Control of Systems that Store Renewable Energy

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ABSTRACT

This paper studies the control of systems that store renewable energy. The problem is to maximize the long-term utility of the energy by controlling how it is used. The methodology for designing the control policy depends on the size of the battery. If the battery is small, the control policy is determined by solving a Markov decision problem. If the battery is large, this problem is complex but one can replace it by a simpler problem where the constraint is on the average power usage. When the battery is large, the average power usage should not exceed the average power harvested by the source. When the battery size is moderate, the control is based on the large deviations of the battery charge. This paper illustrates these methods with a number of examples.

Keywords

Renewable energy; storage; control; large deviations; Markov chain; occupation measure.

1. INTRODUCTION

By increasing the use of renewable energy sources, the energy usage control of the systems that operates with such sources are the great of interest. Unlike conventional power sources, the output power of renewable sources cannot be controlled as there are daily and seasonal fluctuations and inaccurate energy prediction. This makes the control of the systems that operates with such sources challenging [1].

The paper is concerned with systems that utilize renewable energy and are equipped with a battery to adjust for the variability in available power and energy usage. Examples include wireless sensor nodes and buildings.

The problem under study is how to best use the stored energy to maximize the long-term utility. For instance, in the case of a wireless sensor node, the average power used must be less than the average power of the source. However, unless the battery is very large, the variability may cause the battery to go empty even when that condition is met. In such a situation, one suspects that the energy use should take into account the instantaneous amount of energy stored in the battery. One approach is to formulate this problem as a Markov decision in which the state of the system is the amount of stored energy, together with the state of the environment. Unless the battery is small, the size of the state space of this Markov decision problem is very large, which makes the problem difficult to solve. Moreover, this formulation results in a complex control strategy that depends on the stored energy. However, intuition suggests that if the battery is moderate in size, then using energy at an average rate slightly less than the average rate of the source should guarantee that the battery rarely goes empty. This paper explains how to make that intuition precise using the theory of large deviations. The large deviation analysis leads to the constraint for energy usage. The novelty of the analysis is that the source and usage are both variable, in contrast with the theory of effective bandwidth [6] and [7]. Indeed, the usage affects the large deviations of the battery discharge, so that the large deviations appear as constraints for the optimization problem. One contribution of the paper is a formulation that enables the analysis of the large deviations of the battery in a numerically tractable way that can be included in the optimization problem. We compare this approach to the large deviations analysis based on the occupation measure of a Markov chain. We also examine the case when the variability of the energy source and that of the load are independent.

It would be tempting to use a Gaussian approximation [11] to study the large deviations. However, simple examples show that this approximation is very poor.

The paper is structured as follows. We introduce the system model and problem formulation in Section 2. This is followed by Section 3 which is approximating the control policy by replacing the constraint based on large deviation techniques. In Section 4, the large deviation techniques are applied in three ways: direct method which is based on the Chernoff's inequality, a method based on occupation measure and a Gaussian approximation method. Sections 5 and 6 explain the evaluation of the approach for random walk and 2-state Markov chain, respectively. In Section 6, we present the same problem for the case when the variability of the energy source and that of the load are independent. In order to clarify the proposed approach, Section 7 provides

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several examples. Section 8 concludes and summarizes the paper.

1.1 Background and related work

Resource management techniques for energy harvesting systems with uncertain resource availability pose a new set of challenges. These techniques lead to utility maximization considering the energy constraint. For energy harvesting wireless sensor networks EH-WSNs where the resources of interest are energy and data, the transmission rate and data sampling rate maximization satisfying the energy constraint are two important problems. These problems have been addressed in [12]- [17]. The authors in [12] proposed the solution for rate maximization for multiple fading channels of a transmitter. They develop the directional water filling heuristic. The authors in [14] designed a solution for fair and high throughput data extraction from all nodes guaranteeing fairness while maximizing the sampling rate and throughput. Mao et al. in [16] proposed a joint data queue and battery buffer control algorithm, thus the long-term average sensing rate maximization subject to stability of data queue and desired data loss ratio could be achieved. They considered the static channel model and offline knowledge about the energy input. A policy with decoupled admission control and power allocation decisions is developed in [15] that achieves asymptotic optimality for sufficiently large battery capacity to maximum transmission power ratio (explicit bounds are provided). The authors in [17] obtained the energy management policies that are throughput optimal and minimize the mean delay in the queue. They mainly assume that the capacity of the rechargeable battery is large enough; however, they did not consider the consequence of large state space on designing the methodology and algorithms. In this paper, we show that by considering the storage capacity of the system, one can design efficient and simple algorithms. The main advantage of our approach is that the control policy for the energy usage rate does not involve the instantaneous amount of energy stored in the nodes, when the size of the battery is moderate or large.

In this paper, the core idea is to convert the complex Markov decision problem to a simple optimization problem where its constraint is based on large deviation theory. Large deviations theory refers to the collection of techniques to estimate the properties of rare events, such as their frequency and most likely manner of occurrence. Some references on large deviations include Bahadur (1971) [2], Varadhan (1984) [3], Deuschel and Stroock (1989) [5], and Dembo and Zeitouni (1998) [4]. Large deviations are often caused by a large number of unlikely events occurring together, rather than a single event of small probability. The theory of large deviations has been applied to the analysis of Asynchronous Transfer Mode (ATM) networks [6] and [7]. ATM is a packet switching standard that aimed to limit the rate of cell losses due to buffer overflow to negligible values, comparable to losses caused by transmission errors.

In this paper, the state of the system is modeled as a finite Markov chain. There are a few possible approaches to study the large deviations of a Markov chain. One method is based on the occupation measure of Markov chains [9]. The basic idea of this approach is that the most likely way for a Markov chain to have an empirical distribution that differs from the invariant distribution is for it to behave as if it had different transition probabilities consistent with the observed empirical distribution. This is the essence of the contraction mapping theorem [3].

Another approach, that we call the direct method, is to start with Chernoff's inequality and calculate the relevant moment generating functions using the first step equations of a Markov chain.

Yet another approach is to consider a Gaussian approximation for the changes of the Markov chain over a number of steps [10], [11]. However, we explain that this method yields poor estimates of the likelihood that the battery becomes empty for realistic system parameters, which should not be surprising since large deviations typically depend strongly on the higher moments of the distributions.

2. MODEL

A discrete time model of the system is as follows. At time $n \geq 0$, the battery, with capacity B, has accumulated an amount $X_n \in \{0, 1, \ldots, B\}$ of energy, the environment state such as weather condition is Y_n, a Markov chain on some finite state space $\mathcal{Y}$ with a transition probability matrix P, and one uses a control action $U_n \in \mathcal{U}$ where $\mathcal{U}$ is a finite set. The net amount of battery discharge at time n is a function of Y_n and U_n denoted as $g(Y_n, U_n)$. Hence, $E[g(Y_n, U_n)]$ can take positive as well as negative values. A negative value means that the battery tends to recharge more than drain. Also, $r(Y_n, U_n)$ represents the reward of taking action U_n in state Y_n. The action u is possible at time n only if $g(Y_n, u) \leq X_n$. The objective is to choose the control actions to maximize the long term average value of $r(Y_n, U_n)$. That is, the problem is as follows:

$$\begin{array}{rl} \text{Maximize} & E(r(Y_n, U_n)) \\ \text{over} & U_n \\ \text{s.t.} & g(Y_n, U_n) \leq X_n \\ \text{and} & X_{n+1} = [X_n - g(Y_n, U_n)]_0^B. \end{array}$$

Note that in the above problem formulation U_n is the function of state of the system and energy level of the battery. In the last expression, we use the notation

$$[x]_0^B = \max\{0, \min\{x, B\}\}.$$

Since (X_n, Y_n) is a Markov chain controlled by U_n, this is a Markov decision problem. It can be solved by Dynamic Programming. The size of the state space of this problem is $(B+1) \times |\mathcal{Y}|$ and it can be very large unless the battery capacity B is not relatively small. More importantly, the resulting control strategy is complex as it depends on the instantaneous amount of stored energy.

For the purpose of simplifying the solution of the problem and also for deriving some insight into the solution, we examine approximation methods that we explore in the next section.

3. APPROXIMATIONS

If the battery is not too small, the fact that it goes empty is a large deviation under a suitable operating regime. This suggests that one can replace the constraint $g(Y_n, U_n) \leq X_n$ by a constraint on the probability that the battery goes empty. Moreover, this constraint can be guaranteed by using a control strategy that depends only on Y_n and is designed so that the statistics of U_n make it very unlikely to deplete the battery faster than it charges for a duration long enough to empty it. This approach has the benefit of resulting in a much simpler control scheme that does not have to depend on the state of charge of the battery. Moreover, the calculation of the control strategy is also much simpler.

Specifically, we consider the problem

$$\begin{array}{rl} \text{Maximize} & E(r(Y_n, U_n)) \\ \text{over} & q \\ \text{s.t.} & P[U_n = u | Y_n = y] = q(y, u) \\ \text{and} & P(X_n = 0) \leq \beta \\ \text{and} & X_{n+1} = [X_n - g(Y_n, U_n)]_0^B. \end{array}$$

In this formulation, β is a small probability. Also, q defines a stationary control strategy that depends only on Y_n, not on X_n. Thus, we have relaxed the tight constraint $g(Y_n, U_n) \leq X_n$ by replacing it by the constraint $P(X_n = 0) \leq \beta$. We will enforce this constraint by considering the large deviations of the process X_n. Specifically, if $E(g(Y_m, U_m)) < 0$, which is a necessary requirement for the battery to have a small probability of being empty, one can expect the probability, under the stationary distribution, to be on the order of

$$K \exp\{-B\psi(q)\}$$

where K is a constant and $\psi(q)$ depends on the control policy q. That is, the constraint $P(X_n = 0) \leq \beta$ can be replaced by

$$\psi(q) \geq \frac{\delta}{B} \tag{1}$$

where δ is chosen so that $K \exp\{-\delta\} = \beta$.

To determine $\psi(q)$, one argues as follows. The battery becomes empty after $n = B/c$ steps if it discharges at an average rate c for these n steps for some $c > 0$. Thus, one is led to study the probability of such a discharge rate, i.e., the probability

$$P(Z_1 + \cdots + Z_n \geq nc)$$

where

$$Z_m = g(Y_m, U_m).$$

We will show that, when $E(Z_n) < 0$, this probability is approximately equal to

$$\exp\{-n\phi(c, q)\}.$$

Accordingly, with $n = B/c$, we see that this probability is of the order of

$$\exp -B\frac{\phi(c, q)}{c}.$$

Since every $c > 0$ is a possible discharge rate that would empty the battery in B/c steps, the probability that the battery empties is the sum over all $c > 0$ of these probabilities. If B is not too small, this sum is well approximated by the term that corresponds to the smallest exponential rate of decay as a function of B. That is, the probability is well approximated by

$$\exp\{-B\psi(q)\}$$

where

$$\psi(q) := \inf_{c>0} \frac{\phi(c, q)}{c}.$$

To analyze the probabilities, we note that for a given q the random variables (Y_n, U_n) form a Markov chain. Thus, Z_n is a function of a Markov chain. Now, the main concern is how to calculate the value of $\phi(., .)$ and $\psi(.)$. This is explained in next section.

Before proceeding, we review some results about Markov chains.

4. LARGE DEVIATIONS

To develop our estimates, we need to study the large deviations of the process $Z_1 + \cdots + Z_n$ driven by the Markov chain Y_n. To do this, we consider three methods: a direct method, an analysis of the occupation measure of a Markov chain, and a Gaussian approximation. We explain that the direct method is numerically simple and yields good estimates. We use the occupation measure to derive properties of the large deviations. We show that the Gaussian approximation is not satisfactory for our problems.

Direct Method

The direct method is based on Chernoff's inequality and on the first step analysis of a Markov chain.

For $y \in \mathcal{Y}$, $\theta > 0$ and $n \geq 1$, let

$$s_n(y) := E[\exp\{\theta(Z_1 + \cdots + Z_n)\}|Y_1 = y], \forall y \in \mathcal{Y}.$$

Note that (see Appendix A)

$$s_{n+1}(y) = E[\exp\{\theta Z_1\}|Y_1 = y] \sum_{y'} P(y, y') s_n(y'), \forall y \in \mathcal{Y}.$$

Let $\mathbf{s}_n$ be the column vector with components $\{s_n(y), y \in \mathcal{Y}\}$. Then

$$\mathbf{s}_{n+1} = G_\theta \mathbf{s}_n, n \geq 1$$

where

$$G_\theta(y, y') = h_\theta(y) P(y, y')$$

with

$$h_\theta(y) = E[\exp\{\theta Z_1\}|Y_1 = y] = \sum_u q(y, u) \exp\{\theta g(y, u)\}.$$

Consequently,

$$\mathbf{s}_n = G_\theta^n \mathbf{s}_0$$

where $\mathbf{s}_0 = [1, 1, \ldots, 1]'$. Also, from conditional expectation we have,

$$E[\exp\{\theta(Z_1 + \cdots + Z_n)\}] = \pi \mathbf{s}_n = \pi G_\theta^n \mathbf{s}_0 \tag{2}$$

where π is the distribution of Y_1.

Let $\lambda(\theta)$ be the largest eigenvalue of G_θ. We can approximate the mean value above by

$$E[\exp\{\theta(Z_1 + \cdots + Z_n)\}] \approx K\lambda(\theta)^n, n \gg 1$$

where K is a constant. To see this approximation, note that if the eigenvalues of G_θ are distinct, then one can use the eigendecomposition of matrix G_θ

$$G_\theta = VDV^{-1}$$

where D is the diagonal matrix of eigenvalues. Then,

$$G_\theta^n = VD^nV^{-1}$$

and the approximation follows. If the eigenvalues are not distinct, one replaces D by the block Jordan matrix and the same approximation results.

We use that approximation to study the large deviations of Z_n. One has Chernoff's inequality for $\theta > 0$:

$$P(Z_1 + \cdots + Z_n \geq nc) \leq E(\exp\{\theta(Z_1 + \cdots + Z_n - nc)\})$$
$$\approx K\lambda(\theta)^n \exp\{-n\theta c\} = K \exp\{-n(\theta c - \log(\lambda(\theta)))\}.$$

Since this inequality holds for all $\theta > 0$, one can minimize the right-hand side over $\theta > 0$ and find

$$P(Z_1 + \cdots + Z_n \geq nc) \leq K \exp\{-n\phi(c, q)\}$$

where

$$\phi(c, q) = \sup_{\theta>0}\{\theta c - \log(\lambda(\theta))\}.$$

As we explained earlier, $\psi(q) = \inf_{c>0} \phi(c, q)/c$, so that

$$\psi(q) = \inf_{c>0} \frac{\phi(c, q)}{c} = \inf_{c>0} \frac{1}{c} \sup_{\theta>0}\{\theta c - \log(\lambda(\theta))\}. \quad (3)$$

The value of c that minimizes $\frac{\phi(c,q)}{c}$ is the average draining rate which results in the battery to go empty rarely. Moreover, $\psi(q)$ is a strictly decreasing function in terms of our control policy q. Hence, the value of q such that $\psi(q)$ is equal to $\frac{\delta}{B}$ from constraint (1) is the optimum control policy.

Occupation Measure

For the purpose of deriving properties of the large deviations, we consider an estimate based on the occupation measure of the Markov chain $V_n = (Y_n, U_n)$. We use the occupation measure to obtain an expression for the probability that a Markov chain with a given transition matrix behaves as if it had another transition rate matrix over a long period of time.

Consider a Markov chain V_n with transition matrix P_0. For another transition matrix P_1 and a sequence $\mathbf{v} = (v_0, \ldots, v_n)$, let

$$L(\mathbf{v}) = \frac{\pi_0(v_0)P_0(v_0, v_1) \ldots P_0(v_{n-1}, v_n)}{\pi_1(v_0)P_1(v_0, v_1) \ldots P_1(v_{n-1}, v_n)}$$

where π_1 is invariant under P_1 and π_0 is invariant under P_0. Note that

$$\log(L(\mathbf{v})) = \log\left(\frac{\pi_0(v_0)}{\pi_1(v_0)}\right) + \sum_{v,v'} N_n(v, v') \log\left(\frac{P_0(v, v')}{P_1(v, v')}\right) \quad (4)$$

where $N_n(v, v')$ is the number of transitions from v to v' in $\mathbf{v}$. Thus, $L(\mathbf{v})$ is the ratio of the likelihood of $\mathbf{v}$ under P_0 divided by its likelihood under P_1. Note that under P_1, one has

$$N_n(v, v') \approx n\pi_1(v)P_1(v, v').$$

Consequently, for the random sequence $V^n = \{V_1 \ldots, V_n\}$, if we get an exponential from both sides of (4), under P_1 we have,

$$L(V^n) \approx \exp\{-nH(P_1)\} \quad (5)$$

where

$$H(P_1) = -\sum_{v,v'} \pi_1(v)P_1(v, v') \log\left(\frac{P_0(v, v')}{P_1(v, v')}\right).$$

Consider a set A of sequences $\mathbf{v}$ that are typical under P_1. These sequences satisfy the law of large numbers for the Markov chain so that (5) holds and, moreover,

$$P_1(A) \approx 1. \quad (6)$$

We claim that

$$P_0(A) = E_1(1_A(V^n)L(V^n)) \approx \exp\{-nH(P_1)\}. \quad (7)$$

To see the first equality, note that for any function $f(V^n)$ one has

$$E_0(f(V^n)) = \sum_{\mathbf{v}} P_0(\mathbf{v})f(\mathbf{v}) = \sum_{\mathbf{v}} P_1(\mathbf{v})\frac{P_0(\mathbf{v})}{P_1(\mathbf{v})}f(\mathbf{v})$$
$$= \sum_{\mathbf{v}} P_1(\mathbf{v})L(\mathbf{v})f(\mathbf{v}) = E_1(f(V^n)L(V^n)).$$

To get the approximation in (7), we use (5) and (6).

This calculation shows that the likelihood that the Markov chain V_n with transition matrix P_0 behaves as if its transition matrix were P_1 for n steps is exponentially small in n and given by the expression (7).

The next step is to estimate the likelihood $\kappa(\pi_1, n)$ that the empirical distribution of $\{V_1, \ldots, V_n\}$ is π_1. One can use the contraction principle (see e.g., [4] and [3]) to argue that this likelihood is the maximum over P_1 of the probability that the Markov chain behaves as if its transition matrix were P_1, where the maximum is over all P_1 with empirical distribution π_1. Hence, one finds that

$$\kappa(\pi_1, n) = \max_{P_1:\pi_1 P_1=\pi_1} \exp\{-nH(P_1)\} \approx \exp\{-nR(\pi_1)\}$$

where

$$R(\pi_1) := \inf_{P_1:\pi_1 P_1=\pi_1} H(P_1)$$

with $H(P_1)$ as given above.

Now, consider the likelihood that the empirical average value of $\{Z_1, \ldots, Z_n\}$ is $c > 0$, where

$$Z_m = g(V_m), m = 0, 1, \ldots, n.$$

One argues that this likelihood is the maximum of the probabilities that V_n has an empirical distribution π_1, where the maximum is over all π_1 such that

$$\sum_v \pi_1(v)g(v) = c.$$

Thus, this probability is estimated as $\exp\{-n\phi(c, q)\}$ where

$$\phi(c, q) = \min_{\pi_1:\sum_v \pi_1(v)g(v)=c} R(\pi_1). \quad (8)$$

Finally, one argues that the likelihood that the battery discharges is of the order of

$$\exp\{-B\psi(q)\}$$

where

$$\psi(q) = \min_{c>0} \frac{\psi(c, q)}{c}. \quad (9)$$

Gaussian Approximation

The Gaussian approximation considers that

$$Z_1 + \cdots + Z_n \approx \mathcal{N}(n\alpha, n\sigma^2),$$

where $\alpha = E(Z_n)$ is as before and $n\sigma^2 \approx \text{var}(Z_1 + \cdots + Z_n)$.

As we will see below, this approximation is not satisfactory.

5. EVALUATION

We have explained three methods for estimating the likelihood that the battery gets discharged: a direct method, a method based on the occupation measure of the Markov chain, and a Gaussian approximation. In the following subsections, we evaluate these methods for a random walk and two-state Markov chain.

5.1 Evaluation for Random Walk

Let Z_n be i.i.d. with $P(Z_n = 1) = a$ and $P(Z_n = -1) = 1 - a =: b$. We assume that $E(Z_n) = a - b = 2a - 1 < 0$, so that the battery tends to charge more than it discharges. We consider the Markov chain W_n defined by

$$W_{n+1} = (W_n + Z_n)^+, n \geq 0.$$

This is a random walk reflected at 0 that models the discharge process of the battery. The state X_n of charge of the battery can be seen to be essentially $B - W_n$, so that if W_n reaches the value B, the battery gets discharged.

Direct Method

The reflected random walk W_n is a simple Markov chain on $\{0, 1, \ldots\}$ with

$$P(k, k+1) = a \text{ and } P(k+1, k) = b, \forall k \geq 0.$$

Also, $P(0, 0) = b$. We can analyze explicitly this Markov chain without having to resort to Chernoff's bound. If $a < b$, the invariant distribution of X_n is π where

$$\pi(k) = (1-\rho)\rho^k, k \geq 0 \text{ with } \rho := \frac{a}{b}.$$

In particular,

$$P(W_n \geq B) = \sum_{k=B}^{\infty} \pi(k) = \rho^B =: p_Q(B). \qquad (10)$$

Occupation Measure

Using (7), we find that the likelihood that the increments Z_n behave as if $P(Z_n = 1) = a'$ instead of a over n steps is approximately

$$\phi(a') := \exp\{-nH(a')\}$$

where

$$H(a') = -a' \log(\frac{a}{a'}) - (1-a') \log(\frac{1-a}{1-a'}).$$

Thus, according to (8),

$$\phi(c) := \min\{H(a') | E_{a'}(Z_n) = a' - (1-a') \geq c\} = H(\frac{1+c}{2}).$$

Hence, by (9),

$$\psi_O := \inf_{c>0} \frac{\phi(c)}{c} = \log(\frac{1-a}{a}).$$

Finally, we get the estimate for the probability that the battery gets empty as

$$\exp\{-B\psi_O\} = (\frac{a}{1-a})^B,$$

which agrees with (10).

Gaussian Approximation

A Gaussian approximation for this process would work as follows. We argue that for $n \gg 1$,

$$\frac{Z_1 + \cdots + Z_n - n\alpha}{\sqrt{n}} \approx \mathcal{N}(0, \sigma^2)$$

where

$$\sigma^2 = \text{var}(Z_n) = E(Z_n^2) - (E(Z_n))^2 = 1 - \alpha^2.$$

Recall that if W is $\mathcal{N}(0, 1)$, then

$$P(W > x) \leq \frac{1}{x\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}, \forall x > 0.$$

Moreover, this upper bound on the error function is asymptotically tight. Thus, if $V = Z_1 + \cdots + Z_n - n\alpha$, one uses the (poor) approximation $V =_D \sqrt{n\sigma^2} W$, so that

$$P(V > na) = P(W > \frac{na}{\sqrt{n\sigma^2}}) = P(W > \sqrt{n}\frac{a}{\sigma})$$
$$\leq \frac{\sigma}{a\sqrt{n}} \exp\{-n\frac{a^2}{2\sigma^2}\}.$$

Note that this approximation is a bad application of the Central Limit Theorem. Using this approximation, we get

$$P(Z_1 + \cdots + Z_n > n(\alpha + a)) \approx \frac{\sigma}{a\sqrt{n}} \exp\{-n\frac{a^2}{2\sigma^2}\}.$$

This leads to the probability of the battery going empty being of the order of

$$\exp\{-B\psi\}$$

where

$$\psi = \inf_{a: a+\alpha > 0} \frac{1}{a+\alpha} \frac{a^2}{2\sigma^2} = -\frac{2\alpha}{\sigma^2},$$

which gives the following estimate for the probability that the battery goes empty:

$$\exp\{B\frac{2\alpha}{\sigma^2}\} = \exp\{B\frac{2\alpha}{1-\alpha^2}\} =: p_G(B). \qquad (11)$$

Thus, the correct expression is given by (10) and the Gaussian approximation is given (11). Note that

$$\frac{1}{B}\log(p_Q(B)) = \log(\frac{a}{1-a})$$

and

$$\frac{1}{B}\log(p_G(B)) = \frac{2\alpha}{\sigma^2} = \frac{2a-1}{2a(1-a)}.$$

Figure 1 compares these expressions as functions of a. We note that the Gaussian approximation is not very good. This is to be expected since one knows that the Central Limit Theorem provides good estimates of the probability

$$P(Z_1 + \cdots + Z_n > \alpha n + \delta\sqrt{n}),$$

but not of

$$P(Z_1 + \cdots + Z_n > \alpha n + (c - \alpha)n).$$

5.2 Evaluation for 2-state Markov Chain

Next, we compare and validate the estimates obtained by the direct method and from the occupation measure in the case of a $\{-1, 1\}$-Markov chain Z_n with $P(-1, 1) = a$ and

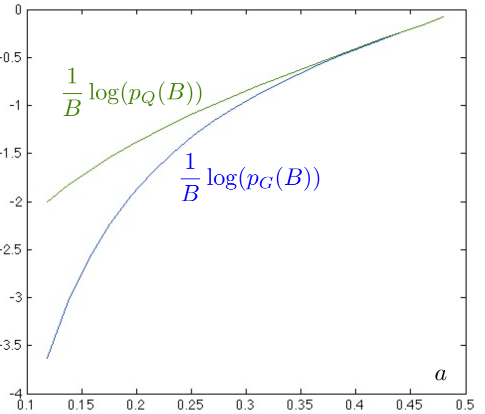

Figure 1: Comparison of (10) and (11). The Gaussian approximation underestimates the probability of large deviations.

$P(1,-1) = b$. The goal is to estimate the probability that the process

$$Z_1 + \cdots + Z_n$$

reaches some large value B. This probability, say $p(B)$ is of the order of $\exp\{-B\psi\}$. We will derive three estimates for ψ: ψ_D, ψ_O and ψ_G using the three methods.

Direct Method for two-state Markov Chain

We find

$$G_\theta = \left[\begin{array}{cc} e^{-\theta}(1-a) & e^{-\theta}a \\ e^{\theta}b & e^{\theta}(1-b) \end{array} \right].$$

We can then evaluate the largest eigenvalue $\lambda(\theta)$ of G_θ and calculate ψ_D using (3).

Occupation Measure

We use (7), (8) and (9) for the two-state Markov chain and we find that the probability of the battery going empty is

$$\exp\{-B\psi_O\}$$

where

$$\psi_O = \inf_{c>0}(\min_{\{P_1 : E_1(Z_n)=c\}} H(a', b'))$$

with

$$H(a',b') = -\frac{a'}{a'+b'}[(1-a')\log(\frac{1-a}{1-a'}) + a'\log(\frac{a}{a'})] - \frac{b'}{a'+b'}[(1-b')\log(\frac{1-b}{1-b'}) + b'\log(\frac{b}{b'})].$$

Occupation Measure vs. Simulations

We compare $p(B)$ measured from simulations to the estimates given by the occupation measure approach.

Figure 2 shows representative results measured by simulating X_n for 10^6 steps for every value of a. The loss rate calculates as the number of times that the battery goes empty over the number of steps (for example here is 10^6). The estimate is based on the large deviation of the occupation measure as explained above.

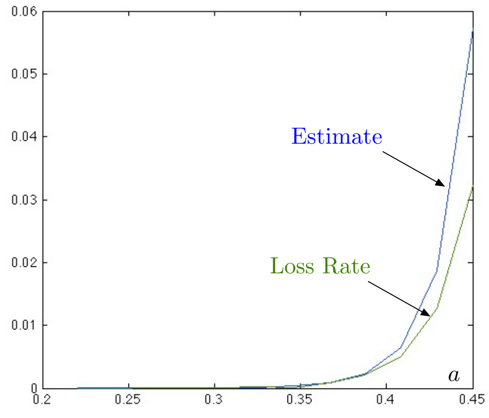

Figure 2: Comparison of actual loss rate and estimate. Here, $b = 0.5$ and $B = 30$.

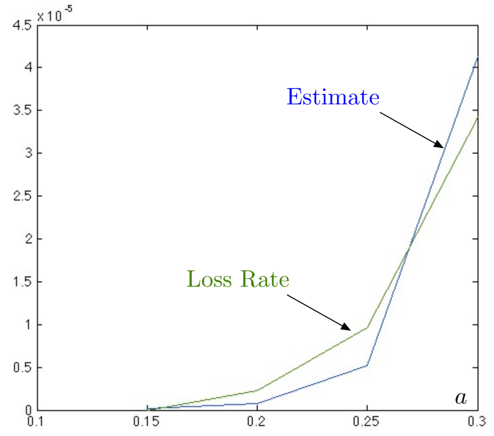

Figure 3: Comparison of actual loss rate and estimate for smaller values of a. Here, $b = 0.5$ and $B = 30$.

Figure 3 shows more results for smaller values of a. Here, the loss rate is measured by simulating X_n for 10^8 for every value of a.

Gaussian Approximation

For this Markov chain, one finds that (see Appendix)

$$\sigma^2 := cd\frac{2-a-b}{a+b} \text{ with } c = \frac{a}{a+b}, d = 1-c.$$

This gives the estimate

$$\exp\{B\frac{2\alpha}{\sigma^2}\} = \exp\{-B\frac{2(b-a)(a+b)^2}{ab(2-a-b)}\}.$$

Comparison

Figure 4 compares the values of ψ for the probability

$$\exp\{-B\psi\}$$

that the battery becomes empty derived using the three methods. The values are shown for $b = 0.5$ and as a function of $a < b$. As in the case of the random walk, we find that the

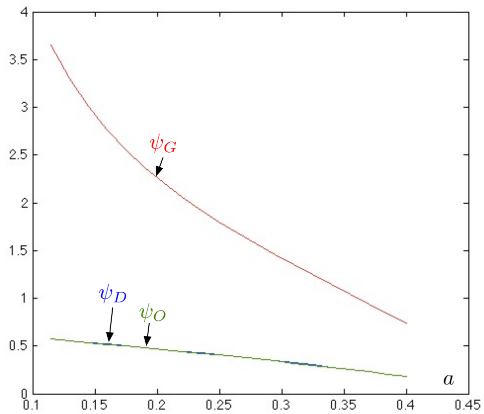

Figure 4: Comparison of estimates with occupation measure, direct method and Gaussian approximation. As before, $b = 0.5$ and $B = 30$.

Gaussian approximation yields poor estimates, which should not be surprising.

6. INDEPENDENT SOURCE AND LOAD

In this section we consider the case where $Y_n = (Y_n^1, Y_n^2)$ and

$$g(Y_n, U_n) = -a(Y_n^1) + b(Y_n^2, U_n).$$

Here, the Markov chains Y_n^1 and Y_n^2 are independent.

For instance, Y_n^1 models the weather that affects the charging rate $a(Y_n^1)$ of the battery and Y_n^2 models the quality of a transmission channel, which affects the reward of transmitting with a given power. We assume that the control policy is defined by q_0 where

$$P[U_n = u | Y_n^1 = y_1, Y_n^2 = y_2] = q_0(y_2, u).$$

The empirical average value of $g(Y_n, U_n)$ differs from its expected value if Y_n^1, Y_n^2 and U_n given Y_n^2 make large deviations. The likelihood of a large deviation where Y_n^1 behaves as if its transition matrix were P^1 instead of P_0^1, Y_n^2 as if its transition matrix were P^2 instead of P_0^2 and U_n given Y_n^2 behaves as it its condition distribution were q instead of q_0 is exponentially small with exponent

$$H(P^1) + H(P^2) + K[q|\pi_2]$$

where

$$H(P^1) = -\sum_{y_1, y_1'} \pi_1(y_1) P^1(y_1, y_1') \log\left(\frac{P_0^1(y_1, y_1')}{P^1(y_1, y_1')}\right)$$

$$H(P^2) = -\sum_{y_2, y_2'} \pi_2(y_2) P^2(y_2, y_2') \log\left(\frac{P_0^2(y_2, y_2')}{P^2(y_2, y_2')}\right)$$

$$K[q|\pi_2] = -\sum_{y_2, u} \pi_2(y_2) q(y_2, u) \log\left(\frac{q_0(y_2, u)}{q(y_2, u)}\right).$$

In these expressions, π_1 is invariant for P^1 and π_2 is invariant for P^2. Thus, the empirical rate of $a(Y_n^1)$ is some value a and the empirical rate of $g(Y_n^2, U_n)$ is some value b with an exponentially small probability with an exponent

$$\phi_1(a) + \phi_2(b).$$

The empirical drain rate of the battery is then $b - a$.

Claim 1. *The likelihood that a battery of size B drains is exponentially small in B with an exponent*

$$\inf_{b>a} \frac{\phi_1(a) + \phi_2(b)}{b - a}.$$

Proof. Assume that there is some value of c such that, for all $a > c$ and $b < c$,

$$\frac{\phi_1(a)}{a - c} \geq \gamma \text{ and } \frac{\phi_2(b)}{c - b} \geq \gamma.$$

Then

$$\phi_1(a) \geq \gamma(a - c) \text{ and } \phi_2(b) \geq \gamma(c - b),$$

so that

$$\frac{\phi_1(a) + \phi_2(b)}{a - b} \geq \gamma.$$

The interpretation of this result is as follows. Assume that there is some constant rate c such that if the battery drains at rate c, its likelihood of getting empty has an exponent γ and also that if the battery recharges at rate c, then the likelihood that the load makes it go empty also has an exponent γ. Then, the combined system with variable charging and discharging rate has rate at least γ. □

A converse of that result is as follows.

Claim 2. *Assume that the combined system has an exponent γ. Then there is some rate c such that each of the two decoupled systems has an exponent γ.*

Proof. To see this, let a^* and b^* be the minimizers of

$$\frac{\phi_1(a) + \phi_2(b)}{b - a}$$

and let γ be the minimum value. The first order conditions are

$$\phi_1'(a^*) = -\phi_2'(b^*) = \gamma.$$

Now, choose c so that

$$\frac{\phi_1(a^*)}{c - a^*} = \gamma.$$

Then we see that

$$\phi_1'(a^*)(c - a^*) = \phi_1(a^*),$$

so that a^* minimizes

$$\frac{\phi_1(a)}{a - c}$$

and the minimum is γ. Similarly, b^* minimizes

$$\frac{\psi_2(b)}{c - b}$$

and the minimum is also γ, which proves the claim. □

7. EXAMPLES

To clarify the analysis, we consider a few simple examples.

No Control

In our first example, $Y_n \in \{0,1\}$ with $P(0,1) = a_0, P(1,0) = b_0, g(0) = -1, g(1) = 1$. We also assume that $U_n = Y_n$, so that there is no randomization of the control. Finally, assume that $a_0 < b_0$, so that

$$E(g(U_n)) = E(g(Y_n)) = \frac{a_0 - b_0}{a_0 + b_0} < 0.$$

Using the occupation method approach, we note that a transition matrix $P(0,1) = a$ and $P'(1,0) = b$ is such that $E(Y_n) = c$ if

$$b = a\frac{1-c}{1+c}.$$

Substituting this value of b in $H(P)$ and minimizing over a, we find

$$\phi(c) = \min_a H(P).$$

We then minimize $\phi(c)/c$ over c. The results is ψ and the likelihood that the battery goes empty is

$$\exp\{-\psi B\}.$$

Numerical examples give the values of ψ, in terms of a and b, shown in Table 1.

a_0	b_0	ψ_O	ψ_D
0.1	0.15	0.057	0.067
0.2	0.3	0.134	0.155
0.3	0.45	0.241	0.282
0.4	0.6	0.406	0.472
0.2	0.4	0.288	0.288
0.3	0.6	0.560	0.561
0.4	0.8	1.099	1.101

Table 1: Values of ψ when $U_n = Y_n$ obtained using the occupation measure (ψ_O) and the direct method (ψ_D)

This table shows that the battery is less likely to get empty (ψ is larger) when b_0 increases or a_0 decreases. Moreover, that is also the case if a_0 and b_0 increase, for a given value of a_0/b_0. Thus, for a given value of $E(g(Y_n))$, the battery is less likely to get empty if Y_n changes faster instead of staying equal to 1 for longer periods of time. This results confirm our intuition.

Using the direct method, we consider the matrix

$$G_\theta(y,y') = e^{\theta y} P(y,y')$$

and define $\lambda(\theta)$ to be its largest eigenvalue. Then

$$\psi_D = \min_{c>0} \frac{1}{c} \sup_{\theta>0} [\theta c - \log(\lambda(\theta))].$$

Control

We now consider the same situation as in the previous example, except that

$$P[U_n = 1|Y_n = 1] = \gamma_0 \text{ and } P[U_n = 1|Y_n = 0] = 0.$$

As a concrete example, say that $a_0 = 0.2$ and $b_0 = 0.3$. We saw that $\psi = 0.134$ if $U_n = Y_n$. This corresponds to a probability of a battery of size 20 going empty that is of the order of

$$\exp\{-20 \times 0.134\} = \exp\{-2.5\} = 0.07,$$

which is not acceptable. Thus, it makes sense to choose the value $U_n = 1$ only a fraction γ_0 of the time that $Y_n = 1$.

A large deviation of $g(U_n)$ occurs when its empirical mean value c is different from its expected value

$$\begin{aligned} E(g(U_n)) &= \gamma_0 P(Y_n = 1) - (1-\gamma_0)P(Y_n = 1) - P(Y_n = 0) \\ &= \frac{2a_0\gamma_0}{a_0 + b_0} - 1. \end{aligned}$$

This can occur as a combination of two events: Y_n can be equal to 1 a fraction of time $\pi(1)$ that differs from $a_0/(a_0 + b_0)$ and the fraction of time that $U_n = 1$ when $Y_n = 1$ can be γ instead of γ_0.

Using the occupation method approach, the resulting empirical mean value of $g(U_n)$ is then

$$\frac{2a\gamma}{a+b} - 1$$

with a probability that is of the order of

$$\exp\{-nH(P) - nK[\gamma|P]\}$$

where $H(P)$ is as before and

$$K[\gamma|P] = -\pi(1)\gamma \log\left(\frac{\gamma_0}{\gamma}\right) - \pi(1)(1-\gamma)\log\left(\frac{1-\gamma_0}{1-\gamma}\right)$$

Using the direct method, one calculates ψ_D from (3). Table 2 shows some numerical results that again confirm the intuition.

a_0	b_0	γ_0	ψ_O	ψ_D
0.1	0.15	0.9	0.096	0.101
0.1	0.15	0.8	0.152	0.155
0.1	0.15	0.6	0.360	0.361
0.2	0.3	0.9	0.215	0.225
0.2	0.3	0.6	0.608	0.607

Table 2: Values of ψ when $P[U_n = 1|Y_n = 1] = \gamma_0$

Optimization

The setup is the same as in the previous example. However, in this example we want to choose γ_0 to maximize

$$E(r(Y_n, U_n))$$

subject to

$$P(W_n = 0) \approx \beta.$$

Assume that $r(0,u) = r(y,0) = 0$ and $r(1,1) = 1$. Thus, we want to maximize γ_0 such that $\psi \geq \beta/B$. The goal is to have a probability of the battery going empty of the order of $\exp\{-\beta\}$.

Say that $\beta = 4.6$, so that $\exp\{-\beta\} = 1\%$. Then, we find the results shown in Table 3 for $a = 0.1$ and $b = 0.15$. (We used the direct method.)

Not surprisingly, if the battery is smaller, one has to be more cautious in using it.

Wireless Sensor Node

Figure 5 illustrates the power flow in a wireless sensor node. The node is equipped with a solar cell that generates a variable amount of power, depending on the state of the weather. Here, for the purpose of illustration, we think of the time

B	γ_0
50	0.92
40	0.87
30	0.80
20	0.70
10	0.54

Table 3: Values of γ_0 for optimization problem

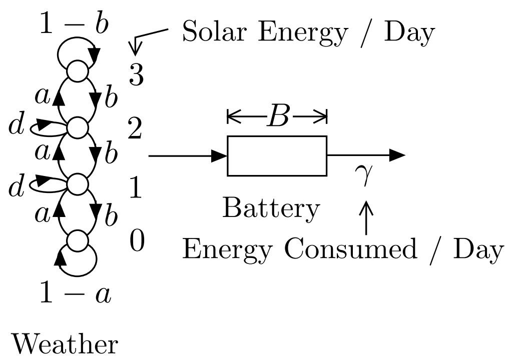

Figure 5: A wireless sensor node equipped with a solar cell.

unit being one day. The system is designed to transmit an amount of energy equal to γ per day. The problem is to determine the maximum value of γ such that the probability that the battery goes empty is about 1%.

We use the direct method, with the model that

$$P[U_n = 1|Y_n = y] = \gamma \text{ and } P[U_n = 0|Y_n = y] = 1 - \gamma.$$

Let $Y_n \in \{0, 1, 2, 3\}$ be the Markov chain that represents the weather. In the figure, $d := 1 - a - b$. The increment in the battery discharge is then

$$Z_n = V_n - Y_n,$$

where the V_n are i.i.d. Bernoulli with mean γ and are independent of the weather.

Using the direct method, we let

$$s_n(y) = E[\exp\{\theta(Z_1 + \cdots + Z_n)\}|Y_1 = y]$$

and we find that

$$\mathbf{s}_{n+1} = G_\theta \mathbf{s}_n$$

where

$$G_\theta(y, y') = h(y)P(y, y')$$

with

$$h(y) = E(\exp\{\theta(V_1 - y)\}) = [\gamma e^\theta + (1 - \gamma)]e^{-y\theta}.$$

We calculate the largest eigenvalue of G_θ then proceed as before, by using (3). Figure 6 shows the exponential rate of decay $\psi(\gamma)$ as a function of γ for relatively sunny and cloudy weathers.

From these curves, one can determine the maximum value of the usage of the sensor node described by γ as a function of the target error probability and of the battery size.

Building with Solar Panels and Variable Load

Figure 7 sketches the power flow of a building with a solar cell, a battery, and a variable load. The control parameter

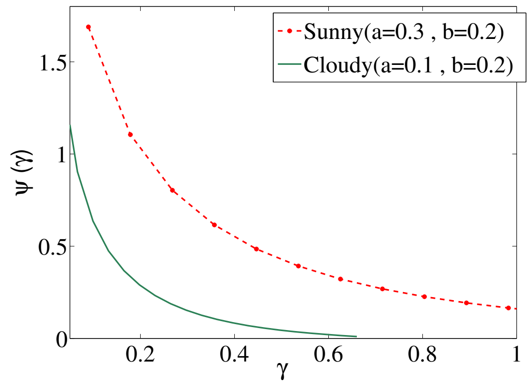

Figure 6: Exponential rate of decay as a function of γ in cloudy and sunny environments.

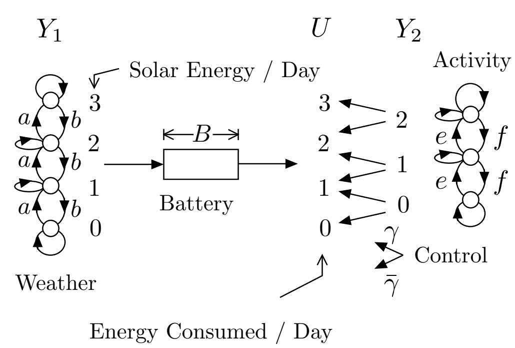

Figure 7: A self-sufficient building with a control parameter γ.

γ is the probability of using a higher rate instead of a lower one, given the level of activity in the building and $\bar{\gamma} = 1 - \gamma$. The problem is to determine the largest possible value of γ so that the probability that the battery gets depleted is acceptably small. As before, we compute $\psi(\gamma)$. The one-day depletion of the battery is

$$Z(n) = U(n) - Y_1(n).$$

As before, we find

$$\mathbf{s}_{n+1} = G_\theta \mathbf{s}_n$$

where

$$G_\theta(y, y') = h(y)P(y, y'),$$

where

$$\begin{aligned} h(y) &= E[\exp\{\theta(U(1) - y_1)\}|Y_2(1) = y_2] \\ &= e^{\theta(y_2 - y_1)}[\gamma e^\theta + 1 - \gamma]. \end{aligned}$$

Figure 8 shows the numerical results.

8. CONCLUSIONS

This paper explored a methodology for addressing the variability of renewable energy and the electric load in the control of systems with energy storage. The main idea is to replace a Markov decision problem formulation by an opti-

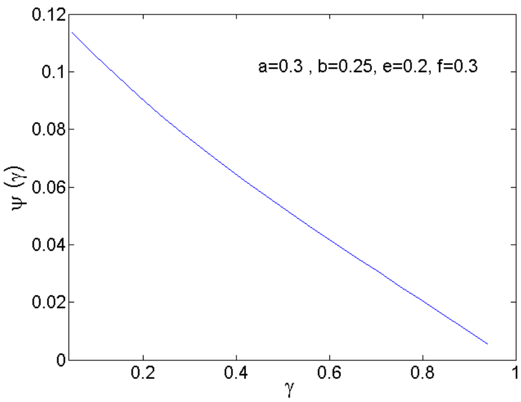

Figure 8: The numerical result for the building model.

mization problem with constraints based on the theory of large deviations.

We compared three methods for evaluating the small probability that the battery goes empty for a given control policy. These methods use the fact that the battery discharge increments are functions of a Markov chain. The three methods are: 1) a direct method based on Chernoff's inequality and the first step equations of a Markov chain; 2) a method based on the analysis of the occupation measure and the contraction principle; 3) a Gaussian approximation method.

Our examples indicate that the direct method and the occupation measure yields essentially the same estimates, but that the first approach is numerically simpler. The examples confirm that the Gaussian approximation usually yields poor estimates that are not satisfactory to address the optimization problems.

Using the occupation measure approach, we could derive properties of the large deviations. We showed a decomposition result when the source and the load are functions of independent Markov chains.

We demonstrated the use of the approach for two simple problems: a wireless sensor node equipped with a solar panel and a self-sufficient building. The methodology applies to much more complex situations. The benefit is that the resulting control law is simple, as it does not depend on the instantaneous charge of the battery.

9. ACKNOWLEDGEMENTS

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10. APPENDIX A: STATEMENT IN DIRECT METHOD

This is to show:

$$s_{n+1}(y) = E[\exp\{\theta Z_1\}|Y_1 = y] \sum_{y'} P(y, y') s_n y', \forall y \in \mathcal{Y}.$$

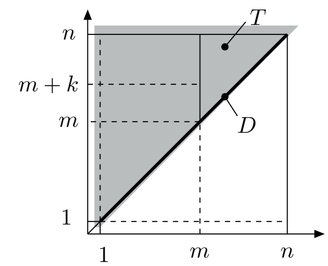

Figure 9: The sum over $(m, n) \in \{1, \ldots, n\}^2$ is decomposed into twice the sum over T minus the sum over D because the terms are symmetric in (m, n).

Starting from LHS, we have,

$$\begin{aligned}
s_{n+1}(y) &= E[\exp\{\theta(Z_1 + \cdots + Z_{n+1})\}|Y_1 = y],\\
&= E[\exp\{\theta Z_1\}.\exp\{\theta(Z_2 + \cdots + Z_{n+1})\}|Y_1 = y],\\
E[\exp\{\theta(Z_2 &+ \cdots + Z_{n+1})\}|Y_1 = y]\\
&= \sum_{y' \in \mathcal{Y}} E[\exp\{\theta(Z_2 + \cdots + Z_{n+1})\}\mathbf{1}\{Y_2 = y'\}|Y_1 = y],\\
&= \sum_{y' \in \mathcal{Y}} E[\exp\{\theta(Z_2 + \cdots + Z_{n+1})\}|Y_2 = y']\\
&\times P[Y_2 = y'|Y_1 = y], \forall y \in \mathcal{Y}.
\end{aligned}$$

11. APPENDIX B: GAUSSIAN APPROXIMATION FOR MARKOV CHAIN

Let $\{X_n\}$ be a $\{0,1\}$-Markov chain with $P(0,1) = a$ and $P(1,0) = b$. We want to show that

$$\mathrm{var}(X_1 + \cdots + X_n) \approx n\sigma^2$$

where

$$\sigma^2 := cd(\frac{2}{a+b} - 1)$$

where $c := a/(a+b)$ and $d := 1 - c$.

We have (see Figure 9)

$$E((\sum_{m=1}^{n} X_m)^2) = 2T - D$$

where

$$T := \sum_{m=1}^{n} \sum_{k=0}^{n-m} E(X_m X_{m+k})$$

and

$$D := \sum_{m=1}^{n} E(X_m^2) = \sum_{m=1}^{n} E(X_m) = nc.$$

One can verify that

$$P^k(1,1) = c + d\lambda^k, \text{ with } \lambda = 1 - a - b.$$

Now,

$$\begin{aligned}T &= \sum_{m=1}^{n}\sum_{k=0}^{n-m} P(X_m = 1)P[X_{m+k} = 1|X_m = 1]\\ &= \sum_{m=1}^{n}\sum_{k=0}^{n-m} cP^k(1,1)\\ &= \sum_{m=1}^{n}\sum_{k=0}^{n-m} c(c + d\lambda^k)\\ &= \sum_{m=1}^{n}\sum_{k=0}^{n-m} c^2 + cd\sum_{m=1}^{n}\sum_{k=0}^{n-m}\lambda^k\\ &= c^2\frac{n^2+n}{2} + cd\sum_{m=1}^{n}(1+\lambda+\cdot+\lambda^{n-m}).\end{aligned}$$

Also,

$$\begin{aligned}\sum_{m=1}^{n}(1+\lambda+\cdots+\lambda^{n-m}) &= \sum_{m=1}^{n}\frac{1-\lambda^{n-m+1}}{1-\lambda}\\ &= \frac{n}{1-\lambda} - \frac{1}{1-\lambda}\sum_{m=1}^{n}\lambda^{n-m+1}\\ &= \frac{n}{1-\lambda} - \frac{\lambda(1-\lambda^n)}{(1-\lambda)^2}.\end{aligned}$$

Hence,

$$\begin{aligned}T &= c^2\frac{n^2+n}{2} + \frac{ncd}{1-\lambda} - \frac{cd\lambda(1-\lambda^n)}{(1-\lambda)^2}\\ &\approx c^2\frac{n^2+n}{2} + \frac{ncd}{1-\lambda}.\end{aligned}$$

Finally, we get

$$E((\sum_{m=1}^{n} X_m)^2) \approx c^2(n^2+n) + \frac{2cdn}{1-\lambda} - nc.$$

Thus,

$$\begin{aligned}\mathrm{var}(X_1+\cdots+X_n) &\approx c^2(n^2+n) + \frac{2cdn}{1-\lambda} - nc - n^2c^2\\ &= n[c^2 + \frac{2cd}{1-\lambda} - c] = ncd\frac{1+\lambda}{1-\lambda}\\ &= ncd\frac{2-a-b}{a+b},\end{aligned}$$

as we wanted to show.

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