Modelling priority systems using local times of Lévy processes

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Motivation: Two-class preemptive priorty service system

High (priority) class is seved in a heavily loaded single-server queue.

Low (priority) class is served only when there is nobody from high class around.

In heavy traffic, we can approximate the workload of the high class, as usual, by a reflected Brownian motion (RBM).

The low class is approximated by a queue fed by the local time of the RBM.

Why?

On the approximation

High priority class: M/M/1 queue in isolation and in heavy traffic:

$$\sqrt{n}(\mu - \lambda) \to c$$

Scaled process converges to BM with drift.

Invariance principle implies that cumulative idle time I_t converges to the local time L_t of the BM.

Low priority class: input $\propto$ cumulative idle time I_t of high priority class.

***Ergo*, the low priority class is, in heavy traffic again, a fluid queue fed by L_t:**

$$Q_t = Q_0 + (L_t - t) - \inf_{0 \le s \le t} [Q_0 + (L_s - s)]$$

(Units are chosen so that service rate of low priority class is 1.)

And so...

We are going to study "queues" fed by the local time of Brownian motion.

Recall that the inverse local time

$$L_x^{-1} = \inf\{t > 0 : L_t > x\}$$

is an increasing process with stationary and independent increments, with discontinuous paths (but continuous in probability) and which is also self-similar.

Mannersalo, Norros and Salminen (QUESTA 2004) derive various characteristics of the low class system in steady-state.

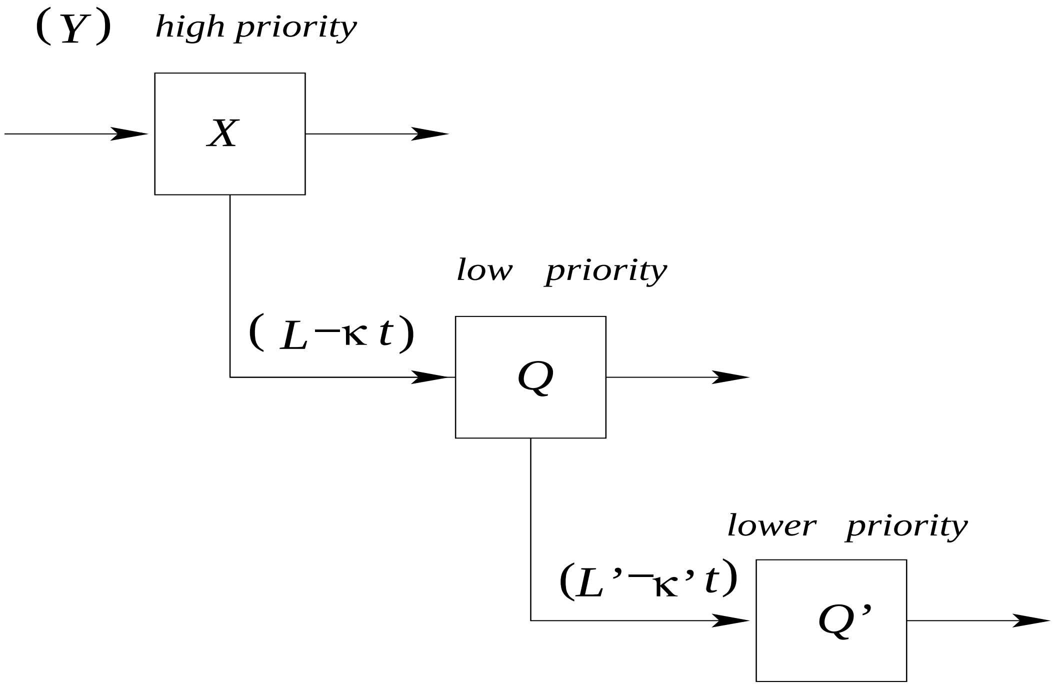
<u>Here</u>: We will replace Brownian motion by Lévy process.

<u>Why</u>: Limit theorems in heavy-tailed case.
(Ref. to prior work on Lévy networks.)

Techniques

- **PALM THEORY**
  esp. from K and Last (Adv. Appl. Prob. 2000)
- **JIM PITMAN'S PAPER ON STATIONARY EXCURSIONS**
  Séminaire de Proba., 1986
- **FLUCTUATION THEORY**
  esp. Wiener-Hopf factorization

Scheme



Y **is a (spectrally one-sided) Lévy process**

X **is the reflection of** Y

L **is the local time at zero of the reflection** X **of** Y

Q **is the reflection of** $L_t - \kappa t$

L' **is the local time at zero of** Q

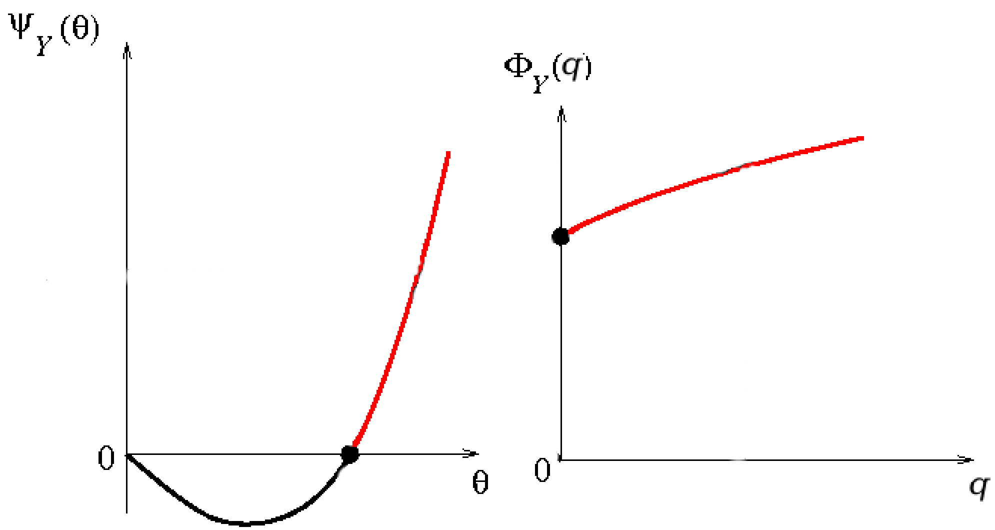
Q' **is the reflection of** $L'_t - \kappa' t$

The driving Lévy process and its stationary reflection

Let Y be a spectrally one-sided Lévy process (NOT a subordinator) with

$$\psi_Y(\theta) := \begin{cases} \log E e^{\theta(Y_{t+1}-Y_t)}, & \textbf{if } Y \textbf{ is spectrally negative} \\ \log E e^{-\theta(Y_{t+1}-Y_t)}, & \textbf{if } Y \textbf{ is spectrally positive} \end{cases}, \quad \theta \geq 0.$$

Let $\Phi_Y(q)$ be the largest nonnegative θ such that $\psi_Y(\theta) = q$.



Assume

$$E(Y_{t+1} - Y_t) < 0$$

and let X be the stationary reflection of Y:

$$X_t = \sup_{-\infty < u \le t} (Y_t - Y_u), \qquad t \in \mathbb{R}.$$

We can define everything on a canonical probability space $(\Omega, \mathscr{F}, P, \vartheta_\cdot)$ [e.g., the one supporting a Poisson random measure η on $\mathbb{R} \times \mathbb{R}_+$ and an independent Brownian motion] where ϑ_t is the natural time-shift by t. As such ϑ_t preserves P and is ergodic.

Then $(X_t, t \in \mathbb{R})$ can be constructed so that

$$X_t = X_0 \circ \vartheta_t,$$

inheriting the underlying stationary-ergodic structure (Loynes' scheme). Furthermore, X is a Markov process. Notation:

$$P_x := P(\cdot | X_0 = x).$$

The local time of the reflected Lévy process and its stationary reflection

Let L be the local time (random measure) of the nonnegative process X at zero, defined by the requirement that it is a stationary and ergodic random measure, i.e.

$$L(s, s+t] = L(0, t] \circ \vartheta_s, \quad t \geq 0, \quad s \in \mathbb{R},$$

that regenerates together with X at every point t such that $X_t = 0$.

Such an L exists and, in most cases, it is unique.

We are now interested in the following process:

$$Q_t = \sup_{-\infty < u \leq t} \{L(u, t] - (t - u)\}, \quad t \in \mathbb{R},$$

which is well-defined if the rate of L is < 1.

Then Q is stationary and ergodic but non-Markovian.

It represents the backlog of the low class in a certain limiting regime.

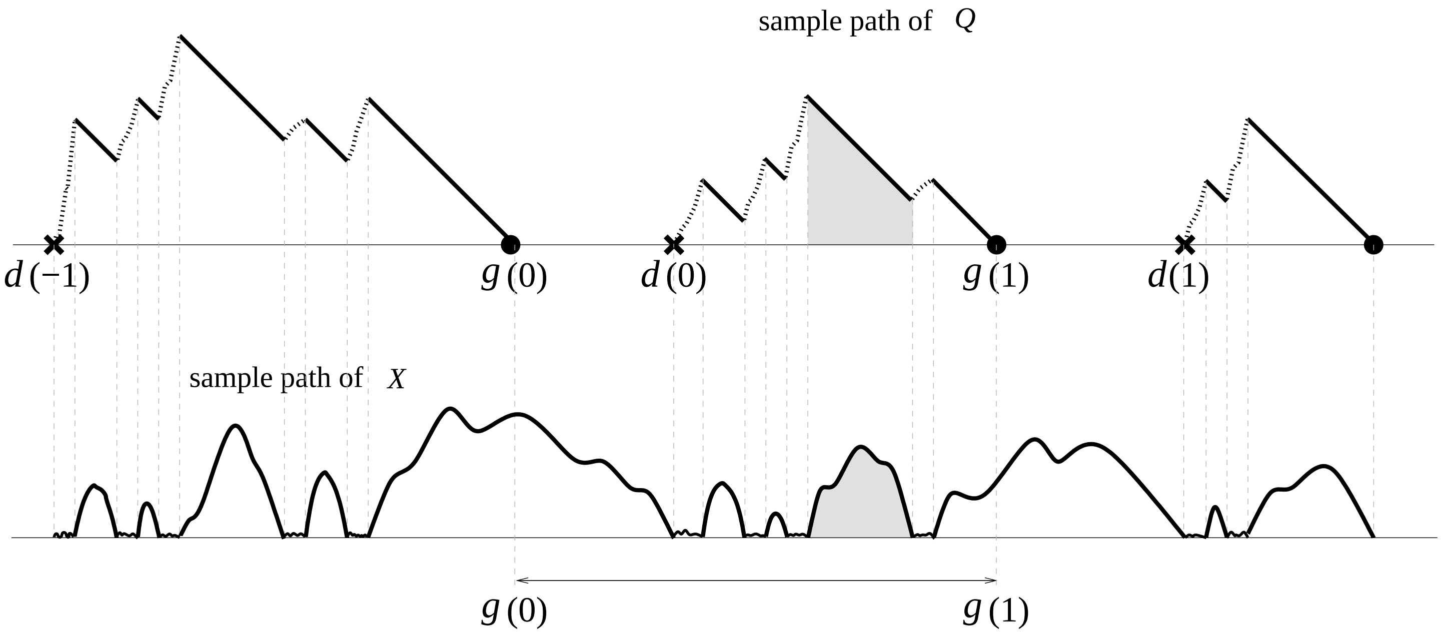
Results

The marginal distribution of Q is exponential with an atom at zero.

The busy periods of Q have computable Laplace transform in terms of ψ_Y, both the typical ones and the ones straddling a deterministic observation time.

Ditto for idle periods.

Exotic examples.



Understanding L

In Stochastic Analysis, L is defined only for the case when

$$P(Y \textbf{ becomes } \le 0 \textbf{ immediately} \mid Y_0 = 0) = 1, \tag{1}$$

and then L is continuous and adapted to the filtration generated by Y.

Adopting the point of view of regenerative processes, we do not need to assume continuity.

Then L exists always.

Spectrally positive Y (but NOT a subordinator)

Then (1) **is always satisfied and**

$$L(s,t] = -\inf_{s<u\leq t}(Y_u - Y_s).$$

If, in addition, Y **has bounded variation paths (and, therefore, has a well-defined drift** d_Y**), then** 0 **is a holding point:**

$$P(Y \textbf{ becomes } > 0 \textbf{ immediately} \mid Y_0 = 0) = 0,$$

and so

$$L(s,t] = |d_Y| \int_s^t \mathbf{1}(X_u = 0)\,du.$$

Spectrally negative Y (but NOT a subordinator)

Then (1) **holds if and only if** Y **has unbounded variation paths. (A tiny Brownian component will do; or** $\int_{|y|>1} |y|\Pi(dy) = \infty$**.)**

Exceptional case

But if Y **is spectrally negative with paths of bounded variation then**

$$J := \{t \in \mathbb{R} : \; X_t = 0\} \equiv \{\mathfrak{t}_j\} \text{ is a discrete set}$$

and if we let n **be the point process with points on this set, we may define** L **by an additional randomization:**

$$L(s,t] = \sum_j \mathfrak{e}_j \mathbf{1}(s < \mathfrak{t}_j \le t),$$

where $\{\mathfrak{e}_j\}$ **is a collection of i.i.d. rate-1 exponentials, independent of everything else.**

Essentially, defining L **in this way, in this special case ensures that the inverse function of the counting process corresponding to** L **is a subordinator, something which holds, automatically, when** L **is continuous.**

Inverse local time

Define the inverse of L with respect to a $t_0 \in \mathbb{R}$ by

$$L^{-1}_{t_0;x} := \inf\{t > 0 : \; L[t_0, t_0 + t] > x\}, \qquad x \geq 0.$$

In particular,

$$L^{-1}_x := L^{-1}_{0,x}.$$

Then $(L^{-1}_x, x \geq 0)$ is a (non-killed) subordinator (because, under our assumptions, X cannot remain positive eventually) with Laplace transform as follows:

$$E_0 e^{-qL^{-1}_x} = \begin{cases} \exp\{-\Phi_Y(q)x\}, & \textbf{if } Y \textbf{ is sp. pos.}(-) \\ \exp\{-xq/\Phi_Y(q)\}, & \textbf{if } Y \textbf{ is sp. neg.}(+) \end{cases}$$

Note: $X_0 = 0 \Rightarrow 0 \in \operatorname{supp} L \Rightarrow L^{-1}_0 = 0$.

Reason for (+): $L^{-1}_x = \inf\{t \geq 0 : \; Y_t = x\}$, P_0**-a.s.**

Reason for (-): The factors in the Wiener-Hopf factorization

$$(T, Y_T) \stackrel{\mathbf{d}}{=} (L^{-1}_{L_T}, Y_{L^{-1}_{L_T}}) \stackrel{\bullet}{+} (M^{-1}_{M_T}, Y_{M^{-1}_{M_T}})$$

are known explicitly when Y is sp. neg.; from this, we can get the Laplace transform of L^{-1}_x as announced [Frisdtedt, 1974].

Rate of L

$\mu := EL(0,1] = EL(t,t+1]$.

If Y is sp$-$ we have

$$\mu = \Phi_Y(0).$$

If Y is sp$+$ we have

$$\mu = \psi_Y'(0).$$

We shall assume that

$$\boxed{\mu < 1}$$

Then

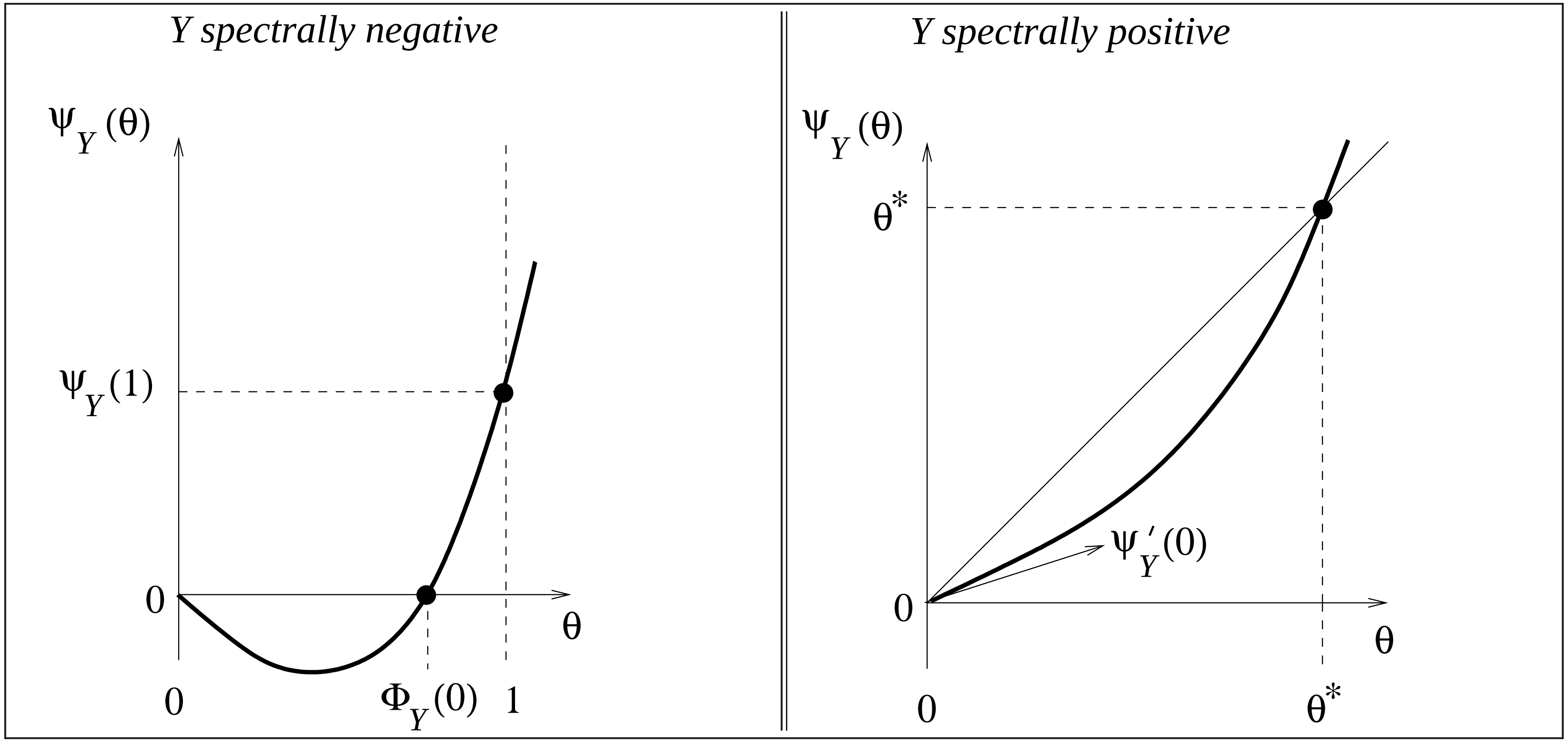
$$Q_t = \sup_{-\infty<u\le t} \{L(u,t] - (t-u)\}, \qquad t \in \mathbb{R},$$

is well-defined and stationary and ergodic.

If Y is sp$+$ with BV paths then its drift $d_Y < 0$ (necessarily). Since $L(s,t] = |d_Y| \int_s^t \mathbf{1}(X_u = 0)du$, if $|d_Y| \le 1$, then $Q \equiv 0$ and the case is trivial. So we assume

$$\boxed{\textbf{drift } d_Y < -1 \textbf{ if } Y \textbf{ is sp+ with BV paths}}$$

Laplace exponents



$$\mu = \Phi_Y(0) \text{ **if** } Y \text{ **is sp**}- \qquad \mu = \psi'_Y(0) \text{ **if** } Y \text{ **is sp**}+.$$

Let L **be a stationary and ergodic random measure on** $(\Omega, \mathscr{F}, P, \vartheta_\cdot)$ **with rate** $\mu < 1$**. Define**

$$Q_t = \sup_{-\infty < u \le t} \{L(u,t] - (t-u)\}, \qquad t \in \mathbb{R},$$

Then for any measurable $\psi : [0,\infty) \to [0,\infty)$

$$E\psi(Q_0) = (1-\mu)\,\psi(0) + \mu E_L \left[\frac{\int_{Q_{0-}}^{Q_0} \psi(x)dx}{Q_0 - Q_{0-}} \right],$$

where

$$P_L(A) = \mu^{-1} E \int_{(0,1]} \mathbf{1}_A \circ \vartheta_t L(dt), \qquad A \in \mathscr{F}.$$

FACT:

$$P_L = P_0$$

in all cases, unless Y **is sp− with BV paths (the exceptional case), and then**

$$P_L(A) = E_0[\mathfrak{e}_0 \mathbf{1}_A].$$

Proof: By Neveu's exchange formula.

By the distributional Little's law

$$P(Q_0 > a) = \mu P_L(Q_0 > a).$$

THEOREM: ***If*** Y ***is sp*** $-$ ***then***

$$P(Q_0 > a) = \mu \exp\{-\psi_Y(1)a\}.$$

If Y ***is sp*** $+$ ***then***

$$P(Q_0 > a) = \mu e^{-\theta^* a}$$

where $\psi_Y(\theta^*) = \theta^*$***.***

Sketch of proof

Express the event $\{Q_0 > a\}$ in terms of L^{-1}, under measure P_0:

$$\{\sup_{t\geq 0}(L_t - t) > a\} = \{\sup_{x\geq 0}(x - L_x^{-1}) > a\}. \tag{2}$$

The reason for this is the following version of the

sloppy formula $h^{-1}(h(t)) \approx t$

true for arbitrary càdlàg increasing functions.

> ***Let*** $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$***, If*** $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ ***is right-continuous and non-decreasing then*** $h^{-1}(x) := \inf\{t : \ h(t) > x\}$***,*** $x \in \mathbb{R}$***, is right-continuous non-decreasing,*** $h^{-1} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$***, and, for all*** $t \in \overline{\mathbb{R}}$***,***
>
> $$h^{-1}(h(t-)-) \leq h^{-1}(h(t)-) \leq t \leq h^{-1}(h(t-)) \leq h^{-1}(h(t)).$$
>
> ***Furthermore,*** $(h^{-1})^{-1} = h$***.***

Once (2) is established, we revert back to familiar ground because

$$\Lambda_x := x - L_x^{-1}$$

is a spectrally negative Lévy process and, since

$$\{\sup_{x\geq 0}(x - L_x^{-1}) > a\} = \{\inf\{x : \ \Lambda_x > a\} < \infty\}$$

the P_0-probability of the latter event can be found easily:

$$P_0(\inf\{x : \ \Lambda_x > a\} < \infty) = \exp\{-\Phi_\Lambda(0)a\}$$

where Φ_Λ is related to ψ_Λ in the same manner that Φ_Y is related to ψ_Y. Thus, $\Phi_\Lambda(0) > 0$ and

$$\psi_\Lambda(\Phi_\Lambda(0)) = 0.$$

Since we know the characteristics of L^{-1} we can easily find ψ_Λ:

$$\psi_\Lambda(\theta) = \begin{cases} \theta - \frac{\theta}{\Phi_Y(\theta)}, & \textbf{if } Y \textbf{ is spectrally negative}, \\ \theta - \Phi_Y(\theta), & \textbf{if } Y \textbf{ is spectrally positive}. \end{cases}$$

Similarly, in the exceptional case.

AIDE-MÉMOIRE:
EXIT TIMES FOR SPECTRALLY NEGATIVE LÉVY PROCESSES

Let S be a spectrally negative Lévy process.

τ_x^+ **= first time t such that $S_t > x$ = first time t such that $S_t = x$**

τ_x^- **= first time t such that $S_t < -x$**

Then

$$E\left[e^{-q\tau_x^+}\right] = e^{-\Phi_S(q)x}, \tag{3}$$

$$E\left[e^{-q\tau_{-x}^-}\right] = Z^{(q)}(x) - \frac{q}{\Phi_S(q)} W^{(q)}(x). \tag{4}$$

where $W^{(q)}, Z^{(q)}$ are the q-scale functions of S, whose analytic definition is:

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi_S(\beta) - q},$$

$$\int_0^\infty e^{-\beta x} Z^{(q)}(x)dx = \frac{\psi_S(\beta)}{\beta(\psi_S(\beta) - q)},$$

EXAMPLE 1: BROWNIAN MOTION WITH DRIFT

$$Y_t = \sigma B_t - \mu t.$$

$$\psi_Y(\theta) = \log E[e^{-\theta Y_1}] = \frac{1}{2}\sigma^2\theta^2 + \mu\theta.$$

The rate of L is $\mu = \psi'(0)$. Assuming $\mu < 1$, we define $(Q_t, t \in \mathbb{R})$ and find

$$P(Q_0 > a) = \mu e^{-\theta^* a}$$

where

$$\theta^* = \psi_Y(\theta^*) = -2(1-\mu)/\sigma^2.$$

Example 2: Compound Poisson process with drift

$$Y_t = S_t - \alpha t, \quad t \in \mathbb{R},$$

where S is a compound Poisson process with only positive jumps, jump rate λ and jump size distribution F. For simplicity, let F be exponential with rate δ. Here Y is sp+ with BV paths and so L is of integral type. We find

$$\psi_Y(\theta) = \alpha\theta - \lambda \int_{[0,\infty)} (1 - e^{-\theta x}) F(dx) = \alpha\theta - \frac{\lambda\theta}{\delta + \theta}, \quad \theta > 0.$$

Rate of L: $\mu = \psi_Y'(0) = \alpha - \lambda/\delta.$

Drift of Y: $d_Y = -\alpha.$

Assume $\mu < 1$ and $d_Y < -1$ and find that

$$P(Q_0 > x) = \mu e^{-\theta^* x}$$

where

$$\theta^* = \psi_Y(\theta^*) = \lambda(\alpha - 1)^{-1} - \delta.$$

Behaviour of sample paths in Example 2

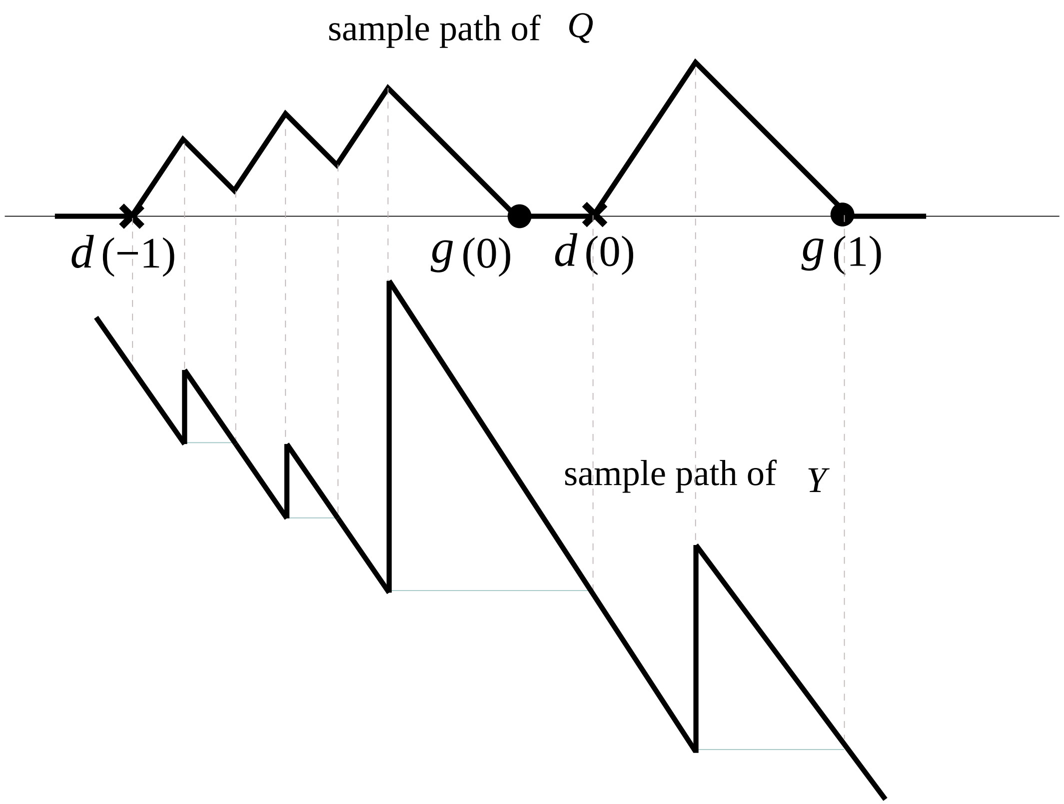


Figure 1: *Typical behaviour of Q and the background Lévy process Y, in case that Y is spectrally positive with bounded variation paths. Here, only the case where the jump part of Y is compound Poisson is depicted. When $Q_t > 0$ and $X_t = 0$, we see that Q_t increases at rate $|d_Y| - 1$, where $d_Y < -1$ is the drift of Y.*

Behaviour of sample paths in case Y is spectrally negative with bounded variation paths (the exceptional case)

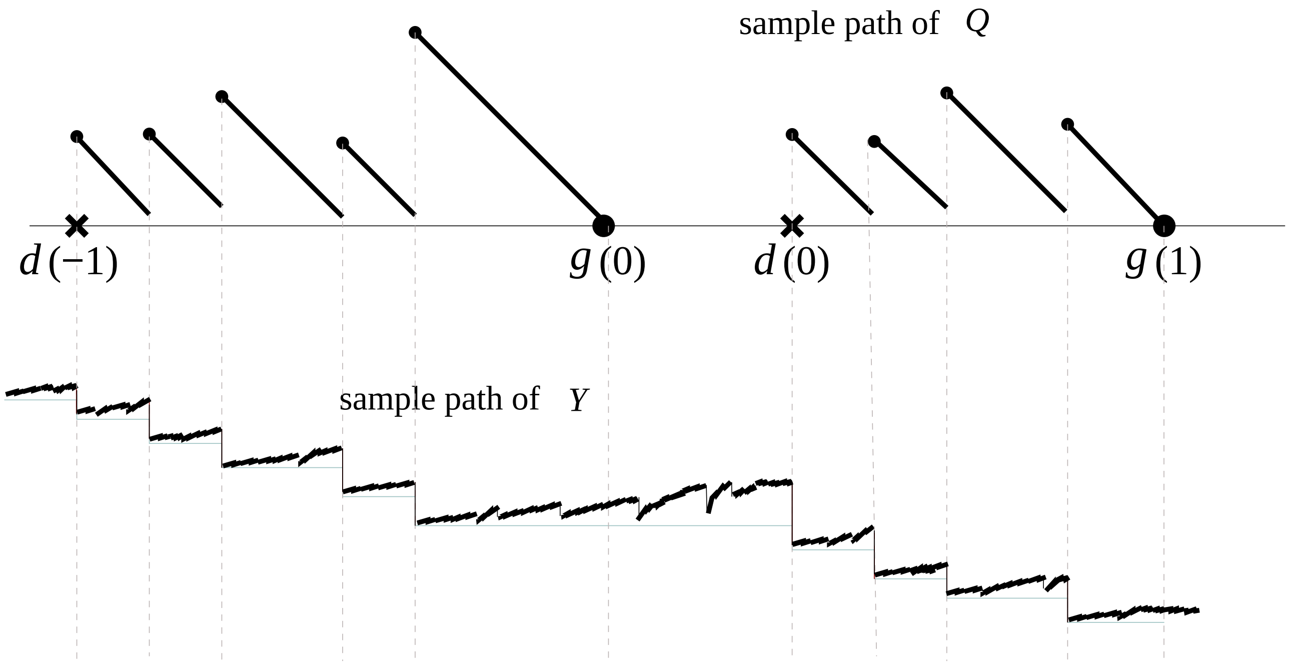


Figure 2: *Typical behaviour of Q and the background Lévy process Y, in case that Y is spectrally negative with bounded variation paths.*

Example 3

$$Y_t = 3bt + \sigma B_t - S_t,$$

B**: standard BM**
S**:** $(1/2)$**-stable subordinator.**

$$\psi_Y(\theta) = \log E e^{\theta(Y_1 - Y_0)} = 3b\theta + \frac{1}{2}\sigma^2\theta^2 - 2c\theta^{1/2}, \qquad \theta > 0,$$

The rate μ **of** L **satisfies** $\psi_Y(\mu) = 0$**. We find:**

$$\mu = 2\left(\frac{c}{\sigma^2}\right)^{2/3} \frac{(\delta^7 + 1)^{1/3} + (\delta^7 - 1)^{1/3}}{\delta^2},$$

where

$$\delta = 1 + b^3\sigma^{-2}c^{-2},$$

Marginal distribution of Q**:**

$$P(Q_0 > x) = \mu e^{-\psi_Y(1)} = \mu \exp\{-(b + \frac{1}{2}\sigma^2 - c)x\},$$

where $\psi_Y(1) > 0$ **because** $\mu < 1$**, by assumption.**

Example 1 again: Brownian motion with drift

Take $\sigma = 1$**. We find**

$$E\left(e^{-\theta(d(0)-g(0))} \,|\, Q_0 = 0\right) = \frac{4\mu}{\sqrt{2\theta+\mu^2}(\sqrt{2\theta+\mu^2}+2-\mu)^2},$$

Taking the inverse Laplace transform, we obtain the density of the length of the idle period $d(0)-g(0)$**, given** $Q_0 > 0$**, as**

$$p_I(v) = 2\mu \mathrm{e}^{-\mu^2 v/2}\left(\sqrt{2v/\pi} - (2-\mu)ve^{(2-\mu)^2 v/2}\mathrm{Erfc}((2-\mu)\sqrt{v/2})\right).$$

where

$$\mathrm{Erfc}(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-t^2}dt.$$

We also note that the density of the busy period straddling a deterministic point is obtained from p_I **by substituting** μ **with** $1-\mu$ **(bandwidth duality).**

Summary

Our work combined several techniques:
– fluctuation theory
– Palm probabilities (Pitman's identity and Neveu exchange formula)
– Markov theory
– pathwise arguments,
in order to analyze a (certain) non-Markovian fluid queue.

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– fluctuation theory
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Open problems

0. Explain the bandwidth duality.

1. Prove the limit theorems!

2. Replace server with more general function.

3. $d > 2$ **classes: Require understanding how a Lévy process hits a boundary defined by a polynomial of degree** $d - 2$**.**

4. Understand the role of reflecting a random measure vs. reflecting its inverse. C.f. also Kallenberg 2003(?)

5. Applications in Financial Maths.