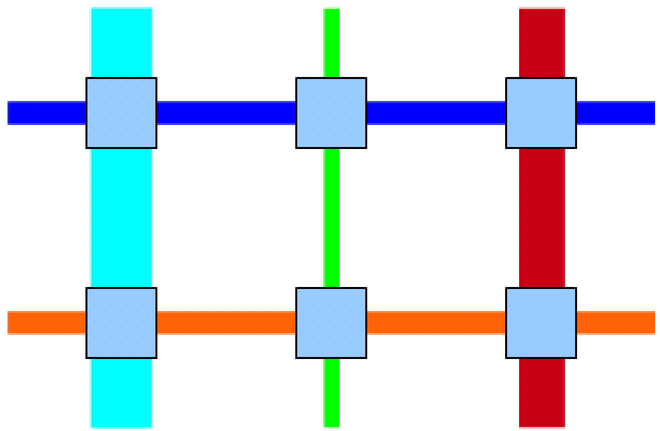

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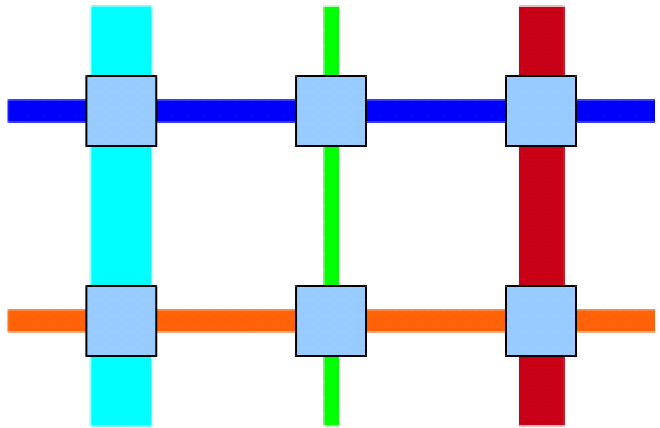

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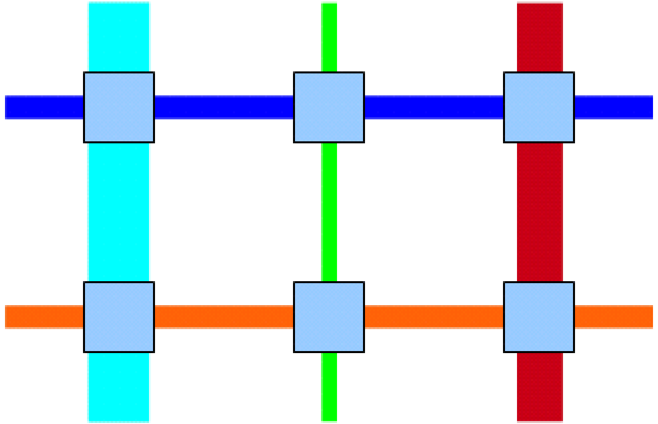

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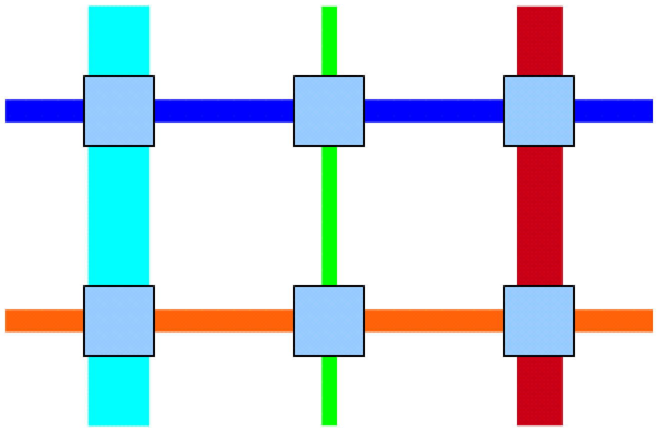

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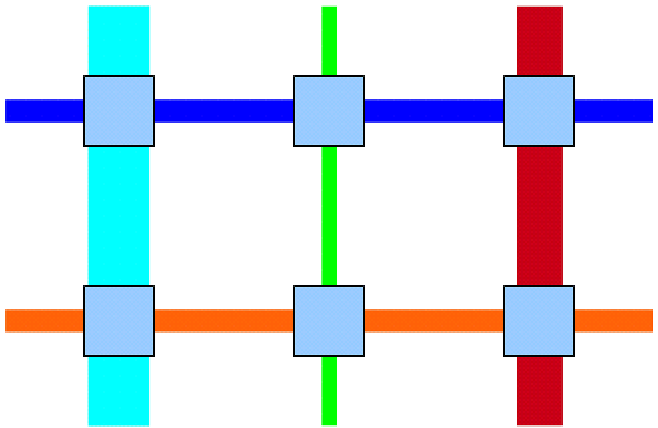

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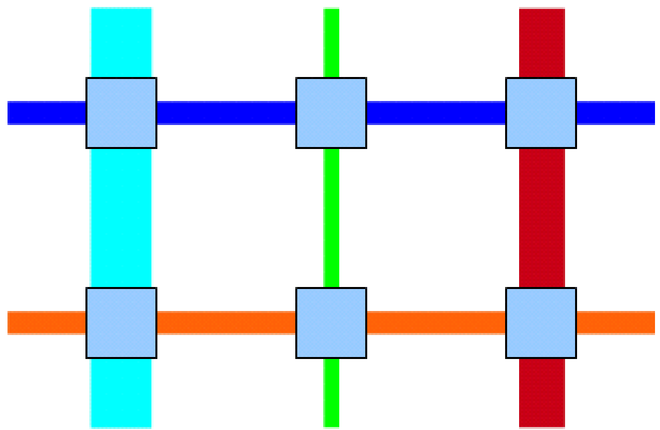

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Utility Optimization in Congested Queueing Networks

Neil Walton

Statistical Laboratory, University of Cambridge.

A Decomposition Theorem

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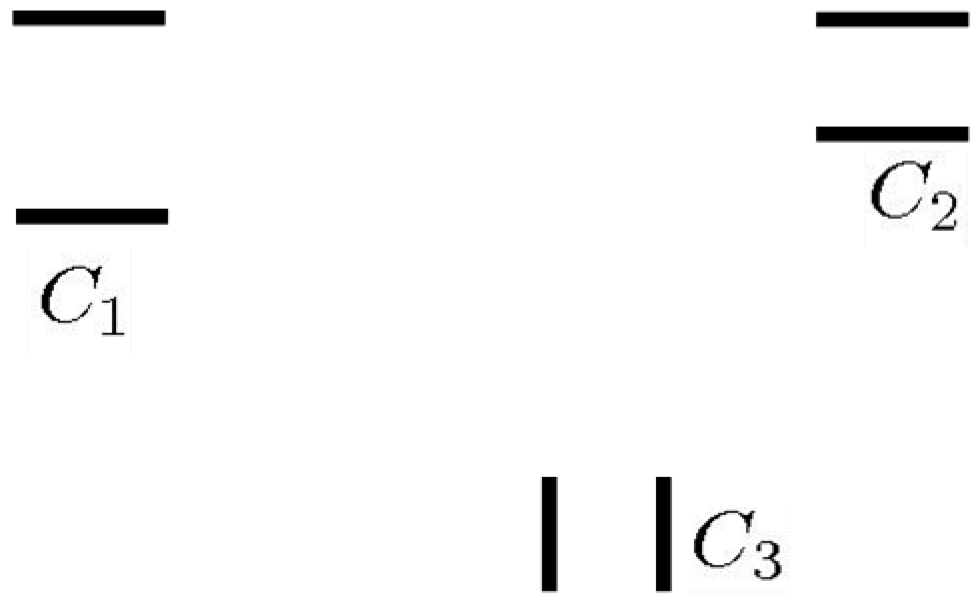

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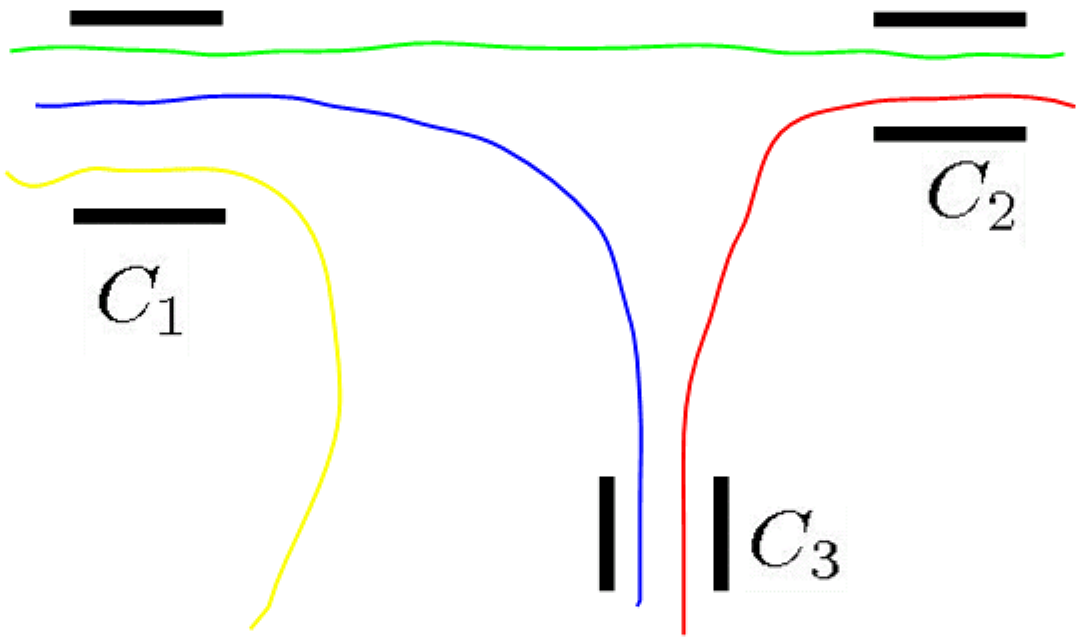

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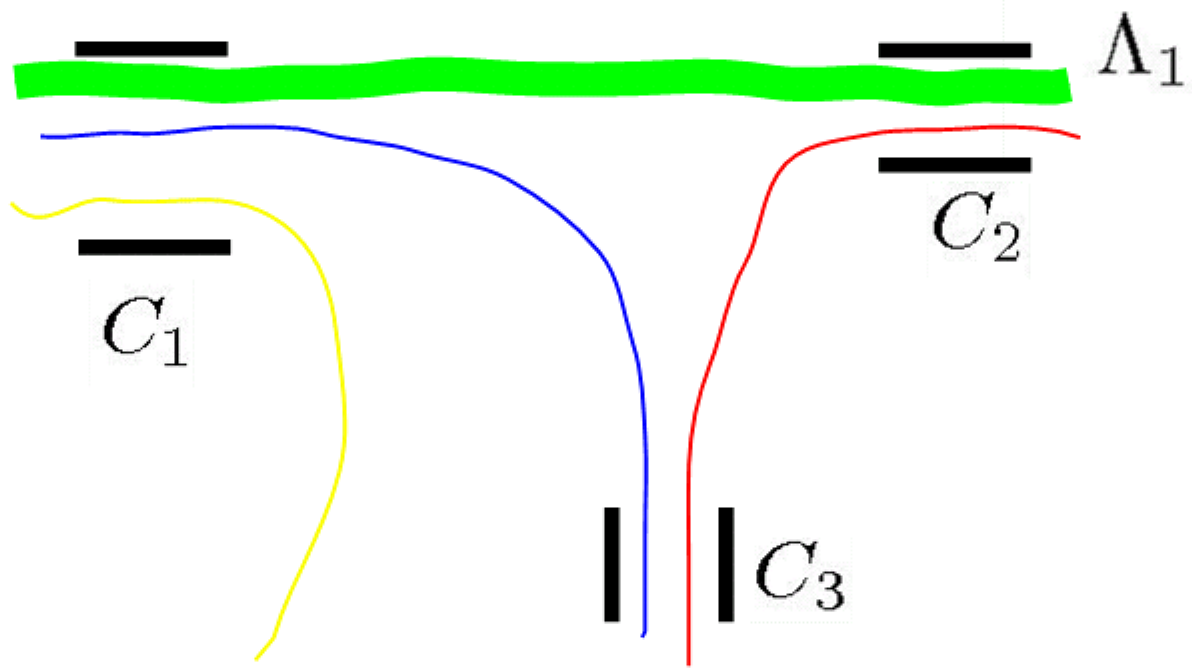

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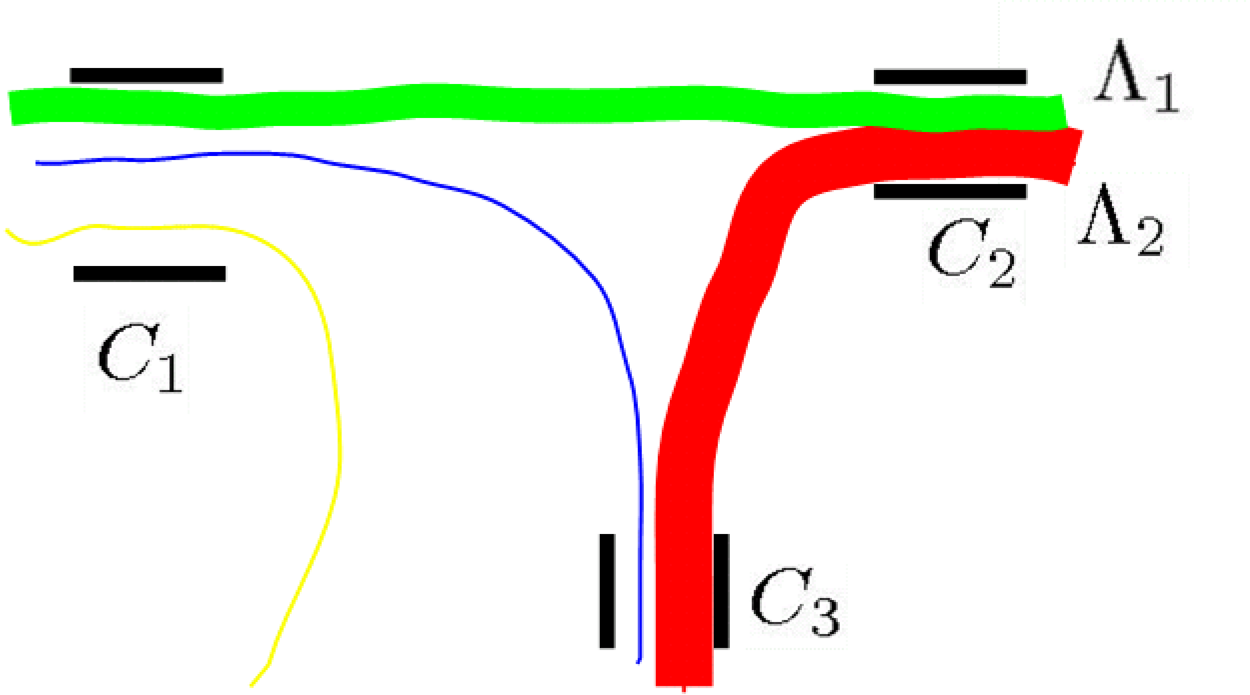

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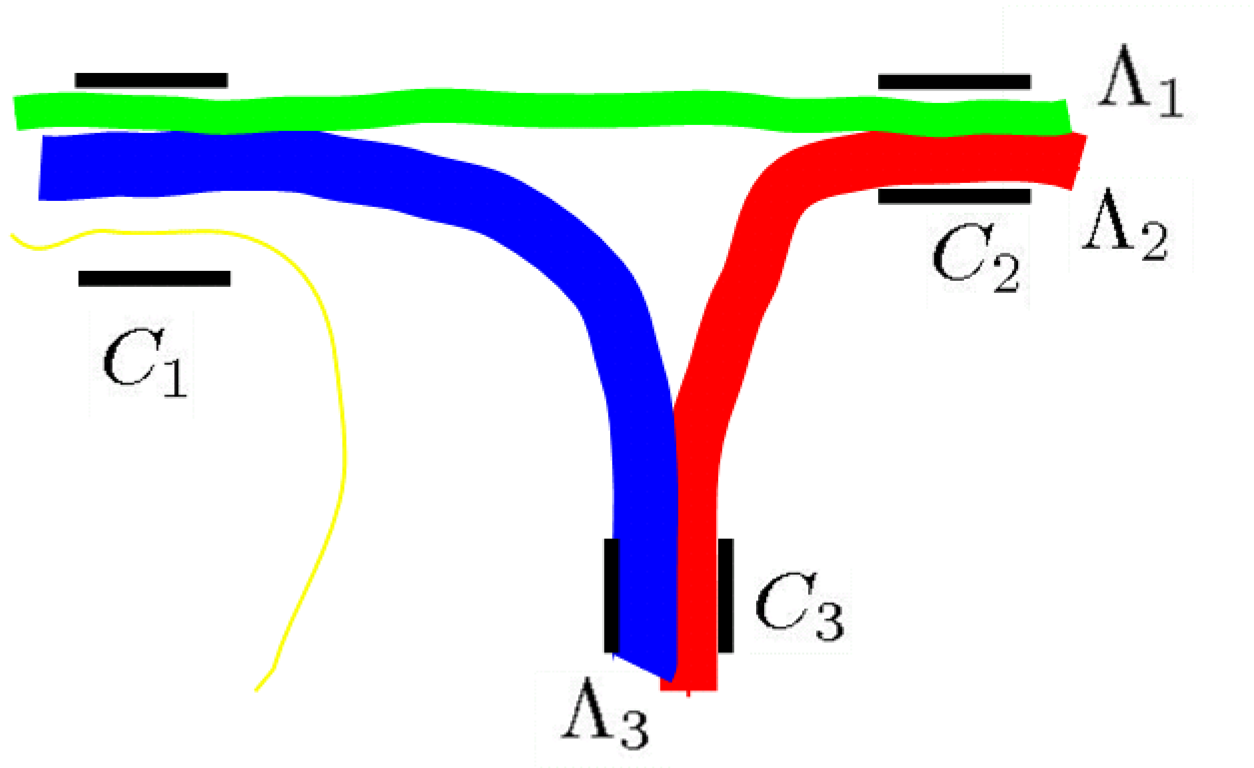

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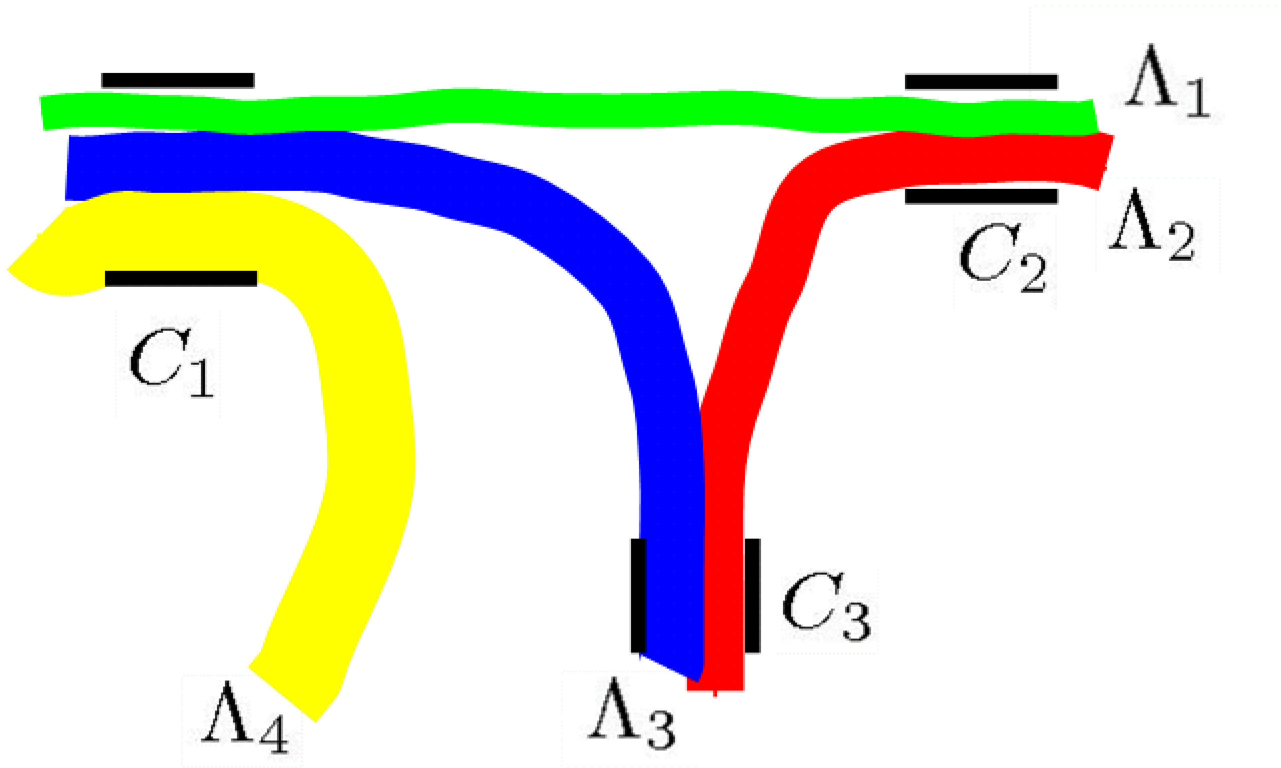

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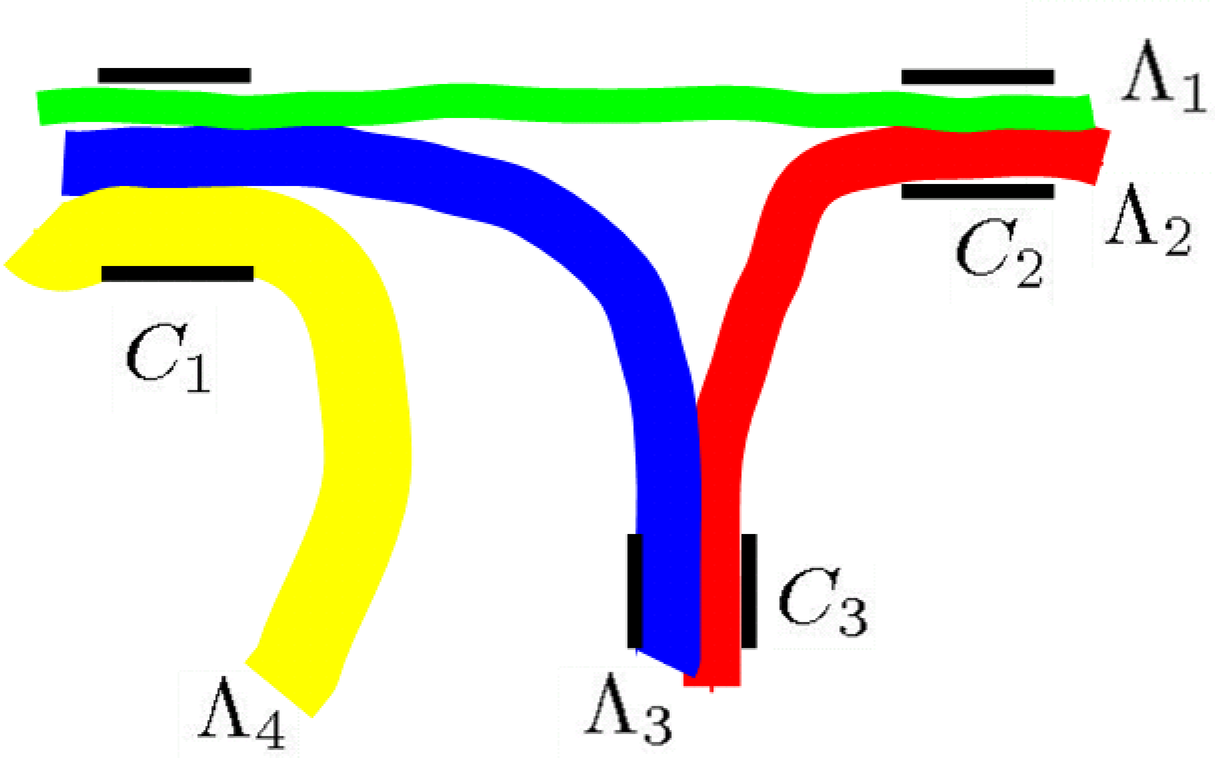

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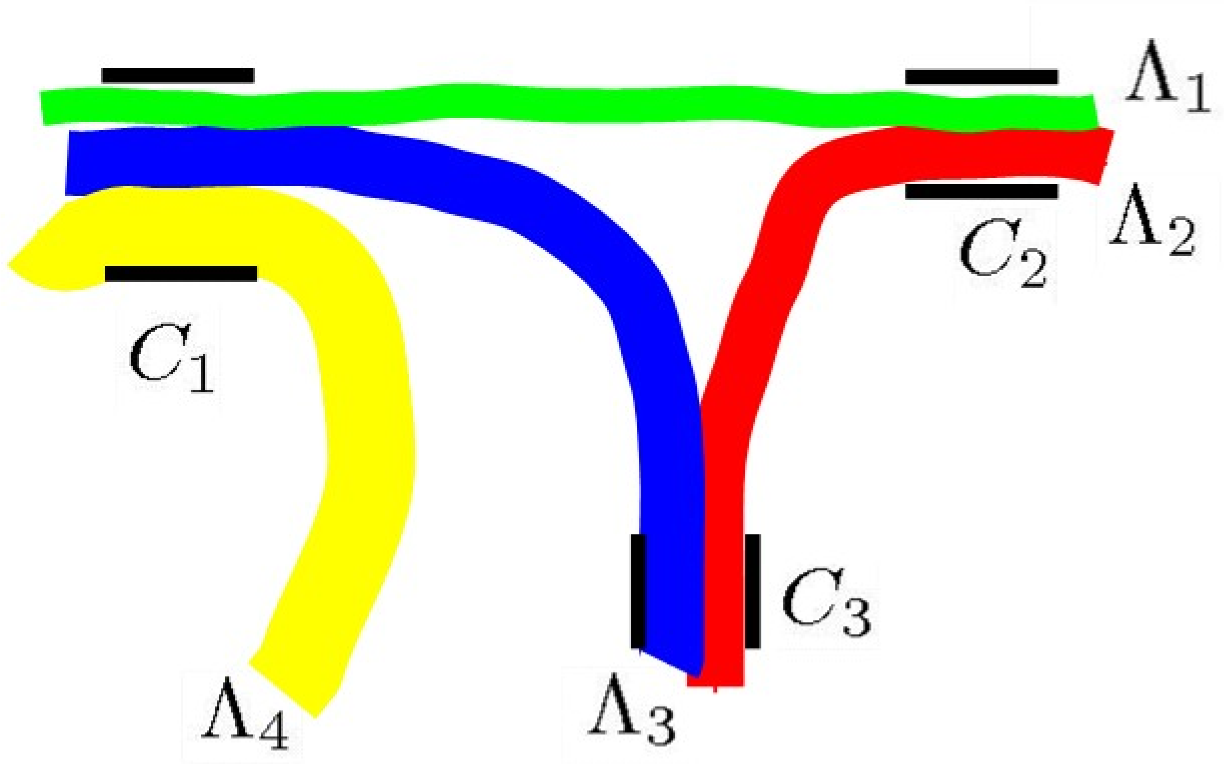

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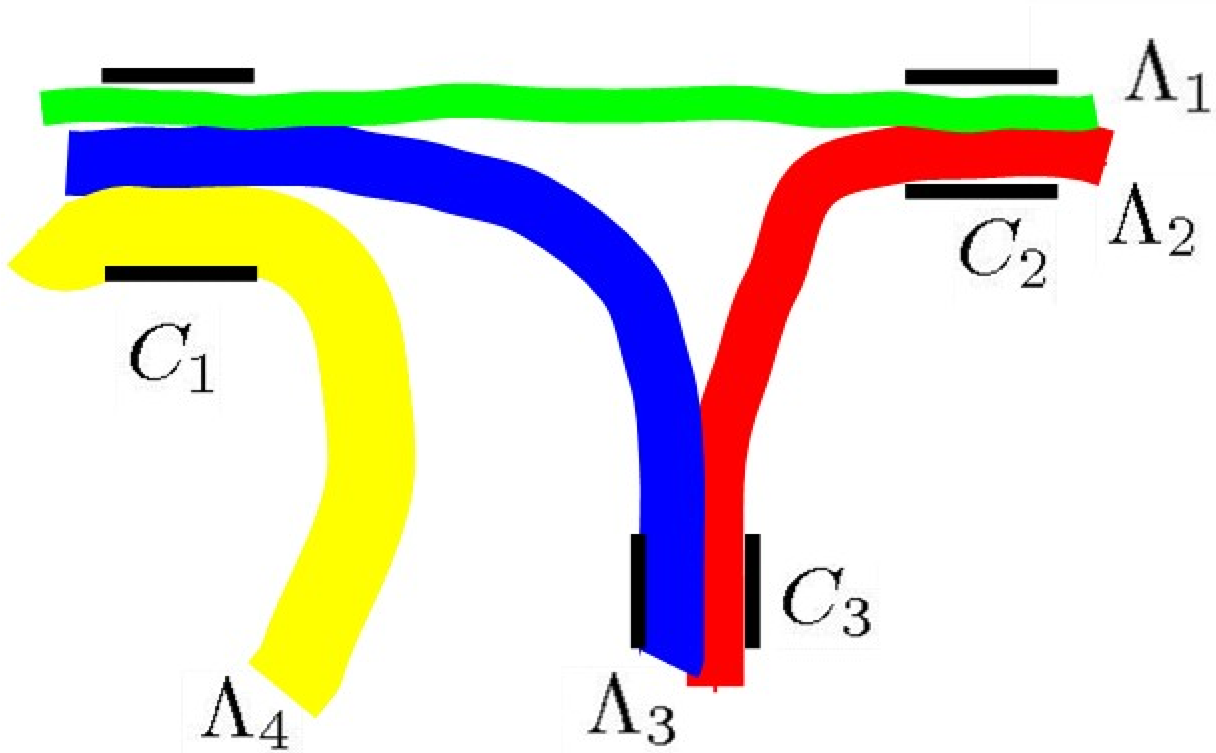

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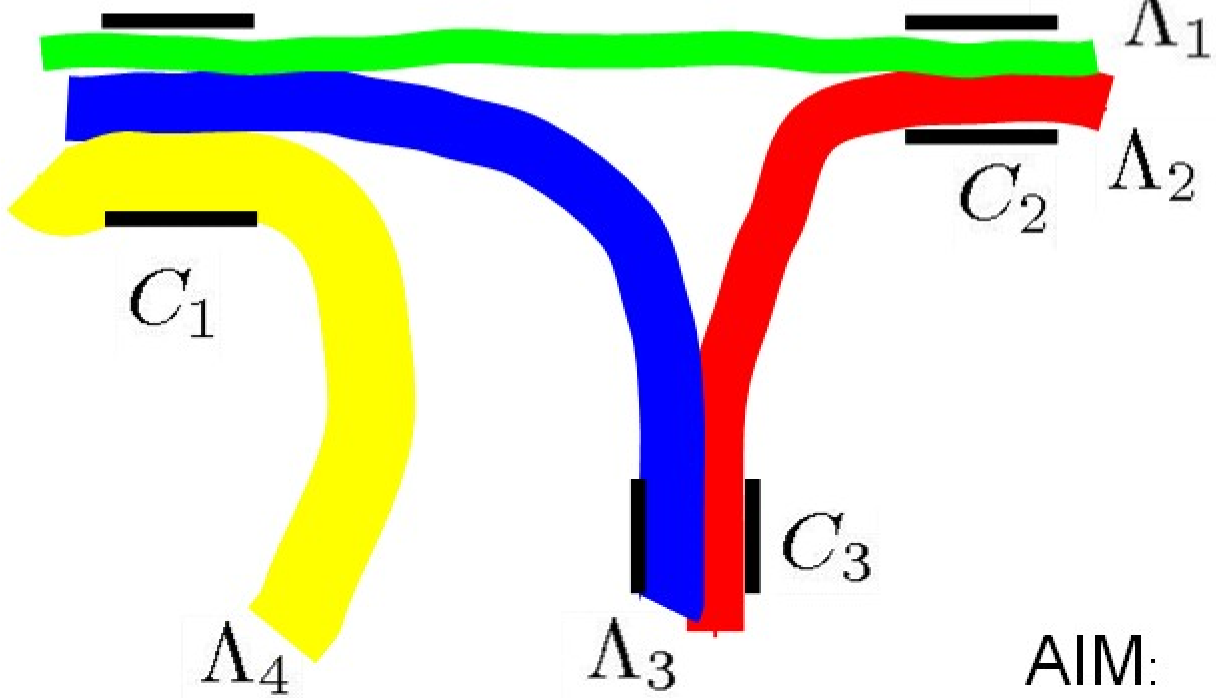

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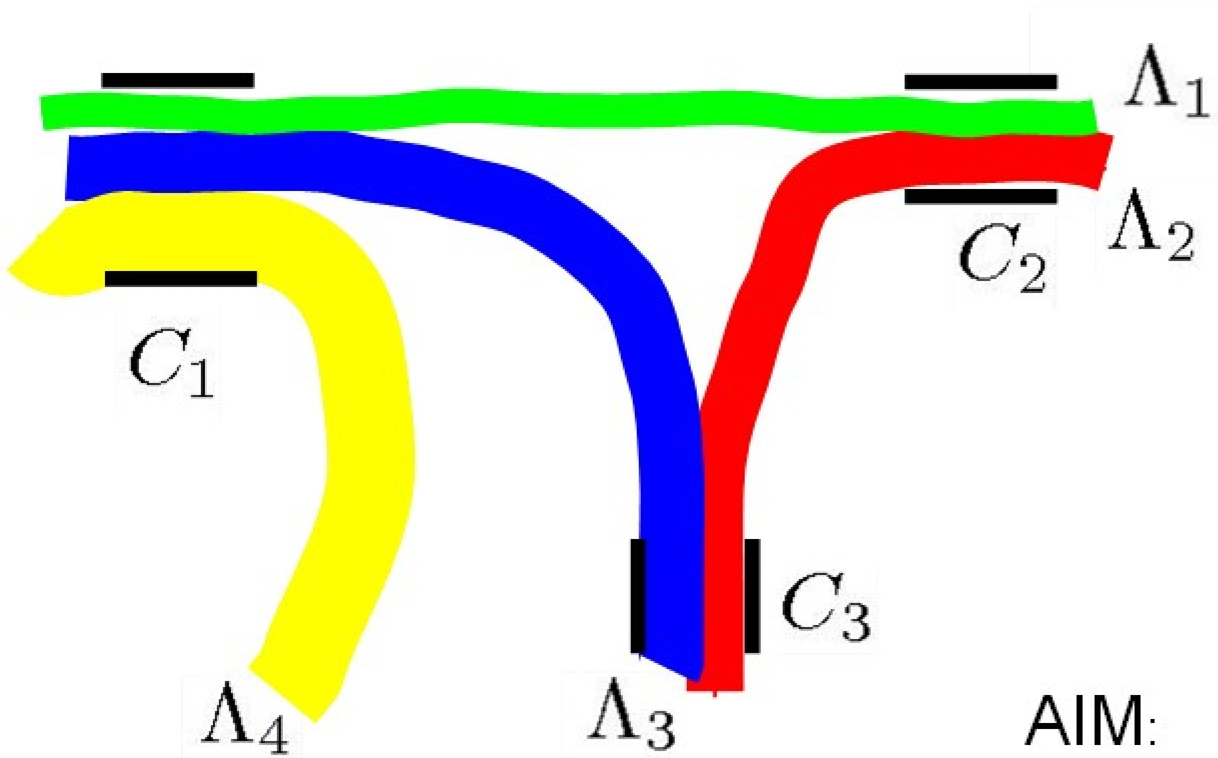

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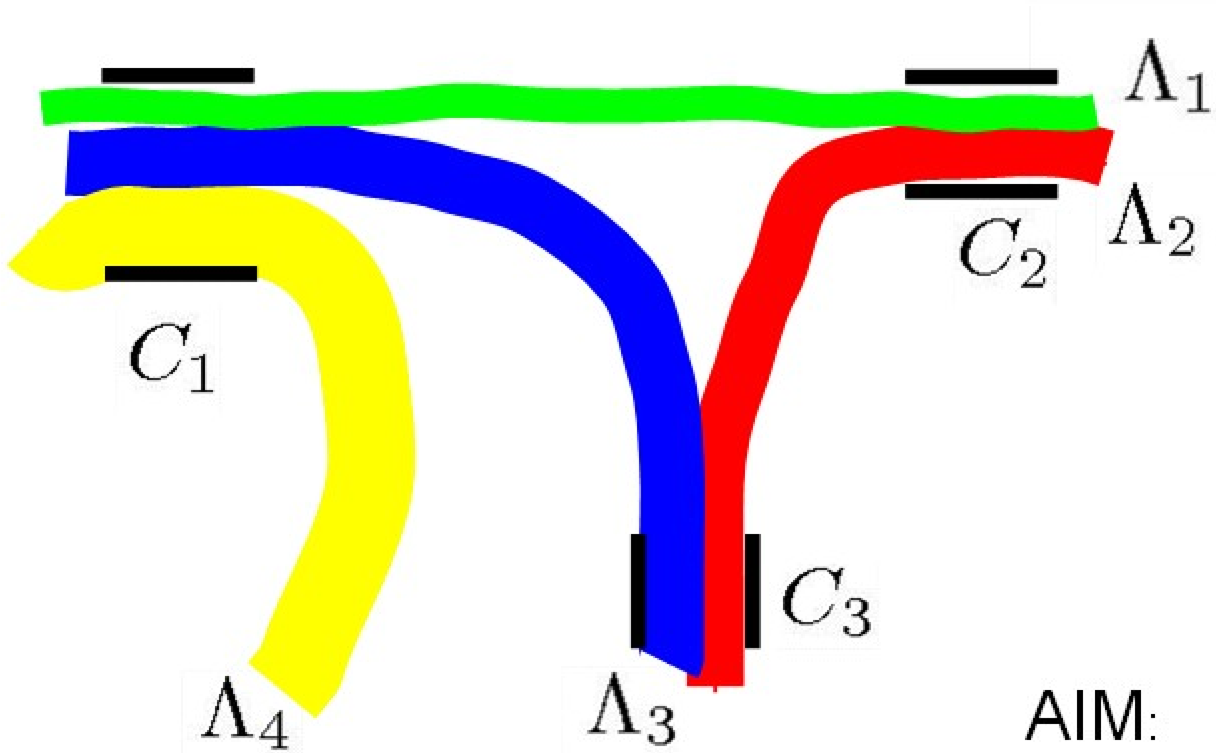

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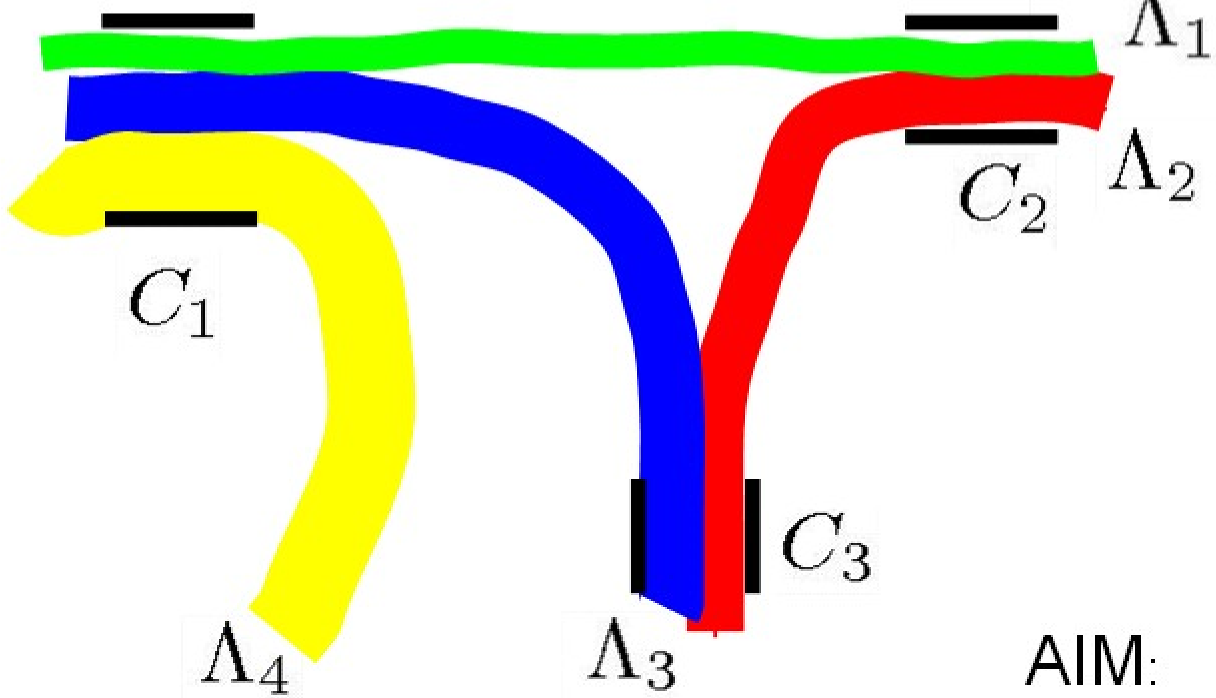

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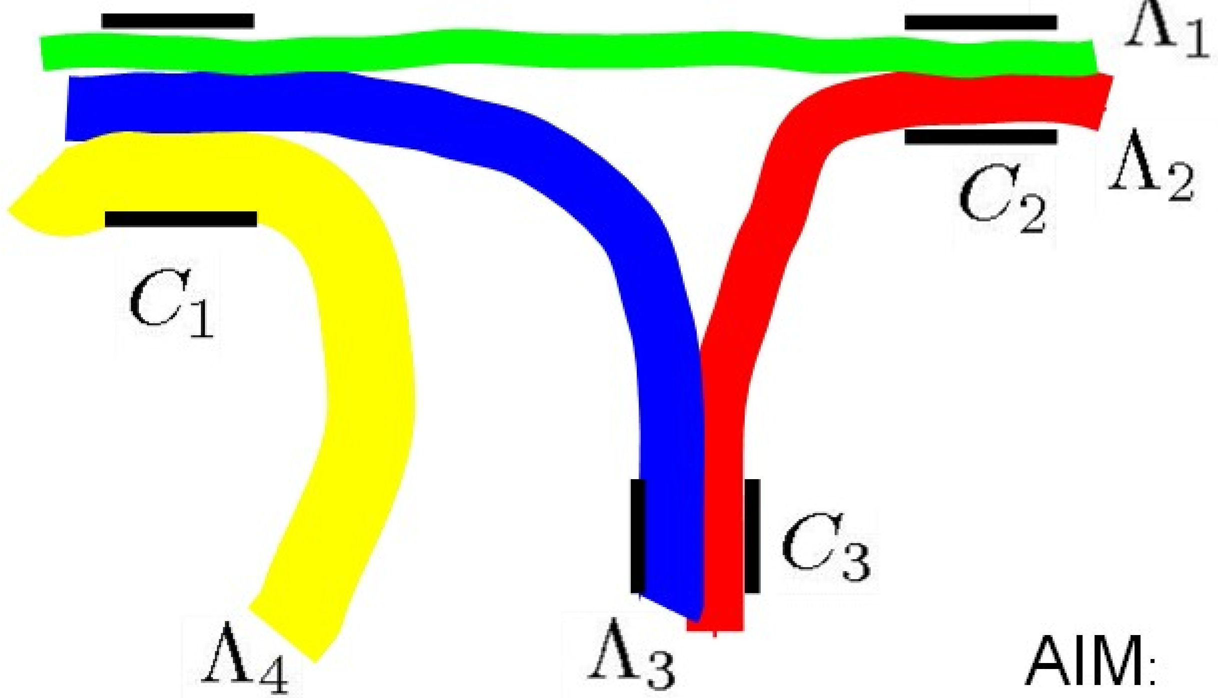

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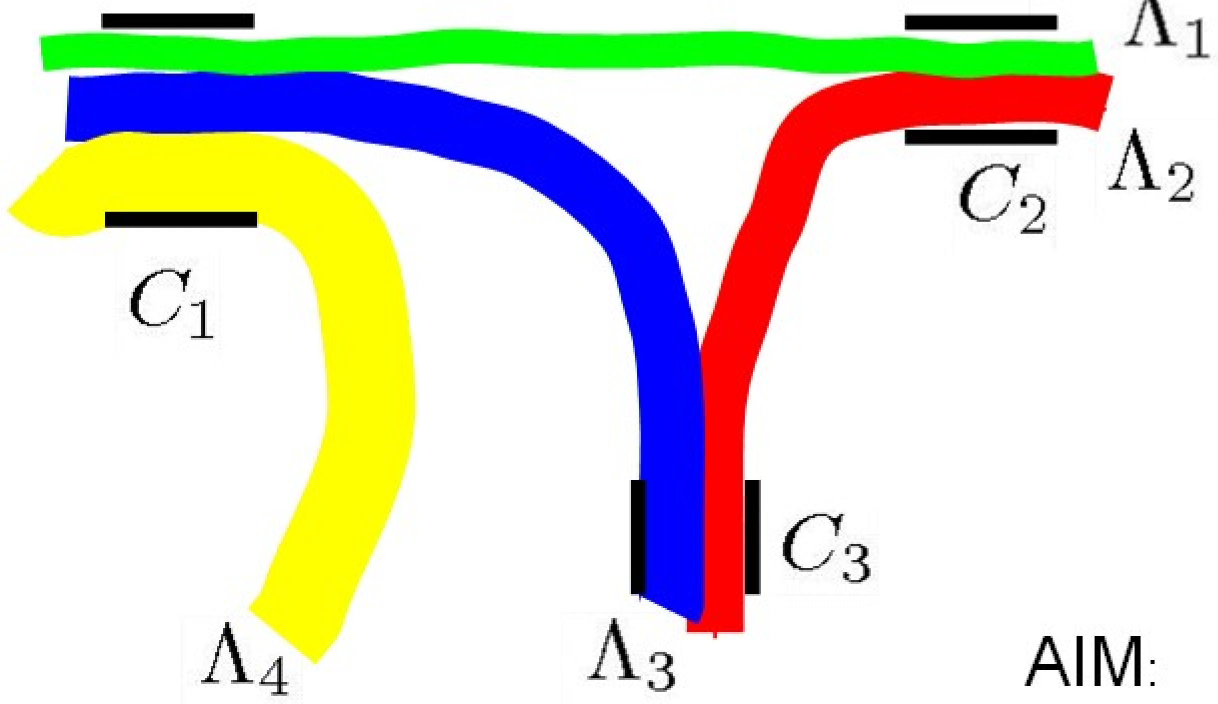

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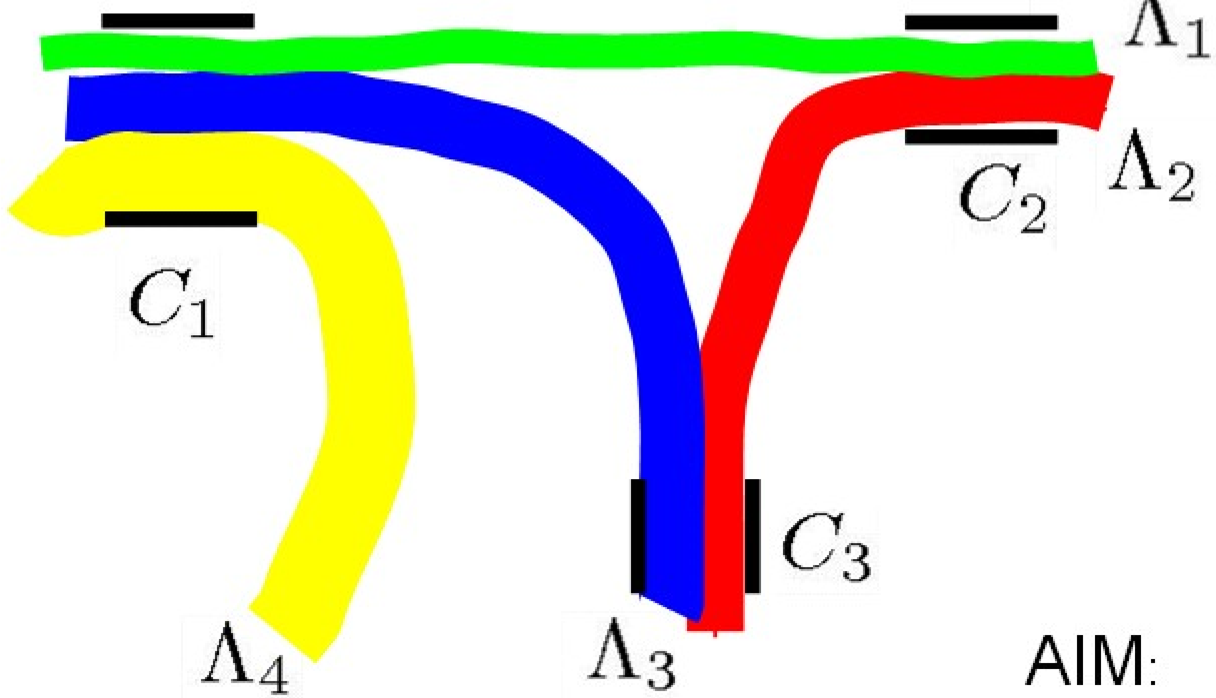

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AIM: SYSTEM PROBLEM

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$$\text{subject to} \quad \sum_{i:j \in i} \Lambda_i \leq C_j, \; j \in \mathcal{J}$$

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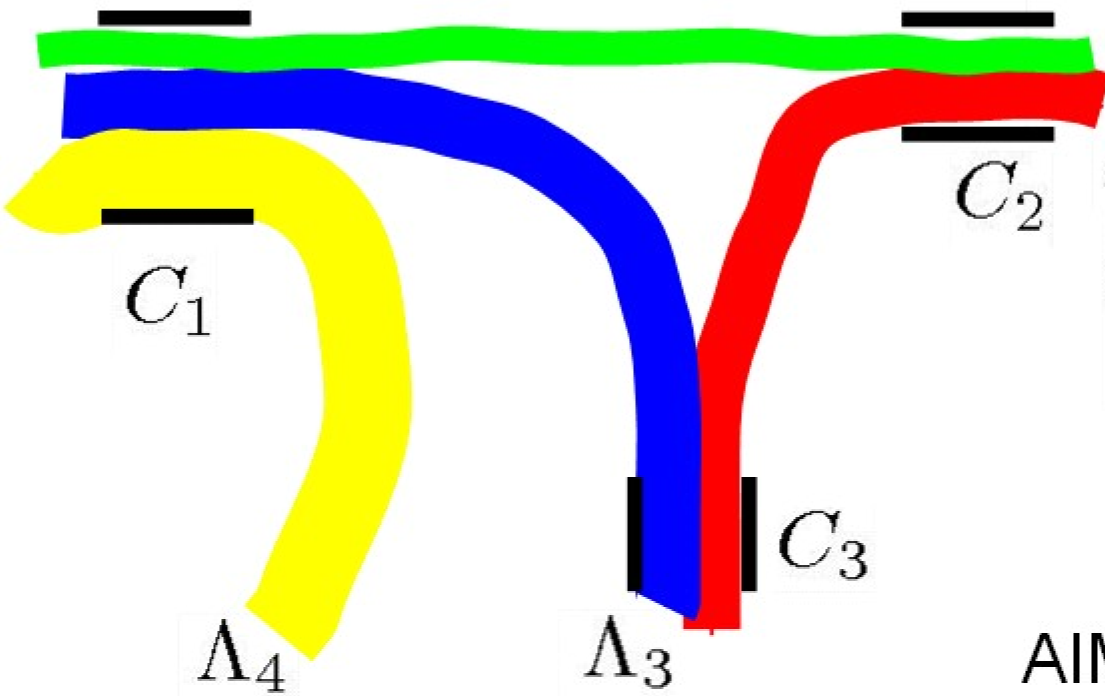
A Decomposition Theorem

F.P. Kelly, “Charging and Rate Control of Elastic Traffic” (1997)



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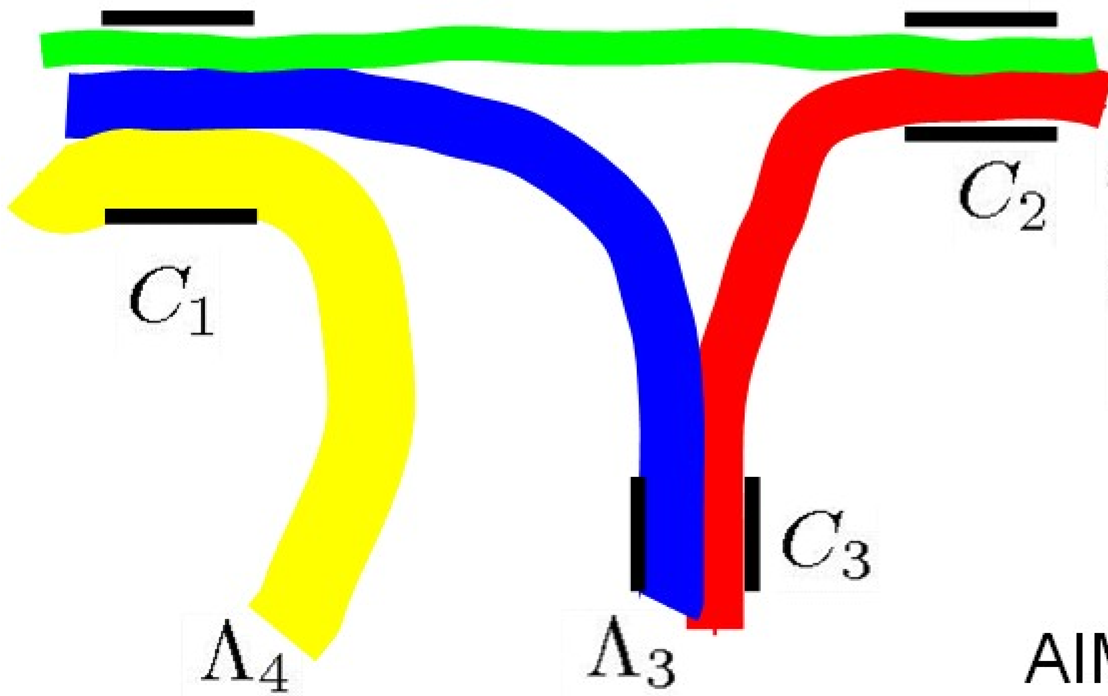

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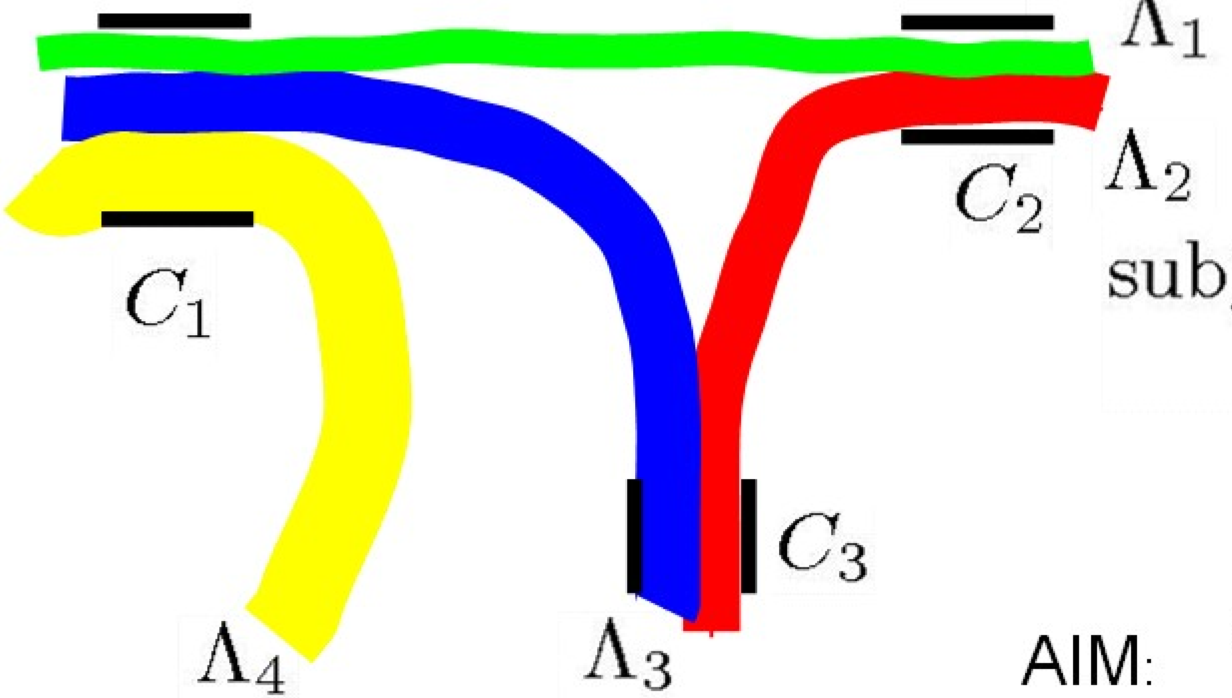

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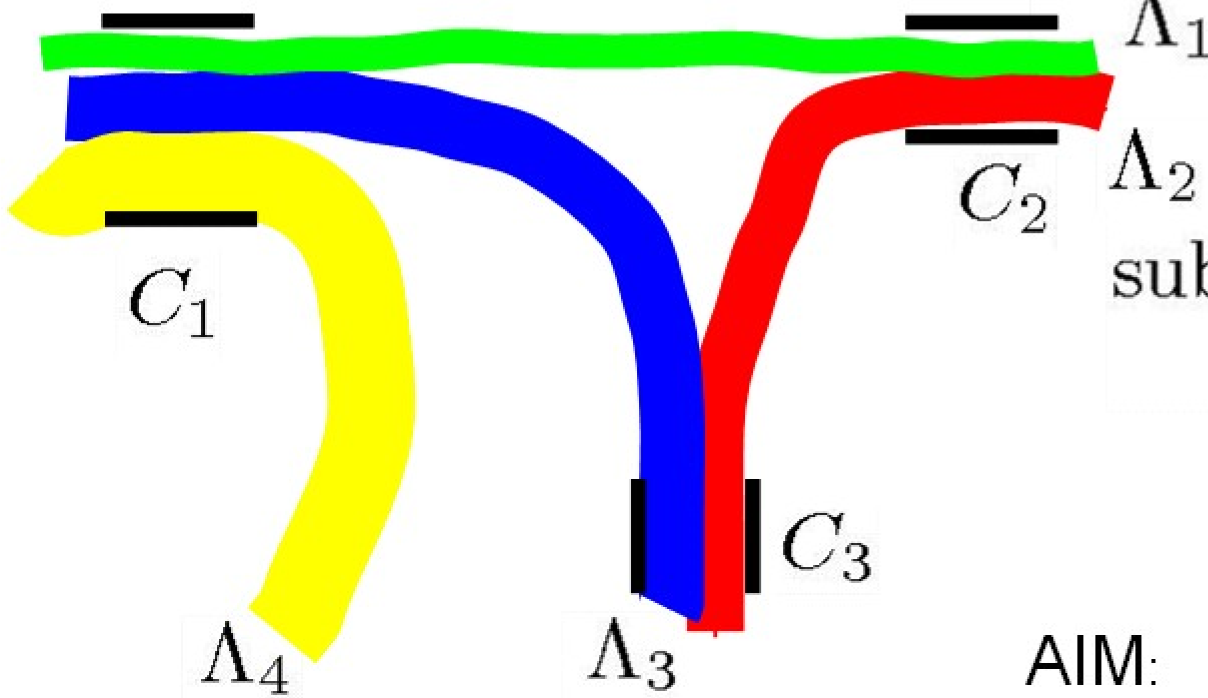

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User *i* choses wealth:

$$\bar{m}_i$$

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Dynamics

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We want a network solve these optimisation problems **implicitly**

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...Differential Equations?

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Kelly, Maulloo, Tan (1998)
Kelly, Gibbens (1999)
Kelly, Key, Zachary (2000)
Johari, Tan (2001)
Raina, Towsley, Wischik (2005)
Strulo, Walker, Wennink (2007)
Yi, Chiang (2008)

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Congestion

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Softer Capacity Constraints

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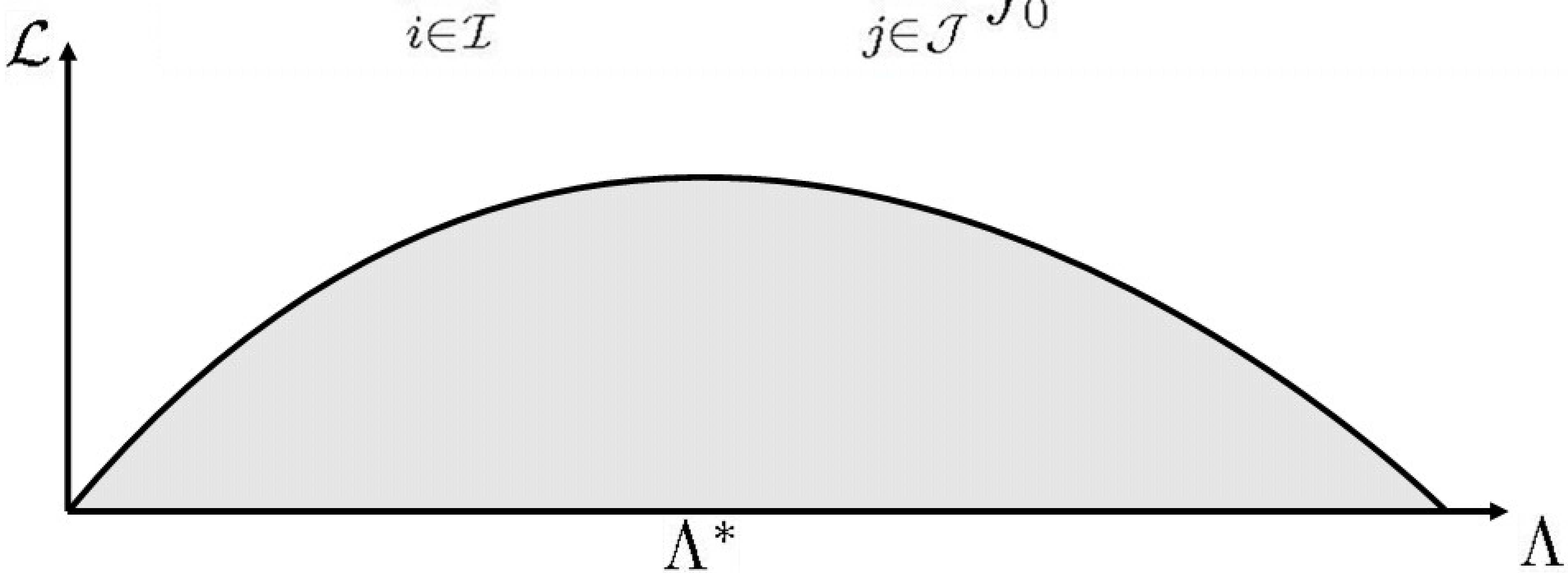

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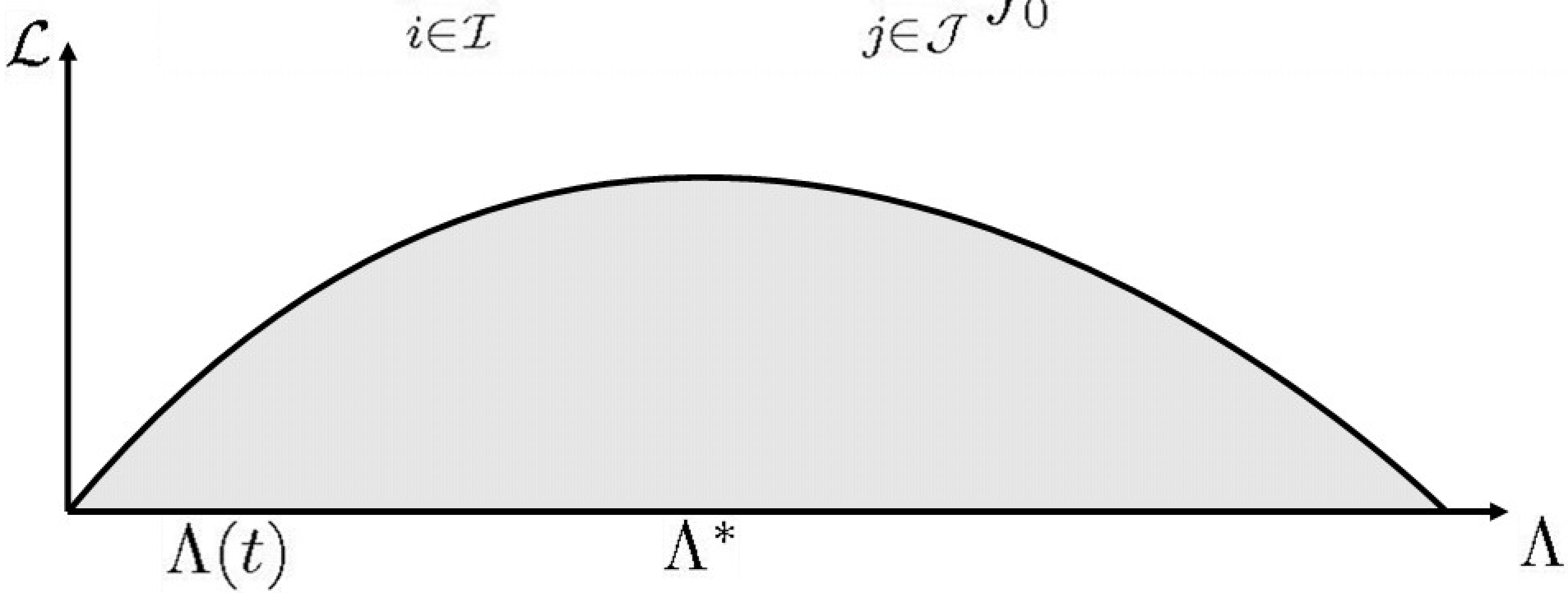

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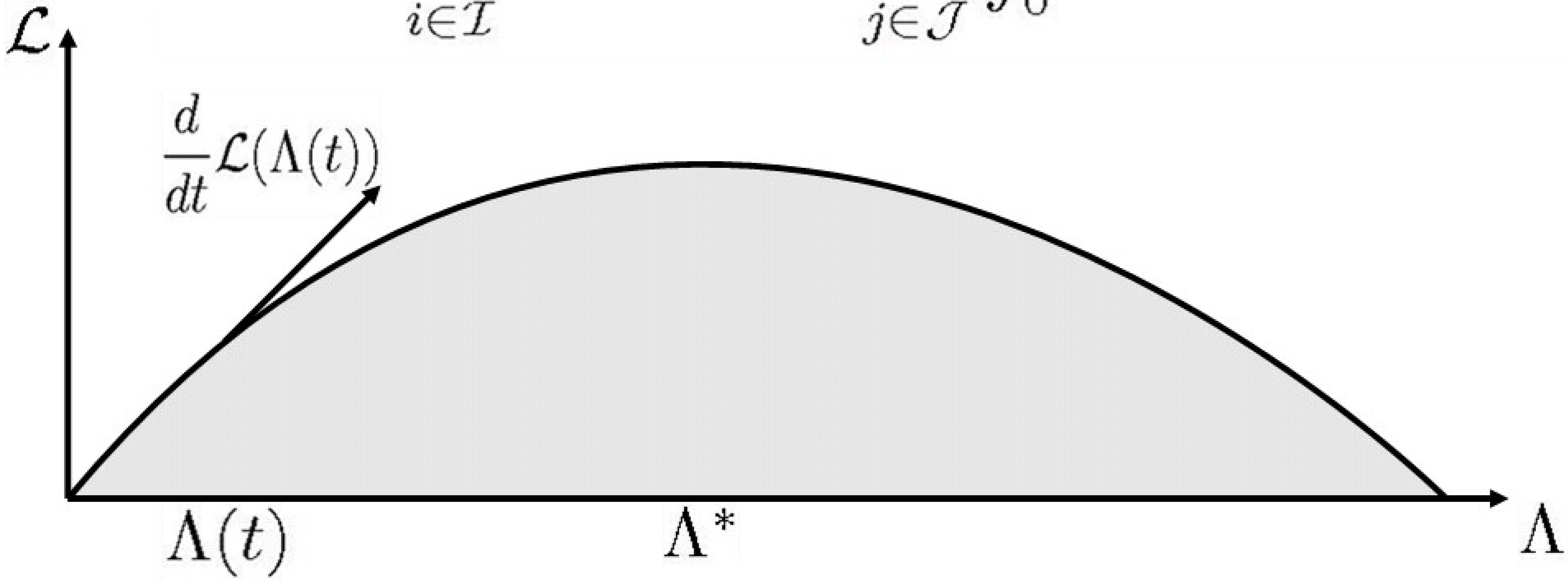

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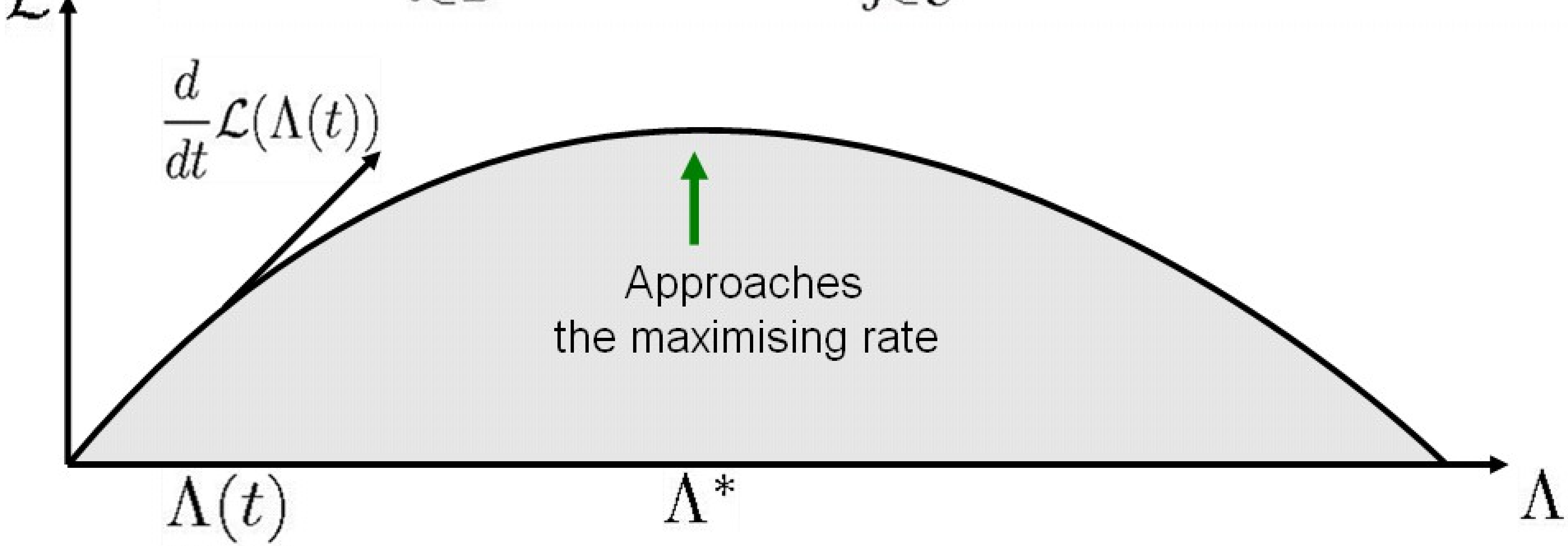

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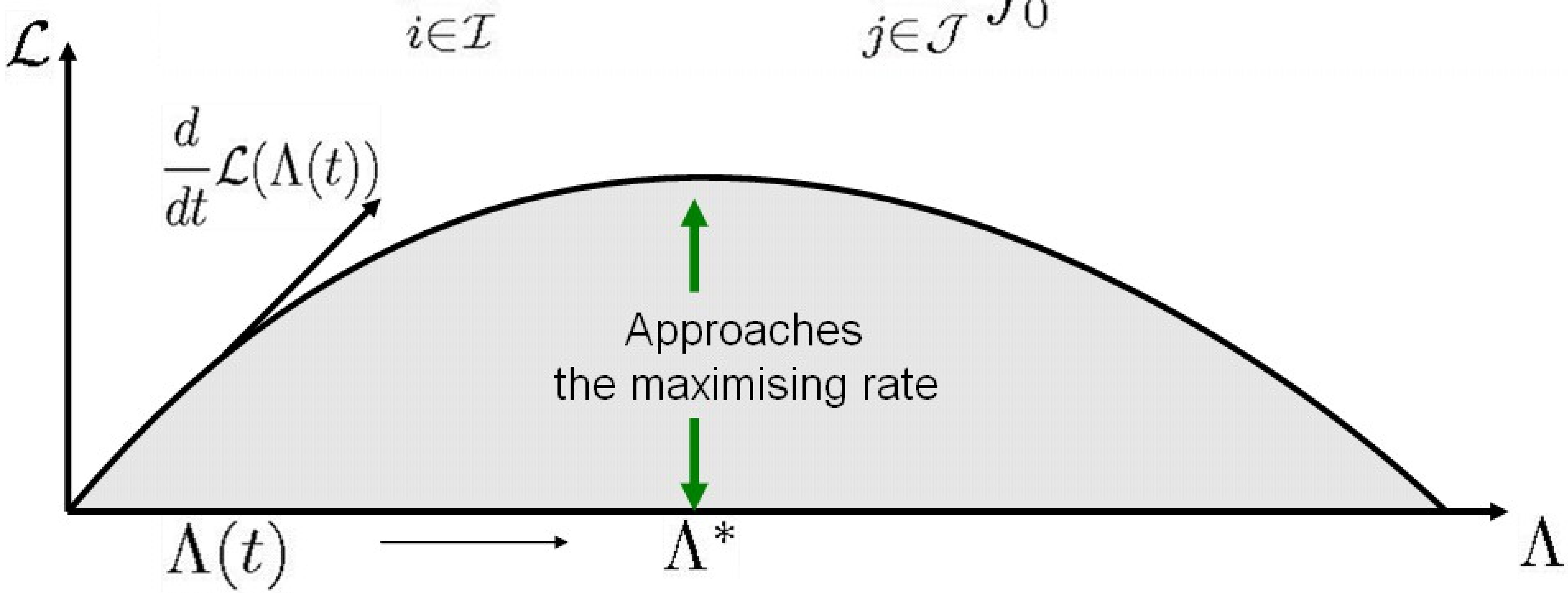

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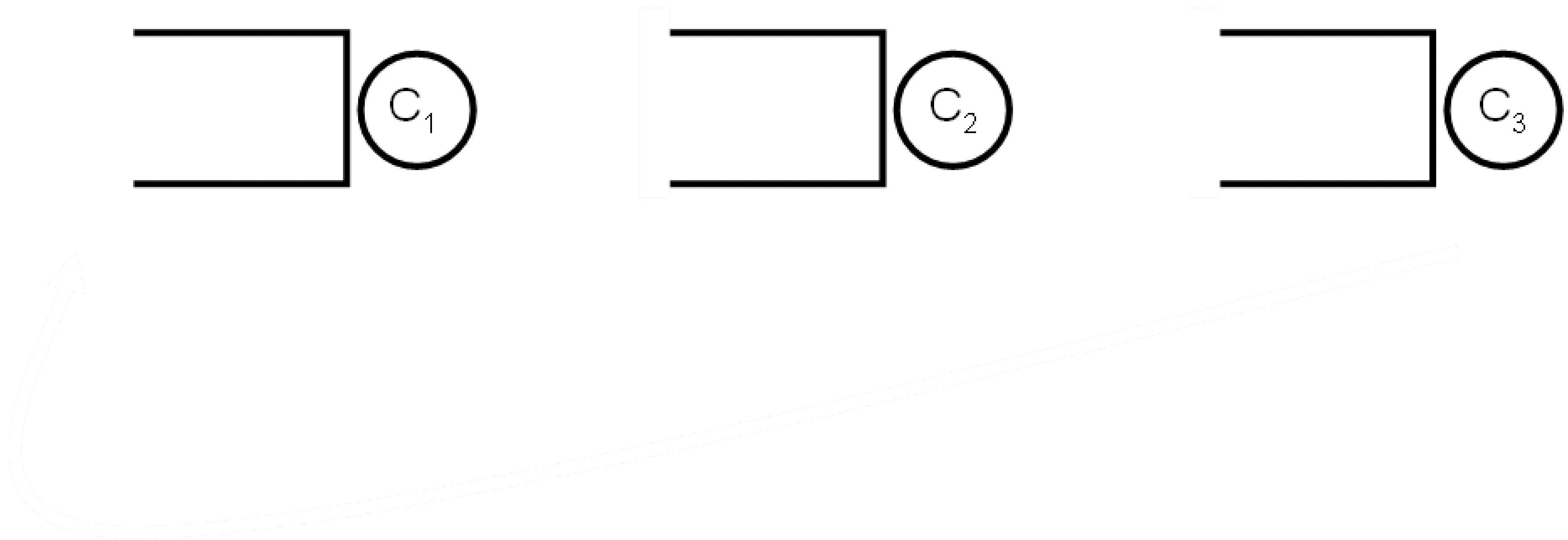
We can add queueing dynamics to the NETWORK PROBLEM…

A Closed Multi-class Queueing Network



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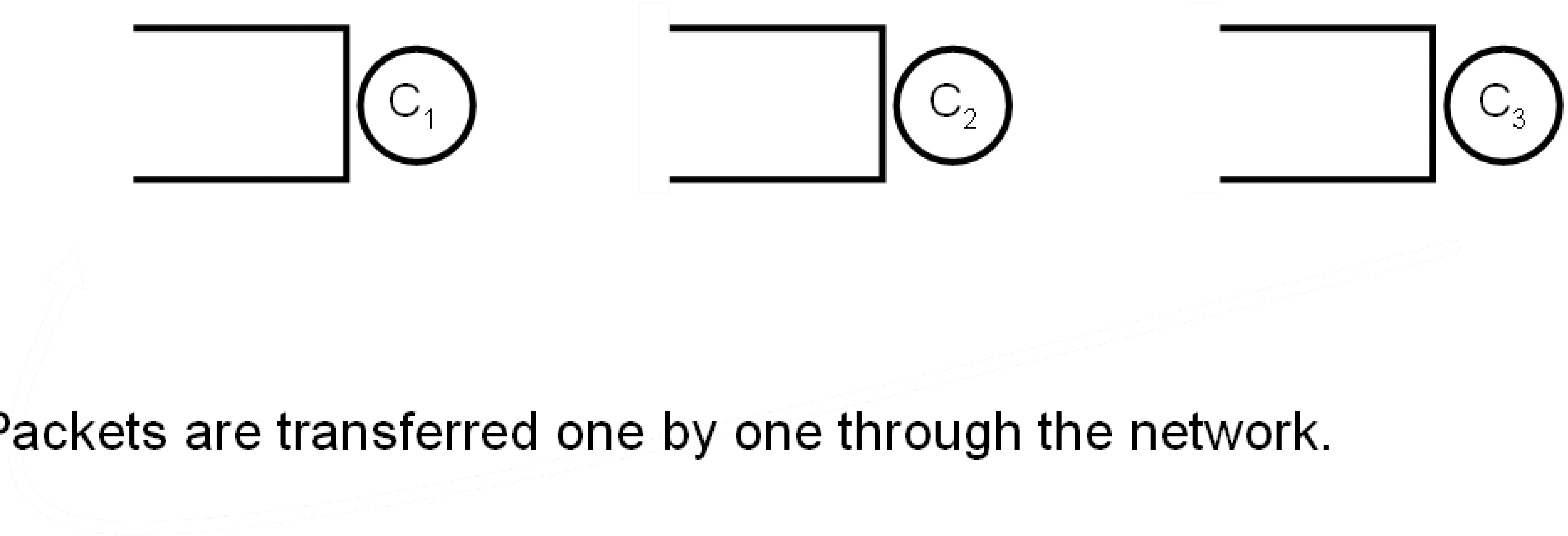

- Packets are transferred one by one through the network.

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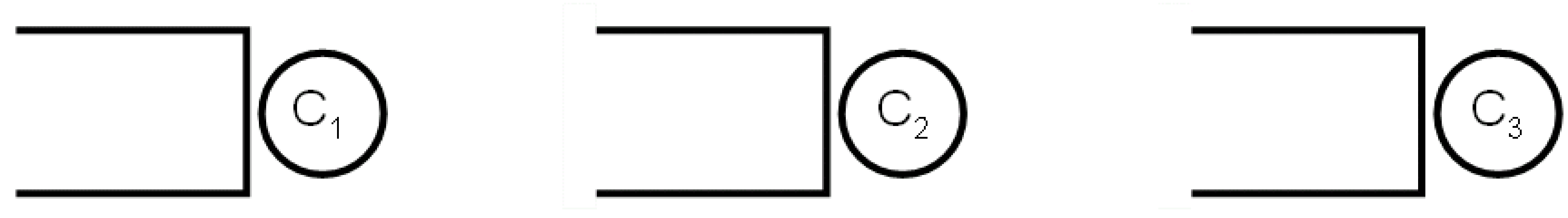

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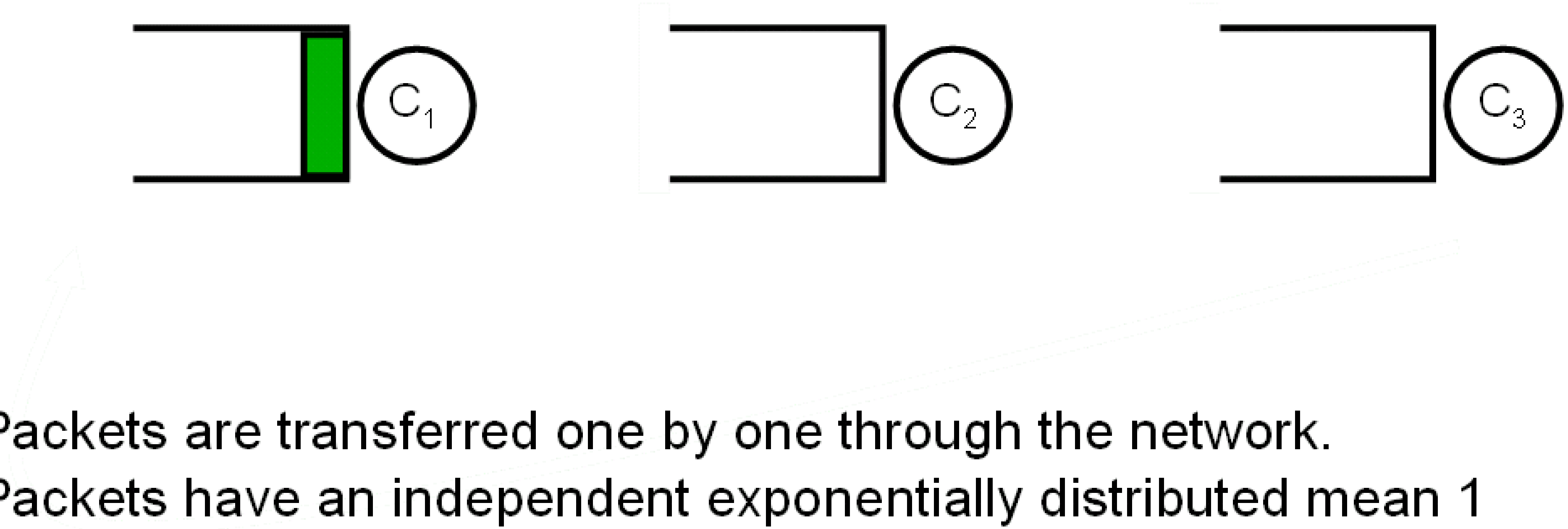

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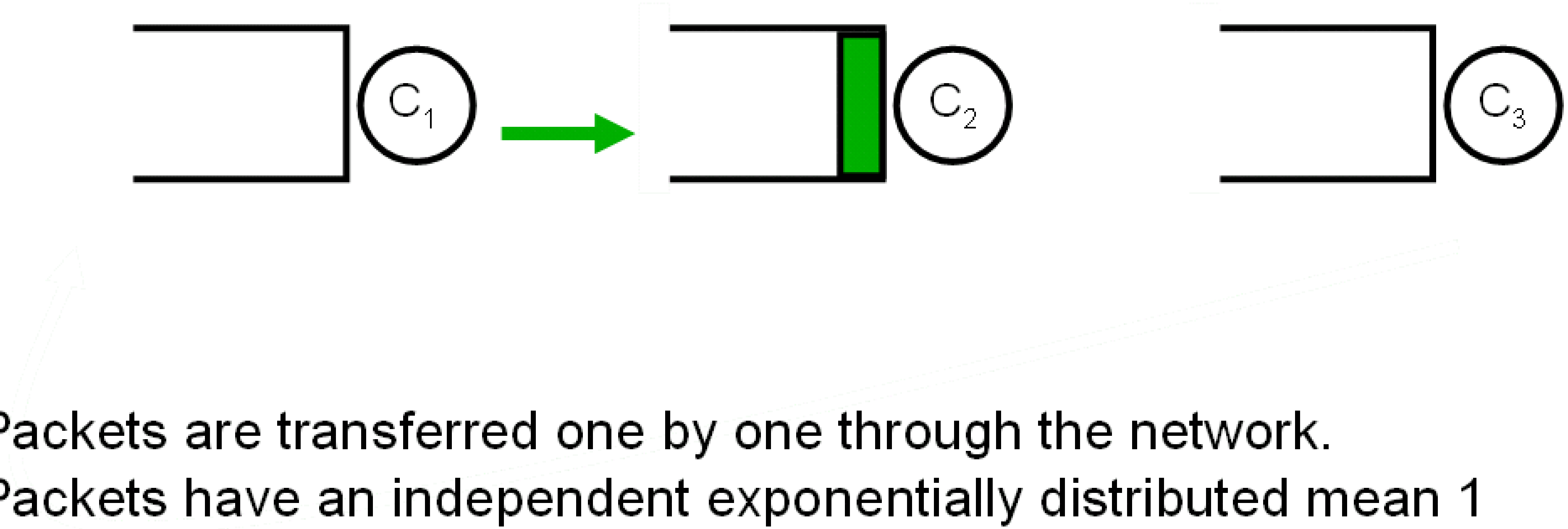

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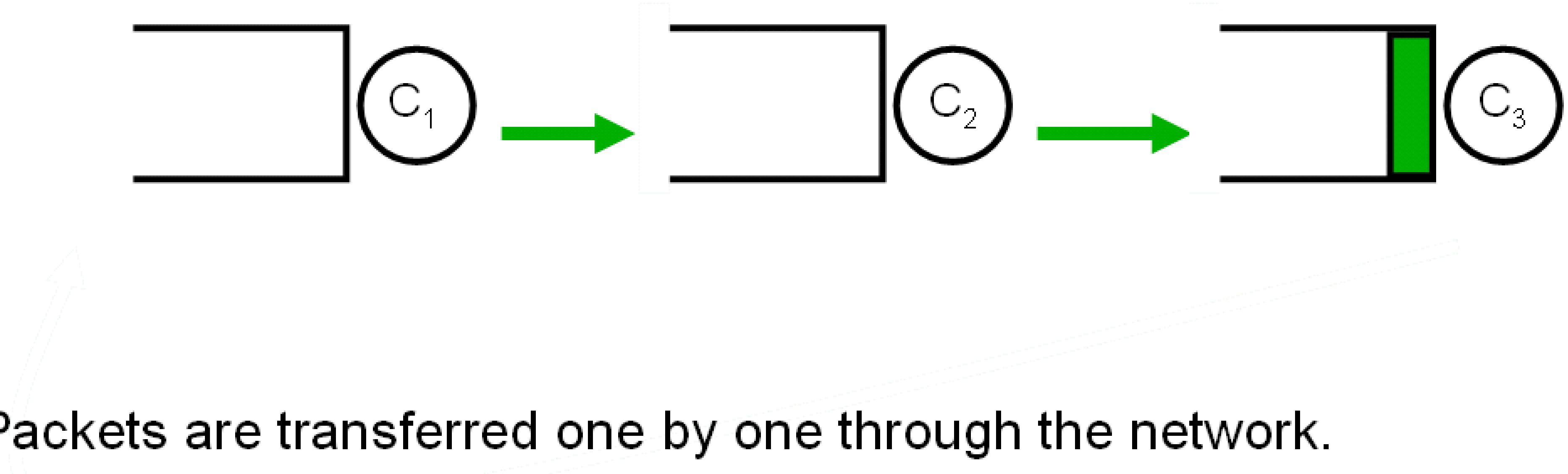

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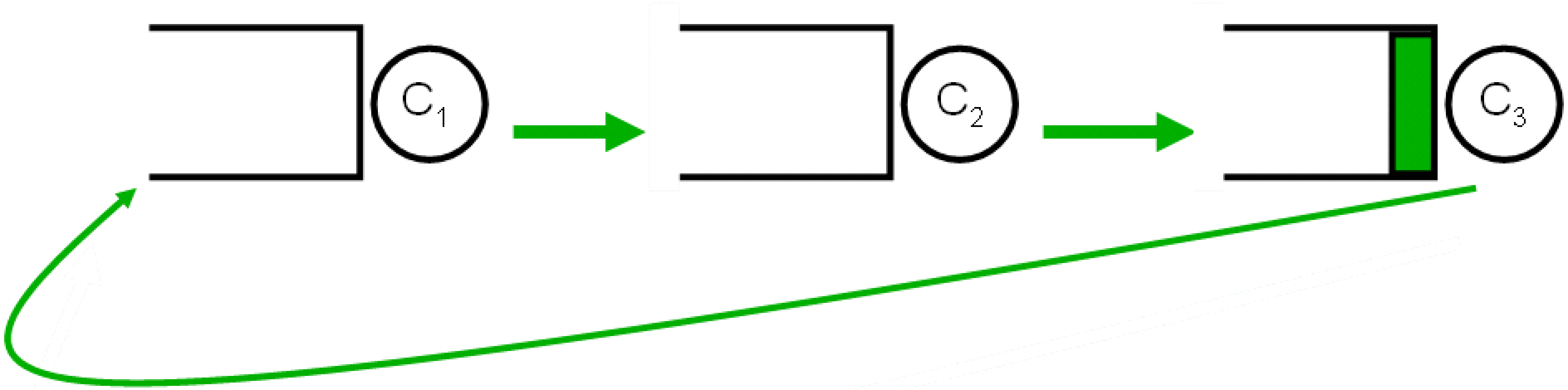

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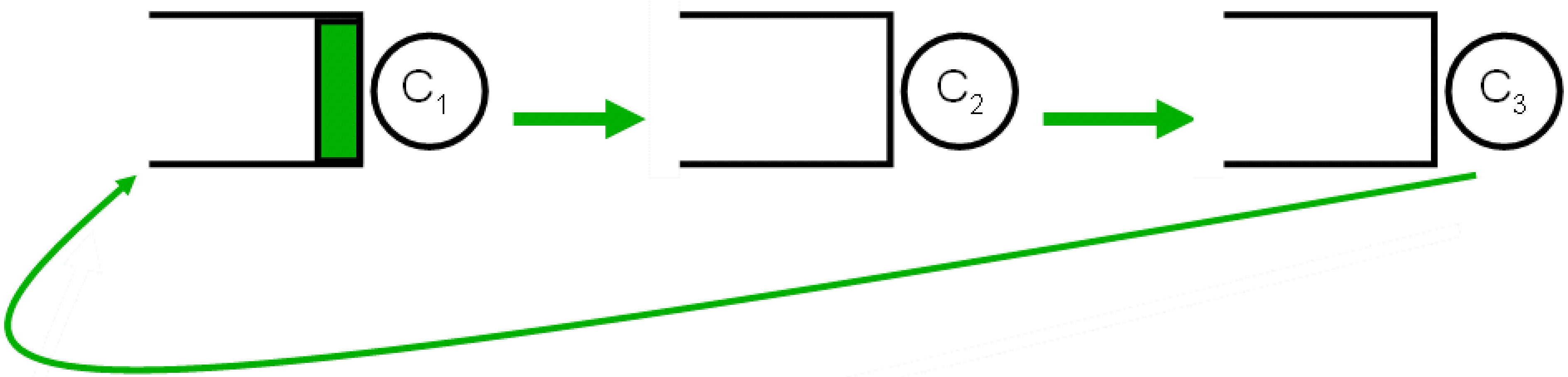

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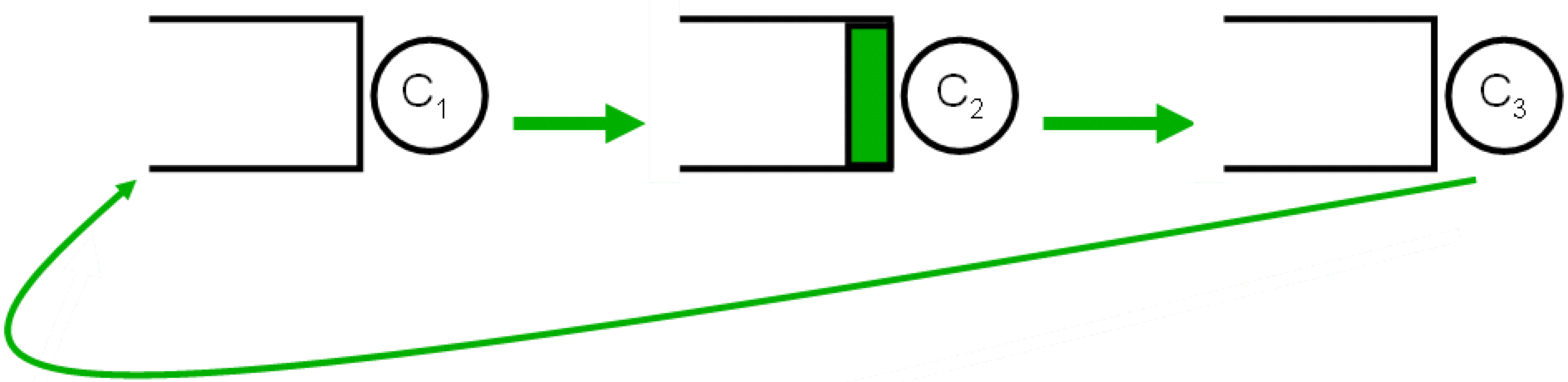

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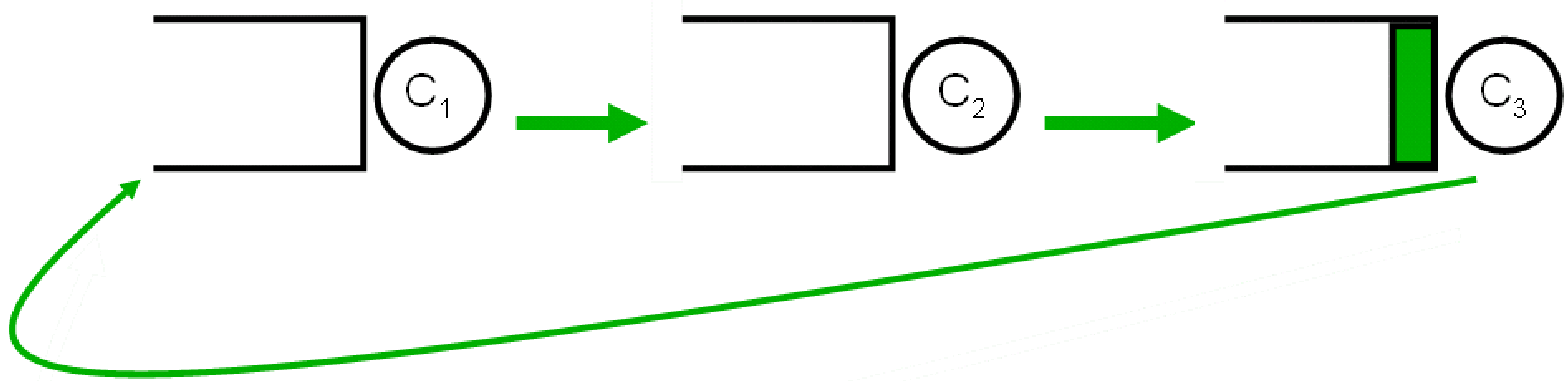

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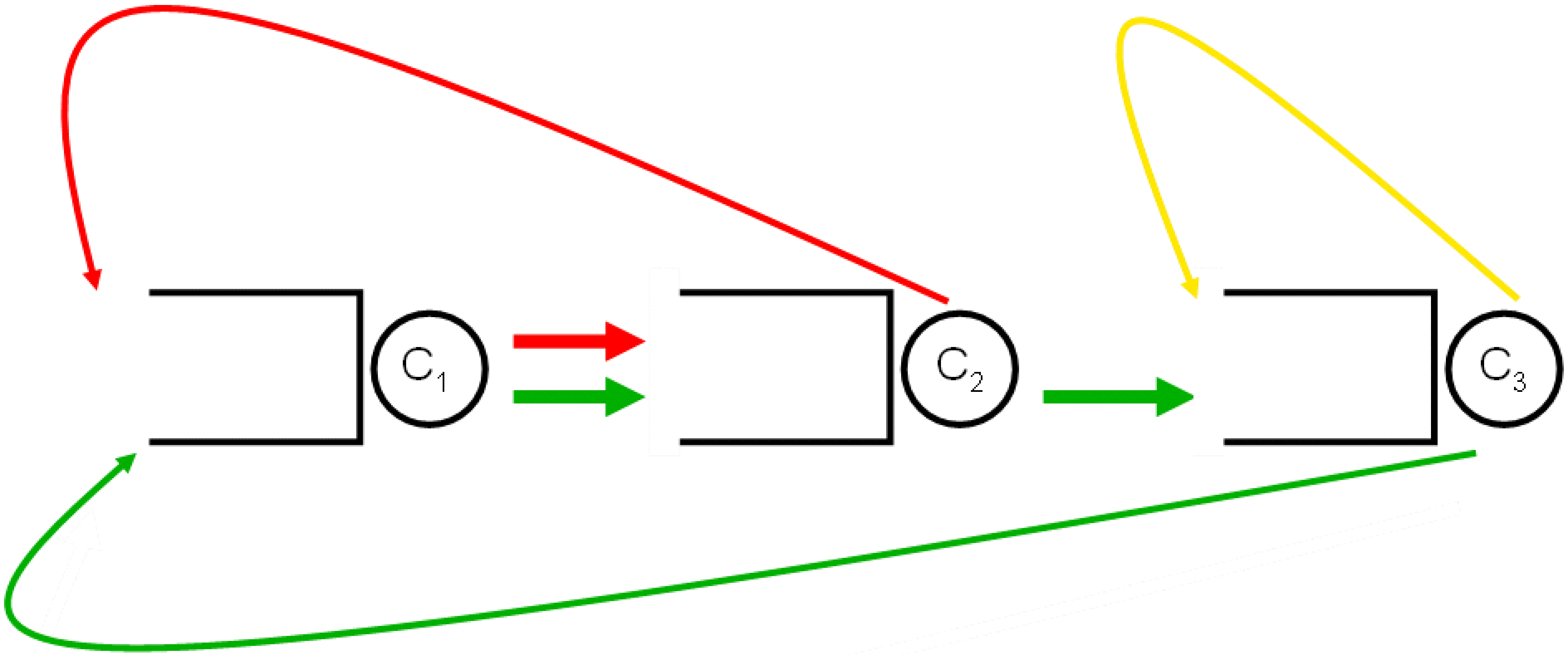

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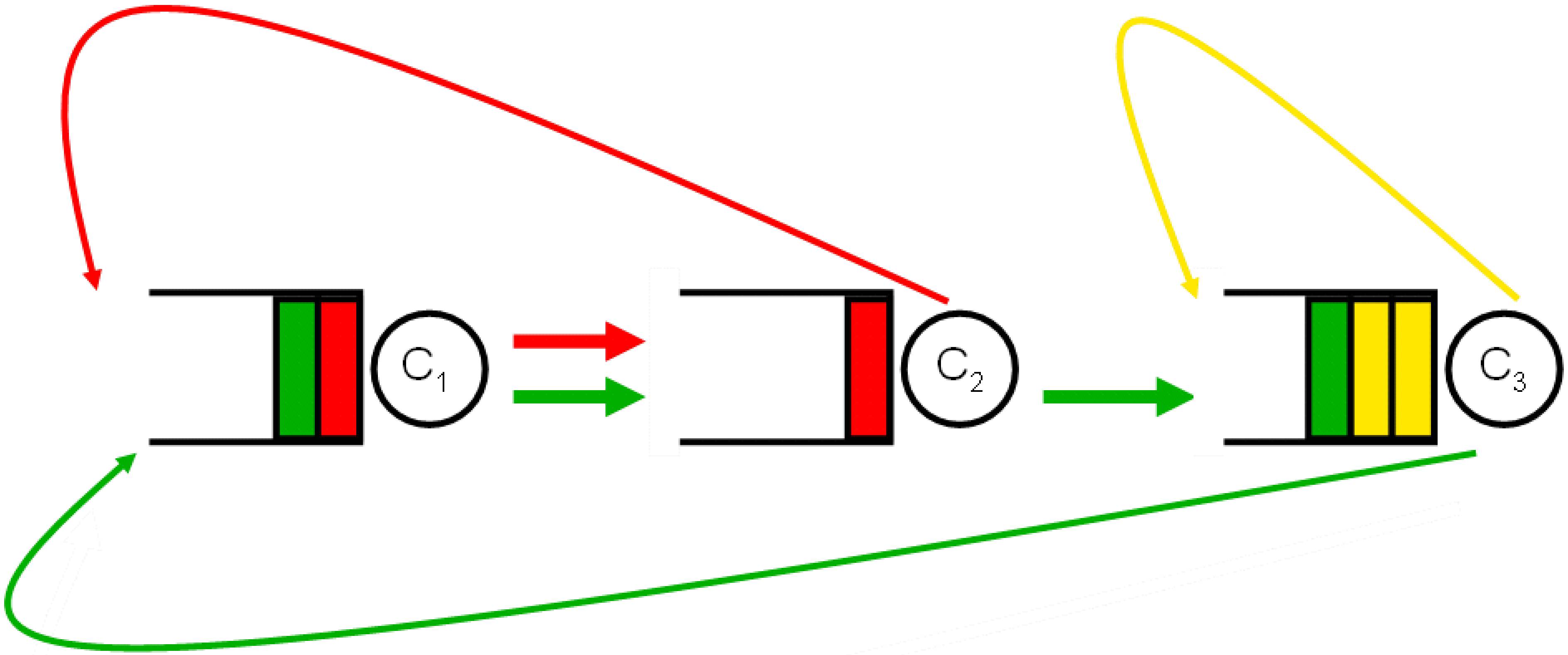

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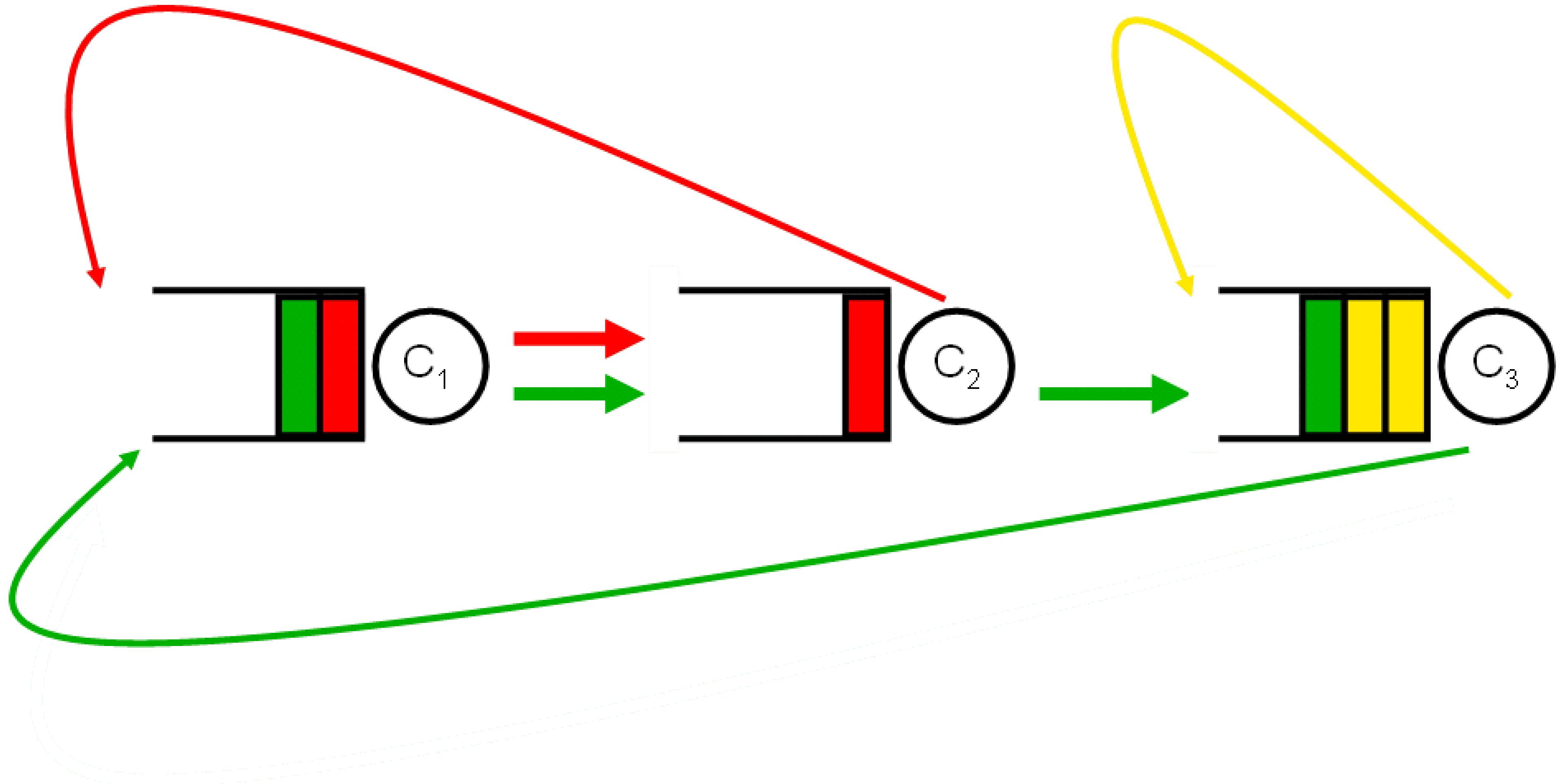

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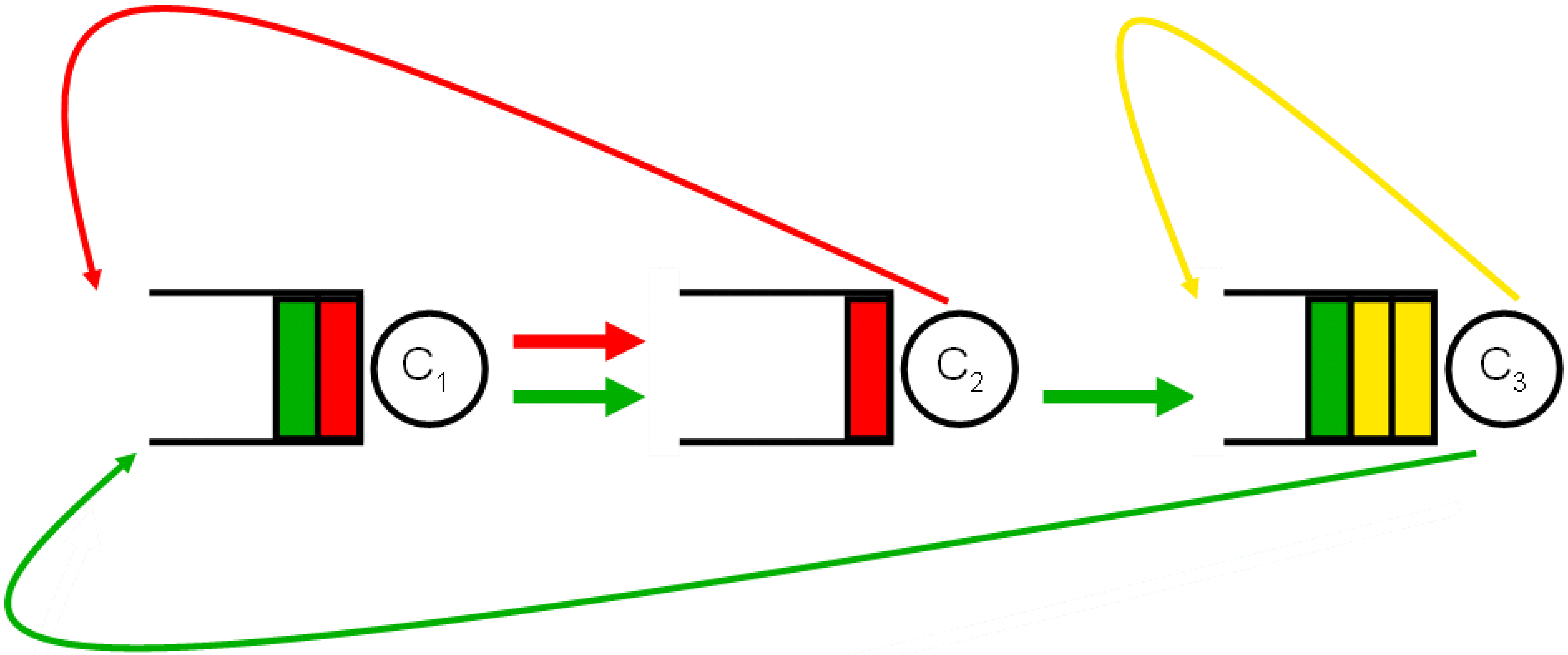

Stationary Distribution:

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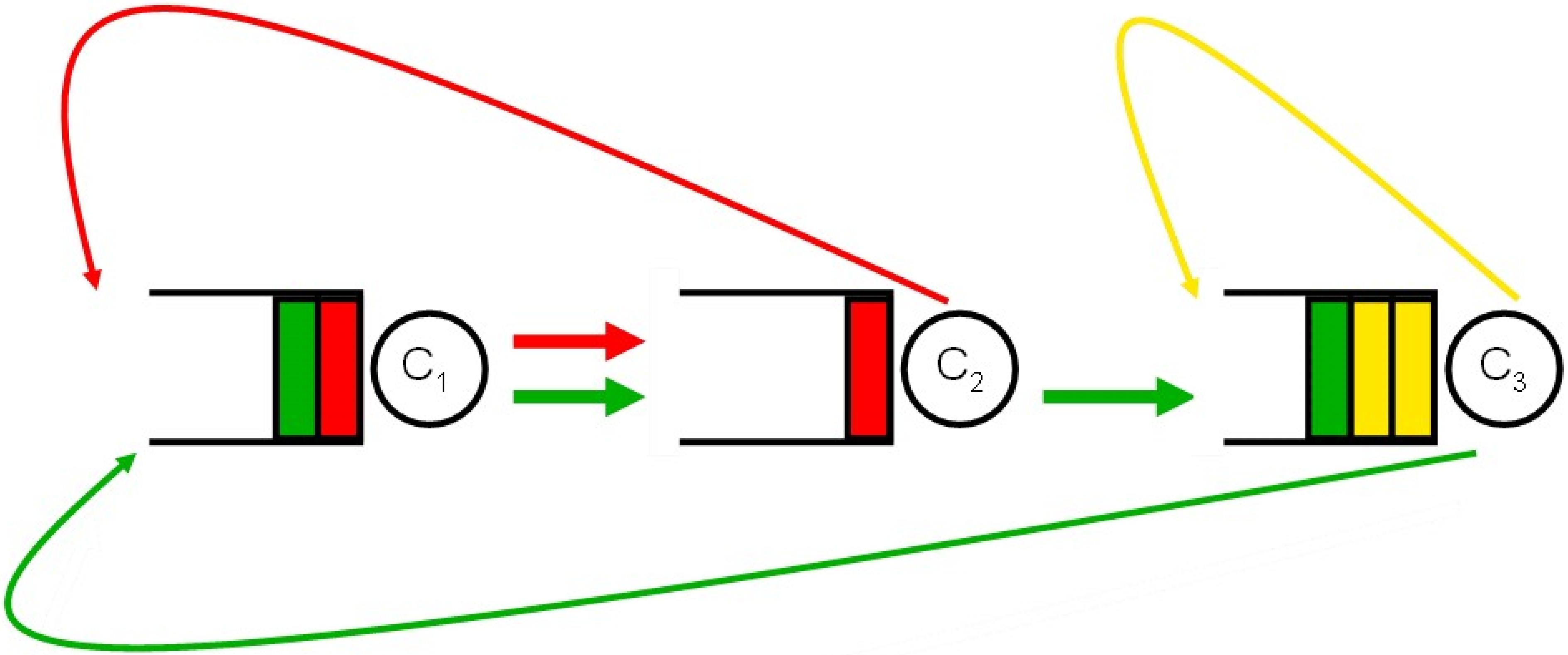

Stationary Distribution:

$$\mathbb{P}(M = m) = \frac{1}{B_n} \prod_{j \in \mathcal{J}} \left(\binom{m_j}{m_{ji} \; : i \ni j} \prod_{i : j \in i} \left(\frac{\rho_i}{C_j} \right)^{m_{ji}} \right)$$

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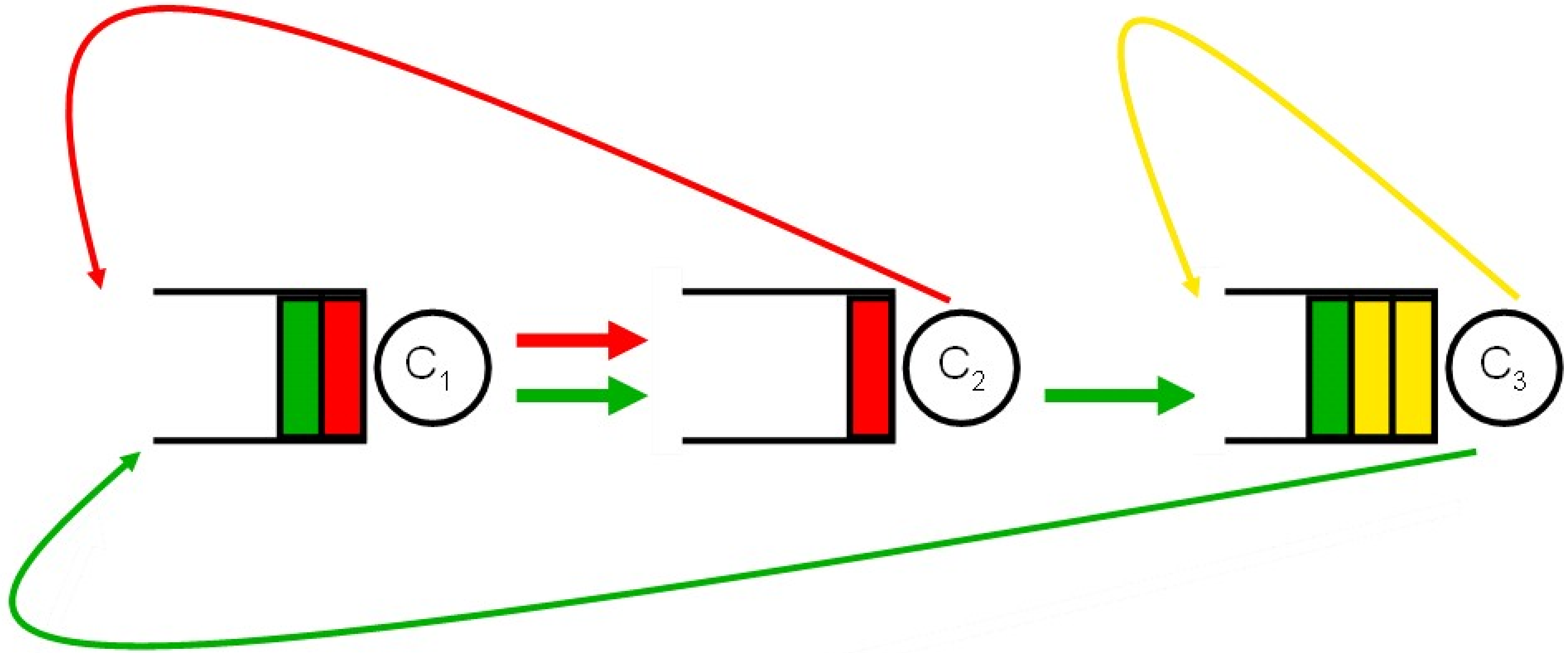

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$$M_{ji} = \#\text{ route } i \text{ packets at queue } j.$$

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A Closed Multi-class Queueing Network

How do we form a large deviations connection with proportional fairness:

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$$\Lambda_i^{SN}(c\bar{m}) \xrightarrow[c \to \infty]{} \Lambda_i^{PF}(\bar{m})$$

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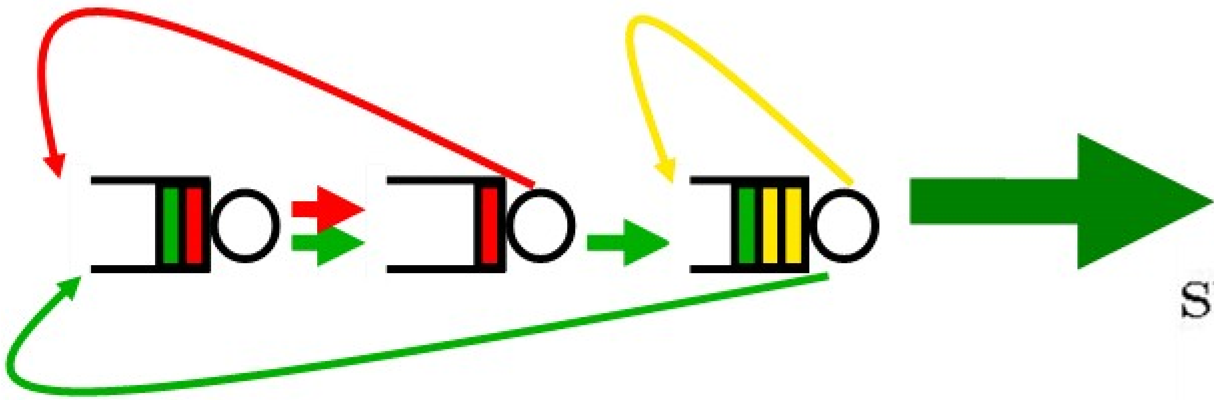

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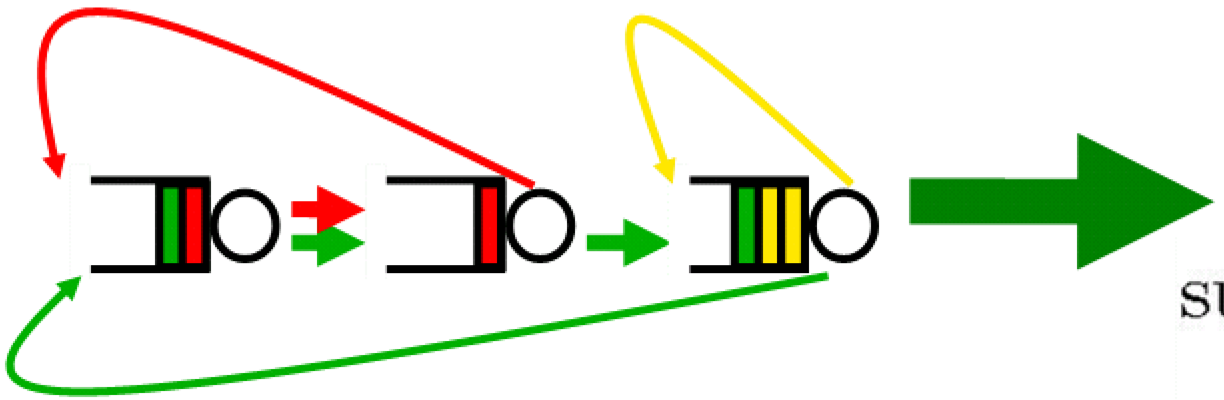

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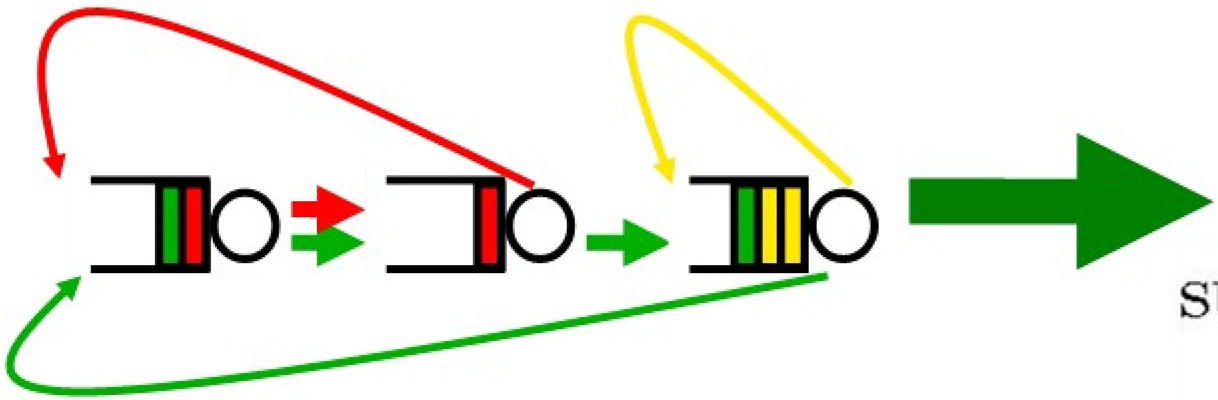

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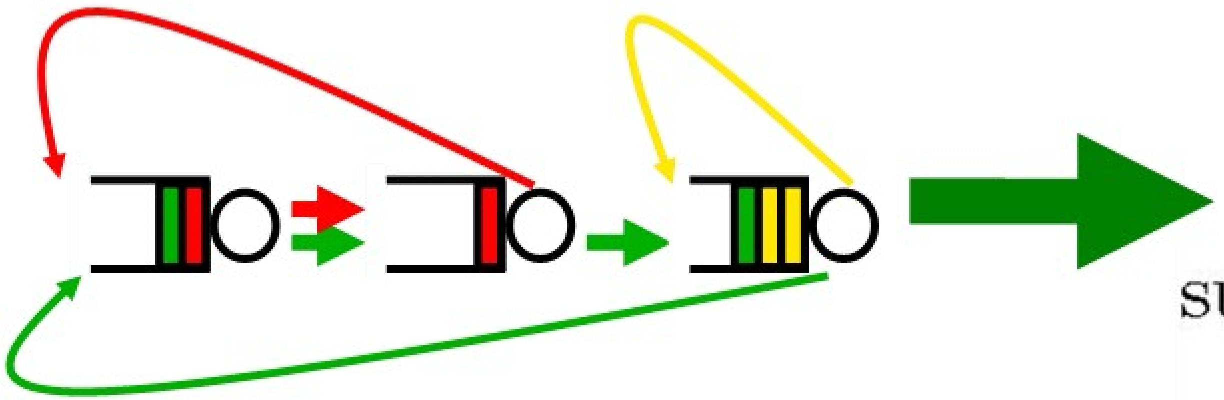

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Looks like a Legendre-Fenchel Transform...

Congestion Windows

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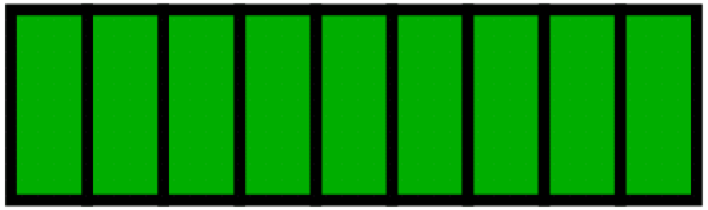

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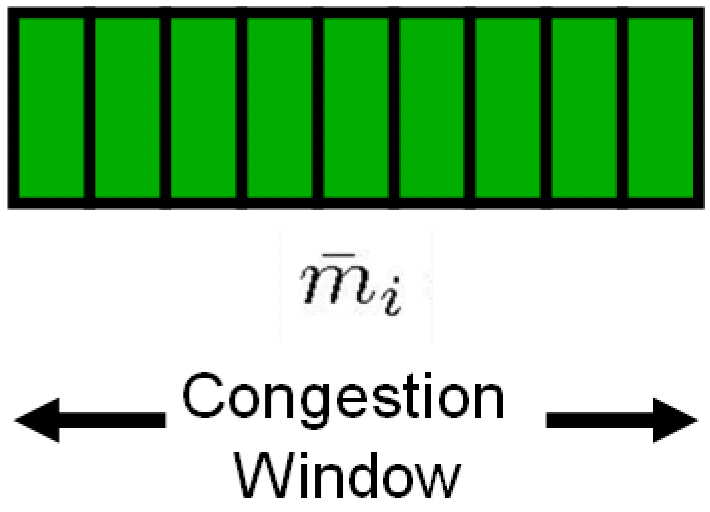

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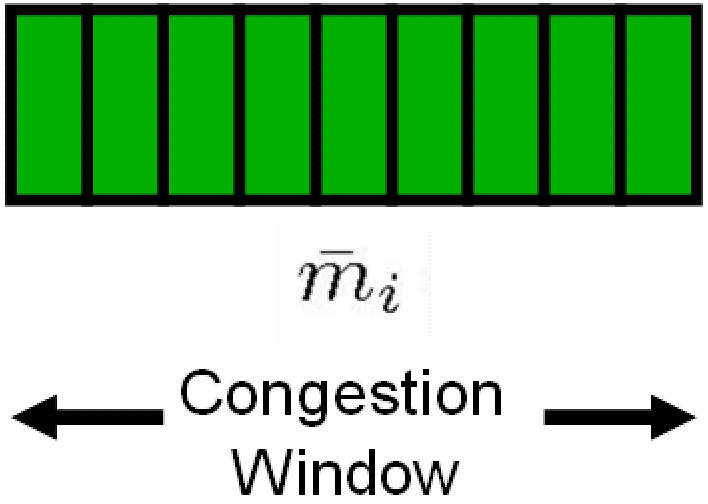
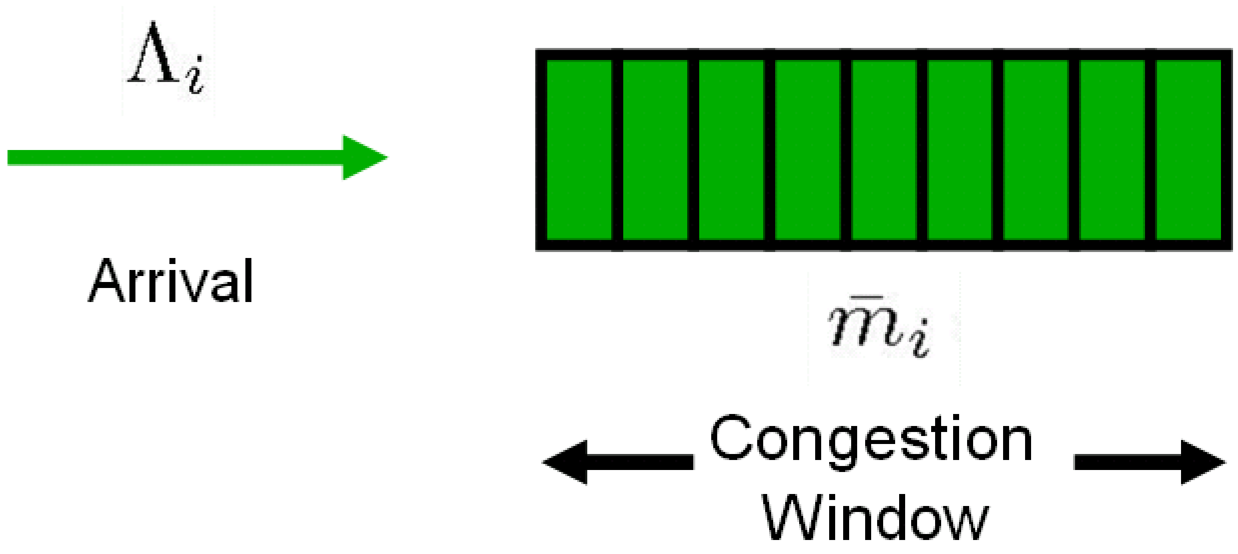


Arrivals acknowledges packets.

Decrease the congestion window size.

Congestion Windows



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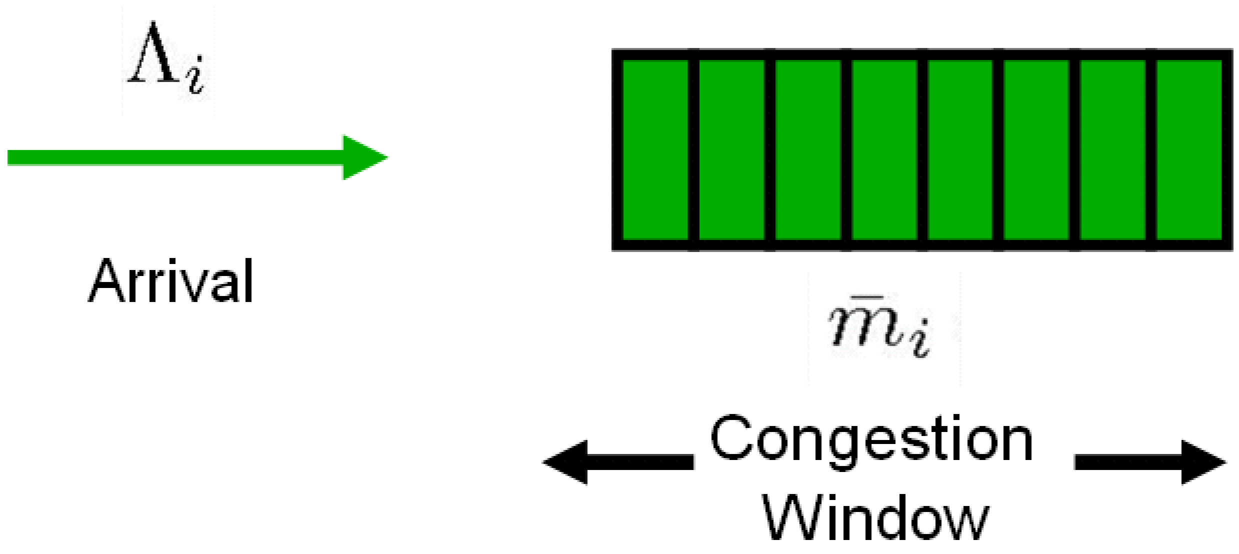

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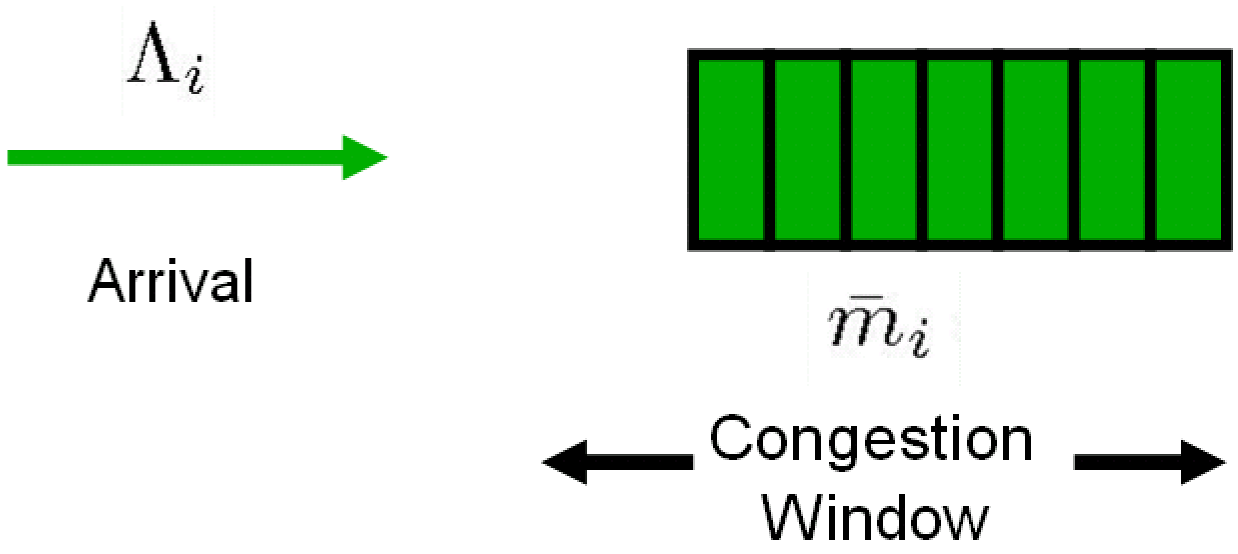

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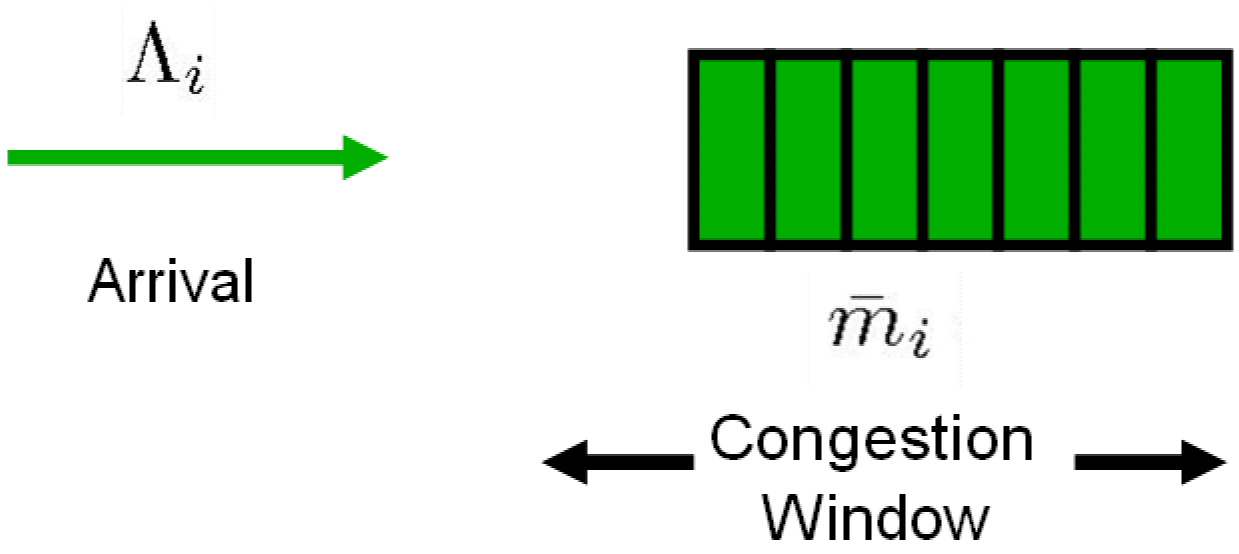

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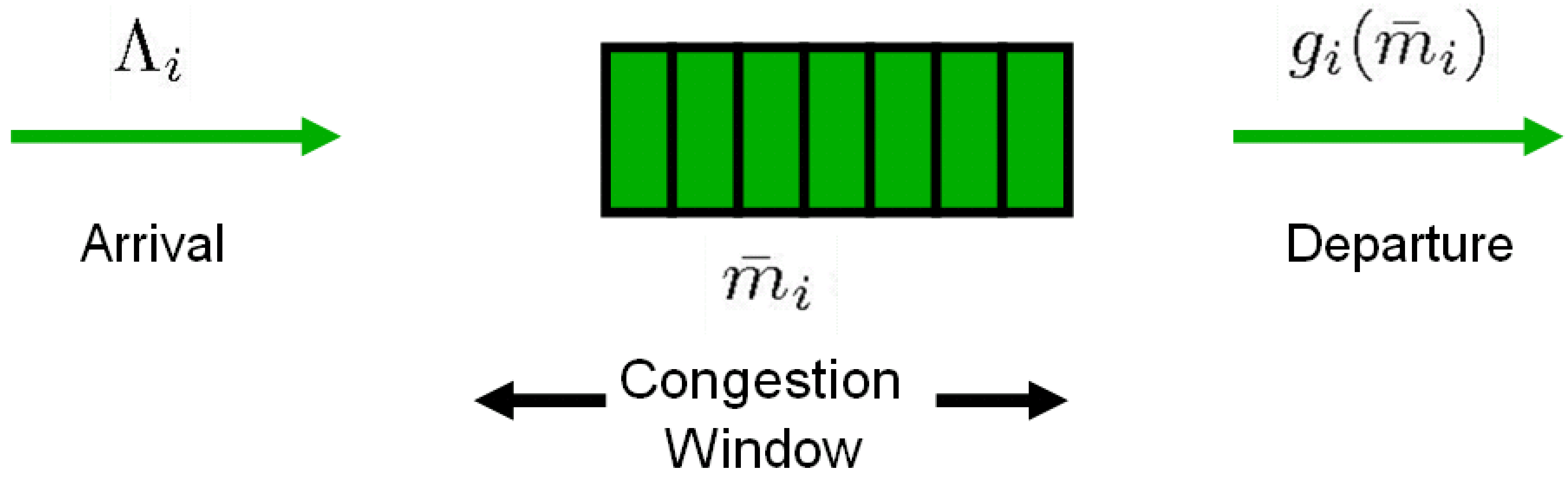

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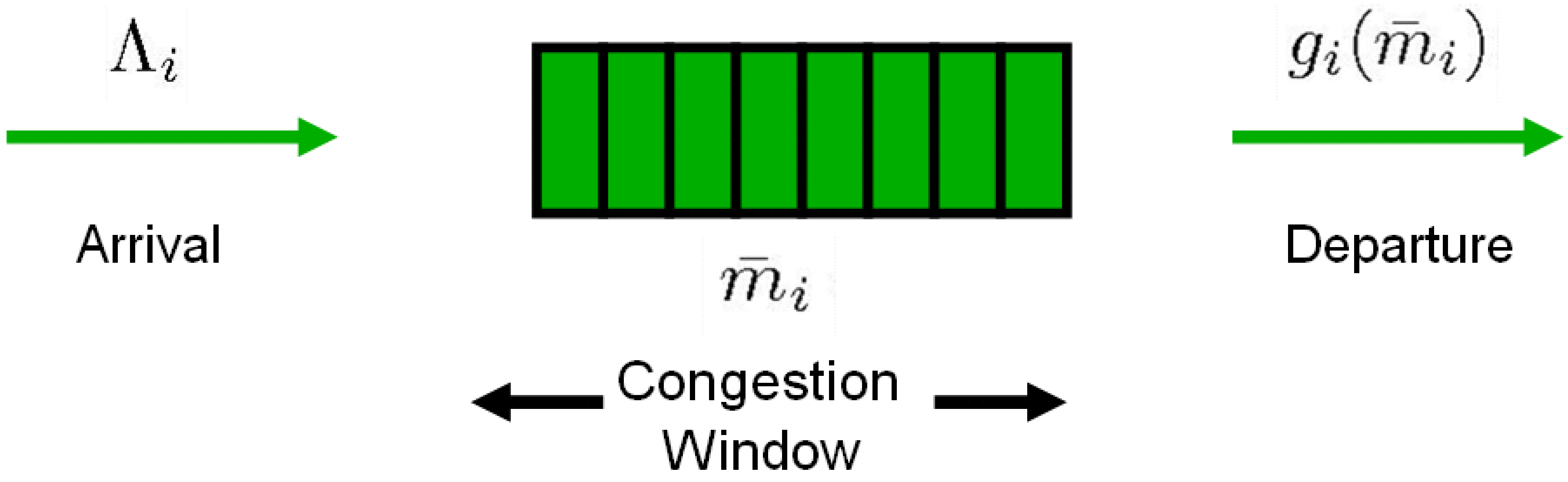

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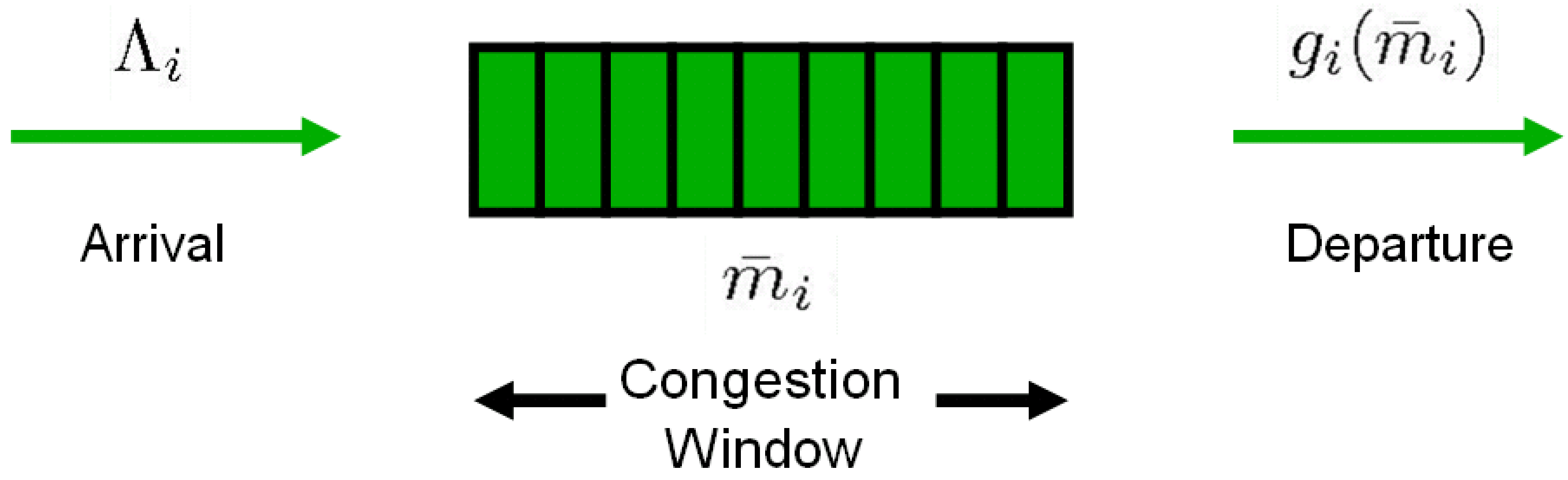

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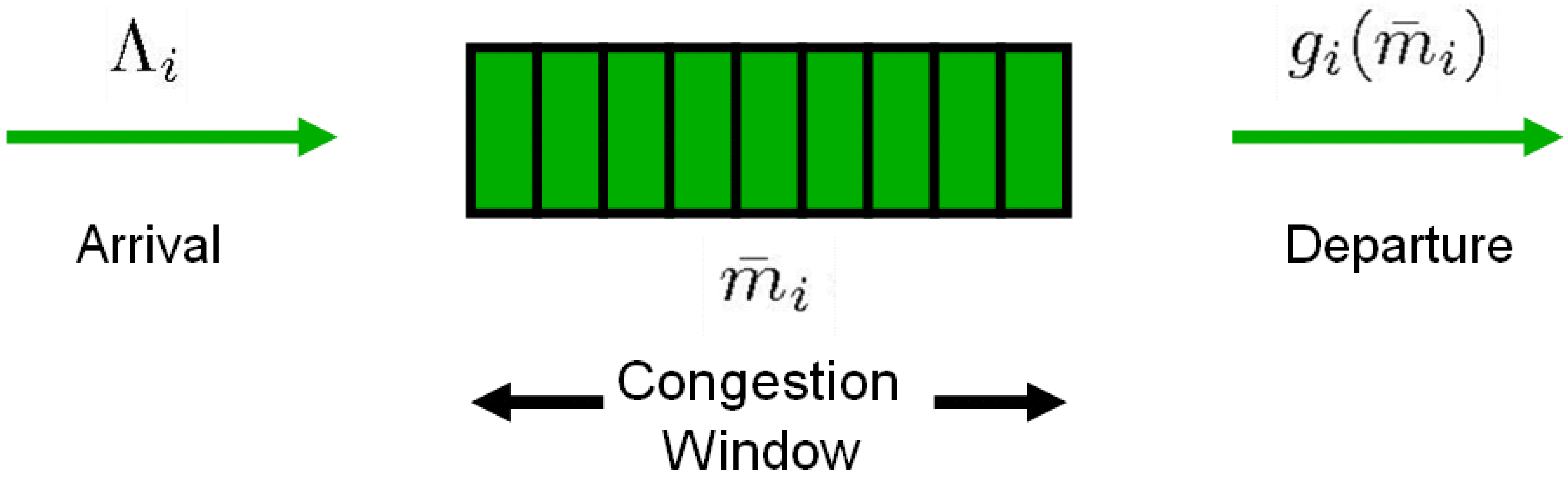

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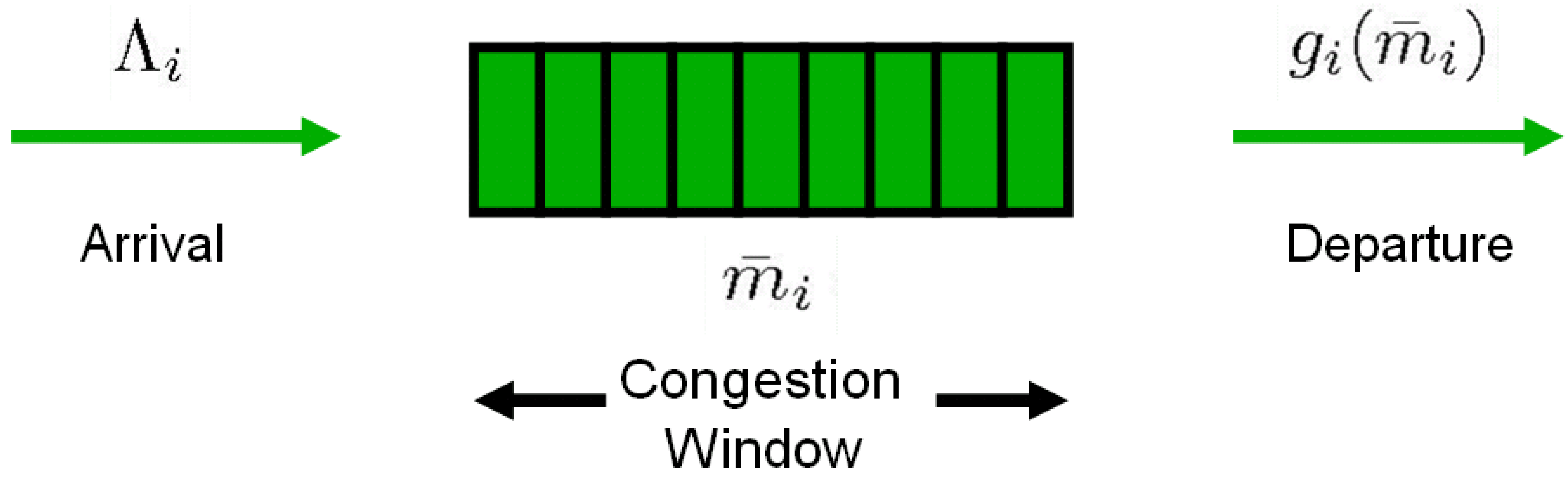
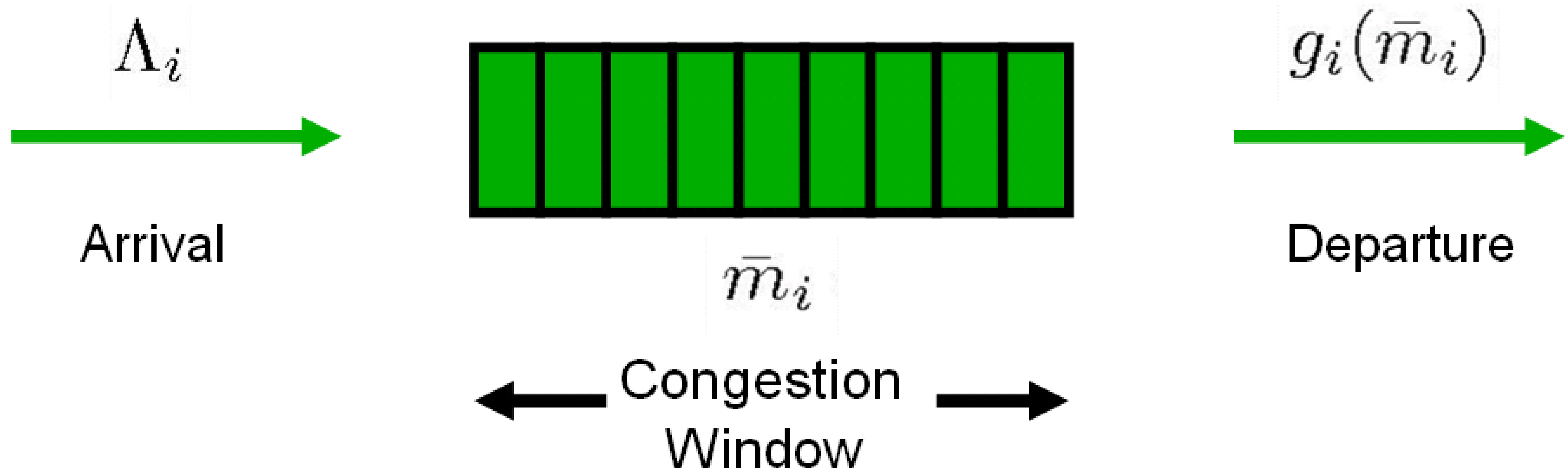


Reversible with Stationary Distribution:

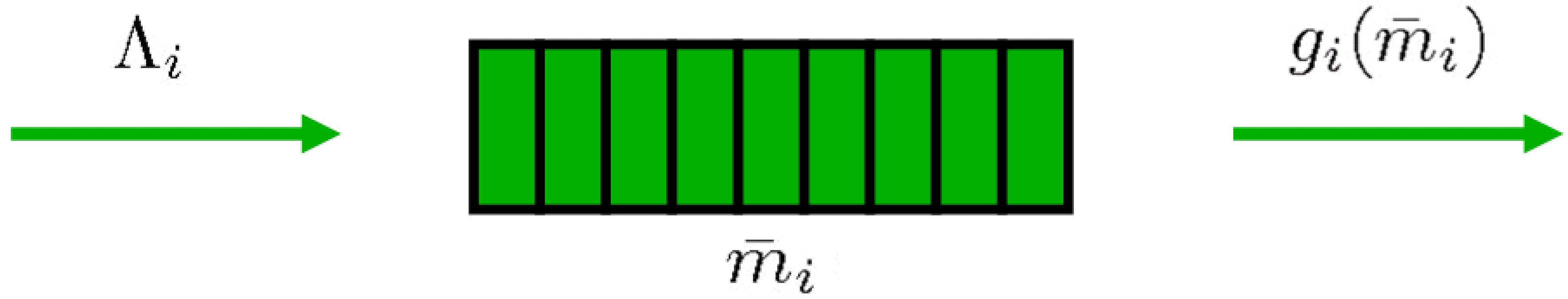
Congestion Windows



Reversible with Stationary Distribution:

$$\pi_i(\bar{m}_i) = \prod_{k=1}^{\bar{m}_i} \frac{g_i(k)}{\Lambda_i}$$

Congestion Windows



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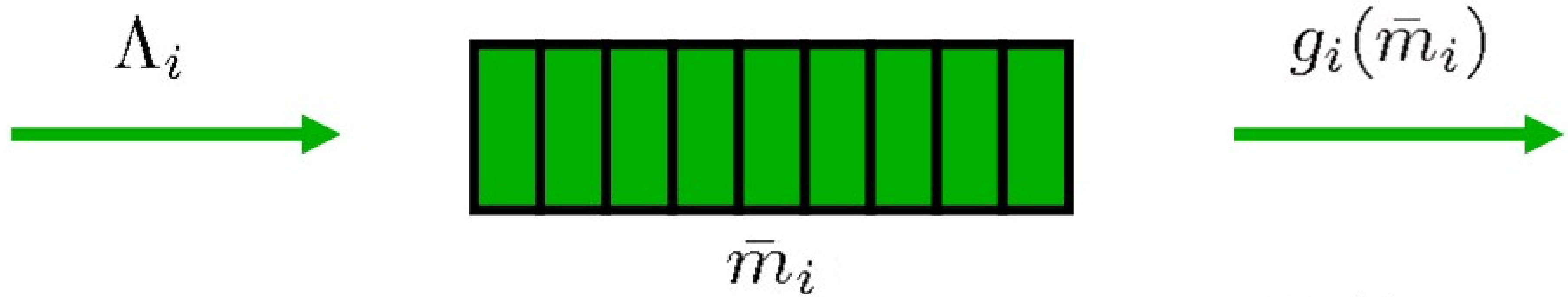
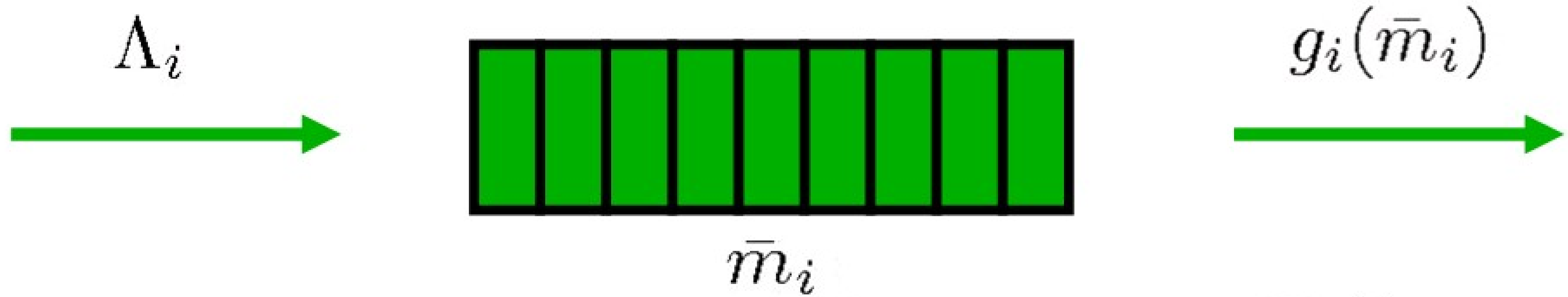


Let $\Lambda_i = e^{\lambda_i}$ and $g_i^{(c)}(\bar{m}_i) = e^{cG_i(\frac{\bar{m}_i+1}{c}) - cG_i(\frac{\bar{m}_i}{c})}$

Congestion Windows



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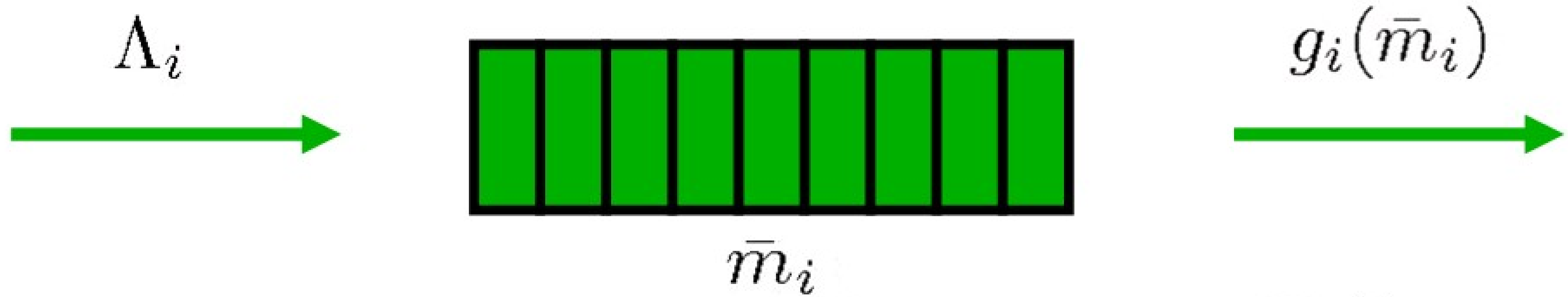
Where we define G_i by a **NEW** USER PROBLEM:

Congestion Windows



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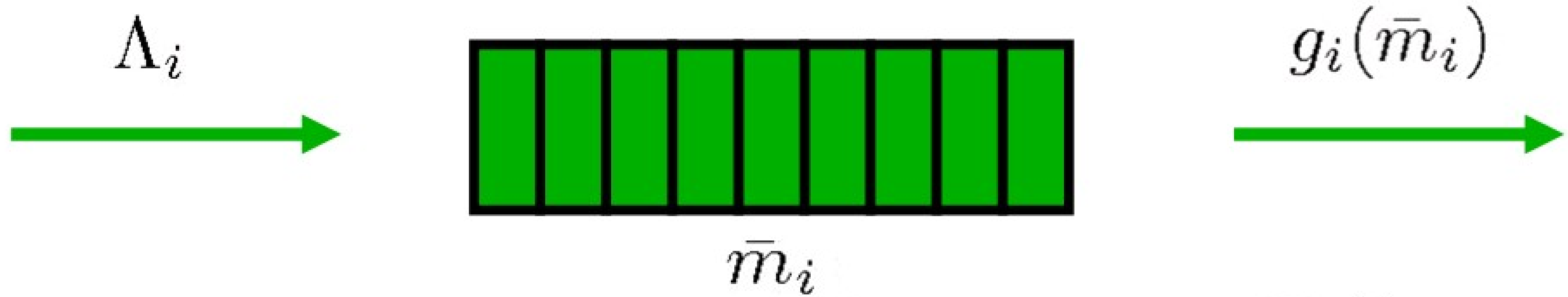

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Where we define G_i by a **NEW** USER PROBLEM:

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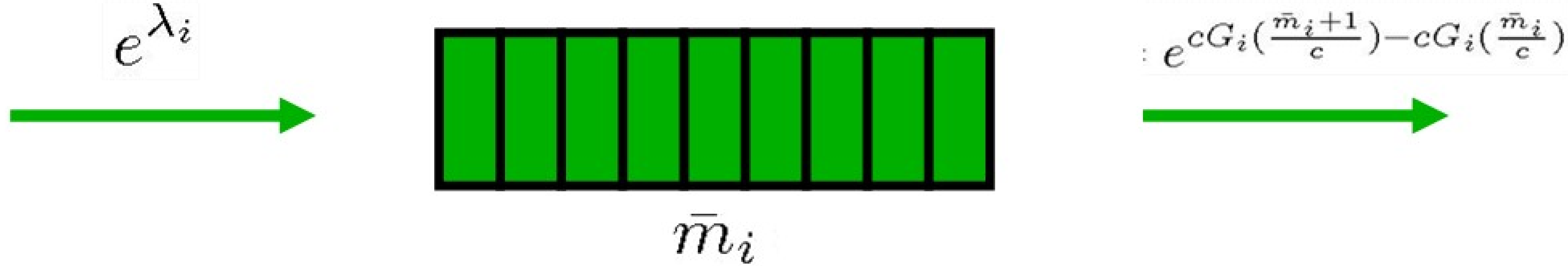
With stationary distribution

$$\pi_i^{(c)}(\bar{m}_i) = e^{cG_i\left(\frac{\bar{m}_i}{c}\right) - \lambda_i \bar{m}_i}$$

Congestion Windows



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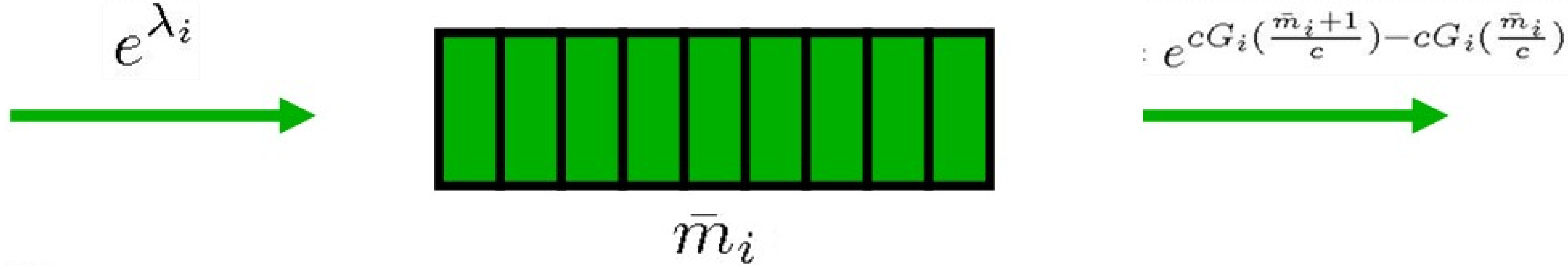
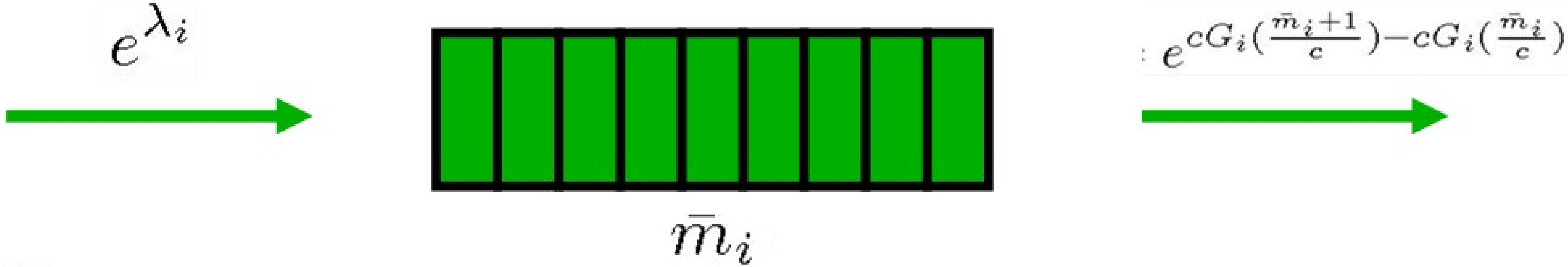


Thus

$$\lim_{c\to\infty}\frac{1}{c}\log\pi^{(c)}(c\bar{m}_i)=G_i(\bar{m}_i)-\lambda_i\bar{m}_i$$

Congestion Windows

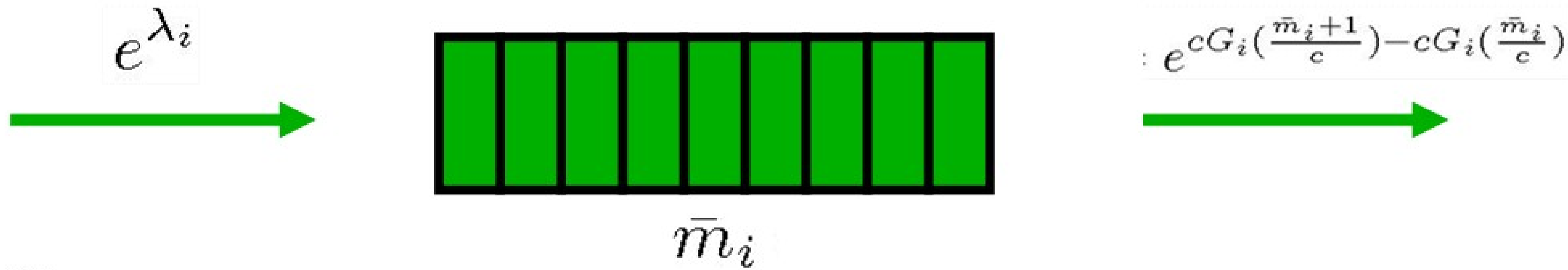


Thus

$$\lim_{c\to\infty}\frac{1}{c}\log\pi^{(c)}(c\bar{m}_i) = G_i(\bar{m}_i) - \lambda_i\bar{m}_i$$

The most likely state is

Congestion Windows



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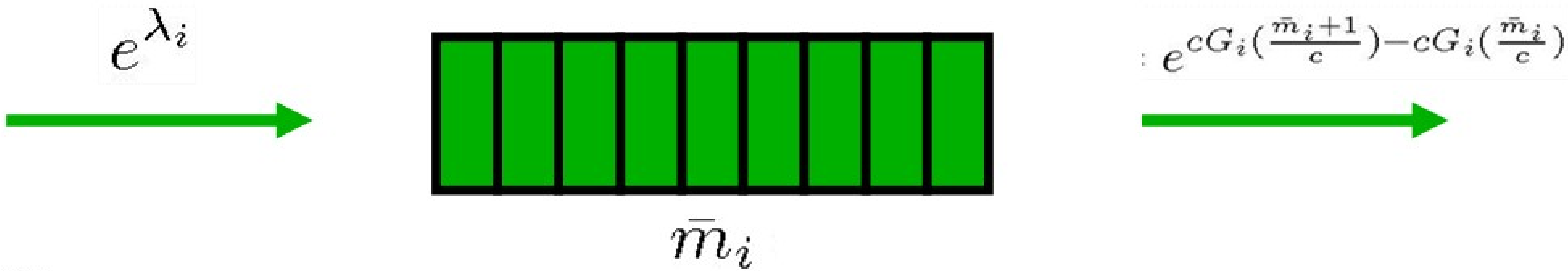
The most likely state is

$$\lim_{c\to\infty}\frac{1}{c}\log\pi^{(c)}(c\bar{m}_i^*)=\max_{\bar{m}_i>0}\{G_i(\bar{m}_i)-\lambda_i\bar{m}_i\}$$

$$= -U_i(e^{\lambda_i})$$

Congestion Windows



Thus

$$\lim_{c\to\infty} \frac{1}{c} \log \pi^{(c)}(c\bar{m}_i) = G_i(\bar{m}_i) - \lambda_i \bar{m}_i$$

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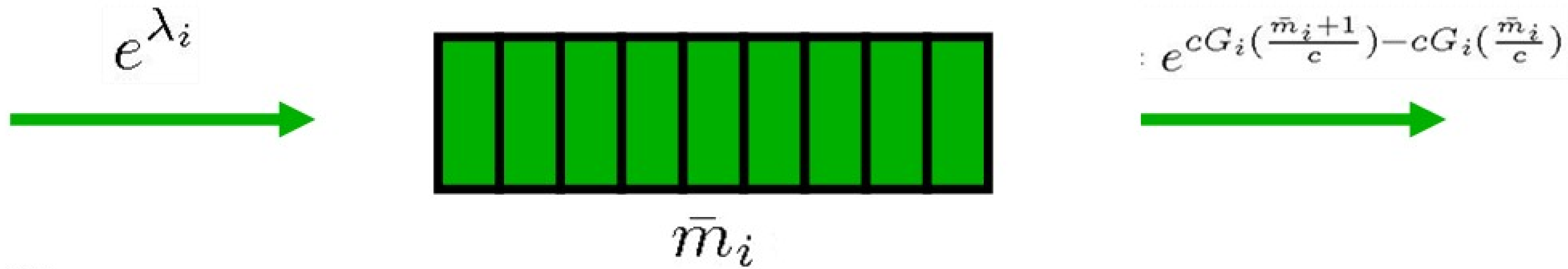
$$= \quad - U_i(e^{\lambda_i})$$

Provided

Assumption: $U_i(e^{\lambda_i})$ is concave.

Congestion Windows



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Provided

Assumption: $U_i(e^{\lambda_i})$ is concave.

For example: weighted alpha fair for $\alpha > 1$

NETWORK PROBLEM is:

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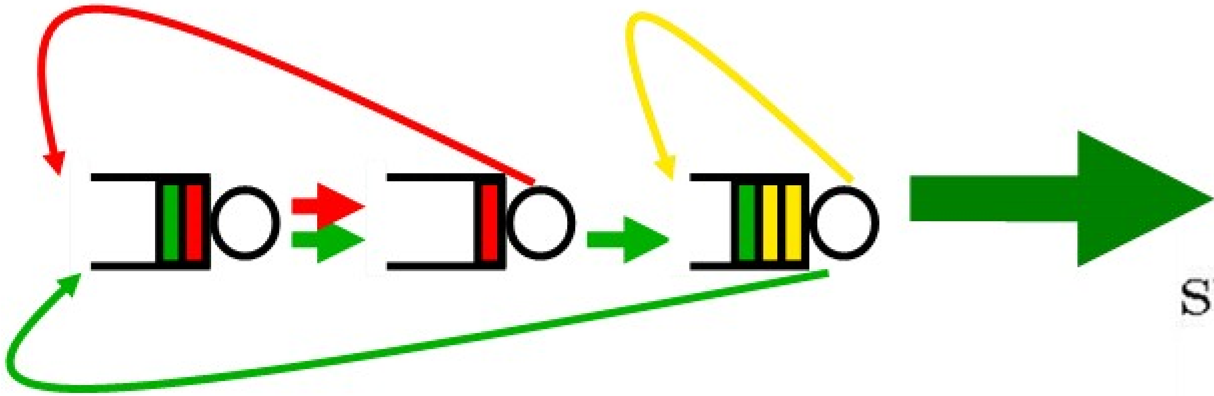

$$\max \sum_{i\in\mathcal{I}} \bar{m}_i \log \Lambda_i$$
$$\text{subject to} \quad \sum_{i:j\in i} \Lambda_i \le C_j,$$

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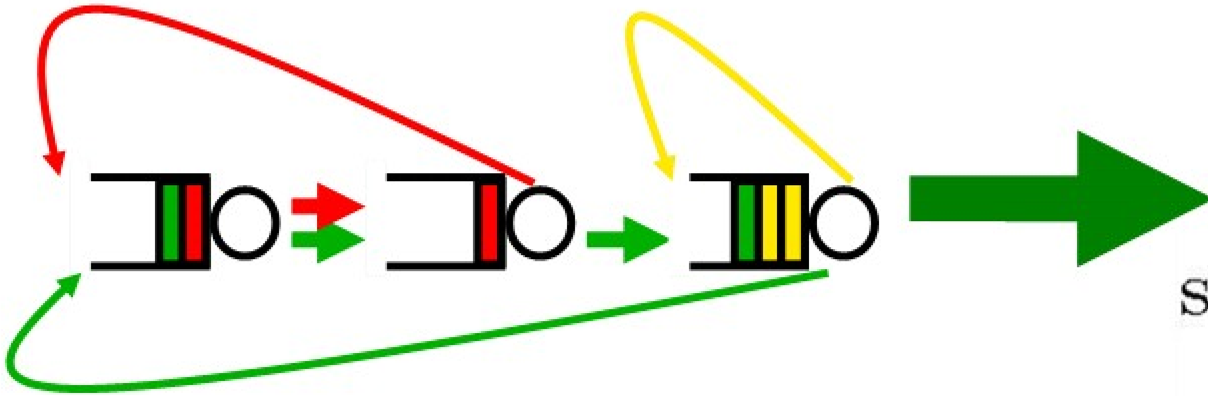

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USER PROBLEM:

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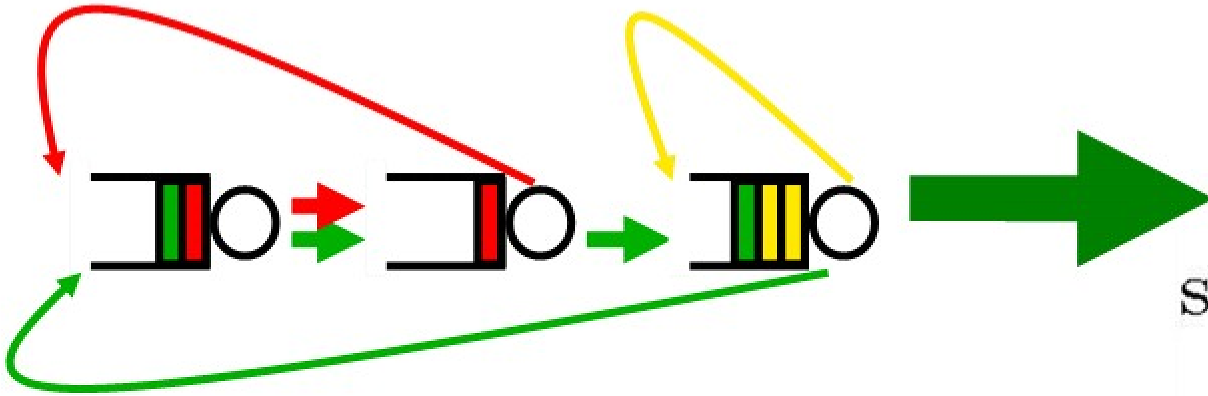

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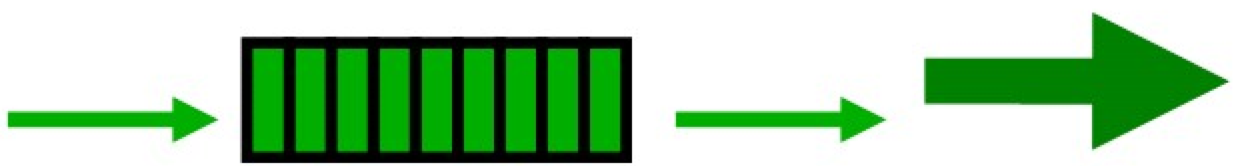
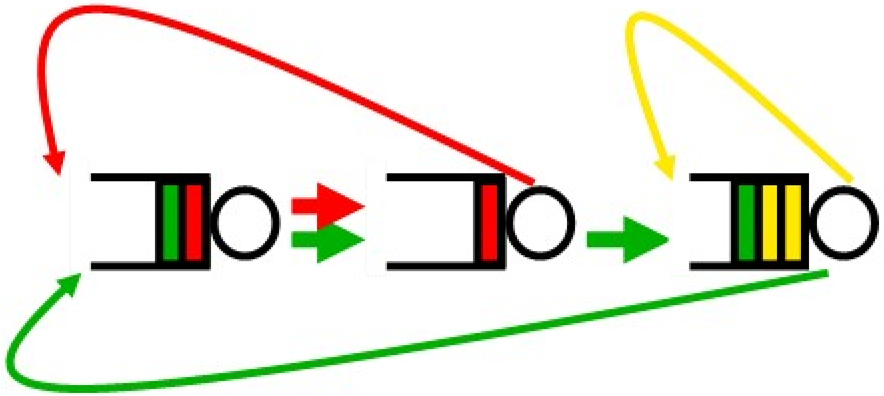


$$\max_{\lambda_i \in \mathbb{R}} \{ U_i(e^{\lambda_i}) - \bar{m}_i \lambda_i \}$$

NETWORK PROBLEM is:



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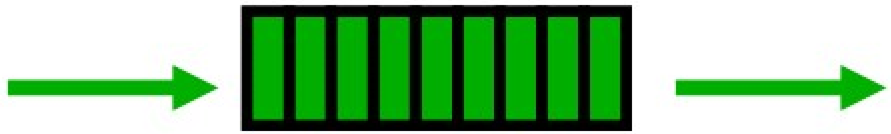

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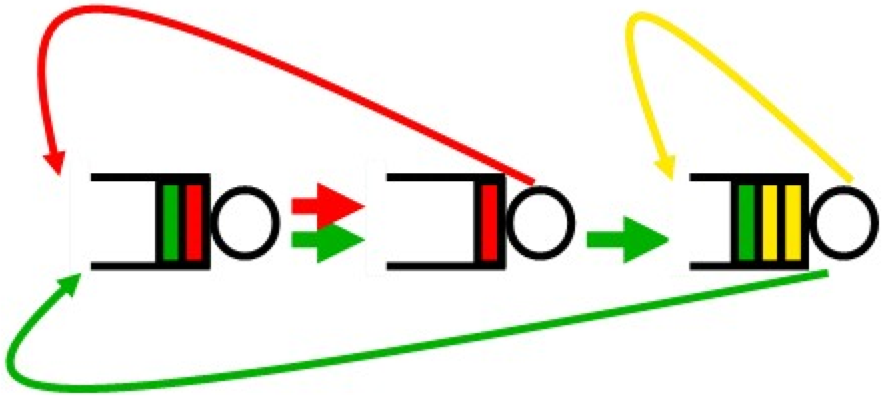
AND PRICES:

NETWORK PROBLEM is:



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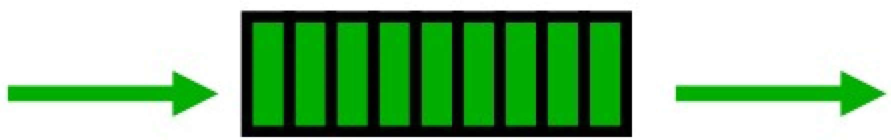

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AND PRICES:

$$\bar{m}_i = \Lambda_i q_i$$

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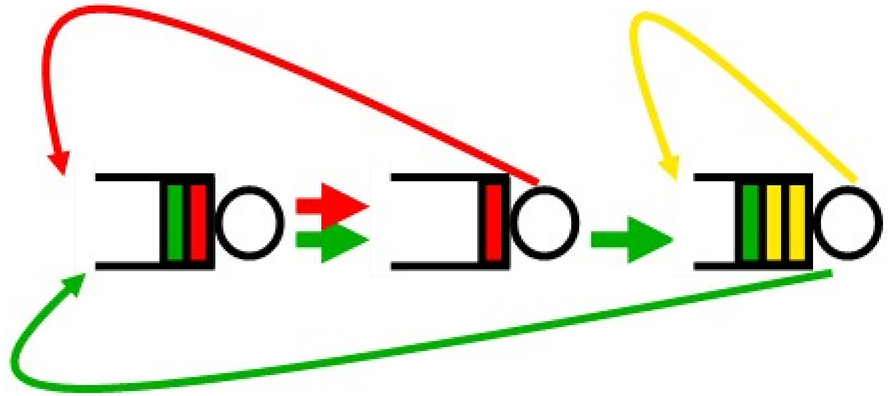

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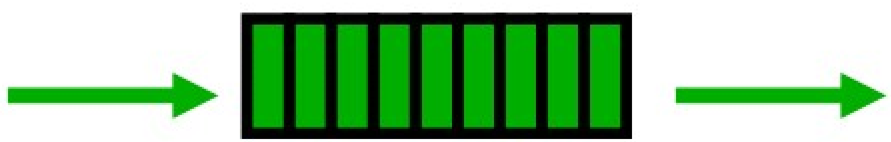

$$\max_{\lambda_i \in \mathbb{R}} \{ U_i(e^{\lambda_i}) - \bar{m}_i \lambda_i \}$$

AND PRICES:

Little's Law

$$\bar{m}_i = \Lambda_i q_i$$

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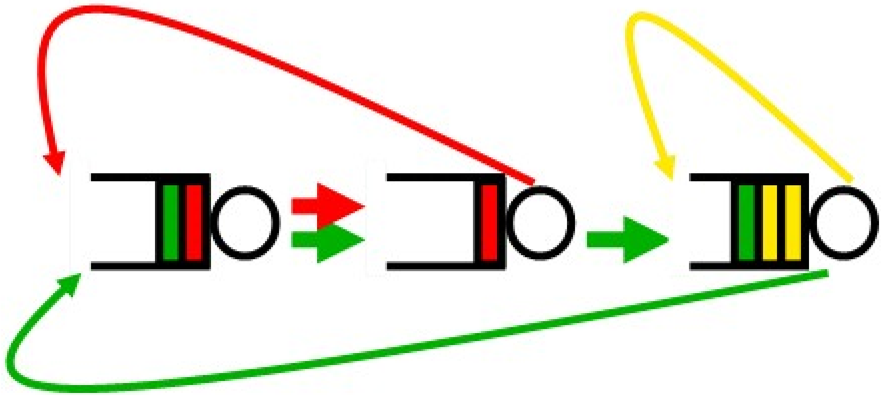

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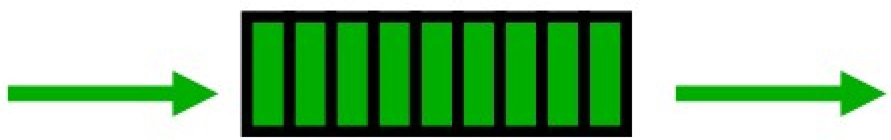

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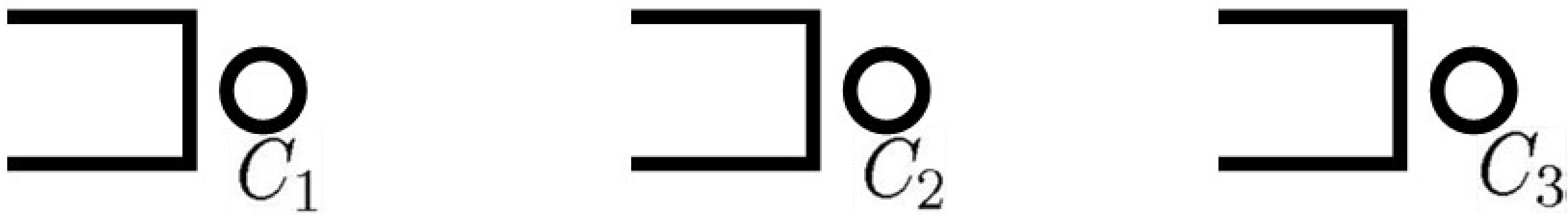
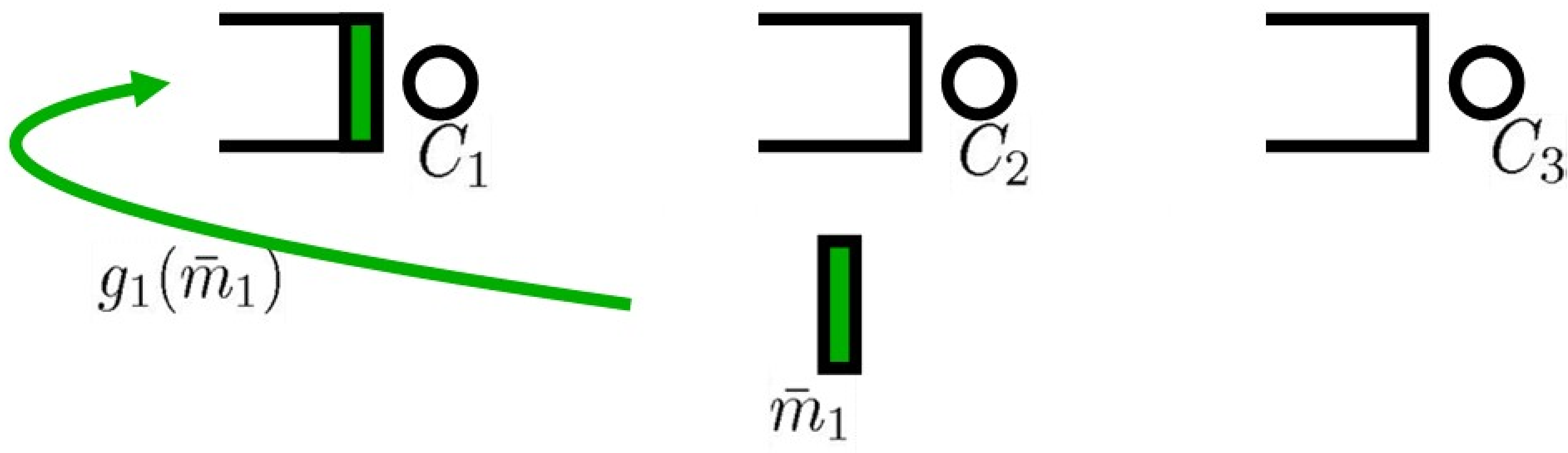
SYSTEM PROBLEM...

A System of Queues



$\bar{m}_1$

A System of Queues



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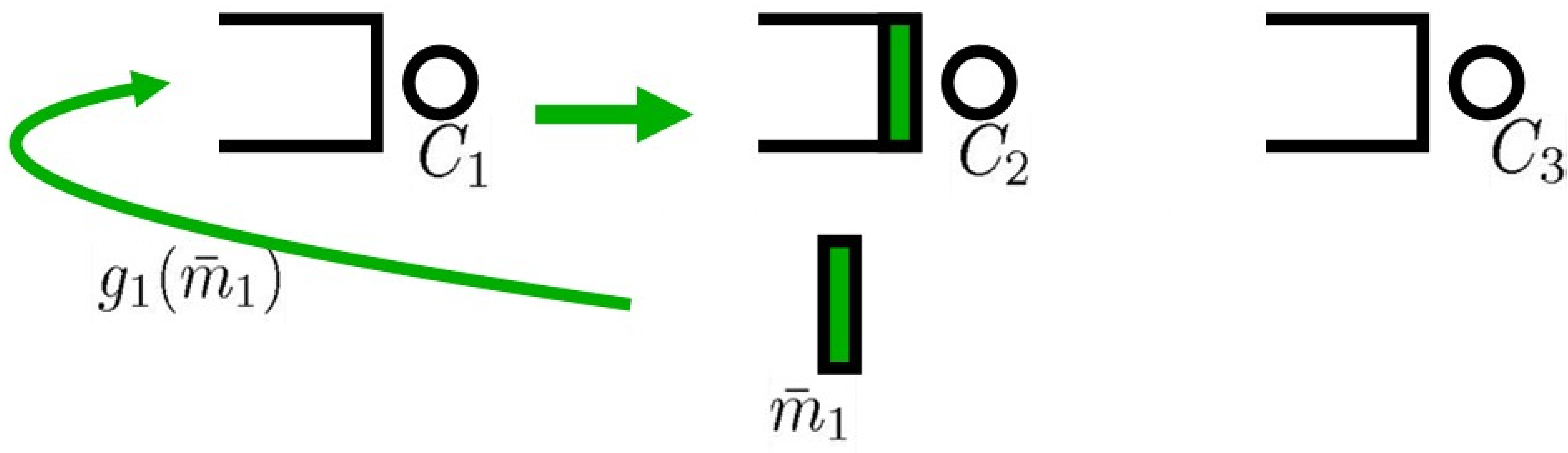

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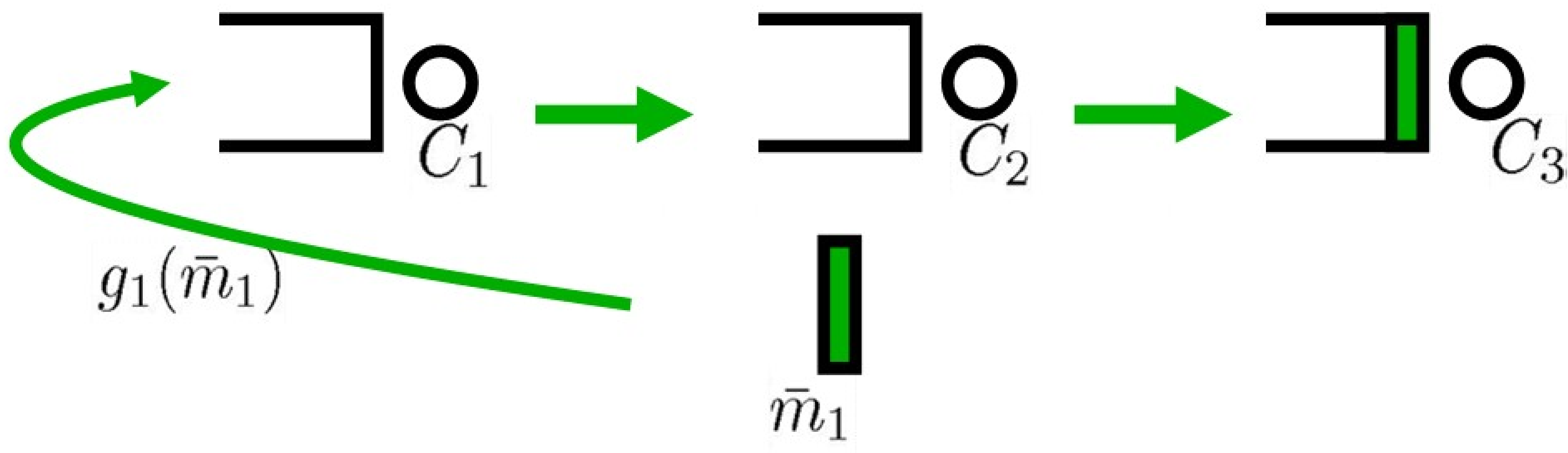

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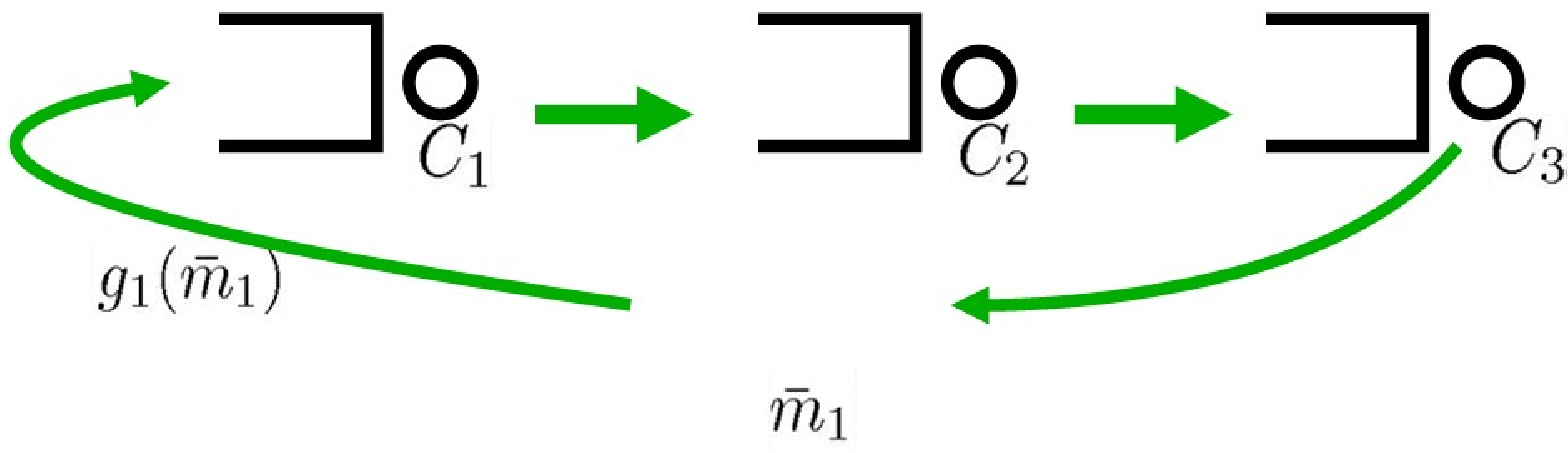

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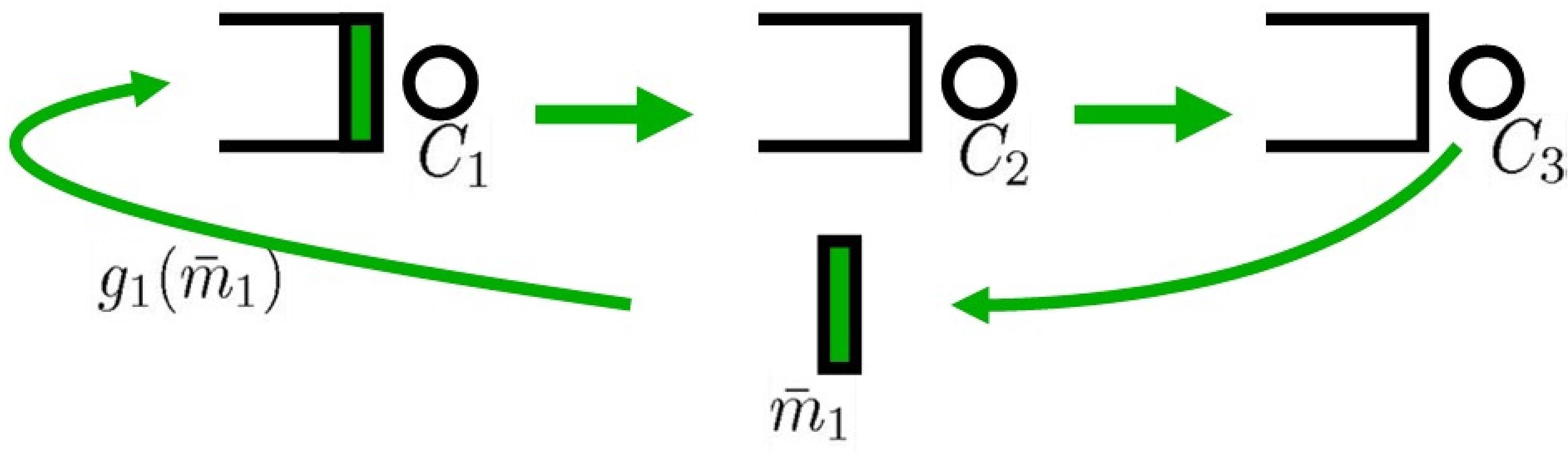

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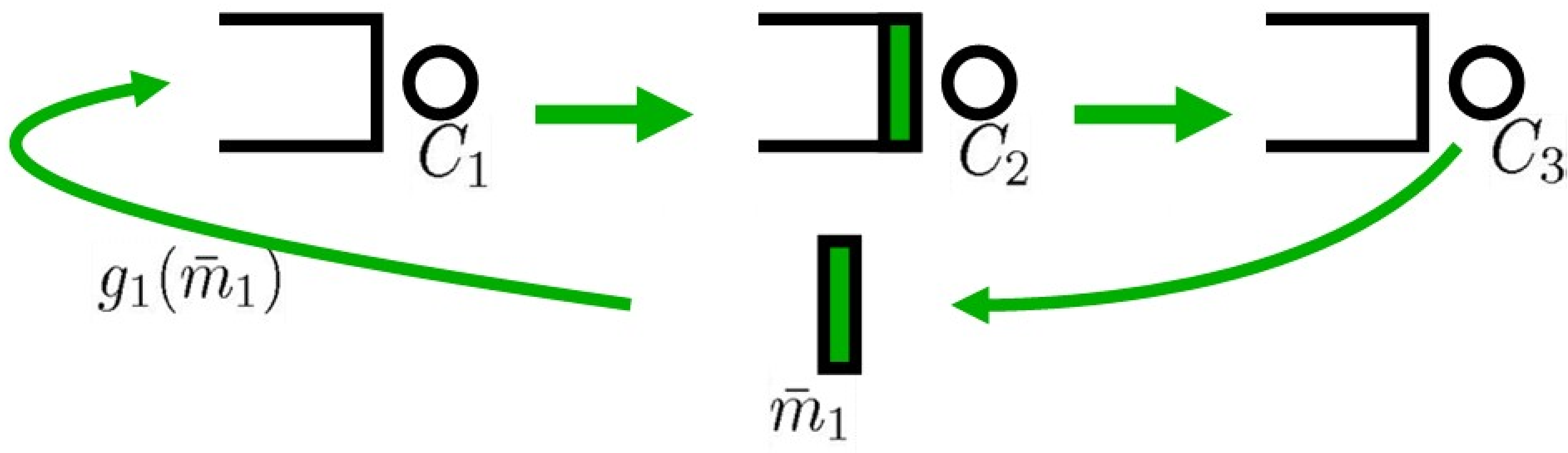

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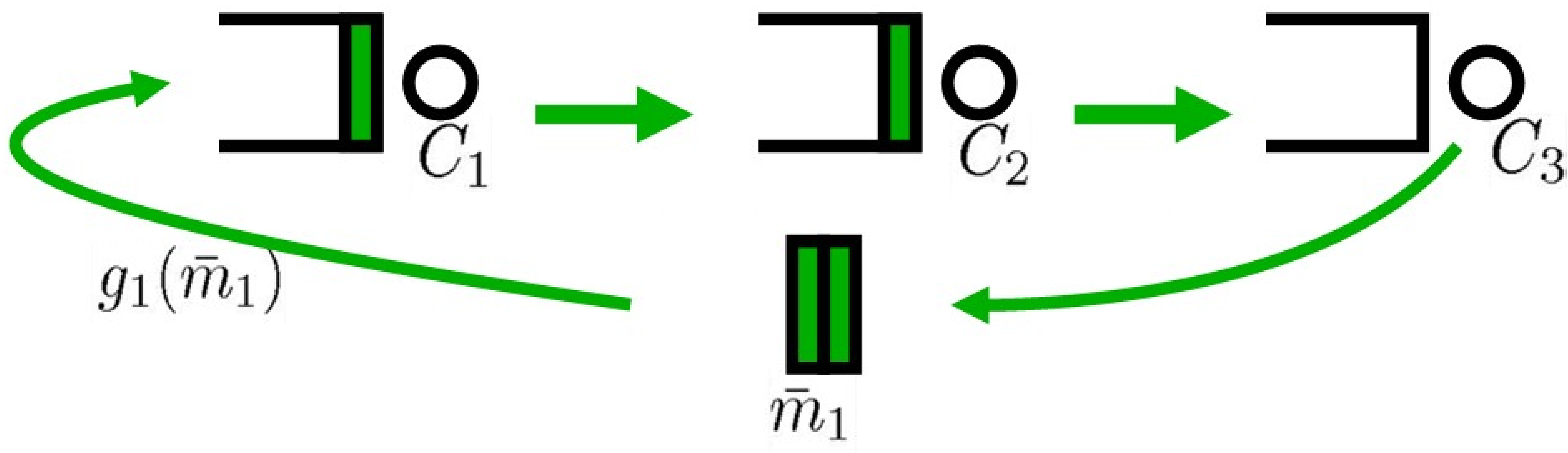

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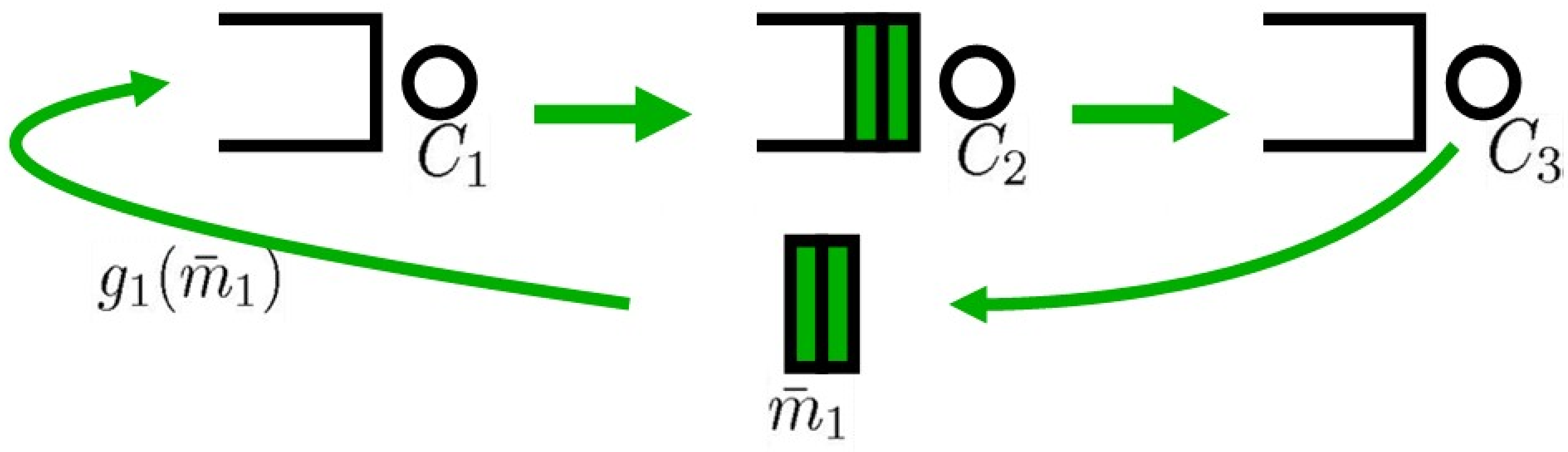

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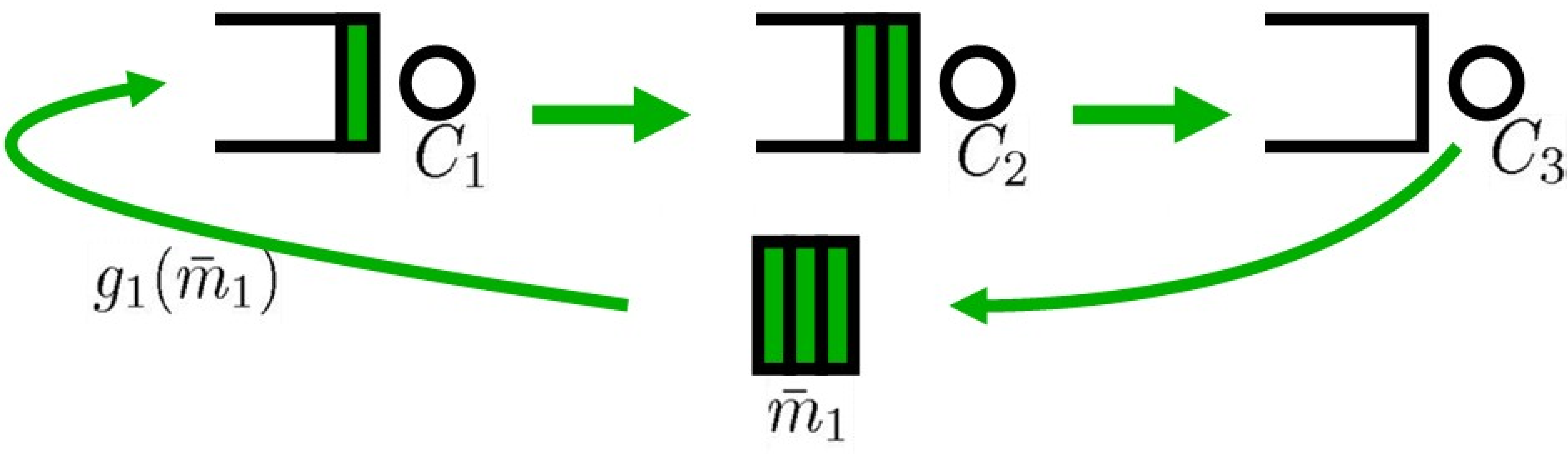

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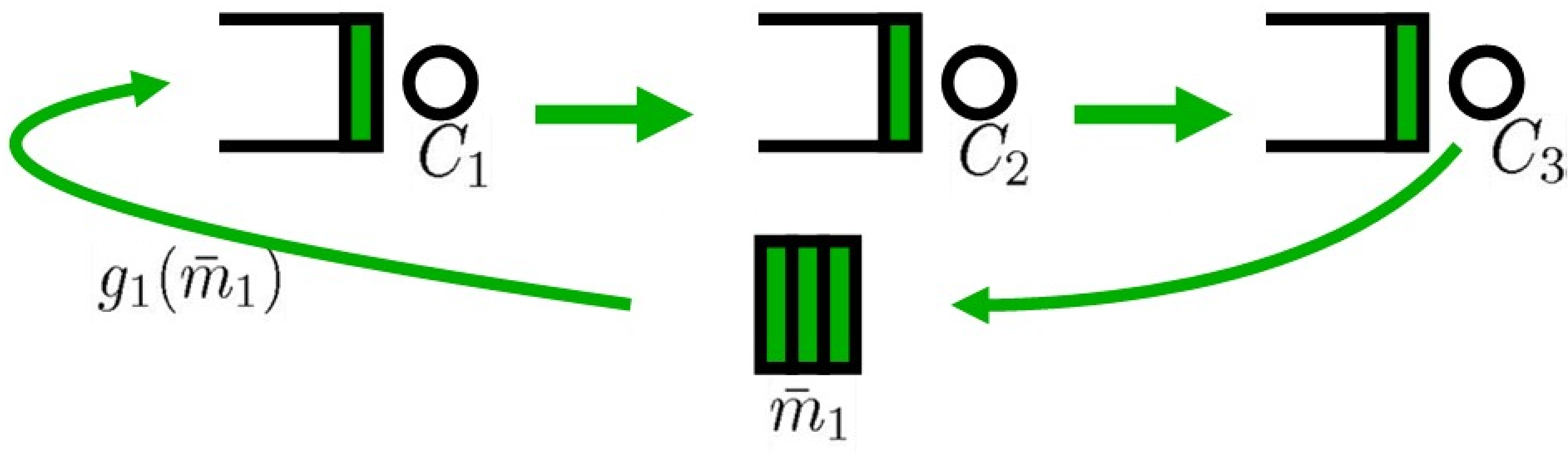

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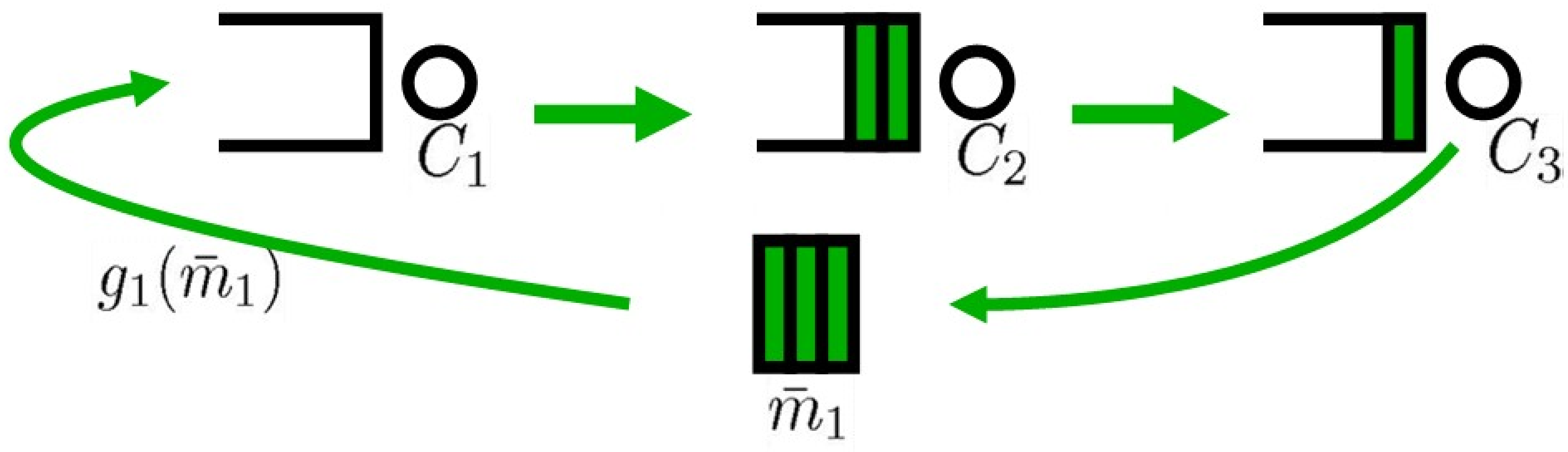

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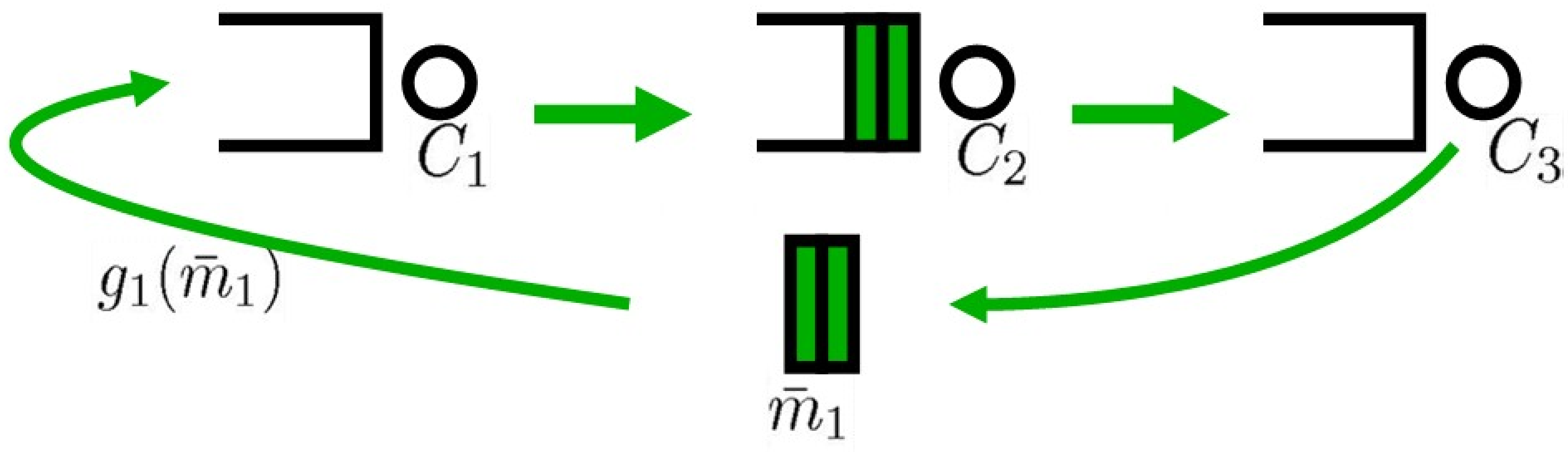

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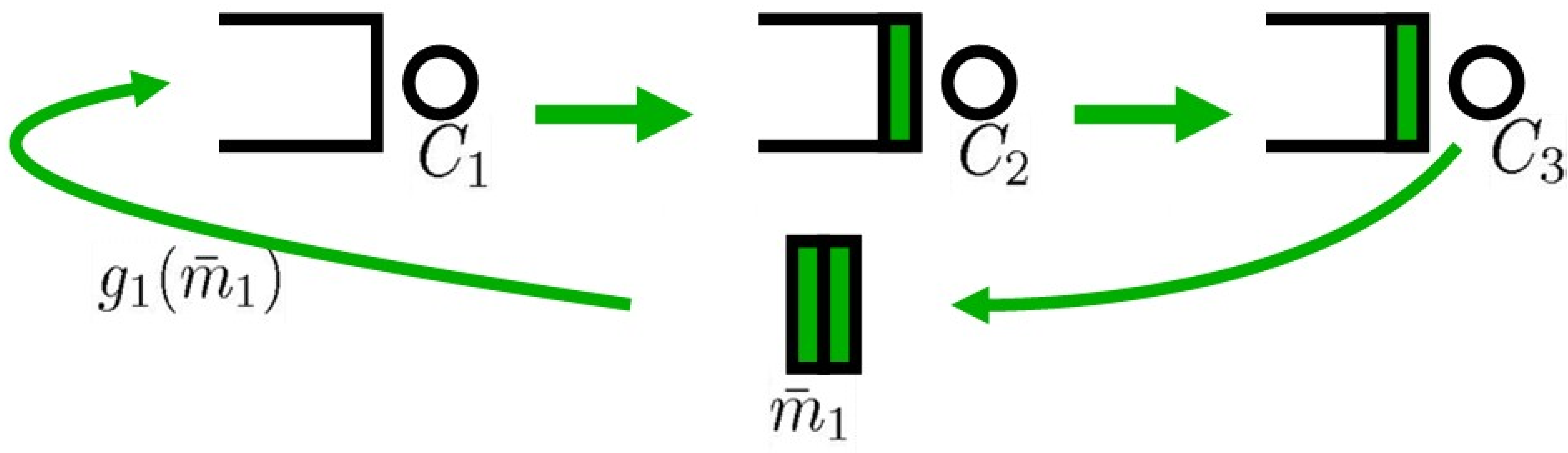

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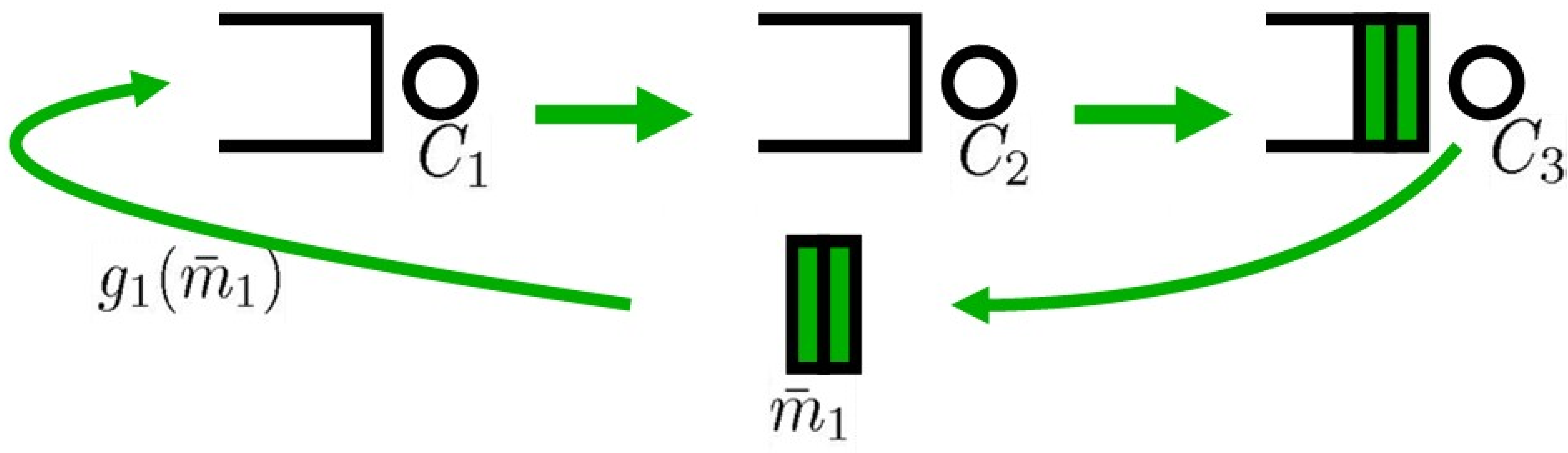

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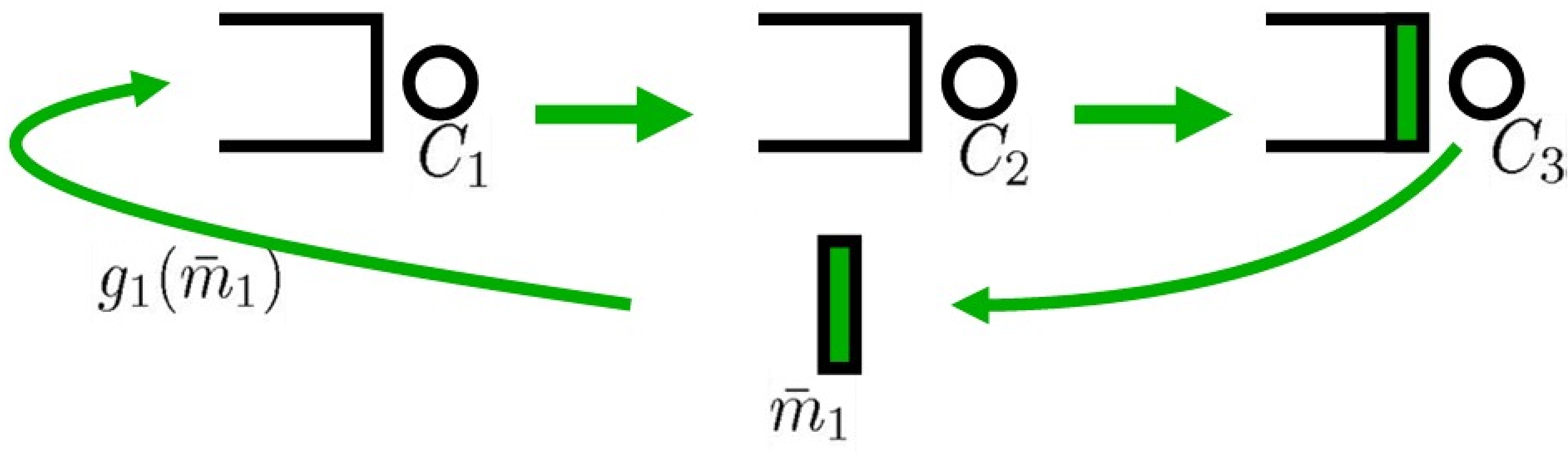

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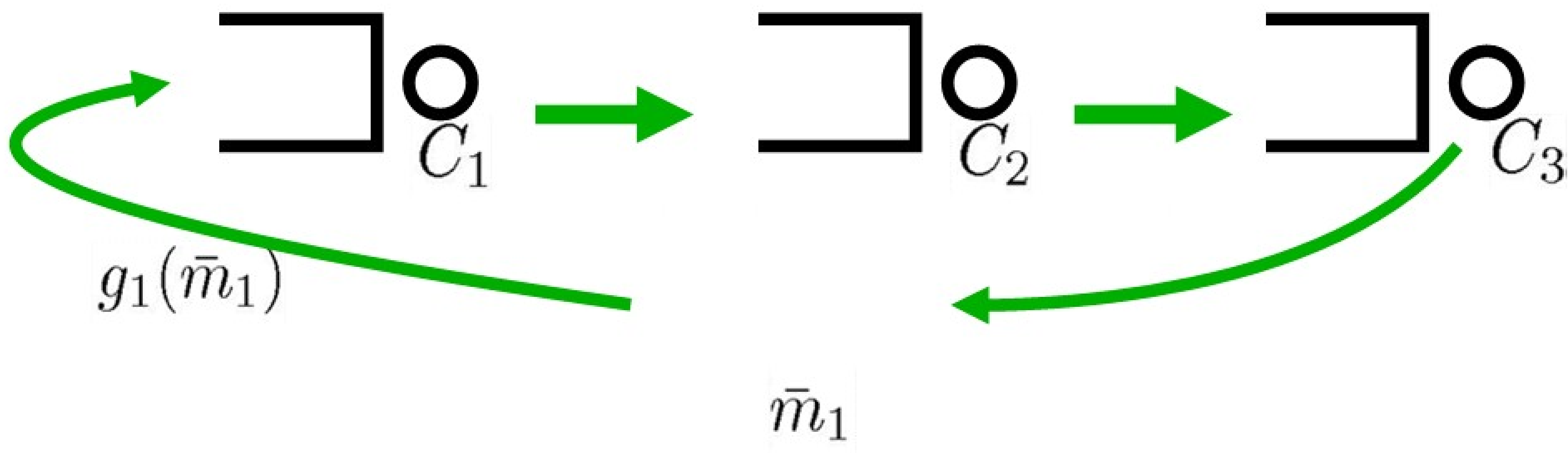

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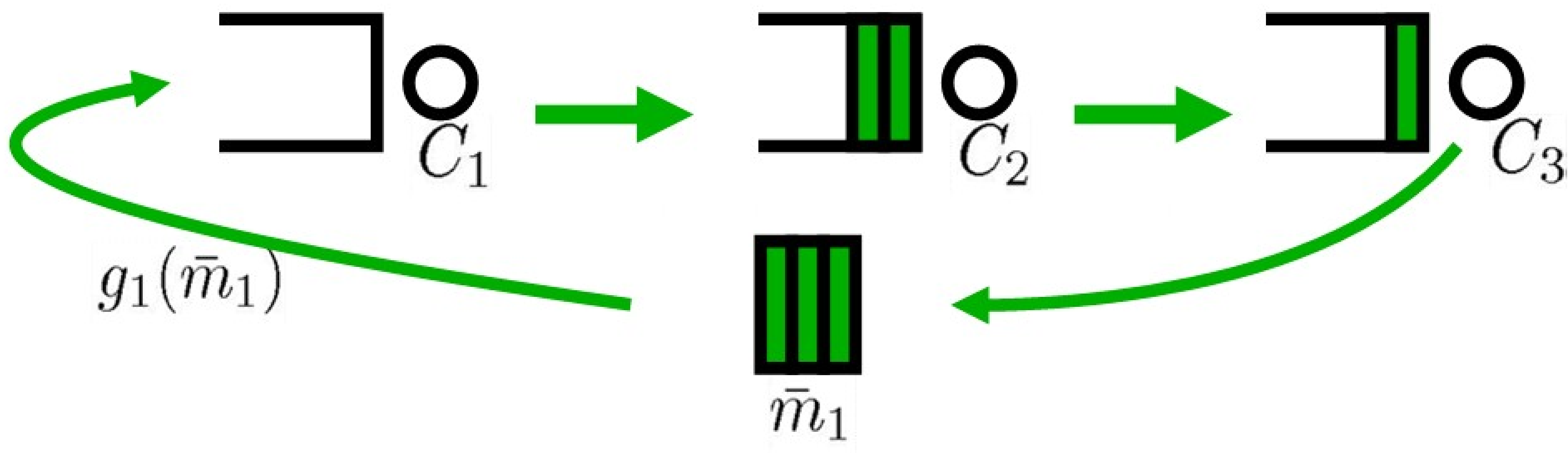

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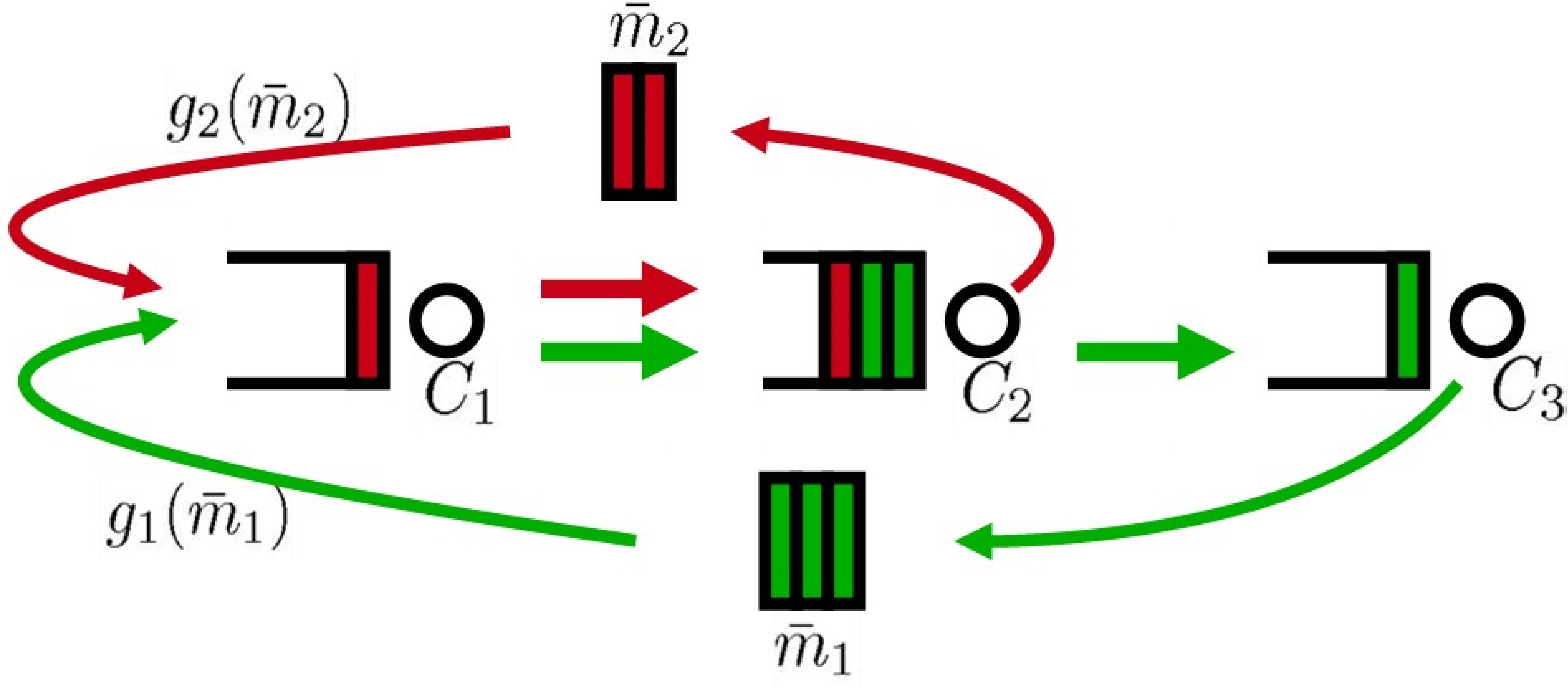

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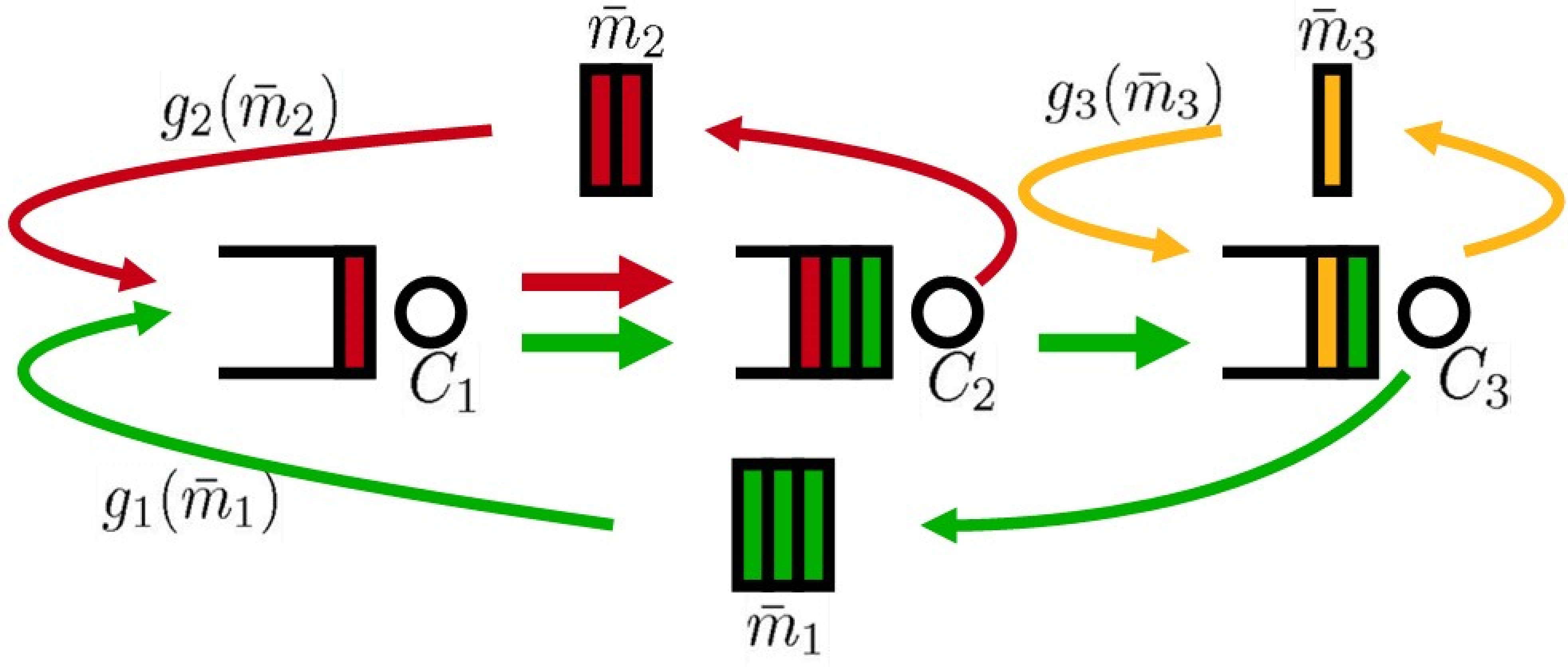

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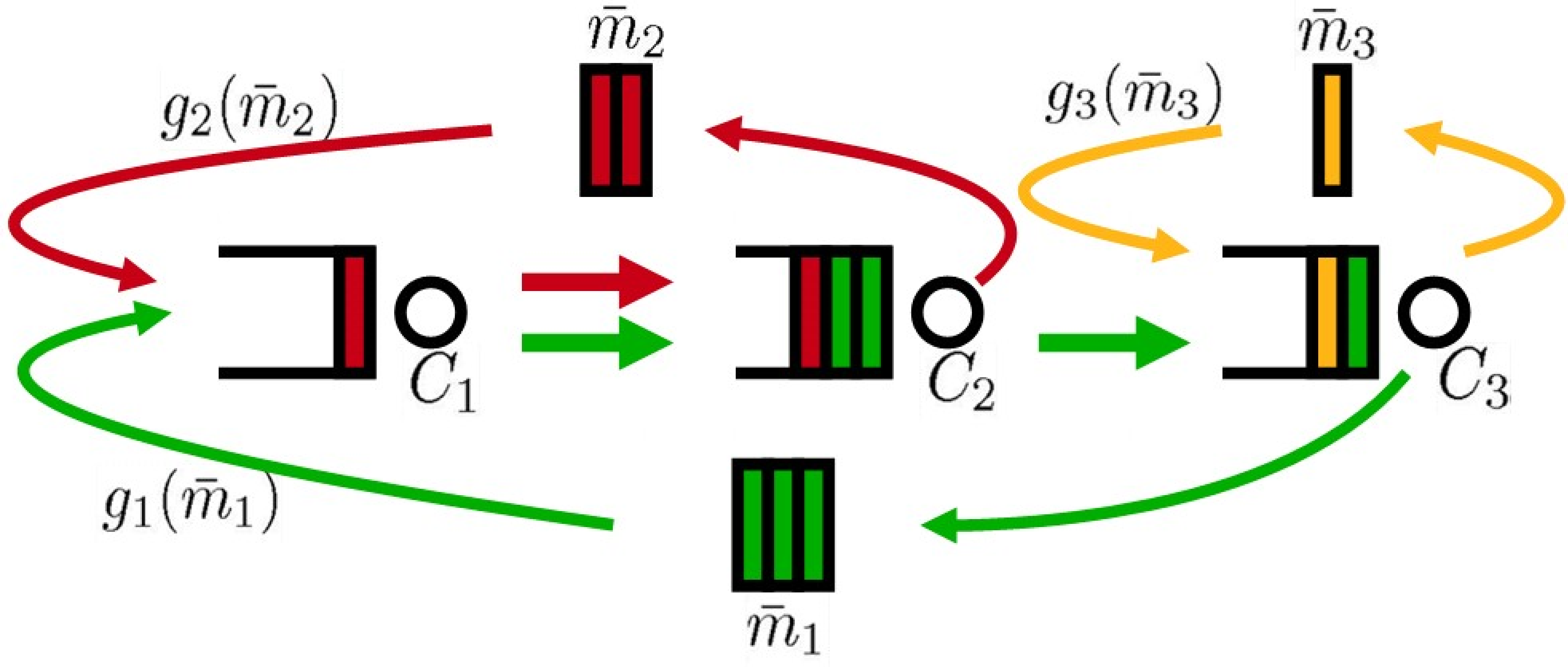
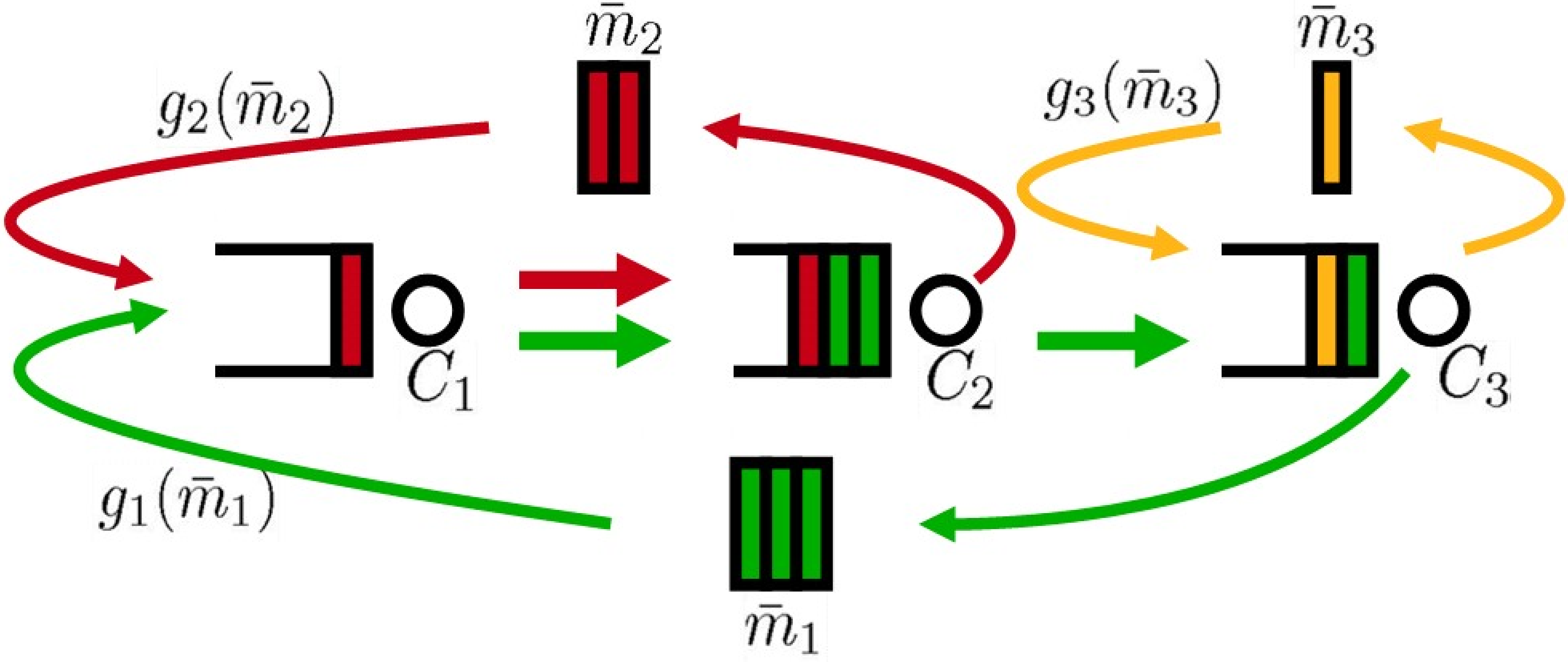


Stationary distribution:

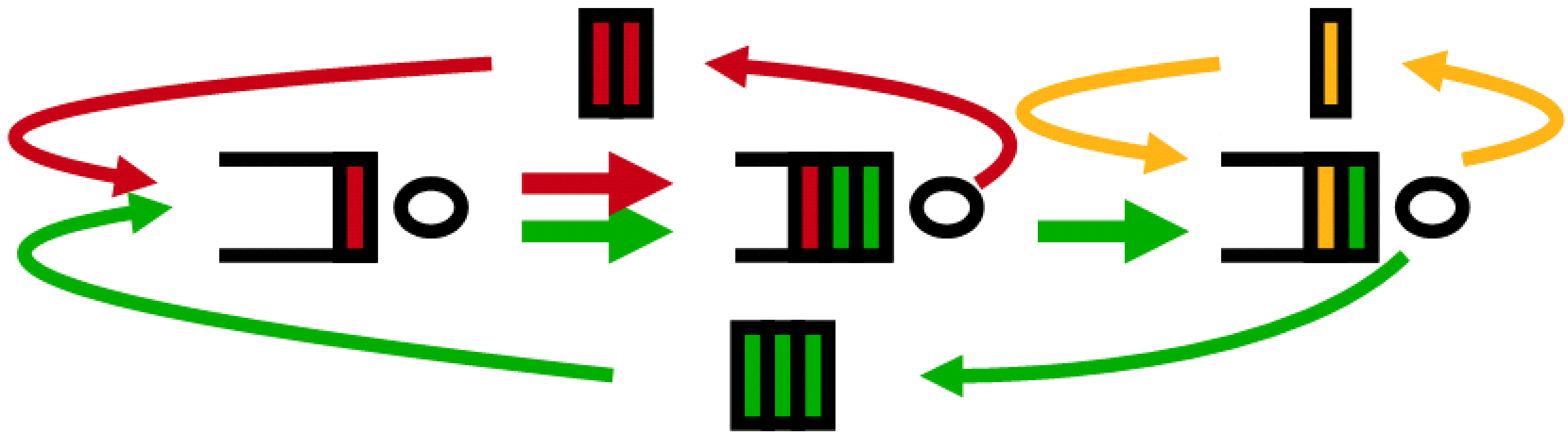
A System of Queues



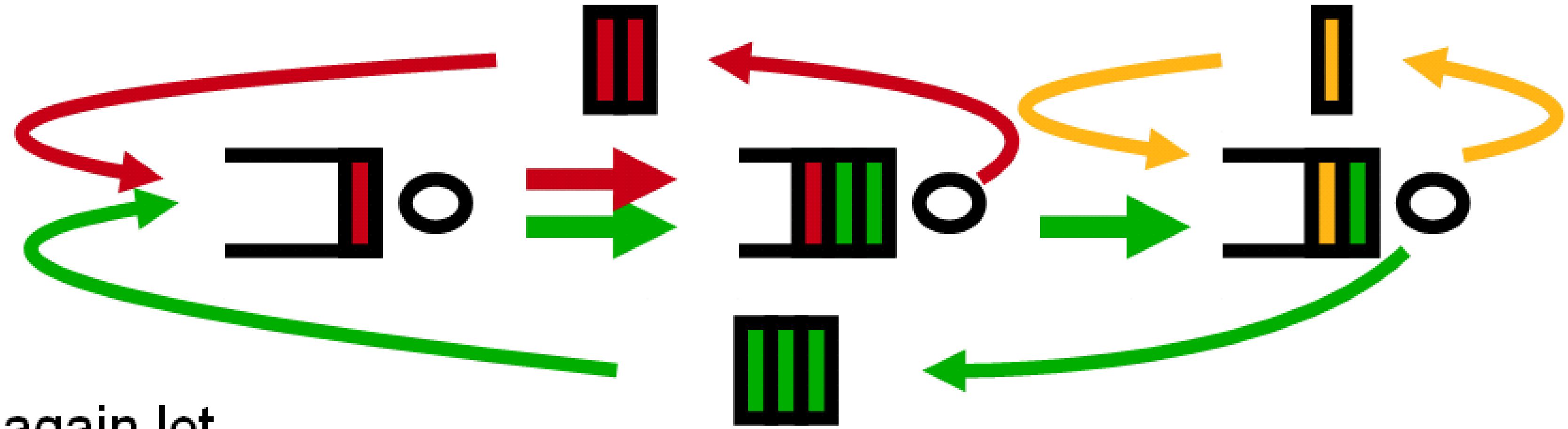
Stationary distribution:

$$\mathbb{P}(M = m) = \frac{1}{B_g} \prod_{j \in \mathcal{J}} \binom{m_j}{m_{ji} \; : \; i \ni j} \frac{1}{C_j^{m_j}} \times \prod_{i \in \mathcal{I}} \prod_{k=1}^{\bar{m}_i} g_i(k)$$

A System of Queues



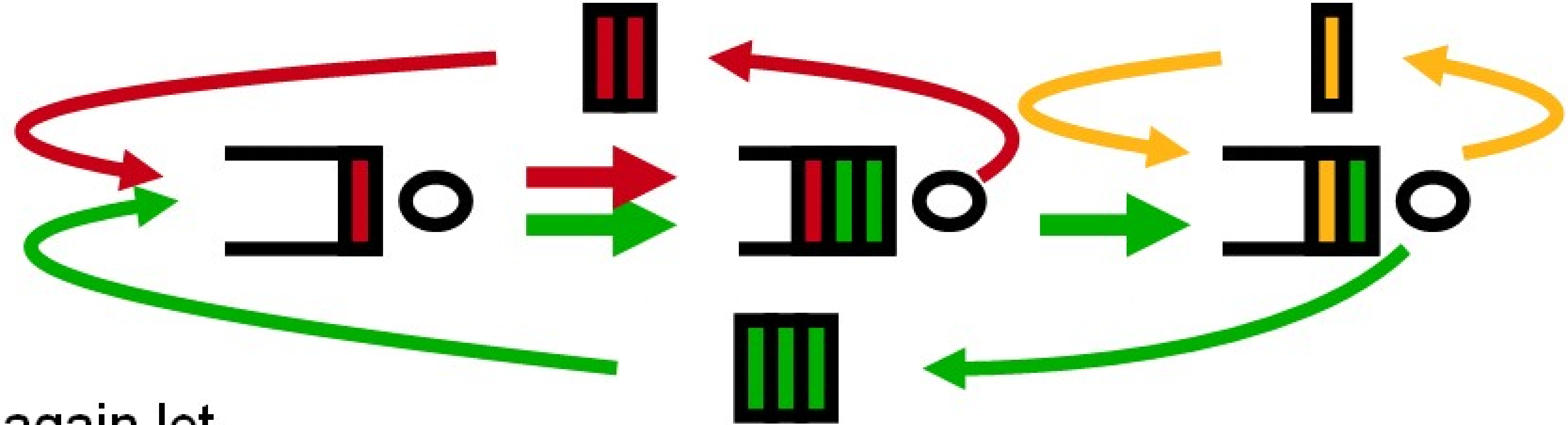
A System of Queues



Once again let

$$g_i^{(c)}(\bar{m}_i) = e^{cG_i\left(\frac{\bar{m}_i+1}{c}\right) - cG_i\left(\frac{\bar{m}_i}{c}\right)}$$

A System of Queues



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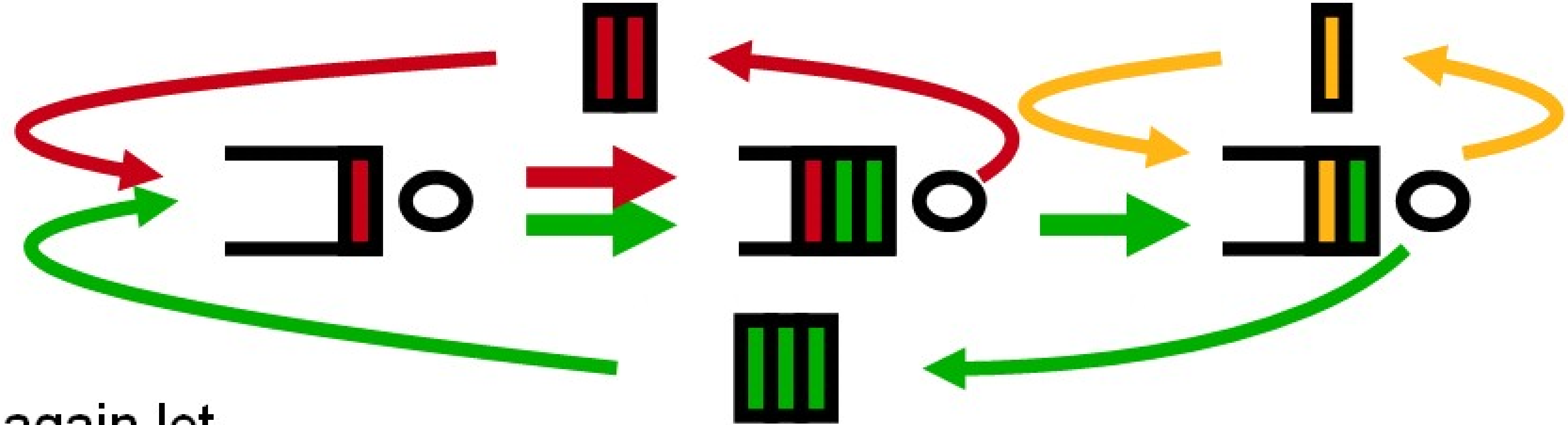
Large Deviations

$$\lim_{c\to\infty} \frac{1}{c} \log \mathbb{P}^{(c)}(M^{(c)} = m) = -\alpha_G(m)$$

A System of Queues



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Large Deviations

$$\lim_{c\to\infty} \frac{1}{c} \log \mathbb{P}^{(c)}(M^{(c)} = m) = -\alpha_G(m)$$

where

$$\alpha_G(m) = \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j \rho_i} - \sum_i G_i(\bar{m}_i)$$

with

$$\bar{m}_i = \sum_j m_{ji}$$

A System of Queues

Most likely state

A System of Queues

Most likely state

$$\min_{m,\bar{m}} \quad \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j} - \sum_i G_i(\bar{m}_i) \quad \text{subject to} \quad \bar{m}_i = \sum_{j \in i} m_{ji} \quad i \in \mathcal{I}.$$

A System of Queues

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Lets calculate its dual

A System of Queues

Most likely state

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Lets calculate its dual

$$\min_{m,\bar{m}} L(m, \bar{m}; \lambda) = \min_{m,\bar{m}} \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j} - \sum_i G_i(\bar{m}_i) + \sum_i \lambda_i \big(\bar{m}_i - \sum_{j \in i} m_{ji}\big)$$

A System of Queues

Most likely state

$$\min_{m,\bar{m}} \quad \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j} - \sum_i G_i(\bar{m}_i) \quad \text{subject to} \quad \bar{m}_i = \sum_{j \in i} m_{ji} \quad i \in \mathcal{I}.$$

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$$= \min_m \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j e^{\lambda_i}} \quad - \quad \sum_i \max_{\bar{m}_i} \{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}$$

A System of Queues

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A System of Queues

Most likely state

$$\min_{m,\bar{m}} \quad \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j} - \sum_i G_i(\bar{m}_i) \quad \text{subject to} \quad \bar{m}_i = \sum_{j \in i} m_{ji} \quad i \in \mathcal{I}.$$

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$$= \min_{m} \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j e^{\lambda_i}} \quad - \quad \sum_i \max_{\bar{m}_i} \{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}$$

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We have dual:

A System of Queues

Most likely state

$$\min_{m,\bar{m}} \quad \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j} - \sum_i G_i(\bar{m}_i) \quad \text{subject to} \quad \bar{m}_i = \sum_{j \in i} m_{ji} \quad i \in \mathcal{I}.$$

Lets calculate its dual

$$\min_{m,\bar{m}} L(m,\bar{m};\lambda) = \min_{m,\bar{m}} \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j} - \sum_i G_i(\bar{m}_i) + \sum_i \lambda_i \Big(\bar{m}_i - \sum_{j \in i} m_{ji}\Big)$$

$$= \min_m \sum_{j,i} m_{ji} \log \frac{m_{ji} C_j}{m_j e^{\lambda_i}} \quad - \quad \sum_i \max_{\bar{m}_i} \{ G_i(\bar{m}_i) - \lambda_i \bar{m}_i \}$$

$$= \begin{cases} \sum_{i \in \mathcal{I}} U_i(e^{\lambda_i}) & \text{if} \quad \sum_{i:j \in i} e^{\lambda_i} \le C_j, \quad j \in \mathcal{J} \\ -\infty & \text{otherwise.} \end{cases}$$

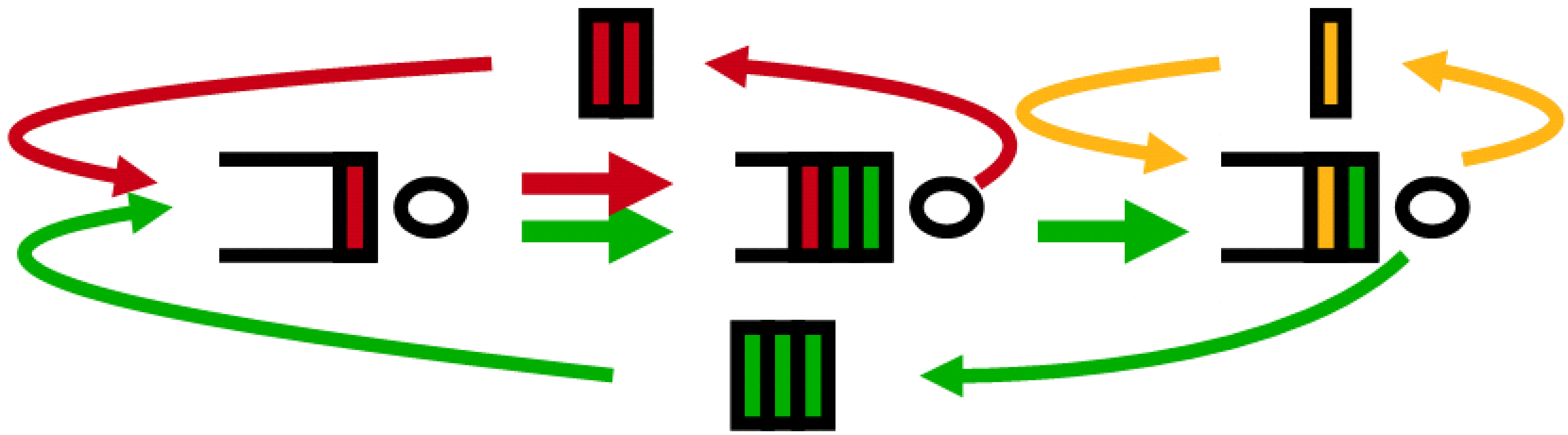
We have dual:

$$\text{maximize} \quad \sum_i U_i(\Lambda_i)$$

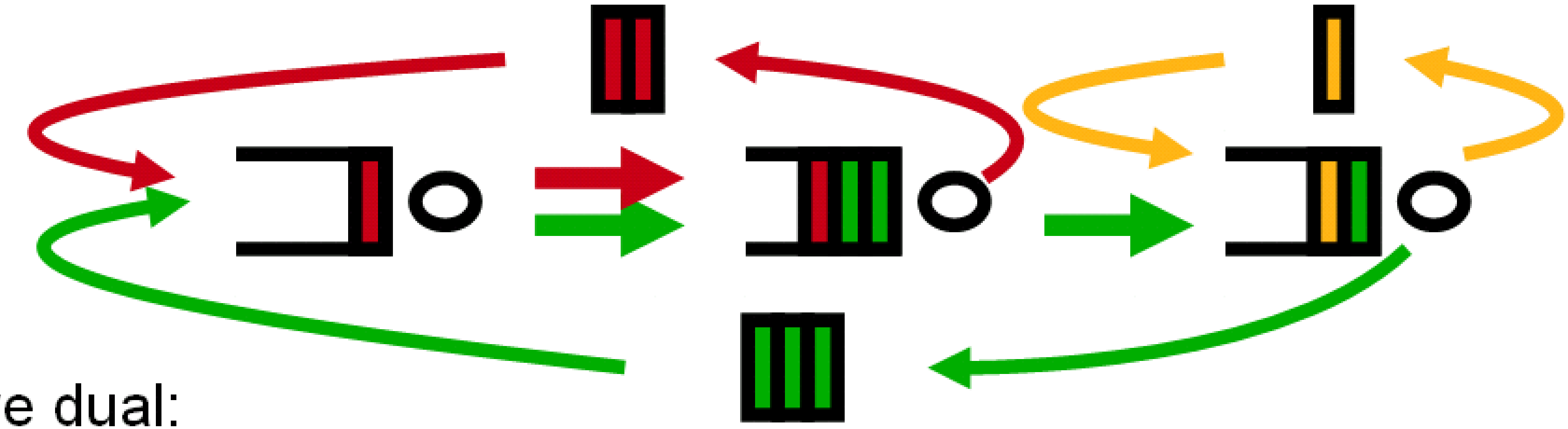
$$\text{subject to} \quad \sum_{i:j \in i} \Lambda_i \le C_j, \quad j \in \mathcal{J}$$

$$\text{over} \quad \Lambda_i \ge 0, \quad i \in \mathcal{I}.$$

A System of Queues

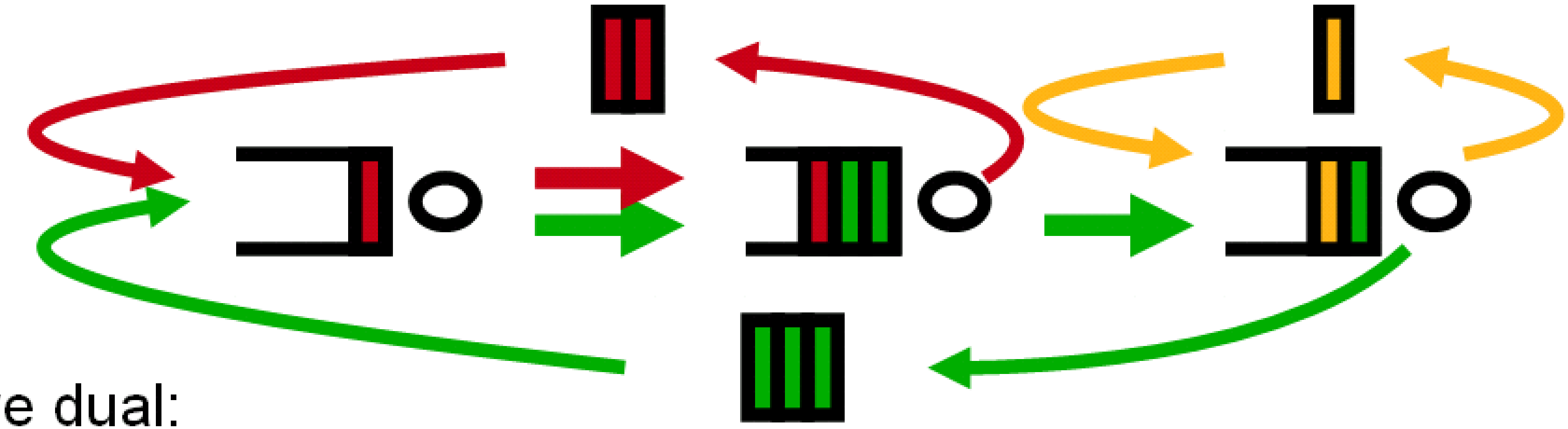


A System of Queues



We have dual:

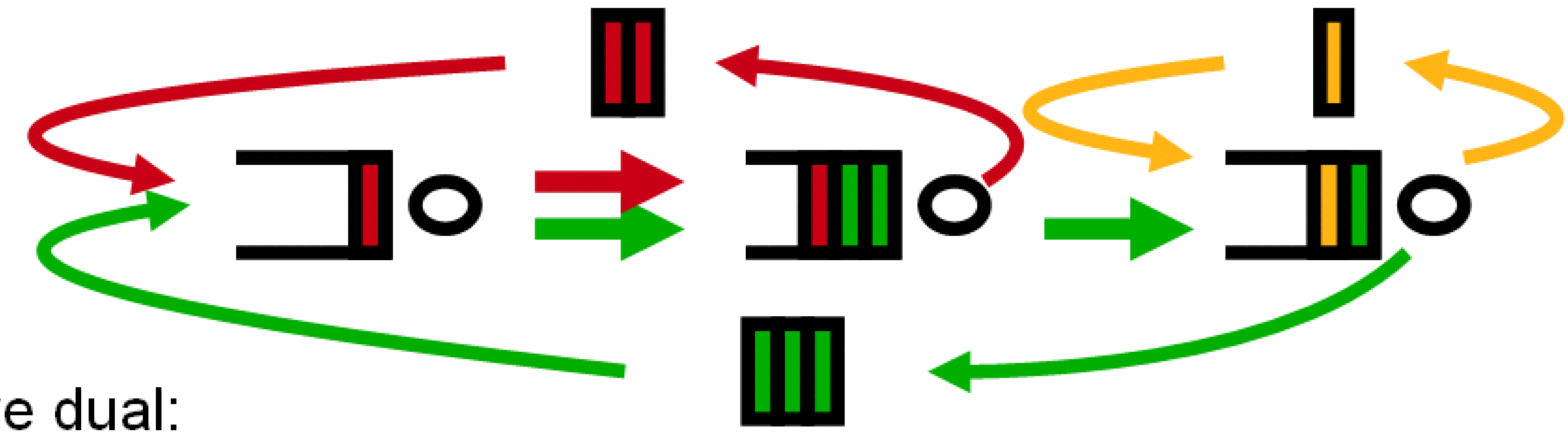
A System of Queues



We have dual:

The SYSYEM PROBLEM

A System of Queues



We have dual:

The SYSYEM PROBLEM

$$\text{maximize} \quad \sum_i U_i(\Lambda_i)$$

$$\text{subject to} \quad \sum_{i:j\in i} \Lambda_i \le C_j, \quad j \in \mathcal{J}$$

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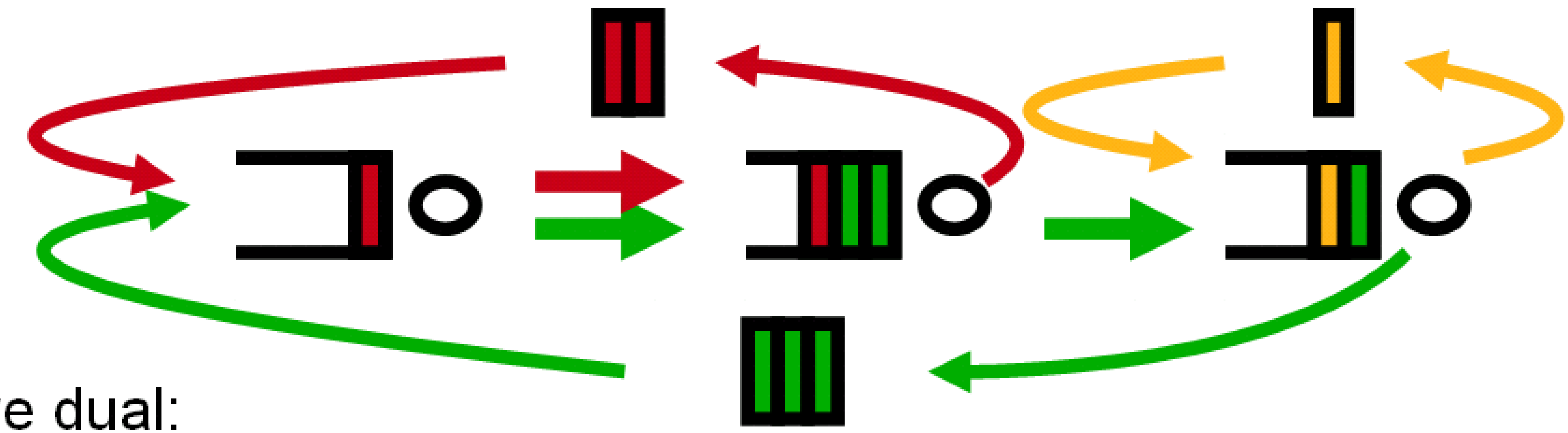
A System of Queues



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The most likely state for the queueing system solves the SYSTEM PROBLEM.

Conclusion

Conclusion

USER PROBLEMS

$$\max_{\bar{m}_i}\{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}$$

Conclusion

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NETWORK PROBLEM

$$\max \sum_{i\in\mathcal{I}} \bar{m}_i \log \Lambda_i$$

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Conclusion

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AND

$$\bar{m}_i = \Lambda_i q_i$$

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Window Size

Conclusion

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$$\max_{\bar{m}_i}\{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}$$

NETWORK PROBLEM

$$\max \sum_{i \in \mathcal{I}} \bar{m}_i \log \Lambda_i$$

$$\text{subject to} \sum_{i: j \in i} \Lambda_i \leq C_j,$$

AND

$$\bar{m}_i = \Lambda_i q_i$$

Window Size

Delay

Conclusion

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$$\max_{\bar{m}_i}\{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}$$

NETWORK PROBLEM

$$\max \sum_{i \in \mathcal{I}} \bar{m}_i \log \Lambda_i$$

$$\text{subject to} \sum_{i: j \in i} \Lambda_i \le C_j,$$

AND

Little's Law

$$\bar{m}_i = \Lambda_i q_i$$

Window Size

Delay

Conclusion

USER PROBLEMS

$$\max_{\bar{m}_i}\{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}$$

NETWORK PROBLEM

$$\max \sum_{i\in\mathcal{I}} \bar{m}_i \log \Lambda_i$$
$$\text{subject to} \sum_{i:j\in i} \Lambda_i \le C_j,$$

Network choses prices with **Queue sizes**:

$$q_i = \sum_{j\in i} q_j$$

AND

Little's Law

$$\bar{m}_i = \Lambda_i q_i$$

Window Size

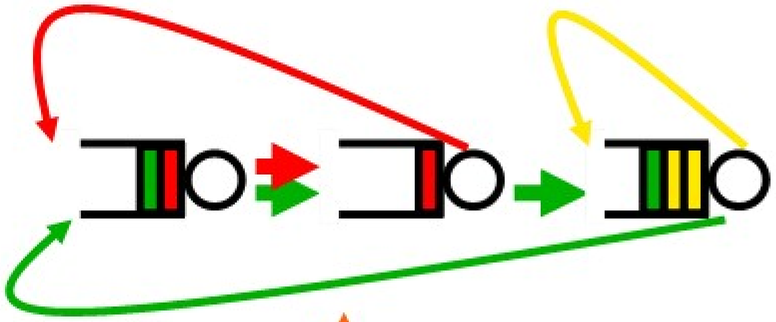
Delay

Conclusion

USER PROBLEMS

$$\max_{\bar{m}_i}\{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}$$

NETWORK PROBLEM



Network choses prices with **Queue sizes**:

$$q_i = \sum_{j \in i} q_j$$

AND

Little's Law

$$\bar{m}_i = \Lambda_i q_i$$

Window Size

Delay

Conclusion

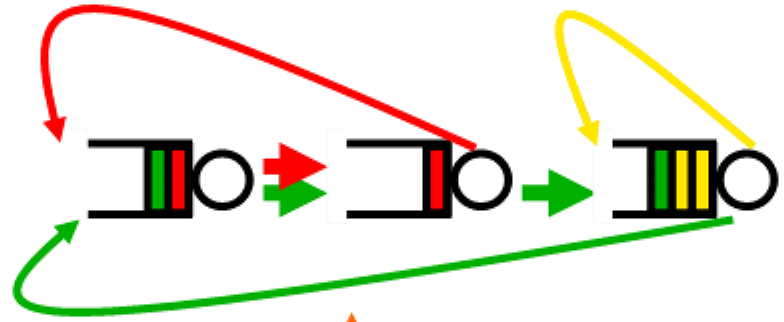
USER PROBLEMS

$$\max_{\bar{m}_i}\{G_i(\bar{m}_i) - \lambda_i \bar{m}_i\}$$

User *i* choses
Congestion window:

$$\bar{m}_i$$

NETWORK PROBLEM



Network choses prices
with **Queue sizes**:

$$q_i = \sum_{j \in i} q_j$$

AND

Little's Law

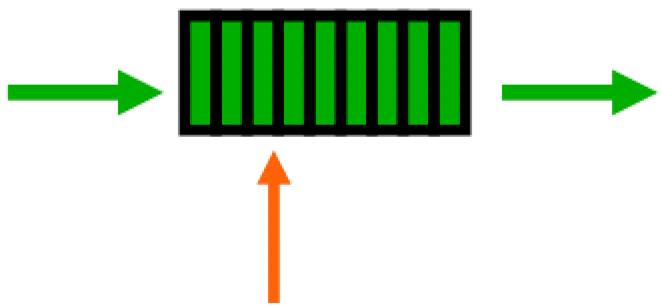
$$\bar{m}_i = \Lambda_i q_i$$

Window
Size

Delay

Conclusion

USER PROBLEMS



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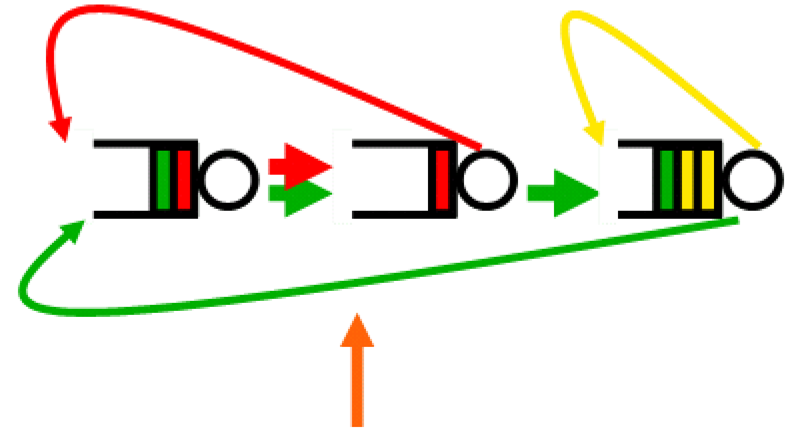
NETWORK PROBLEM



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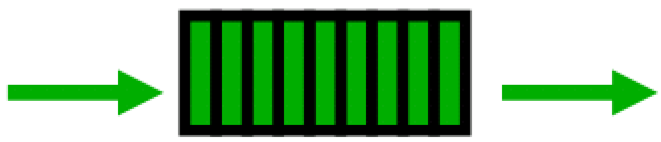
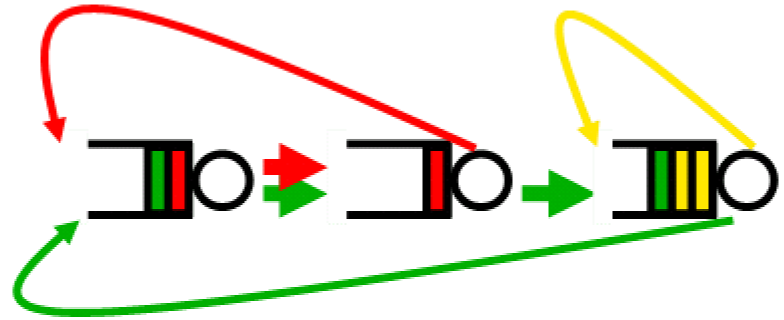
USER PROBLEMS



NETWORK PROBLEM



AND

Little's Law

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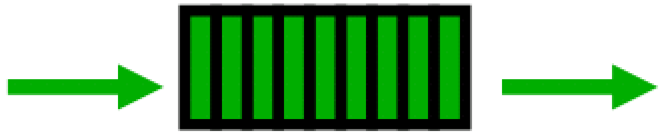
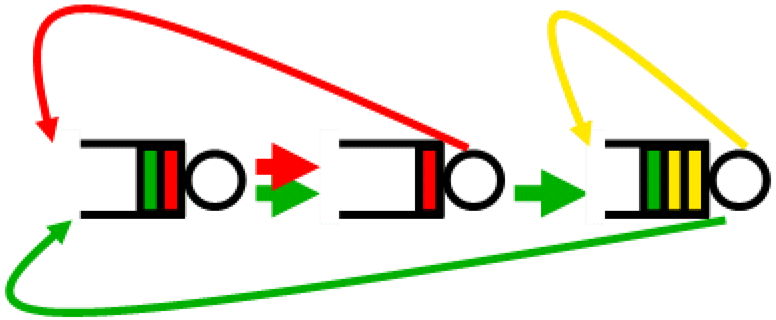
Conclusion

USER PROBLEMS



NETWORK PROBLEM



AND

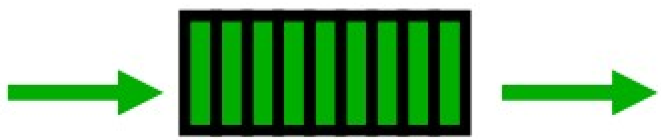
Little's Law

$$\bar{m}_i = \Lambda_i q_i$$

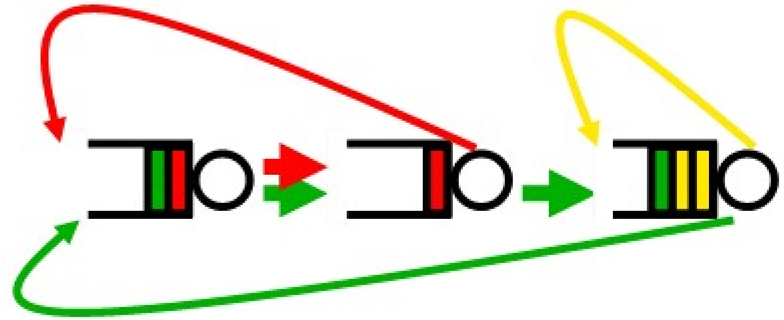
Solves

Conclusion

USER PROBLEMS



NETWORK PROBLEM



AND

Little's Law

$$\bar{m}_i = \Lambda_i q_i$$

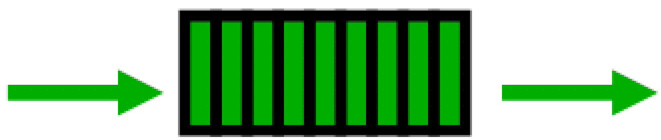
Solves

SYSTEM PROBLEM

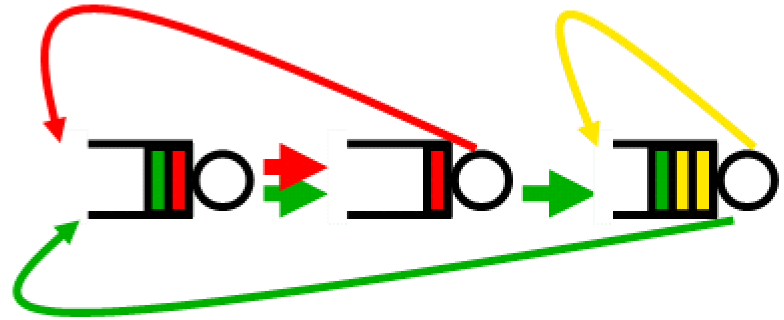
$$\begin{aligned} &\max && \sum_{i \in \mathcal{I}} U_i(\Lambda_i) \\ &\text{subject to} && \sum_{i : j \in i} \Lambda_i \leq C_j, \; j \in \mathcal{J} \\ &\text{over} && \Lambda_i \geq 0, \quad i \in \mathcal{I}. \end{aligned}$$

Conclusion

USER PROBLEMS



NETWORK PROBLEM



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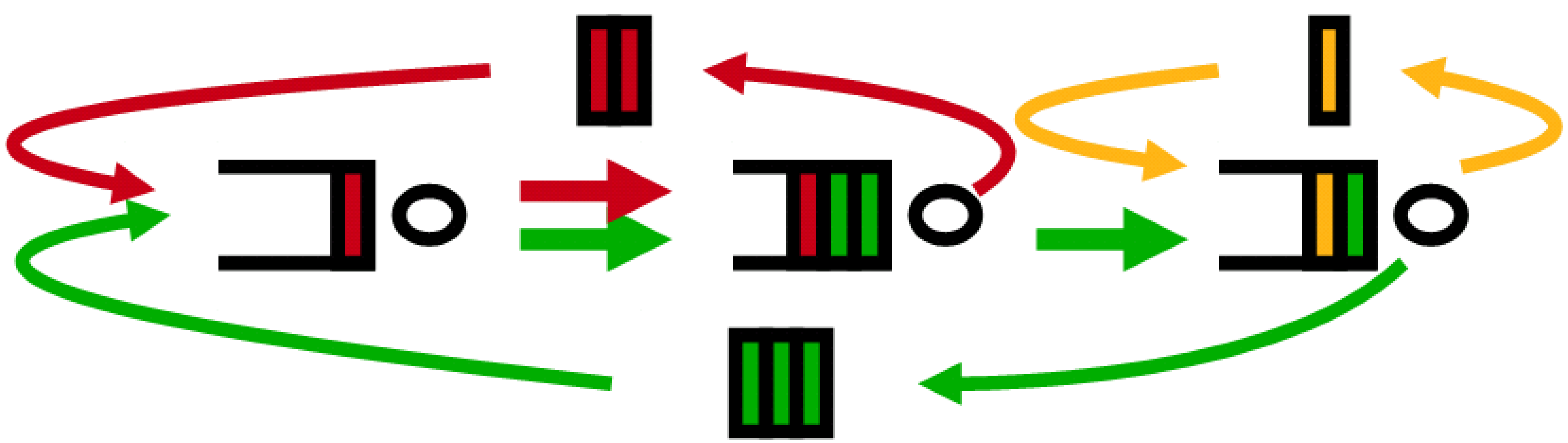
Solves

SYSTEM PROBLEM



THANK YOU FOR LISTENING!