Metrics for temporal graphs

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Overview

Classical network data

1^{st} Unit	2^{nd} Unit	**(Weight)**
1	2	**3**
1	4	**1**
2	3	**5**
2	4	**2**
2	5	**7**
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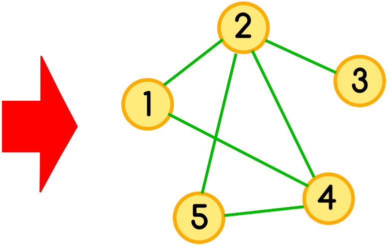

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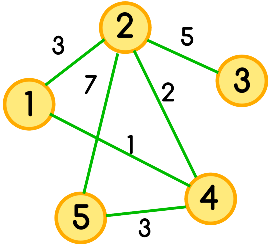

Classical graph metrics

- Connectedness and components

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$\Longrightarrow$ Processes on networks (percolation, communication, spreading, synchronisation, opinions, etc.)

Time-resolved data

1st Unit	2nd Unit	Start	Length
2	4	0	40
2	5	50	10
2	3	70	20
4	5	60	50
1	2	130	15
1	4	140	35
2	3	220	20

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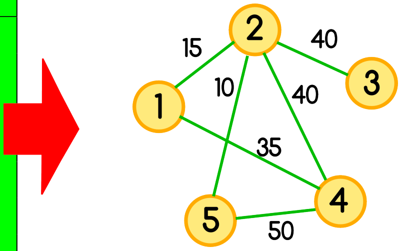

Shortcomings of aggregated graphs

- Loss of temporal correlations and time-dependence

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- Overestimation of the number of available walks and paths

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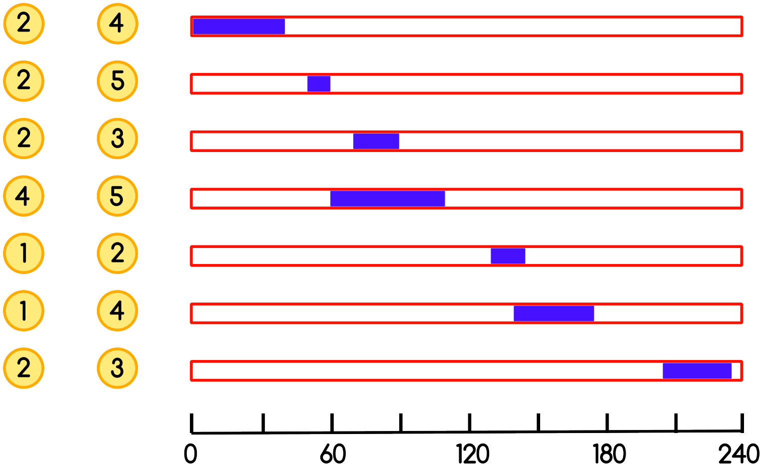
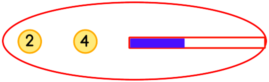


Contacts

- $c = (i, j, t, \delta t)$ is a contact
-
-
-



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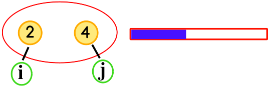

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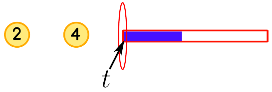

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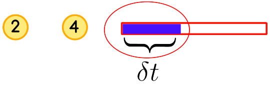

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$$\Rightarrow$$

$$\begin{bmatrix} 0 & 3 & 0 & 1 & 0 \\ 3 & 0 & 5 & 2 & 7 \\ 0 & 5 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & 7 & 0 & 3 & 0 \end{bmatrix}$$

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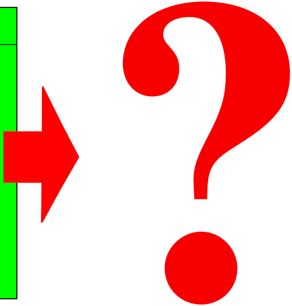

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- $[t, t + \Delta t] \Longrightarrow G_t$ contains all the contacts $(\cdot, \cdot, \tau_i, \delta\tau_i)$ overlapping with $[t, t + \Delta t]$, i.e. such that:

$$t \leq \tau_i < t + \Delta t \quad \text{or} \tag{1}$$

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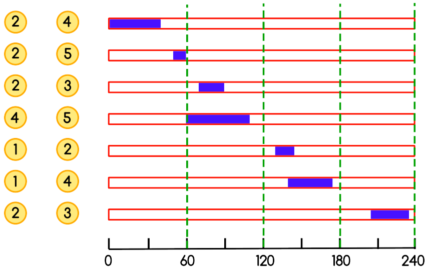
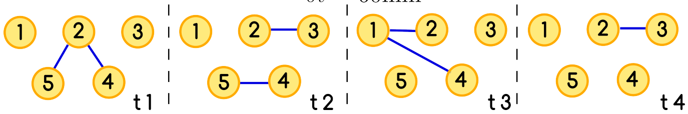
- G_t is a snapshot of the system in $[t, t + \Delta t]$.
- The sequence $\mathcal{G}_{0,T} = \{G_0, G_{\Delta t}, \dots G_T\}$ of M snapshots over N nodes is a time-varying graph.

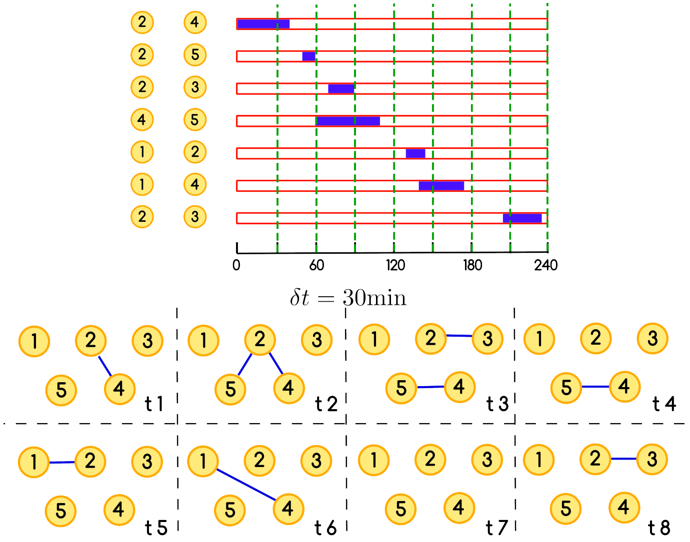
Time scales (1)



$\delta t = 60\text{min}$

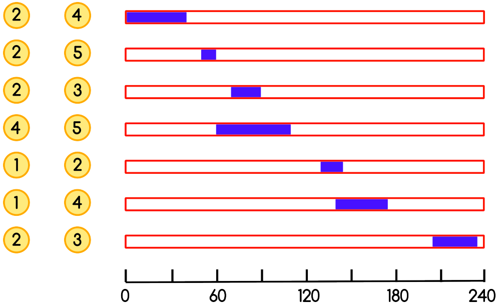


Time scales (2)



Reachability

From node 5 to node 1



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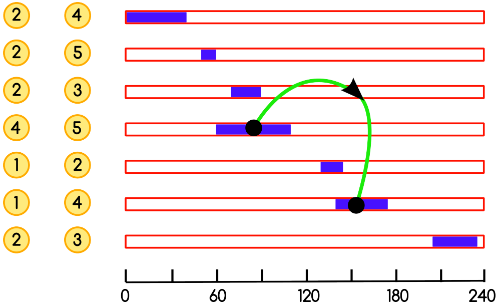

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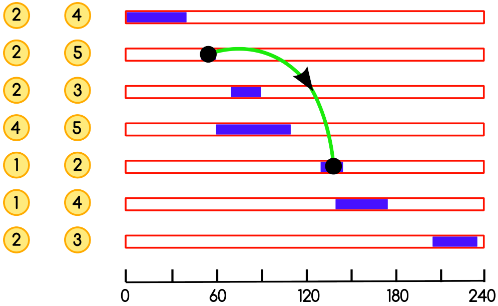

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- Temporal connectedness IS NEITHER symmetric NOR transitive.

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Given a node i we define:

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- the temporal IN-component of i (nodes j for which there is a TW from j to i)
- the temporal strongly connected component of i (nodes j which are both in $\mathrm{IN}(i)$ and in $\mathrm{OUT}(i)$)
- i and j are strongly connected if $i \in \mathrm{IN}(j)$ and $i \in \mathrm{OUT}(j)$

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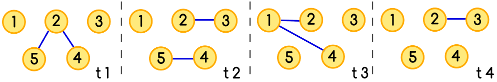
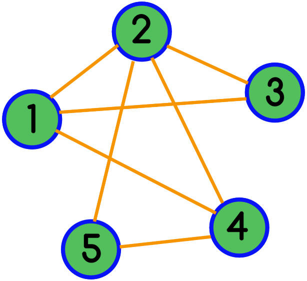
- The strongly connected components of G are the maximal-cliques of $G_{\mathcal{G}}$
- Finding the largest strongly connected component of a TVG takes exponential time in the number of edges of the affine graph!

Affine graphs





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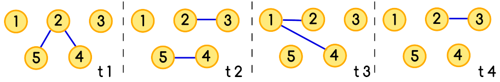

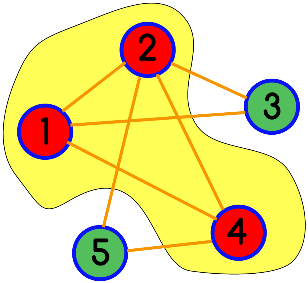

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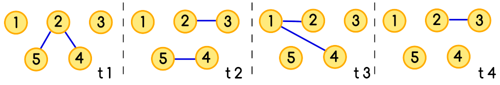

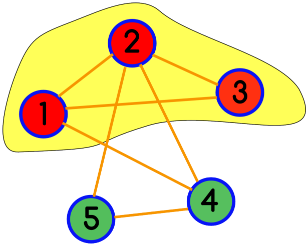

Application: Facebook

$\sim$ 100.000 profiles in Santa Barbara (CA) (2009)
1 week of messages

- Friendship network (static graph)
- Communication network (TVG – $\Delta t = 1$ hour)

Week	K	S	C
1	43491	22	12000
2	48404	20	13998
3	43400	16	12773
4	60853	41	17933
5	65703	23	19973
6	70282	27	20976
7	60666	28	18537
8	73772	46	20256
9	79645	38	21990
10	66849	18	20425
11	55040	27	18266
12	51418	28	15667

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A time-respecting path has many different "lengths", namely:

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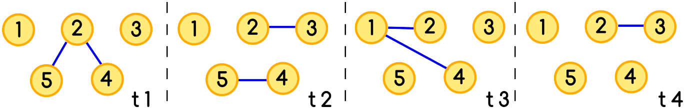
temporal distance $d_{i,j}$ is the temporal length of the temporal shortest path from i to j.


1
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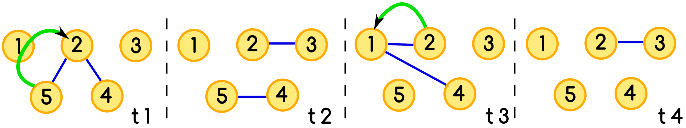
4
t 4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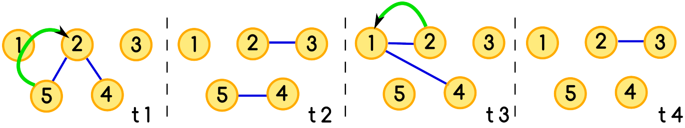

- Topological length: 2
- Temporal length: $3\Delta t$

Length-related metrics

Average temporal length

$$L = \frac{1}{N(N-1)} \sum_{ij} d_{ij} \tag{4}$$

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Temporal efficiency:

$$\mathcal{E} = \frac{1}{N(N-1)} \sum_{ij} \frac{1}{d_{ij}} \tag{6}$$

Application: node percolation

- Damage: $D\%$ of the nodes are removed (percolated) from the network $\Longrightarrow$ new graph $\mathcal{G}_{\mathcal{D}}$

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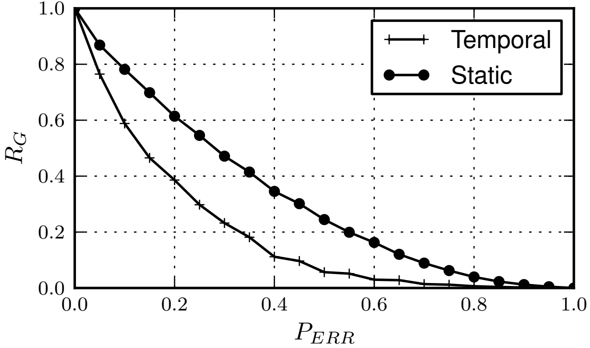
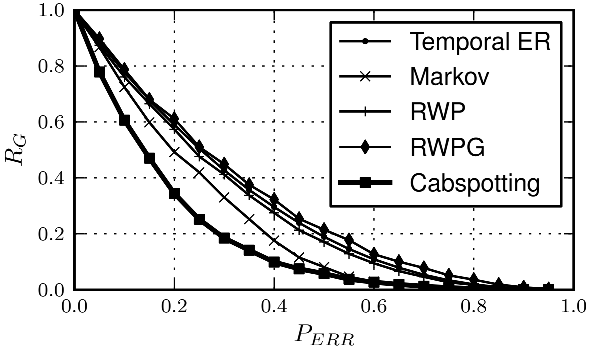
- Robustness:

$$R = \frac{E_{\mathcal{G}_D}}{E_{\mathcal{G}}} \tag{7}$$

Cabspotting: aggregated vs TVG



Cabspotting: TVG vs random models



Temporal Clustering

Topological overlap of the neighbourhood of i in $[t_m, t_{m+1}]$:

$$C_i(t_m, t_{m+1}) = \frac{\sum_j a_{ij}(t_m) a_{ij}(t_{m+1})}{\sqrt{\left[\sum_j a_{ij}(t_m)\right]\left[\sum_j a_{ij}(t_{m+1})\right]}} \tag{8}$$

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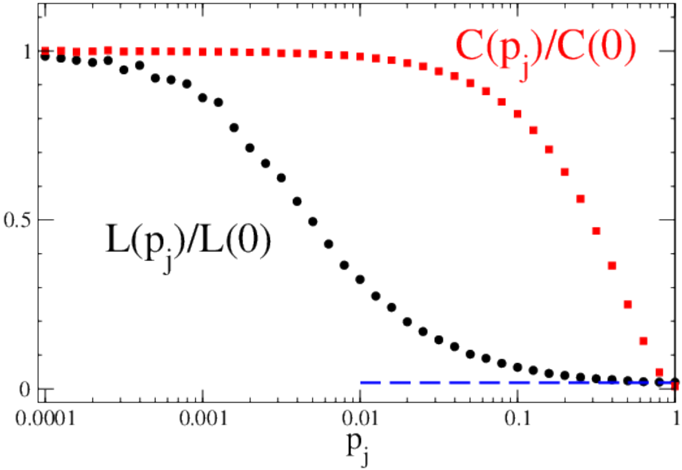
Temporal correlation coefficient

$$C = \frac{1}{N} \sum_i C_i \tag{10}$$

Temporal small-world effect

	C	C^{rand}	L	L^{rand}	E	E^{rand}
α	0.44	0.18 (0.03)	3.9	4.2	0.50	0.48
β	0.40	0.17 (0.002)	6.0	3.6	0.41	0.45
γ	0.48	0.13 (0.003)	12.2	8.7	0.39	0.37
δ	0.44	0.17 (0.003)	2.2	2.4	0.57	0.56
d1	0.80	0.44 (0.01)	8.84	6.00	0.192	0.209
d2	0.78	0.35 (0.01)	5.04	4.01	0.293	0.298
d3	0.81	0.38 (0.01)	9.06	6.76	0.134	0.141
d4	0.83	0.39 (0.01)	21.42	15.55	0.019	0.028
Mar	0.044	0.007 (0.0002)	456	451	0.000183	0.000210
Jun	0.046	0.006 (0.0002)	380	361	0.000047	0.000057
Sep	0.046	0.006 (0.0002)	414	415	0.000058	0.000074
Dec	0.049	0.006 (0.0002)	403	395	0.000047	0.000059


C(p_j)/C(0)
L(p_j)/L(0)
1
0.5
0
0.0001
0.001
0.01
0.1
1
p_j

Betweenness and closeness centrality

Temporal betweenness centrality of a node at time t_m:

$$C_i^B(t_m) = \frac{1}{(N-1)(N-2)} \sum_{j \neq i} \sum_{\substack{k \neq j \\ k \neq i}} \frac{U(i, t_m, j, k)}{\sigma_{jk}} \tag{11}$$

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Average temporal betweenness of node i:

$$C_i^B = \frac{1}{M} \sum_m C_i^B(t_m) \tag{12}$$

Betweenness and closeness centrality

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Average temporal betweenness of node i:

$$C_i^B = \frac{1}{M} \sum_m C_i^B(t_m) \tag{12}$$

Average temporal closeness of i:

$$C_i^C = \frac{N-1}{\sum_j d_{ij}} \tag{13}$$

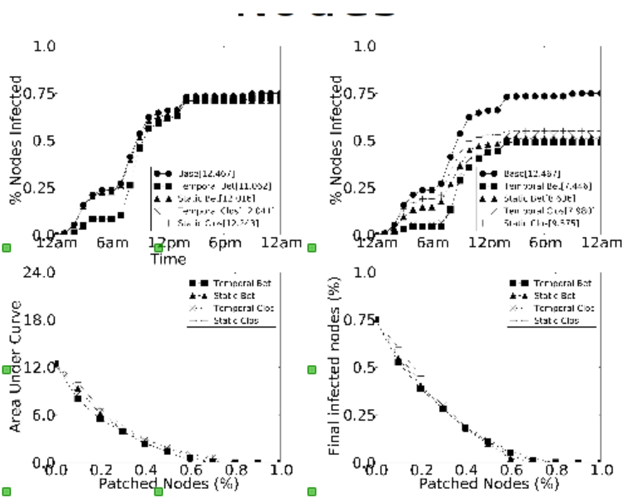
Application: information spreading & success

ID	Name	Role
9	Stephanie Panus	(Unknown)
13	Marie Heard	Legal
17	Mike Grigsby	Manager
48	Tana Jones	Executive
53	John Lavorato	Trader
54	Greg Whalley	President
67	Sara Shackleton	Vice President
73	Jeff Dasovich	Trader
75	Gerald Nemec	Director of Trading
107	Louise Kitchen	Trader
122	Sally Beck	Managing Director
127	Kenneth Lay	Manager
139	Mary Hain	Director
147	Carol Clair	Trader
150	Liz Taylor	Secretary

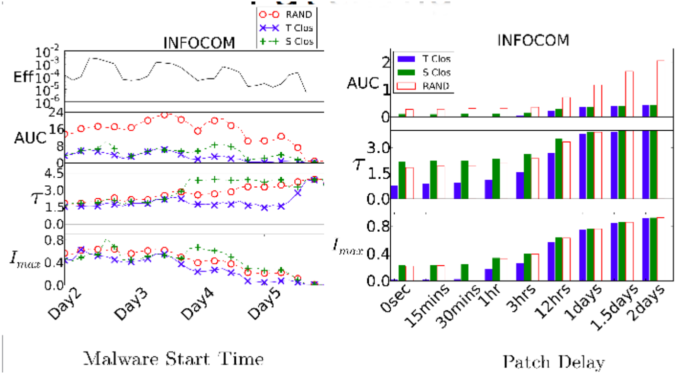
Top bonuses awarded

John Lavorato: $5 million
Louise Kitchen: $2 million
Jeffrey McMahon: $1.5 million
James Fallon: $1.5 million
Raymond Bowen Jr.: $750,000
Mark Haedicke: $750,000
Gary Hickerson: $700,000
Wesley Colwell: $600,000
Richard Dimichele: $600,000

Application: mobile malware containment



Application: mobile malware containment



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