Non-uniform B-Spline Subdivision Using Refine and Smooth

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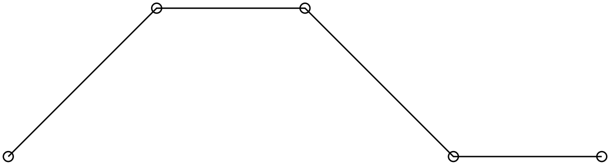
Twelfth IMA Conference on
The Mathematics of Surfaces

Outline

1. Introduction
   - Refine and smooth
   - Motivation
   - Blossoming
2. Approaches to non-uniform refine and smooth
   - Degree independent
   - Schaefer's algorithm
   - Symmetric algorithm

Refine and smooth subdivision

- Lane-Riesenfeld – uniform B-spline subdivision



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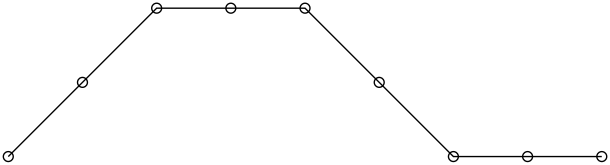

- Refine
    - polygon lengthened by adding points

Refine and smooth subdivision

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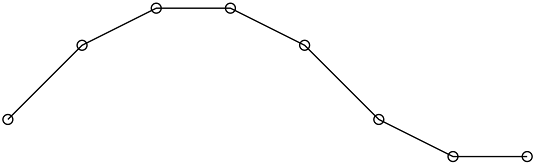
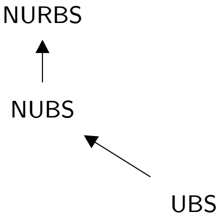


- Refine
  - polygon lengthened by adding points
- and Smooth
  - each step creates another polygon
  - points moved using local filters
- More smoothing steps for higher degree

Why non-uniform B-splines?



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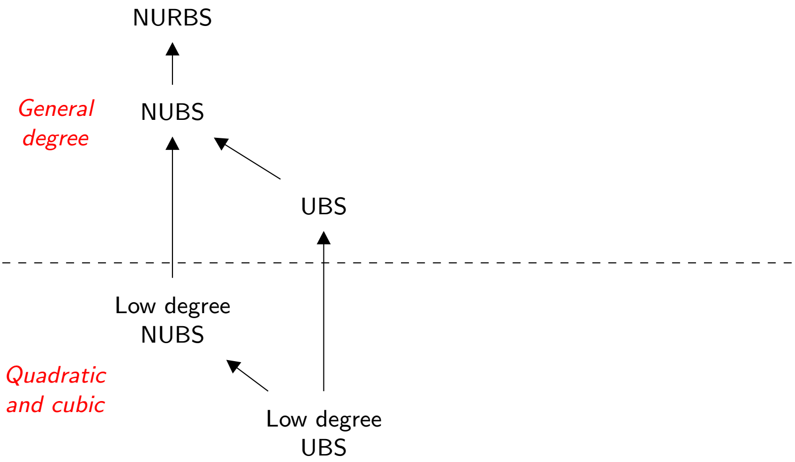

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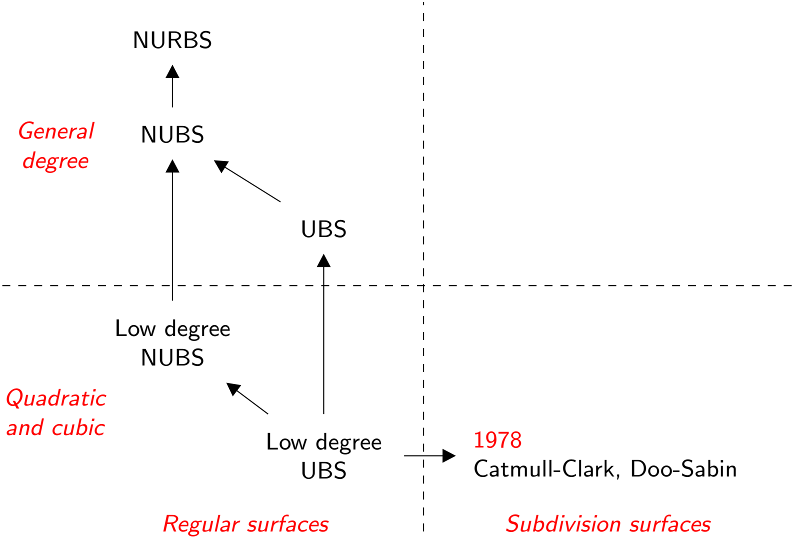

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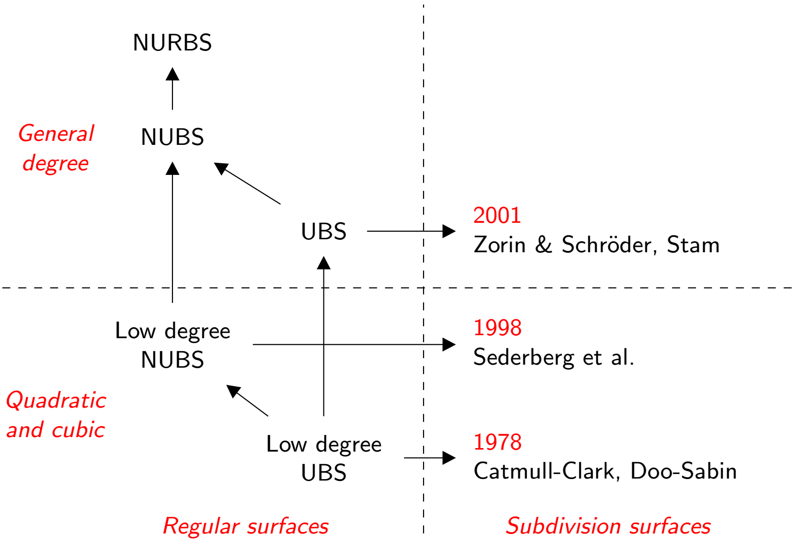

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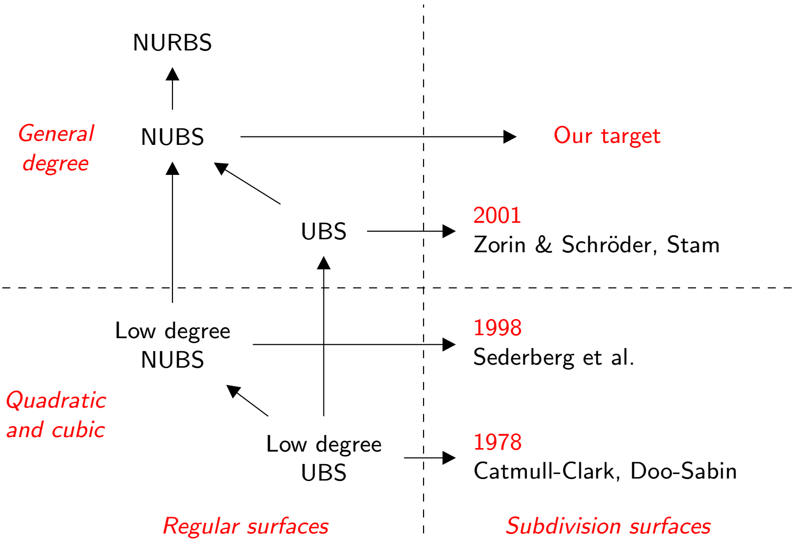

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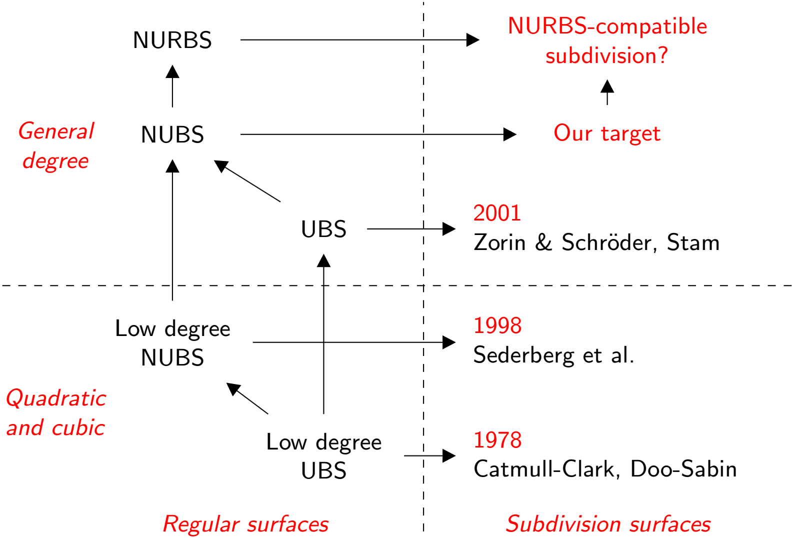

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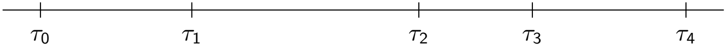
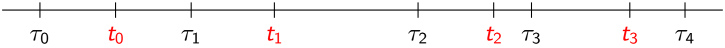
- Our aim: a knot insertion algorithm that is
    - non-uniform,
    - general degree, and uses
    - refine and smooth

Blossoming



Blossoming

Insert knots t_i: $\tau_i \leq t_i \leq \tau_{i+1}$



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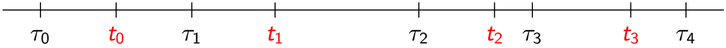

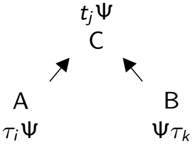

$$C = \frac{\tau_k - t_j}{\tau_k - \tau_i} A + \frac{t_j - \tau_i}{\tau_k - \tau_i} B$$

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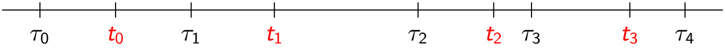

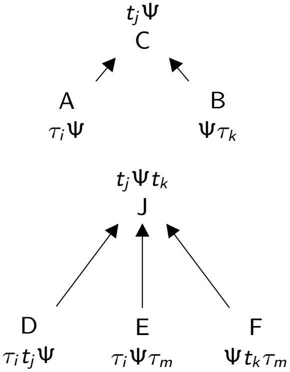

$$C = \frac{\tau_k - t_j}{\tau_k - \tau_i}A + \frac{t_j - \tau_i}{\tau_k - \tau_i}B$$

$$J = \frac{\tau_m - t_k}{\tau_m - \tau_i}D + \frac{t_k - t_j}{\tau_m - \tau_i}E + \frac{t_j - \tau_i}{\tau_m - \tau_i}F$$

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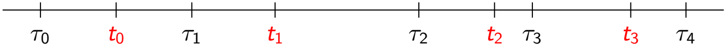

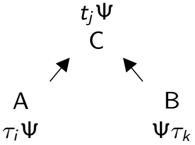

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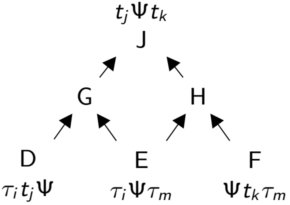

$$G = \frac{\tau_m - t_k}{\tau_m - x} D + \frac{t_k - x}{\tau_m - x} E$$

$$H = \frac{x - t_j}{x - \tau_i} E + \frac{t_j - \tau_i}{x - \tau_i} F$$

$$J = \frac{\tau_m - x}{\tau_m - \tau_i} G + \frac{x - \tau_i}{\tau_m - \tau_i} H$$

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Adapting Lane-Riesenfeld

For Lane-Riesenfeld...

- Subdivision uses one refine and multiple smooth steps
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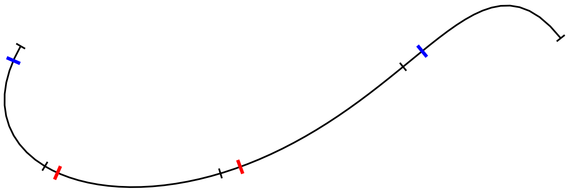
Maintaining degree independence

- But intermediate polygons that compute lower degree subdivision introduce constraints on $\boldsymbol{\tau}$ and $\mathbf{t}$



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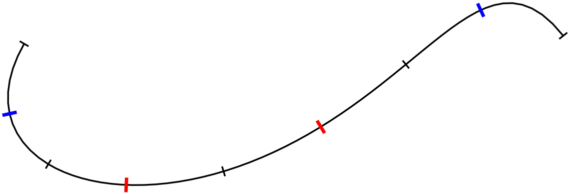

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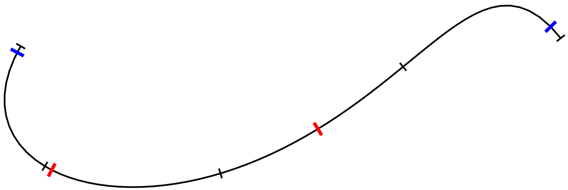

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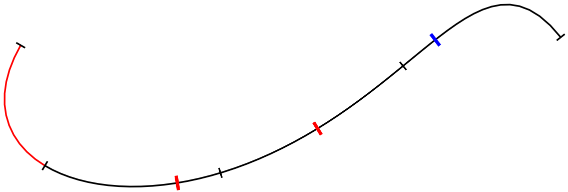

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We want a non-uniform algorithm, where. . .

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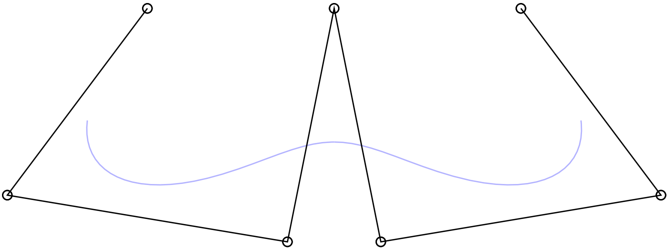

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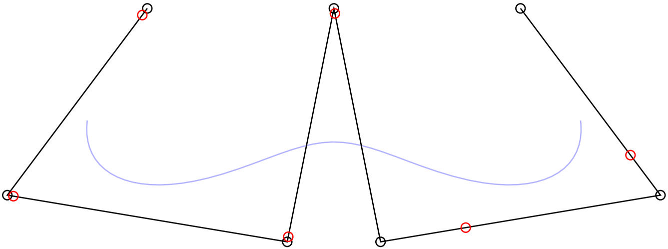

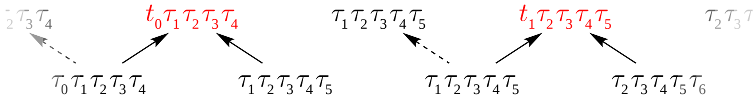
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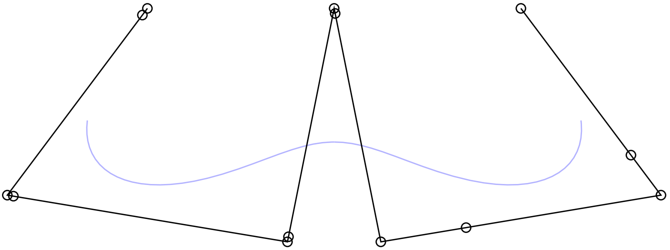

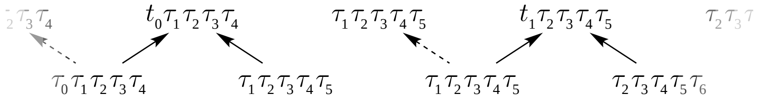
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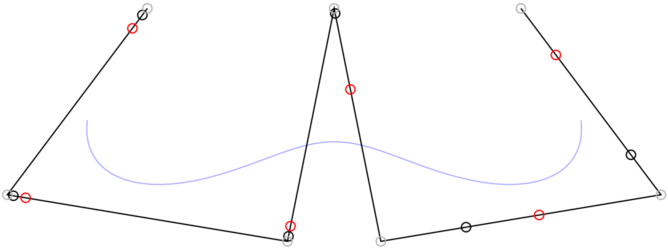

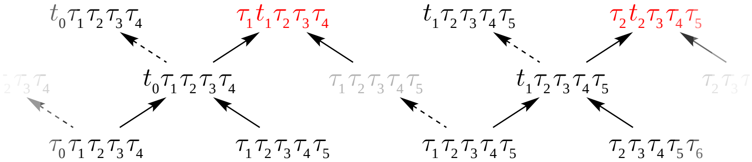
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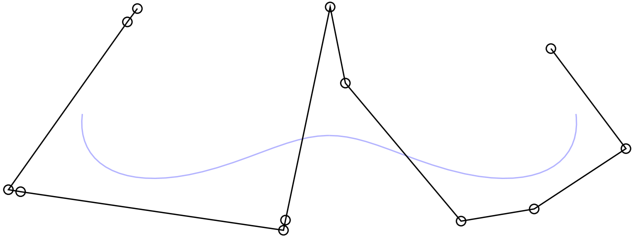

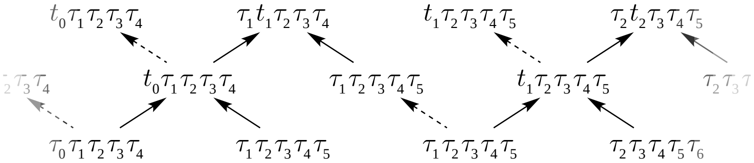
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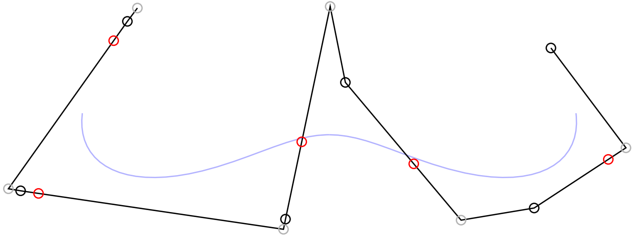

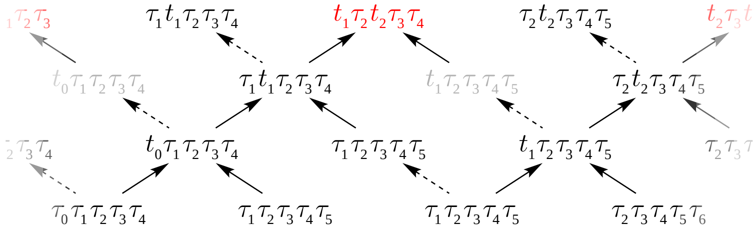
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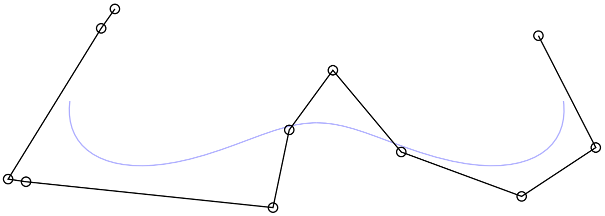

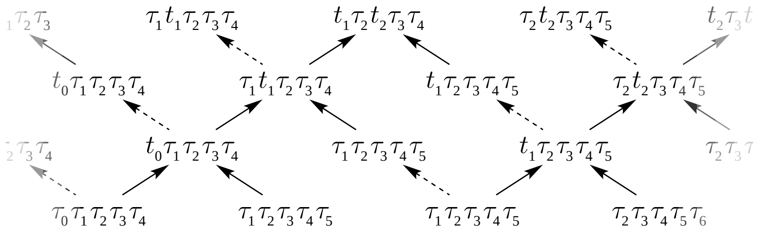

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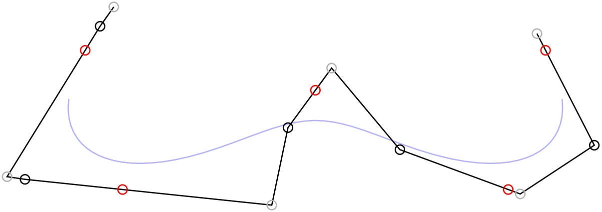

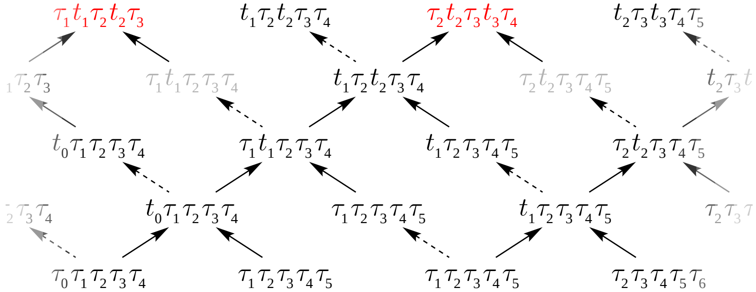
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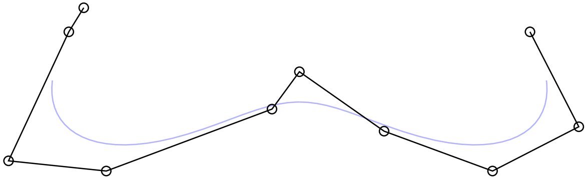

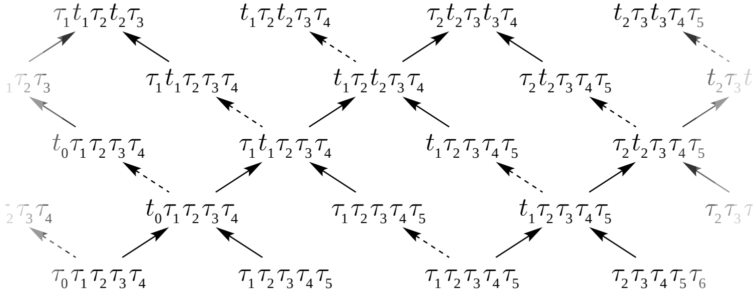
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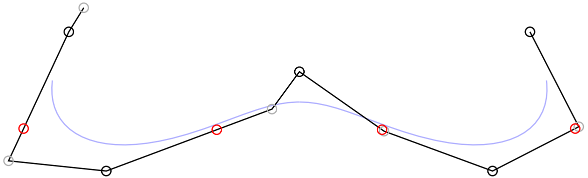

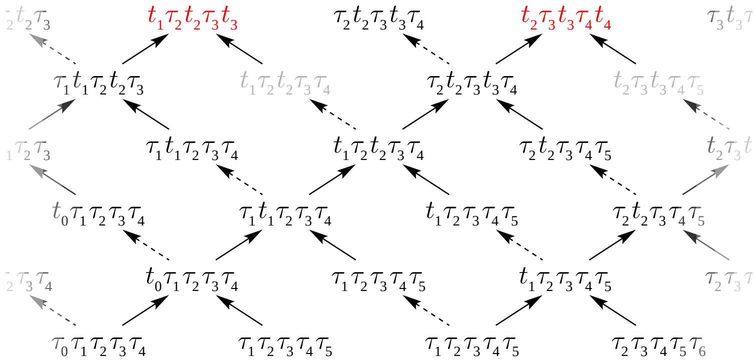
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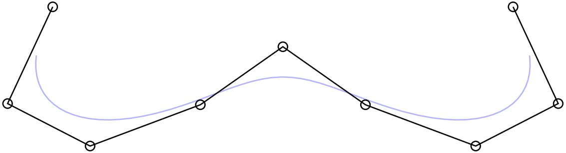

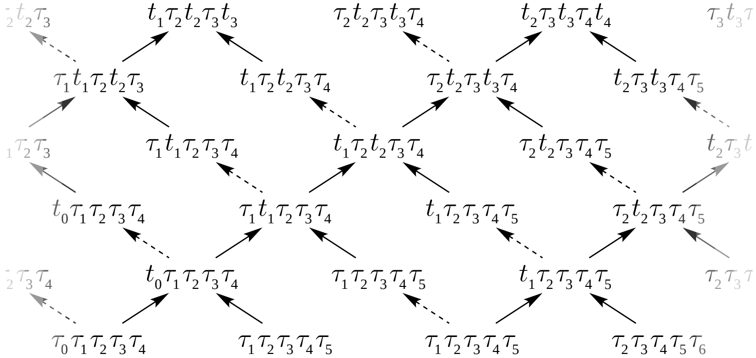
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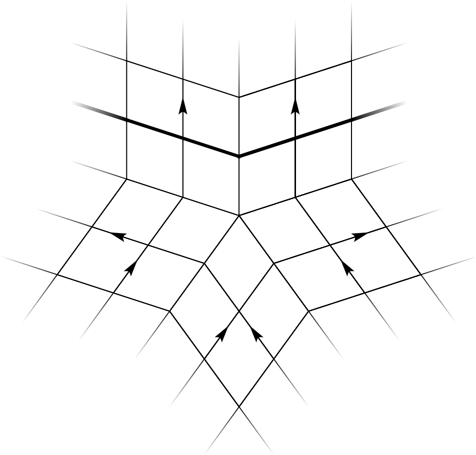
Schaefer's algorithm is asymmetric

The asymmetry in Schaefer's algorithm makes it hard to use on surfaces



Revisiting our requirements

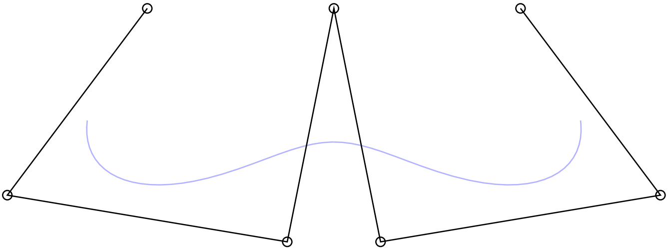
We want a non-uniform algorithm, where...

- Subdivision uses one refine and multiple smooth steps
- Smoothing filters compute a new point from two old points

Revisiting our requirements

We want a non-uniform algorithm, where...

- Subdivision uses one refine and multiple smooth steps
- Smoothing filters compute a new point from two old points
- Intermediate smoothing steps are symmetric



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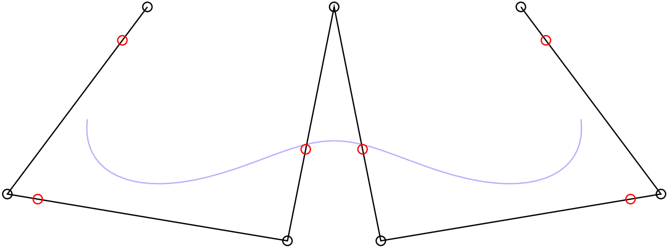

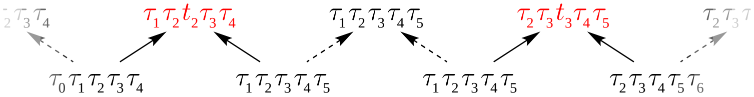
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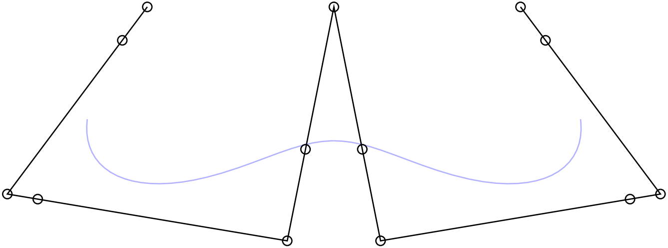

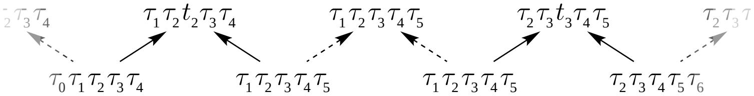
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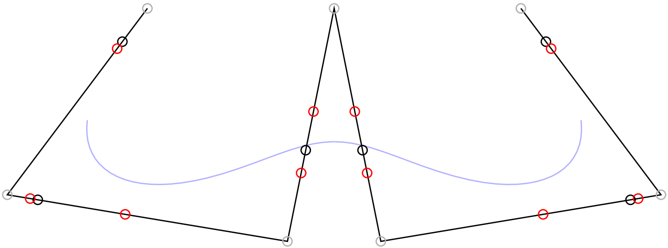

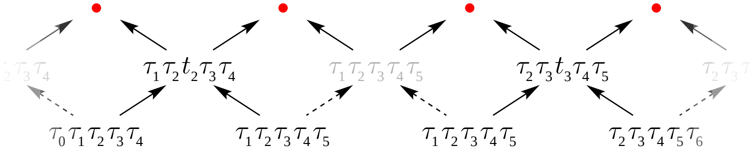
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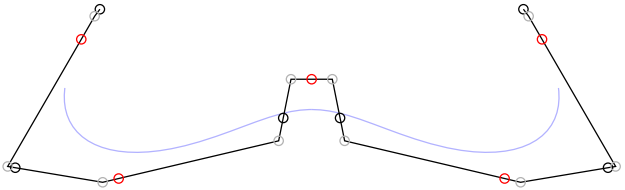

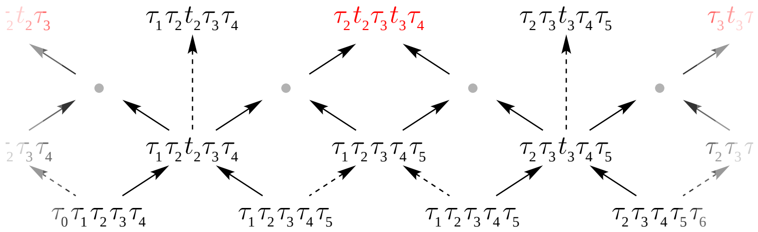
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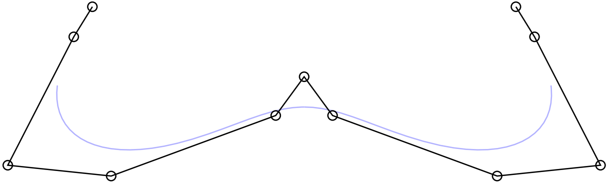

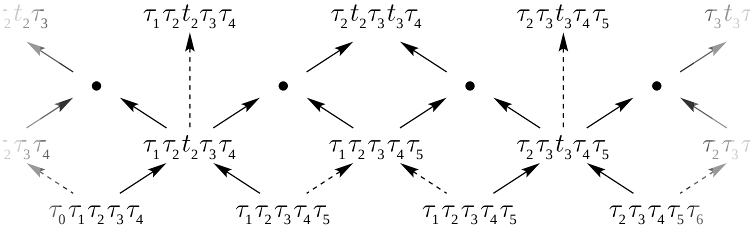
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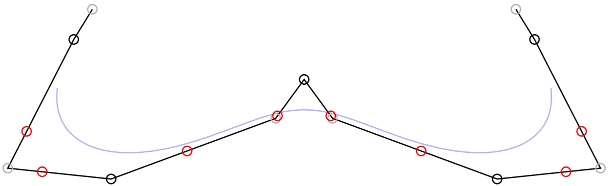

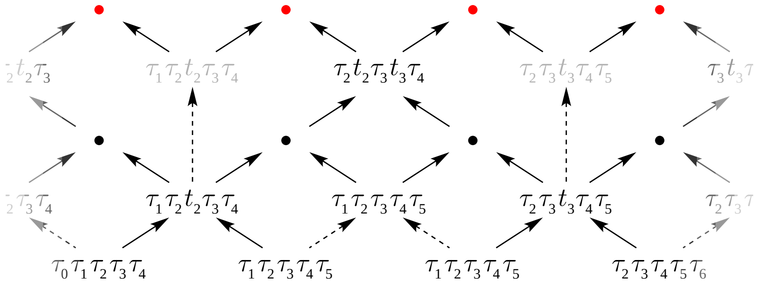
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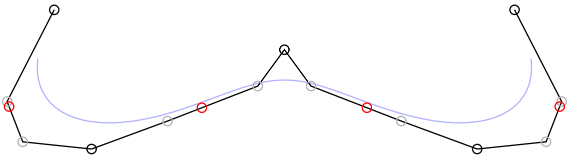

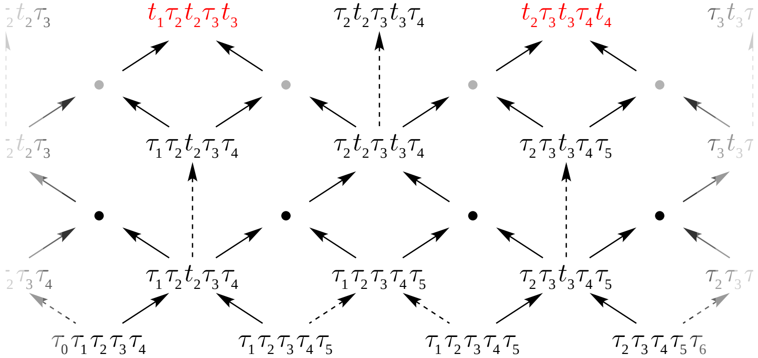
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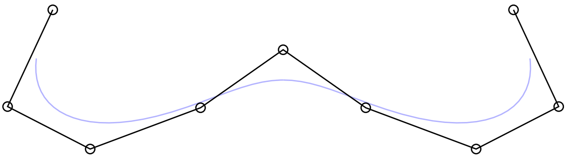

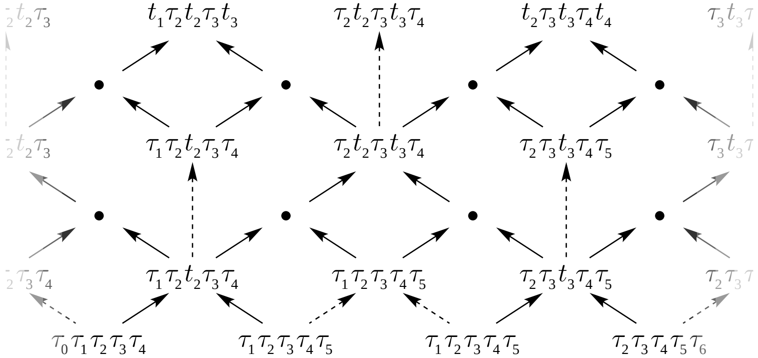
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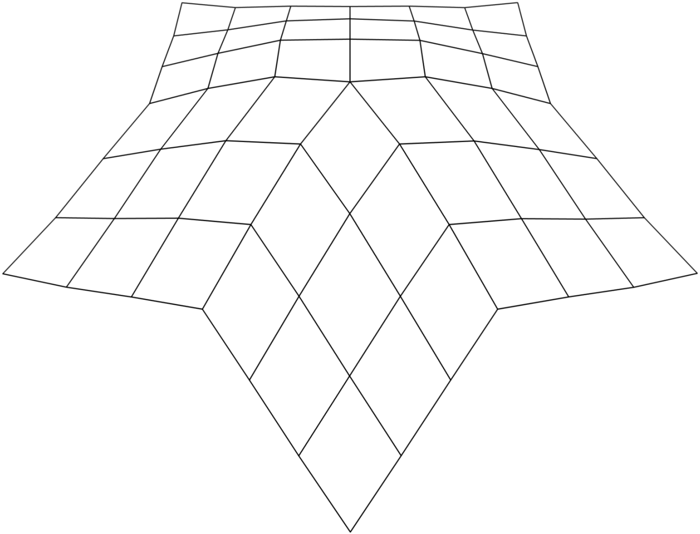
Summary

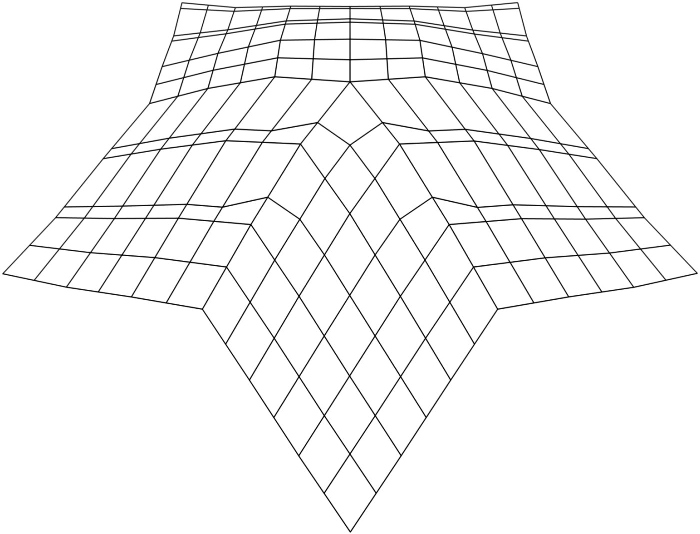
- There are non-uniform analogues of the Lane-Riesenfeld refine and smooth algorithm
- Different requirements lead to different approaches
- A symmetric algorithm may lead to subdivision schemes generalising NURBS

- Taking this work further
  - Elegantly handling multiple knots
  - Extraordinary points

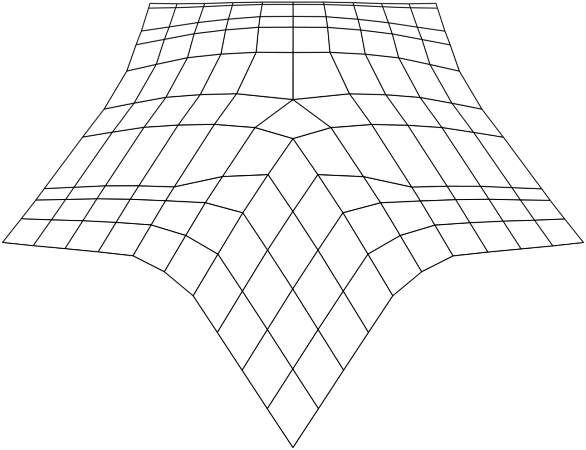
Extraordinary points



Extraordinary points



Extraordinary points



Extraordinary points