Isabelle/HOLCF — Higher-Order Logic of Computable Functions

January 18, 2026

Contents

1 Partial orders **3**
- 1.1 Type class for partial orders . . . 3
- 1.2 Upper bounds . . . 4
- 1.3 Least upper bounds . . . 5
- 1.4 Countable chains . . . 6
- 1.5 Finite chains . . . 7

2 Classes cpo and pcpo **8**
- 2.1 Complete partial orders . . . 8
- 2.2 Pointed cpos . . . 10
- 2.3 Chain-finite and flat cpos . . . 10
- 2.4 Discrete cpos . . . 11

3 Continuity and monotonicity **11**
- 3.1 Definitions . . . 11
- 3.2 Equivalence of alternate definition . . . 12
- 3.3 Collection of continuity rules . . . 13
- 3.4 Continuity of basic functions . . . 13
- 3.5 Finite chains and flat pcpos . . . 13

4 Admissibility and compactness **14**
- 4.1 Definitions . . . 14
- 4.2 Admissibility on chain-finite types . . . 15
- 4.3 Admissibility of special formulae and propagation . . . 15
- 4.4 Compactness . . . 16

5 Class instances for the full function space **17**
- 5.1 Full function space is a partial order . . . 17
- 5.2 Full function space is chain complete . . . 18
- 5.3 Full function space is pointed . . . 18
- 5.4 Propagation of monotonicity and continuity . . . 19

6 The cpo of cartesian products **19**
6.1 Unit type is a pcpo . . . 19
6.2 Product type is a partial order . . . 20
6.3 Monotonicity of *Pair*, *fst*, *snd* . . . 20
6.4 Product type is a cpo . . . 21
6.5 Product type is pointed . . . 21
6.6 Continuity of *Pair*, *fst*, *snd* . . . 22
6.7 Compactness and chain-finiteness . . . 23

7 Discrete cpo types **24**
7.1 Discrete cpo class instance . . . 24
7.2 *undiscr* . . . 24

8 Subtypes of pcpos **24**
8.1 Proving a subtype is a partial order . . . 24
8.2 Proving a subtype is finite . . . 25
8.3 Proving a subtype is chain-finite . . . 25
8.4 Proving a subtype is complete . . . 25
8.4.1 Continuity of *Rep* and *Abs* . . . 26
8.5 Proving subtype elements are compact . . . 26
8.6 Proving a subtype is pointed . . . 27
8.6.1 Strictness of *Rep* and *Abs* . . . 27
8.7 Proving a subtype is flat . . . 28
8.8 HOLCF type definition package . . . 28

9 The type of continuous functions **28**
9.1 Definition of continuous function type . . . 28
9.2 Syntax for continuous lambda abstraction . . . 29
9.3 Continuous function space is pointed . . . 29
9.4 Basic properties of continuous functions . . . 30
9.4.1 Beta-reduction simproc . . . 30
9.5 Continuity of application . . . 31
9.6 Continuity simplification procedure . . . 32
9.7 Miscellaneous . . . 33
9.8 Continuous injection-retraction pairs . . . 33
9.9 Identity and composition . . . 34
9.10 Strictified functions . . . 35
9.11 Continuity of let-bindings . . . 35

10 Continuous deflations and ep-pairs **36**
10.1 Continuous deflations . . . 36
10.2 Deflations with finite range . . . 37
10.3 Continuous embedding-projection pairs . . . 38
10.4 Uniqueness of ep-pairs . . . 39

10.5 Composing ep-pairs 40

11 The type of strict products **40**
11.1 Definition of strict product type 40
11.2 Definitions of constants 41
11.3 Case analysis 41
11.4 Properties of *spair* 42
11.5 Properties of *sfst* and *ssnd* 43
11.6 Compactness 43
11.7 Properties of *ssplit* 44
11.8 Strict product preserves flatness 44

12 The type of lifted values **44**
12.1 Definition of new type for lifting 44
12.2 Ordering on lifted cpo 44
12.3 Lifted cpo is a partial order 45
12.4 Lifted cpo is a cpo 45
12.5 Lifted cpo is pointed 45
12.6 Continuity of *Iup* and *Ifup* 45
12.7 Continuous versions of constants 46

13 Lifting types of class type to flat pcpo's **47**
13.1 Lift as a datatype 48
13.2 Lift is flat 48
13.3 Continuity of *case-lift* 48
13.4 Further operations 48

14 The type of lifted booleans **49**
14.1 Type definition and constructors 49
14.2 Case analysis 50
14.3 Boolean connectives 51
14.4 Rewriting of HOLCF operations to HOL functions 52
14.5 Compactness 52

15 The type of strict sums **53**
15.1 Definition of strict sum type 53
15.2 Definitions of constructors 53
15.3 Properties of *sinl* and *sinr* 54
15.4 Case analysis 55
15.5 Case analysis combinator 55
15.6 Strict sum preserves flatness 56

16 The Strict Function Type **56**

17 Map functions for various types **57**
17.1 Map operator for continuous function space 57
17.2 Map operator for product type 58
17.3 Map function for lifted cpo 59
17.4 Map function for strict products 59
17.5 Map function for strict sums 60
17.6 Map operator for strict function space 61

18 The cpo of cartesian products **62**
18.1 Continuous case function for unit type 62
18.2 Continuous version of split function 62
18.3 Convert all lemmas to the continuous versions 62

19 Profinite and bifinite cpos **63**
19.1 Chains of finite deflations 63
19.2 Omega-profinite and bifinite domains 63
19.3 Building approx chains 64
19.4 Class instance proofs 65

20 Defining algebraic domains by ideal completion **66**
20.1 Ideals over a preorder 66
20.2 Lemmas about least upper bounds 68
20.3 Locale for ideal completion 68
20.3.1 Principal ideals approximate all elements 69
20.4 Defining functions in terms of basis elements 69

21 A universal bifinite domain **70**
21.1 Basis for universal domain 70
21.1.1 Basis datatype 70
21.1.2 Basis ordering 71
21.1.3 Generic take function 71
21.2 Defining the universal domain by ideal completion 72
21.3 Compact bases of domains 73
21.4 Universality of *udom* 74
21.4.1 Choosing a maximal element from a finite set 74
21.4.2 Compact basis take function 75
21.4.3 Rank of basis elements 76
21.4.4 Sequencing basis elements 77
21.4.5 Embedding and projection on basis elements 78
21.4.6 EP-pair from any bifinite domain into *udom* 80
21.5 Chain of approx functions for type *udom* 81

22 Algebraic deflations **82**
22.1 Type constructor for finite deflations . . . 82
22.2 Defining algebraic deflations by ideal completion . . . 83
22.3 Applying algebraic deflations . . . 84
22.4 Deflation combinators . . . 84

23 Representable domains **85**
23.1 Class of representable domains . . . 85
23.2 Domains are bifinite . . . 86
23.3 Universal domain ep-pairs . . . 87
23.4 Type combinators . . . 87
23.5 Class instance proofs . . . 88
23.5.1 Universal domain . . . 88
23.5.2 Lifted cpo . . . 89
23.5.3 Strict function space . . . 89
23.5.4 Continuous function space . . . 90
23.5.5 Strict product . . . 91
23.5.6 Cartesian product . . . 91
23.5.7 Unit type . . . 92
23.5.8 Discrete cpo . . . 93
23.5.9 Strict sum . . . 93
23.5.10 Lifted HOL type . . . 94

24 The unit domain **95**

25 Fixed point operator and admissibility **96**
25.1 Iteration . . . 96
25.2 Least fixed point operator . . . 97
25.3 Fixed point induction . . . 98
25.4 Fixed-points on product types . . . 99

26 Package for defining recursive functions in HOLCF **99**
26.1 Pattern-match monad . . . 99
26.1.1 Run operator . . . 100
26.1.2 Monad plus operator . . . 100
26.2 Match functions for built-in types . . . 101
26.3 Mutual recursion . . . 102
26.4 Initializing the fixrec package . . . 103

27 Domain package **103**
27.1 Continuous isomorphisms . . . 103
27.2 Proofs about take functions . . . 105
27.3 Finiteness . . . 105
27.4 Proofs about constructor functions . . . 107

27.5 ML setup . . . 108
27.6 Representations of types . . . 108
27.7 Deflations as sets . . . 109
27.8 Proving a subtype is representable . . . 109
27.9 Isomorphic deflations . . . 110
27.10Setting up the domain package . . . 112

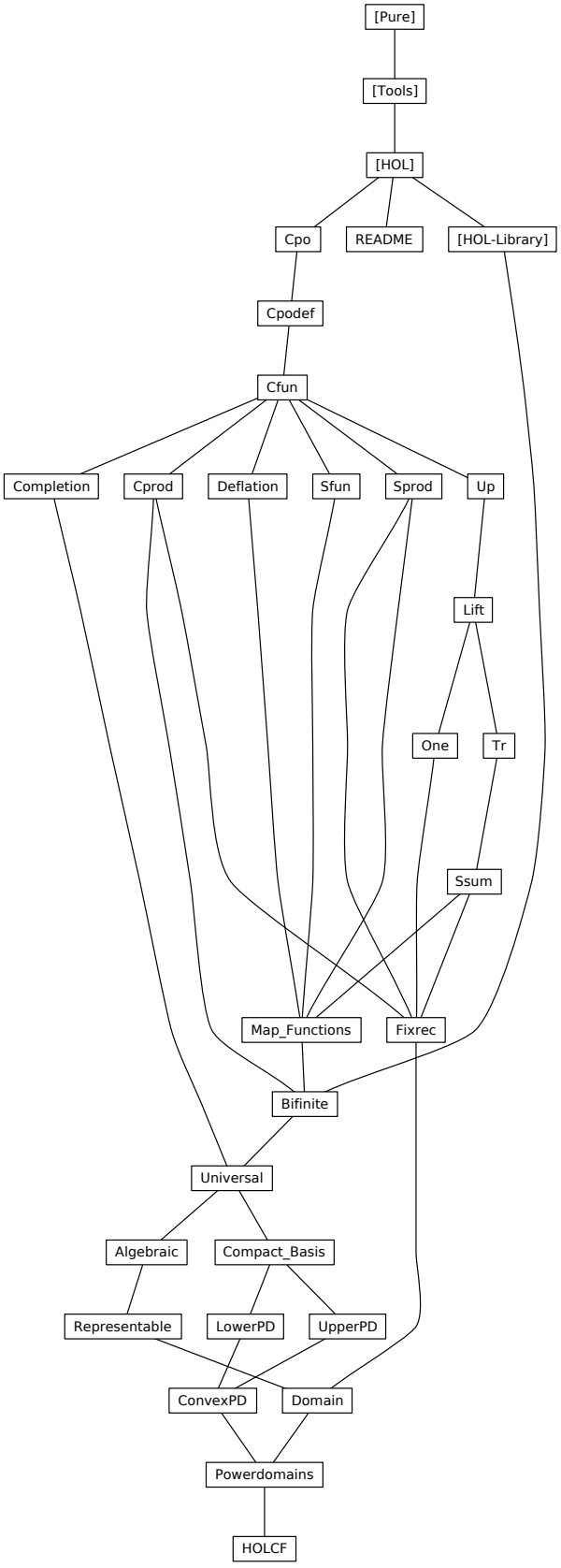
28 A compact basis for powerdomains **113**
28.1 A compact basis for powerdomains . . . 113
28.2 Unit and plus constructors . . . 113
28.3 Fold operator . . . 114

29 Upper powerdomain **115**
29.1 Basis preorder . . . 115
29.2 Type definition . . . 116
29.3 Monadic unit and plus . . . 116
29.4 Induction rules . . . 118
29.5 Monadic bind . . . 119
29.6 Map . . . 120
29.7 Upper powerdomain is bifinite . . . 121
29.8 Join . . . 121

30 Lower powerdomain **122**
30.1 Basis preorder . . . 122
30.2 Type definition . . . 123
30.3 Monadic unit and plus . . . 124
30.4 Induction rules . . . 126
30.5 Monadic bind . . . 126
30.6 Map . . . 127
30.7 Lower powerdomain is bifinite . . . 128
30.8 Join . . . 128

31 Convex powerdomain **129**
31.1 Basis preorder . . . 129
31.2 Type definition . . . 130
31.3 Monadic unit and plus . . . 131
31.4 Induction rules . . . 132
31.5 Monadic bind . . . 133
31.6 Map . . . 134
31.7 Convex powerdomain is bifinite . . . 135
31.8 Join . . . 135
31.9 Conversions to other powerdomains . . . 136

32 Powerdomains **138**
32.1 Universal domain embeddings 138
32.2 Deflation combinators 138
32.3 Domain class instances 139
32.4 Isomorphic deflations 140
32.5 Domain package setup for powerdomains 141


[Pure]
[Tools]
[HOL]
Cpo
README
[HOL-Library]
Cpodef
Cfun
Completion
Cprod
Deflation
Sfun
Sprod
Up
Lift
One
Tr
Ssum
Map_Functions
Fixrec
Bifinite
Universal
Algebraic
Compact_Basis
Representable
LowerPD
UpperPD
ConvexPD
Domain
Powerdomains
HOLCF

theory *Cpo*
imports *Main*
begin

1 Partial orders

declare [[*typedef-overloaded*]]

1.1 Type class for partial orders

class *below* =
fixes *below* :: $'a \Rightarrow 'a \Rightarrow bool$
begin

notation (*ASCII*)
below (**infix** ‹<<› *50*)

notation
below (**infix** ‹⊑› *50*)

abbreviation *not-below* :: $'a \Rightarrow 'a \Rightarrow bool$ (**infix** ‹⋢› *50*)
where *not-below x y* ≡ ¬ *below x y*

notation (*ASCII*)
not-below (**infix** ‹~<<› *50*)

lemma *below-eq-trans*: $a \sqsubseteq b \Longrightarrow b = c \Longrightarrow a \sqsubseteq c$
⟨*proof*⟩

lemma *eq-below-trans*: $a = b \Longrightarrow b \sqsubseteq c \Longrightarrow a \sqsubseteq c$
⟨*proof*⟩

end

class *po* = *below* +
assumes *below-refl* [*iff*]: $x \sqsubseteq x$
assumes *below-trans*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq z \Longrightarrow x \sqsubseteq z$
assumes *below-antisym*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow x = y$
begin

lemma *eq-imp-below*: $x = y \Longrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *box-below*: $a \sqsubseteq b \Longrightarrow c \sqsubseteq a \Longrightarrow b \sqsubseteq d \Longrightarrow c \sqsubseteq d$
⟨*proof*⟩

lemma *po-eq-conv*: $x = y \longleftrightarrow x \sqsubseteq y \wedge y \sqsubseteq x$

⟨*proof*⟩

lemma *rev-below-trans*: $y \sqsubseteq z \Longrightarrow x \sqsubseteq y \Longrightarrow x \sqsubseteq z$
⟨*proof*⟩

lemma *not-below2not-eq*: $x \not\sqsubseteq y \Longrightarrow x \neq y$
⟨*proof*⟩

end

lemmas *HOLCF-trans-rules* [*trans*] =
below-trans
below-antisym
below-eq-trans
eq-below-trans

context *po*
begin

1.2 Upper bounds

definition *is-ub* :: $'a\ set \Rightarrow 'a \Rightarrow bool$ (**infix** ‹<|› 55)
where $S <| x \longleftrightarrow (\forall y{\in}S.\ y \sqsubseteq x)$

lemma *is-ubI*: $(\bigwedge x.\ x \in S \Longrightarrow x \sqsubseteq u) \Longrightarrow S <| u$
⟨*proof*⟩

lemma *is-ubD*: $[\![S <| u;\ x \in S]\!] \Longrightarrow x \sqsubseteq u$
⟨*proof*⟩

lemma *ub-imageI*: $(\bigwedge x.\ x \in S \Longrightarrow f\,x \sqsubseteq u) \Longrightarrow (\lambda x.\ f\,x)\ `\ S <| u$
⟨*proof*⟩

lemma *ub-imageD*: $[\![f\ `\ S <| u;\ x \in S]\!] \Longrightarrow f\,x \sqsubseteq u$
⟨*proof*⟩

lemma *ub-rangeI*: $(\bigwedge i.\ S\,i \sqsubseteq x) \Longrightarrow range\ S <| x$
⟨*proof*⟩

lemma *ub-rangeD*: $range\ S <| x \Longrightarrow S\,i \sqsubseteq x$
⟨*proof*⟩

lemma *is-ub-empty* [*simp*]: $\{\} <| u$
⟨*proof*⟩

lemma *is-ub-insert* [*simp*]: $(insert\ x\ A) <| y = (x \sqsubseteq y \wedge A <| y)$
⟨*proof*⟩

lemma *is-ub-upward*: $[\![S <| x;\ x \sqsubseteq y]\!] \Longrightarrow S <| y$

⟨*proof*⟩

1.3 Least upper bounds

definition *is-lub* :: *'a set* ⇒ *'a* ⇒ *bool* (**infix** ‹<<|› *55*)
  where $S <<| x \longleftrightarrow S <| x \wedge (\forall u.\ S <| u \longrightarrow x \sqsubseteq u)$

definition *lub* :: *'a set* ⇒ *'a*
  where *lub S* = (*THE x*. *S* <<| *x*)

end

syntax (*ASCII*)
  -BLub :: [*pttrn*, *'a set*, *'b*] ⇒ *'b* (‹(‹*indent=3 notation=‹binder LUB››LUB* -:-./ -)› [*0*,*0*, *10*] *10*)

syntax
  -BLub :: [*pttrn*, *'a set*, *'b*] ⇒ *'b* (‹(‹*indent=3 notation=‹binder* ⨆››⨆-∈-./ -)› [*0*,*0*, *10*] *10*)

syntax-consts
  -BLub ⇌ *lub*

translations
  LUB x:*A*. *t* ⇌ *CONST lub* ((λ*x*. *t*) ‘ *A*)

context *po*
begin

abbreviation *Lub* (**binder** ‹⨆› *10*)
  where ⨆*n*. *t n* ≡ *lub* (*range t*)

notation (*ASCII*)
  Lub (**binder** ‹*LUB* › *10*)

access to some definition as inference rule

lemma *is-lubD1*: *S* <<| *x* ⟹ *S* <| *x*
  ⟨*proof*⟩

lemma *is-lubD2*: ⟦*S* <<| *x*; *S* <| *u*⟧ ⟹ *x* ⊑ *u*
  ⟨*proof*⟩

lemma *is-lubI*: ⟦*S* <| *x*; ⋀*u*. *S* <| *u* ⟹ *x* ⊑ *u*⟧ ⟹ *S* <<| *x*
  ⟨*proof*⟩

lemma *is-lub-below-iff*: *S* <<| *x* ⟹ *x* ⊑ *u* ⟷ *S* <| *u*
  ⟨*proof*⟩

lubs are unique

lemma *is-lub-unique*: $S <<| x \Longrightarrow S <<| y \Longrightarrow x = y$
⟨*proof*⟩

technical lemmas about *lub* and $(<<|)$

lemma *is-lub-lub*: $M <<| x \Longrightarrow M <<| lub\ M$
⟨*proof*⟩

lemma *lub-eqI*: $M <<| l \Longrightarrow lub\ M = l$
⟨*proof*⟩

lemma *is-lub-singleton* [*simp*]: $\{x\} <<| x$
⟨*proof*⟩

lemma *lub-singleton* [*simp*]: $lub\ \{x\} = x$
⟨*proof*⟩

lemma *is-lub-bin*: $x \sqsubseteq y \Longrightarrow \{x, y\} <<| y$
⟨*proof*⟩

lemma *lub-bin*: $x \sqsubseteq y \Longrightarrow lub\ \{x, y\} = y$
⟨*proof*⟩

lemma *is-lub-maximal*: $S <| x \Longrightarrow x \in S \Longrightarrow S <<| x$
⟨*proof*⟩

lemma *lub-maximal*: $S <| x \Longrightarrow x \in S \Longrightarrow lub\ S = x$
⟨*proof*⟩

1.4 Countable chains

definition *chain* :: $(nat \Rightarrow 'a) \Rightarrow bool$
where — Here we use countable chains and I prefer to code them as functions!
$chain\ Y = (\forall i.\ Y\ i \sqsubseteq Y\ (Suc\ i))$

lemma *chainI*: $(\bigwedge i.\ Y\ i \sqsubseteq Y\ (Suc\ i)) \Longrightarrow chain\ Y$
⟨*proof*⟩

lemma *chainE*: $chain\ Y \Longrightarrow Y\ i \sqsubseteq Y\ (Suc\ i)$
⟨*proof*⟩

chains are monotone functions

lemma *chain-mono-less*: $chain\ Y \Longrightarrow i < j \Longrightarrow Y\ i \sqsubseteq Y\ j$
⟨*proof*⟩

lemma *chain-mono*: $chain\ Y \Longrightarrow i \leq j \Longrightarrow Y\ i \sqsubseteq Y\ j$
⟨*proof*⟩

lemma *chain-shift*: $chain\ Y \Longrightarrow chain\ (\lambda i.\ Y\ (i + j))$
⟨*proof*⟩

technical lemmas about (least) upper bounds of chains

lemma *is-lub-rangeD1*: *range S <<| x ⟹ S i ⊑ x*
⟨*proof*⟩

lemma *is-ub-range-shift*: *chain S ⟹ range (λi. S (i + j)) <| x = range S <| x*
⟨*proof*⟩

lemma *is-lub-range-shift*: *chain S ⟹ range (λi. S (i + j)) <<| x = range S <<| x*
⟨*proof*⟩

the lub of a constant chain is the constant

lemma *chain-const* [*simp*]: *chain (λi. c)*
⟨*proof*⟩

lemma *is-lub-const*: *range (λx. c) <<| c*
⟨*proof*⟩

lemma *lub-const* [*simp*]: *(⨆ i. c) = c*
⟨*proof*⟩

1.5 Finite chains

definition *max-in-chain* :: *nat ⇒ (nat ⇒ 'a) ⇒ bool*
where — finite chains, needed for monotony of continuous functions
max-in-chain i C ⟷ (∀ j. i ≤ j ⟶ C i = C j)

definition *finite-chain* :: *(nat ⇒ 'a) ⇒ bool*
where *finite-chain C = (chain C ∧ (∃ i. max-in-chain i C))*

results about finite chains

lemma *max-in-chainI*: *(⋀j. i ≤ j ⟹ Y i = Y j) ⟹ max-in-chain i Y*
⟨*proof*⟩

lemma *max-in-chainD*: *max-in-chain i Y ⟹ i ≤ j ⟹ Y i = Y j*
⟨*proof*⟩

lemma *finite-chainI*: *chain C ⟹ max-in-chain i C ⟹ finite-chain C*
⟨*proof*⟩

lemma *finite-chainE*: *⟦finite-chain C; ⋀i. ⟦chain C; max-in-chain i C⟧ ⟹ R⟧ ⟹ R*
⟨*proof*⟩

lemma *lub-finch1*: *chain C ⟹ max-in-chain i C ⟹ range C <<| C i*
⟨*proof*⟩

lemma *lub-finch2*: *finite-chain C ⟹ range C <<| C (LEAST i. max-in-chain i C)*

⟨*proof*⟩

lemma *finch-imp-finite-range*: *finite-chain Y* ⟹ *finite* (*range Y*)
⟨*proof*⟩

lemma *finite-range-has-max*:
fixes *f* :: *nat* ⇒ *'a*
and *r* :: *'a* ⇒ *'a* ⇒ *bool*
assumes *mono*: ⋀*i j*. *i* ≤ *j* ⟹ *r* (*f i*) (*f j*)
assumes *finite-range*: *finite* (*range f*)
shows ∃ *k*. ∀ *i*. *r* (*f i*) (*f k*)
⟨*proof*⟩

lemma *finite-range-imp-finch*: *chain Y* ⟹ *finite* (*range Y*) ⟹ *finite-chain Y*
⟨*proof*⟩

lemma *bin-chain*: *x* ⊑ *y* ⟹ *chain* (λ*i*. *if i=0 then x else y*)
⟨*proof*⟩

lemma *bin-chainmax*: *x* ⊑ *y* ⟹ *max-in-chain* (*Suc 0*) (λ*i*. *if i=0 then x else y*)
⟨*proof*⟩

lemma *is-lub-bin-chain*: *x* ⊑ *y* ⟹ *range* (λ*i*::*nat*. *if i=0 then x else y*) <<| *y*
⟨*proof*⟩

the maximal element in a chain is its lub

lemma *lub-chain-maxelem*: *Y i* = *c* ⟹ ∀ *i*. *Y i* ⊑ *c* ⟹ *lub* (*range Y*) = *c*
⟨*proof*⟩

end

2 Classes cpo and pcpo

2.1 Complete partial orders

The class cpo of chain complete partial orders

class *cpo* = *po* +
assumes *cpo*: *chain S* ⟹ ∃ *x*. *range S* <<| *x*

default-sort *cpo*

context *cpo*
begin

in cpo's everthing equal to THE lub has lub properties for every chain

lemma *cpo-lubI*: *chain S* ⟹ *range S* <<| (⨆ *i*. *S i*)
⟨*proof*⟩

lemma *thelubE*: $[\![chain\ S;\ (\bigsqcup i.\ S\ i) = l]\!] \Longrightarrow range\ S <<|\ l$
⟨*proof*⟩

Properties of the lub

lemma *is-ub-thelub*: $chain\ S \Longrightarrow S\ x \sqsubseteq (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

lemma *is-lub-thelub*: $[\![chain\ S;\ range\ S <|\ x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
⟨*proof*⟩

lemma *lub-below-iff*: $chain\ S \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x \longleftrightarrow (\forall i.\ S\ i \sqsubseteq x)$
⟨*proof*⟩

lemma *lub-below*: $[\![chain\ S;\ \bigwedge i.\ S\ i \sqsubseteq x]\!] \Longrightarrow (\bigsqcup i.\ S\ i) \sqsubseteq x$
⟨*proof*⟩

lemma *below-lub*: $[\![chain\ S;\ x \sqsubseteq S\ i]\!] \Longrightarrow x \sqsubseteq (\bigsqcup i.\ S\ i)$
⟨*proof*⟩

lemma *lub-range-mono*: $[\![range\ X \subseteq range\ Y;\ chain\ Y;\ chain\ X]\!] \Longrightarrow (\bigsqcup i.\ X\ i)$
$\sqsubseteq (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *lub-range-shift*: $chain\ Y \Longrightarrow (\bigsqcup i.\ Y\ (i + j)) = (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *maxinch-is-thelub*: $chain\ Y \Longrightarrow max\text{-}in\text{-}chain\ i\ Y = ((\bigsqcup i.\ Y\ i) = Y\ i)$
⟨*proof*⟩

the $\sqsubseteq$ relation between two chains is preserved by their lubs

lemma *lub-mono*: $[\![chain\ X;\ chain\ Y;\ \bigwedge i.\ X\ i \sqsubseteq Y\ i]\!] \Longrightarrow (\bigsqcup i.\ X\ i) \sqsubseteq (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

the $=$ relation between two chains is preserved by their lubs

lemma *lub-eq*: $(\bigwedge i.\ X\ i = Y\ i) \Longrightarrow (\bigsqcup i.\ X\ i) = (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *ch2ch-lub*:
assumes *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
assumes *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
shows $chain\ (\lambda i.\ \bigsqcup j.\ Y\ i\ j)$
⟨*proof*⟩

lemma *diag-lub*:
assumes *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
assumes *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
shows $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup i.\ Y\ i\ i)$
⟨*proof*⟩

lemma *ex-lub*:
assumes *1*: $\bigwedge j.\ chain\ (\lambda i.\ Y\ i\ j)$
assumes *2*: $\bigwedge i.\ chain\ (\lambda j.\ Y\ i\ j)$
shows $(\bigsqcup i.\ \bigsqcup j.\ Y\ i\ j) = (\bigsqcup j.\ \bigsqcup i.\ Y\ i\ j)$
⟨*proof*⟩

end

2.2 Pointed cpos

The class pcpo of pointed cpos

class *pcpo* = *cpo* +
assumes *least*: $\exists\, x.\ \forall\, y.\ x \sqsubseteq y$
begin

definition *bottom* :: *'a* (‹⊥›)
where $bottom = (THE\ x.\ \forall\, y.\ x \sqsubseteq y)$

lemma *minimal* [*iff*]: $\bot \sqsubseteq x$
⟨*proof*⟩

end

Old "UU" syntax:

abbreviation (*input*) $UU \equiv bottom$

Simproc to rewrite $\bot = x$ to $x = \bot$.

⟨*ML*⟩
useful lemmas about $\bot$

lemma *below-bottom-iff* [*simp*]: $x \sqsubseteq \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *eq-bottom-iff*: $x = \bot \longleftrightarrow x \sqsubseteq \bot$
⟨*proof*⟩

lemma *bottomI*: $x \sqsubseteq \bot \Longrightarrow x = \bot$
⟨*proof*⟩

lemma *lub-eq-bottom-iff*: $chain\ Y \Longrightarrow (\bigsqcup i.\ Y\ i) = \bot \longleftrightarrow (\forall\, i.\ Y\ i = \bot)$
⟨*proof*⟩

2.3 Chain-finite and flat cpos

further useful classes for HOLCF domains

class *chfin* = *po* +
assumes *chfin*: $chain\ Y \Longrightarrow \exists\, n.\ max\text{-}in\text{-}chain\ n\ Y$
begin

subclass *cpo*
⟨*proof*⟩

lemma *chfin2finch*: *chain* $Y \Longrightarrow$ *finite-chain* Y
⟨*proof*⟩

end

class *flat* = *pcpo* +
assumes *ax-flat*: $x \sqsubseteq y \Longrightarrow x = \bot \vee x = y$
begin

subclass *chfin*
⟨*proof*⟩

lemma *flat-below-iff*: $x \sqsubseteq y \longleftrightarrow x = \bot \vee x = y$
⟨*proof*⟩

lemma *flat-eq*: $a \neq \bot \Longrightarrow a \sqsubseteq b = (a = b)$
⟨*proof*⟩

end

2.4 Discrete cpos

class *discrete-cpo* = *below* +
assumes *discrete-cpo* [*simp*]: $x \sqsubseteq y \longleftrightarrow x = y$
begin

subclass *po*
⟨*proof*⟩

In a discrete cpo, every chain is constant

lemma *discrete-chain-const*:
assumes S: *chain* S
shows $\exists x.\ S = (\lambda i.\ x)$
⟨*proof*⟩

subclass *chfin*
⟨*proof*⟩

end

3 Continuity and monotonicity

3.1 Definitions

definition *monofun* :: $('a{::}po \Rightarrow 'b{::}po) \Rightarrow bool$ — monotonicity

where *monofun f* $\longleftrightarrow$ $(\forall x\ y.\ x \sqsubseteq y \longrightarrow f\ x \sqsubseteq f\ y)$

definition *cont* :: $('a \Rightarrow 'b) \Rightarrow bool$
where *cont f* = $(\forall Y.\ chain\ Y \longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i))$

lemma *contI*: $(\bigwedge Y.\ chain\ Y \Longrightarrow range\ (\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i)) \Longrightarrow cont\ f$
⟨*proof*⟩

lemma *contE*: *cont f* $\Longrightarrow$ *chain Y* $\Longrightarrow$ *range* $(\lambda i.\ f\ (Y\ i)) <<|\ f\ (\bigsqcup i.\ Y\ i)$
⟨*proof*⟩

lemma *monofunI*: $(\bigwedge x\ y.\ x \sqsubseteq y \Longrightarrow f\ x \sqsubseteq f\ y) \Longrightarrow monofun\ f$
⟨*proof*⟩

lemma *monofunE*: *monofun f* $\Longrightarrow$ $x \sqsubseteq y \Longrightarrow f\ x \sqsubseteq f\ y$
⟨*proof*⟩

3.2 Equivalence of alternate definition

monotone functions map chains to chains

lemma *ch2ch-monofun*: *monofun f* $\Longrightarrow$ *chain Y* $\Longrightarrow$ *chain* $(\lambda i.\ f\ (Y\ i))$
⟨*proof*⟩

monotone functions map upper bound to upper bounds

lemma *ub2ub-monofun*: *monofun f* $\Longrightarrow$ *range Y* $<|$ *u* $\Longrightarrow$ *range* $(\lambda i.\ f\ (Y\ i)) <|$ *f u*
⟨*proof*⟩

a lemma about binary chains

lemma *binchain-cont*: *cont f* $\Longrightarrow$ $x \sqsubseteq y \Longrightarrow$ *range* $(\lambda i{::}nat.\ f\ (if\ i = 0\ then\ x\ else\ y)) <<|\ f\ y$
⟨*proof*⟩

continuity implies monotonicity

lemma *cont2mono*: *cont f* $\Longrightarrow$ *monofun f*
⟨*proof*⟩

lemmas *cont2monofunE* = *cont2mono* [*THEN monofunE*]

lemmas *ch2ch-cont* = *cont2mono* [*THEN ch2ch-monofun*]

continuity implies preservation of lubs

lemma *cont2contlubE*: *cont f* $\Longrightarrow$ *chain Y* $\Longrightarrow$ *f* $(\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ f\ (Y\ i))$
⟨*proof*⟩

lemma *contI2*:
fixes *f* :: $'a \Rightarrow 'b$
assumes *mono*: *monofun f*

assumes *below*: $\bigwedge Y.\ [\![chain\ Y;\ chain\ (\lambda i.\ f\ (Y\ i))]\!] \Longrightarrow f\ (\bigsqcup i.\ Y\ i) \sqsubseteq (\bigsqcup i.\ f\ (Y\ i))$
shows *cont f*
⟨*proof*⟩

3.3 Collection of continuity rules

named-theorems *cont2cont continuity intro rule*

3.4 Continuity of basic functions

The identity function is continuous

lemma *cont-id* [*simp*, *cont2cont*]: *cont* $(\lambda x.\ x)$
⟨*proof*⟩

constant functions are continuous

lemma *cont-const* [*simp*, *cont2cont*]: *cont* $(\lambda x.\ c)$
⟨*proof*⟩

application of functions is continuous

lemma *cont-apply*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c$ **and** $t :: 'a \Rightarrow 'b$
assumes *1*: *cont* $(\lambda x.\ t\ x)$
assumes *2*: $\bigwedge x.$ *cont* $(\lambda y.\ f\ x\ y)$
assumes *3*: $\bigwedge y.$ *cont* $(\lambda x.\ f\ x\ y)$
shows *cont* $(\lambda x.\ (f\ x)\ (t\ x))$
⟨*proof*⟩

lemma *cont-compose*: *cont c* $\Longrightarrow$ *cont* $(\lambda x.\ f\ x) \Longrightarrow$ *cont* $(\lambda x.\ c\ (f\ x))$
⟨*proof*⟩

Least upper bounds preserve continuity

lemma *cont2cont-lub* [*simp*]:
assumes *chain*: $\bigwedge x.$ *chain* $(\lambda i.\ F\ i\ x)$
and *cont*: $\bigwedge i.$ *cont* $(\lambda x.\ F\ i\ x)$
shows *cont* $(\lambda x.\ \bigsqcup i.\ F\ i\ x)$
⟨*proof*⟩

if-then-else is continuous

lemma *cont-if* [*simp*, *cont2cont*]: *cont f* $\Longrightarrow$ *cont g* $\Longrightarrow$ *cont* $(\lambda x.$ *if b then f x else g x*$)$
⟨*proof*⟩

3.5 Finite chains and flat pcpos

Monotone functions map finite chains to finite chains.

lemma *monofun-finch2finch*: *monofun f* $\Longrightarrow$ *finite-chain Y* $\Longrightarrow$ *finite-chain* $(\lambda n.\ f\ (Y\ n))$

⟨*proof*⟩

The same holds for continuous functions.

lemma *cont-finch2finch*: *cont f* ⟹ *finite-chain Y* ⟹ *finite-chain* (λ*n*. *f* (*Y n*))
⟨*proof*⟩

All monotone functions with chain-finite domain are continuous.

lemma *chfindom-monofun2cont*: *monofun f* ⟹ *cont f*
for *f* :: *'a*::*chfin* ⇒ *'b*
⟨*proof*⟩

All strict functions with flat domain are continuous.

lemma *flatdom-strict2mono*: *f* ⊥ = ⊥ ⟹ *monofun f*
for *f* :: *'a*::*flat* ⇒ *'b*::*pcpo*
⟨*proof*⟩

lemma *flatdom-strict2cont*: *f* ⊥ = ⊥ ⟹ *cont f*
for *f* :: *'a*::*flat* ⇒ *'b*::*pcpo*
⟨*proof*⟩

All functions with discrete domain are continuous.

lemma *cont-discrete-cpo* [*simp*, *cont2cont*]: *cont f*
for *f* :: *'a*::*discrete-cpo* ⇒ *'b*
⟨*proof*⟩

4 Admissibility and compactness

4.1 Definitions

context *cpo*
begin

definition *adm* :: (*'a* ⇒ *bool*) ⇒ *bool*
where *adm P* ⟷ (∀ *Y*. *chain Y* ⟶ (∀ *i*. *P* (*Y i*)) ⟶ *P* (⨆ *i*. *Y i*))

lemma *admI*: (⋀*Y*. ⟦*chain Y*; ∀ *i*. *P* (*Y i*)⟧ ⟹ *P* (⨆ *i*. *Y i*)) ⟹ *adm P*
⟨*proof*⟩

lemma *admD*: *adm P* ⟹ *chain Y* ⟹ (⋀*i*. *P* (*Y i*)) ⟹ *P* (⨆ *i*. *Y i*)
⟨*proof*⟩

lemma *admD2*: *adm* (λ*x*. ¬ *P x*) ⟹ *chain Y* ⟹ *P* (⨆ *i*. *Y i*) ⟹ ∃ *i*. *P* (*Y i*)
⟨*proof*⟩

lemma *triv-admI*: ∀ *x*. *P x* ⟹ *adm P*
⟨*proof*⟩

end

4.2 Admissibility on chain-finite types

For chain-finite (easy) types every formula is admissible.

lemma *adm-chfin* [*simp*]: *adm P* **for** *P* :: *'a*::*chfin* ⇒ *bool*
⟨*proof*⟩

4.3 Admissibility of special formulae and propagation

context *cpo*
begin

lemma *adm-const* [*simp*]: *adm* (λ*x*. *t*)
⟨*proof*⟩

lemma *adm-conj* [*simp*]: *adm* (λ*x*. *P x*) ⟹ *adm* (λ*x*. *Q x*) ⟹ *adm* (λ*x*. *P x* ∧ *Q x*)
⟨*proof*⟩

lemma *adm-all* [*simp*]: (⋀*y*. *adm* (λ*x*. *P x y*)) ⟹ *adm* (λ*x*. ∀ *y*. *P x y*)
⟨*proof*⟩

lemma *adm-ball* [*simp*]: (⋀*y*. *y* ∈ *A* ⟹ *adm* (λ*x*. *P x y*)) ⟹ *adm* (λ*x*. ∀ *y*∈*A*. *P x y*)
⟨*proof*⟩

Admissibility for disjunction is hard to prove. It requires 2 lemmas.

lemma *adm-disj-lemma1*:
assumes *adm*: *adm P*
assumes *chain*: *chain Y*
assumes *P*: ∀ *i*. ∃ *j*≥*i*. *P* (*Y j*)
shows *P* (⨆ *i*. *Y i*)
⟨*proof*⟩

lemma *adm-disj-lemma2*: ∀ *n*::*nat*. *P n* ∨ *Q n* ⟹ (∀ *i*. ∃ *j*≥*i*. *P j*) ∨ (∀ *i*. ∃ *j*≥*i*. *Q j*)
⟨*proof*⟩

lemma *adm-disj* [*simp*]: *adm* (λ*x*. *P x*) ⟹ *adm* (λ*x*. *Q x*) ⟹ *adm* (λ*x*. *P x* ∨ *Q x*)
⟨*proof*⟩

lemma *adm-imp* [*simp*]: *adm* (λ*x*. ¬ *P x*) ⟹ *adm* (λ*x*. *Q x*) ⟹ *adm* (λ*x*. *P x* ⟶ *Q x*)
⟨*proof*⟩

lemma *adm-iff* [*simp*]: *adm* (λ*x*. *P x* ⟶ *Q x*) ⟹ *adm* (λ*x*. *Q x* ⟶ *P x*) ⟹ *adm* (λ*x*. *P x* ⟷ *Q x*)
⟨*proof*⟩

end

admissibility and continuity

lemma *adm-below* [*simp*]: *cont* $(\lambda x.\ u\ x) \Longrightarrow$ *cont* $(\lambda x.\ v\ x) \Longrightarrow$ *adm* $(\lambda x.\ u\ x \sqsubseteq v\ x)$
⟨*proof*⟩

lemma *adm-eq* [*simp*]: *cont* $(\lambda x.\ u\ x) \Longrightarrow$ *cont* $(\lambda x.\ v\ x) \Longrightarrow$ *adm* $(\lambda x.\ u\ x = v\ x)$
⟨*proof*⟩

lemma *adm-subst*: *cont* $(\lambda x.\ t\ x) \Longrightarrow$ *adm* $P \Longrightarrow$ *adm* $(\lambda x.\ P\ (t\ x))$
⟨*proof*⟩

lemma *adm-not-below* [*simp*]: *cont* $(\lambda x.\ t\ x) \Longrightarrow$ *adm* $(\lambda x.\ t\ x \not\sqsubseteq u)$
⟨*proof*⟩

4.4 Compactness

context *cpo*
begin

definition *compact* :: $'a \Rightarrow$ *bool*
where *compact* $k =$ *adm* $(\lambda x.\ k \not\sqsubseteq x)$

lemma *compactI*: *adm* $(\lambda x.\ k \not\sqsubseteq x) \Longrightarrow$ *compact* k
⟨*proof*⟩

lemma *compactD*: *compact* $k \Longrightarrow$ *adm* $(\lambda x.\ k \not\sqsubseteq x)$
⟨*proof*⟩

lemma *compactI2*: $(\bigwedge Y.\ [\![\mathit{chain}\ Y; x \sqsubseteq (\bigsqcup i.\ Y\ i)]\!] \Longrightarrow \exists i.\ x \sqsubseteq Y\ i) \Longrightarrow$ *compact* x
⟨*proof*⟩

lemma *compactD2*: *compact* $x \Longrightarrow$ *chain* $Y \Longrightarrow x \sqsubseteq (\bigsqcup i.\ Y\ i) \Longrightarrow \exists i.\ x \sqsubseteq Y\ i$
⟨*proof*⟩

lemma *compact-below-lub-iff*: *compact* $x \Longrightarrow$ *chain* $Y \Longrightarrow x \sqsubseteq (\bigsqcup i.\ Y\ i) \longleftrightarrow (\exists i.\ x \sqsubseteq Y\ i)$
⟨*proof*⟩

end

lemma *compact-chfin* [*simp*]: *compact* x **for** $x :: 'a$::*chfin*
⟨*proof*⟩

lemma *compact-imp-max-in-chain*: *chain* $Y \Longrightarrow$ *compact* $(\bigsqcup i.\ Y\ i) \Longrightarrow \exists i.$ *max-in-chain* $i\ Y$
⟨*proof*⟩

admissibility and compactness

lemma *adm-compact-not-below* [*simp*]:
compact k $\Longrightarrow$ *cont* ($\lambda x.\ t\ x$) $\Longrightarrow$ *adm* ($\lambda x.\ k \not\sqsubseteq t\ x$)
⟨*proof*⟩

lemma *adm-neq-compact* [*simp*]: *compact k* $\Longrightarrow$ *cont* ($\lambda x.\ t\ x$) $\Longrightarrow$ *adm* ($\lambda x.\ t\ x \neq k$)
⟨*proof*⟩

lemma *adm-compact-neq* [*simp*]: *compact k* $\Longrightarrow$ *cont* ($\lambda x.\ t\ x$) $\Longrightarrow$ *adm* ($\lambda x.\ k \neq t\ x$)
⟨*proof*⟩

lemma *compact-bottom* [*simp*, *intro*]: *compact* $\bot$
⟨*proof*⟩

Any upward-closed predicate is admissible.

lemma *adm-upward*:
assumes *P*: $\bigwedge x\ y.\ [\![P\ x;\ x \sqsubseteq y]\!] \Longrightarrow P\ y$
shows *adm P*
⟨*proof*⟩

lemmas *adm-lemmas* =
adm-const adm-conj adm-all adm-ball adm-disj adm-imp adm-iff
adm-below adm-eq adm-not-below
adm-compact-not-below adm-compact-neq adm-neq-compact

5 Class instances for the full function space

5.1 Full function space is a partial order

instantiation *fun* :: (*type*, *below*) *below*
begin

definition *below-fun-def*: $(\sqsubseteq) \equiv (\lambda f\ g.\ \forall x.\ f\ x \sqsubseteq g\ x)$

instance ⟨*proof*⟩
end

instance *fun* :: (*type*, *po*) *po*
⟨*proof*⟩

lemma *fun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall x.\ f\ x \sqsubseteq g\ x)$
⟨*proof*⟩

lemma *fun-belowI*: $(\bigwedge x.\ f\ x \sqsubseteq g\ x) \Longrightarrow f \sqsubseteq g$
⟨*proof*⟩

lemma *fun-belowD*: $f \sqsubseteq g \Longrightarrow f\ x \sqsubseteq g\ x$
⟨*proof*⟩

5.2 Full function space is chain complete

Properties of chains of functions.

lemma *fun-chain-iff*: *chain* $S \longleftrightarrow (\forall x.\ \mathit{chain}\ (\lambda i.\ S\ i\ x))$
⟨*proof*⟩

lemma *ch2ch-fun*: *chain* $S \Longrightarrow \mathit{chain}\ (\lambda i.\ S\ i\ x)$
⟨*proof*⟩

lemma *ch2ch-lambda*: $(\bigwedge x.\ \mathit{chain}\ (\lambda i.\ S\ i\ x)) \Longrightarrow \mathit{chain}\ S$
⟨*proof*⟩

Type $'a \Rightarrow 'b$ is chain complete

lemma *is-lub-lambda*: $(\bigwedge x.\ \mathit{range}\ (\lambda i.\ Y\ i\ x) <<|\ f\ x) \Longrightarrow \mathit{range}\ Y <<|\ f$
⟨*proof*⟩

lemma *is-lub-fun*: *chain* $S \Longrightarrow \mathit{range}\ S <<|\ (\lambda x.\ \bigsqcup i.\ S\ i\ x)$
for $S :: \mathit{nat} \Rightarrow 'a{::}\mathit{type} \Rightarrow 'b$
⟨*proof*⟩

lemma *lub-fun*: *chain* $S \Longrightarrow (\bigsqcup i.\ S\ i) = (\lambda x.\ \bigsqcup i.\ S\ i\ x)$
for $S :: \mathit{nat} \Rightarrow 'a{::}\mathit{type} \Rightarrow 'b$
⟨*proof*⟩

instance *fun* :: (*type*, *cpo*) *cpo*
⟨*proof*⟩

instance *fun* :: (*type*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

5.3 Full function space is pointed

lemma *minimal-fun*: $(\lambda x.\ \bot) \sqsubseteq f$
⟨*proof*⟩

instance *fun* :: (*type*, *pcpo*) *pcpo*
⟨*proof*⟩

lemma *inst-fun-pcpo*: $\bot = (\lambda x.\ \bot)$
⟨*proof*⟩

lemma *app-strict* [*simp*]: $\bot\ x = \bot$
⟨*proof*⟩

lemma *lambda-strict*: $(\lambda x.\ \bot) = \bot$
⟨*proof*⟩

5.4 Propagation of monotonicity and continuity

The lub of a chain of monotone functions is monotone.

lemma *adm-monofun*: *adm monofun*
⟨*proof*⟩

The lub of a chain of continuous functions is continuous.

lemma *adm-cont*: *adm cont*
⟨*proof*⟩

Function application preserves monotonicity and continuity.

lemma *mono2mono-fun*: *monofun f* ⟹ *monofun* (λ*x*. *f x y*)
⟨*proof*⟩

lemma *cont2cont-fun*: *cont f* ⟹ *cont* (λ*x*. *f x y*)
⟨*proof*⟩

lemma *cont-fun*: *cont* (λ*f*. *f x*)
⟨*proof*⟩

⟨*ML*⟩

lemma *cont* (λ*f*. *f x*) **and** *cont* (λ*f*. *f x y*) **and** *cont* (λ*f*. *f x y z*)
⟨*proof*⟩

Lambda abstraction preserves monotonicity and continuity. (Note (λ*x*. λ*y*. *f x y*) = *f*.)

lemma *mono2mono-lambda*: (⋀*y*. *monofun* (λ*x*. *f x y*)) ⟹ *monofun f*
⟨*proof*⟩

lemma *cont2cont-lambda* [*simp*]:
assumes *f*: ⋀*y*. *cont* (λ*x*. *f x y*)
shows *cont f*
⟨*proof*⟩

What D.A.Schmidt calls continuity of abstraction; never used here

lemma *contlub-lambda*: (⋀*x*. *chain* (λ*i*. *S i x*)) ⟹ (λ*x*. ⨆*i*. *S i x*) = (⨆*i*. (λ*x*. *S i x*))
for *S* :: *nat* ⇒ *'a*::*type* ⇒ *'b*
⟨*proof*⟩

6 The cpo of cartesian products

6.1 Unit type is a pcpo

instantiation *unit* :: *discrete-cpo*
begin

definition *below-unit-def* [*simp*]: $x \sqsubseteq (y::unit) \longleftrightarrow True$

instance
⟨*proof*⟩

end

instance *unit* :: *pcpo*
⟨*proof*⟩

6.2 Product type is a partial order

instantiation *prod* :: (*below*, *below*) *below*
begin

definition *below-prod-def*: $(\sqsubseteq) \equiv \lambda p1\ p2.\ (fst\ p1 \sqsubseteq fst\ p2 \wedge snd\ p1 \sqsubseteq snd\ p2)$

instance ⟨*proof*⟩

end

instance *prod* :: (*po*, *po*) *po*
⟨*proof*⟩

6.3 Monotonicity of *Pair*, *fst*, *snd*

lemma *prod-belowI*: $fst\ p \sqsubseteq fst\ q \Longrightarrow snd\ p \sqsubseteq snd\ q \Longrightarrow p \sqsubseteq q$
⟨*proof*⟩

lemma *Pair-below-iff* [*simp*]: $(a, b) \sqsubseteq (c, d) \longleftrightarrow a \sqsubseteq c \wedge b \sqsubseteq d$
⟨*proof*⟩

Pair (-,-) is monotone in both arguments

lemma *monofun-pair1*: $monofun\ (\lambda x.\ (x, y))$
⟨*proof*⟩

lemma *monofun-pair2*: $monofun\ (\lambda y.\ (x, y))$
⟨*proof*⟩

lemma *monofun-pair*: $x1 \sqsubseteq x2 \Longrightarrow y1 \sqsubseteq y2 \Longrightarrow (x1, y1) \sqsubseteq (x2, y2)$
⟨*proof*⟩

lemma *ch2ch-Pair* [*simp*]: $chain\ X \Longrightarrow chain\ Y \Longrightarrow chain\ (\lambda i.\ (X\ i, Y\ i))$
⟨*proof*⟩

fst and *snd* are monotone

lemma *fst-monofun*: $x \sqsubseteq y \Longrightarrow fst\ x \sqsubseteq fst\ y$
⟨*proof*⟩

lemma *snd-monofun*: $x \sqsubseteq y \Longrightarrow snd\ x \sqsubseteq snd\ y$
⟨*proof*⟩

lemma *monofun-fst*: *monofun fst*
⟨*proof*⟩

lemma *monofun-snd*: *monofun snd*
⟨*proof*⟩

lemmas *ch2ch-fst* [*simp*] = *ch2ch-monofun* [*OF monofun-fst*]

lemmas *ch2ch-snd* [*simp*] = *ch2ch-monofun* [*OF monofun-snd*]

lemma *prod-chain-cases*:
assumes *chain*: *chain Y*
obtains *A B*
where *chain A* **and** *chain B* **and** $Y = (\lambda i.\ (A\ i,\ B\ i))$
⟨*proof*⟩

6.4 Product type is a cpo

lemma *is-lub-Pair*: *range A* <<| *x* ⟹ *range B* <<| *y* ⟹ *range* $(\lambda i.\ (A\ i,\ B\ i))$ <<| $(x,\ y)$
⟨*proof*⟩

lemma *lub-Pair*: *chain A* ⟹ *chain B* ⟹ $(\bigsqcup i.\ (A\ i,\ B\ i)) = (\bigsqcup i.\ A\ i, \bigsqcup i.\ B\ i)$
for $A :: nat \Rightarrow 'a$ **and** $B :: nat \Rightarrow 'b$
⟨*proof*⟩

lemma *is-lub-prod*:
fixes $S :: nat \Rightarrow ('a \times 'b)$
assumes *chain S*
shows *range S* <<| $(\bigsqcup i.\ fst\ (S\ i), \bigsqcup i.\ snd\ (S\ i))$
⟨*proof*⟩

lemma *lub-prod*: *chain S* ⟹ $(\bigsqcup i.\ S\ i) = (\bigsqcup i.\ fst\ (S\ i), \bigsqcup i.\ snd\ (S\ i))$
for $S :: nat \Rightarrow 'a \times 'b$
⟨*proof*⟩

instance *prod* :: (*cpo*, *cpo*) *cpo*
⟨*proof*⟩

instance *prod* :: (*discrete-cpo*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

6.5 Product type is pointed

lemma *minimal-prod*: $(\bot,\ \bot) \sqsubseteq p$
⟨*proof*⟩

instance *prod* :: (*pcpo*, *pcpo*) *pcpo*
⟨*proof*⟩

lemma *inst-prod-pcpo*: $\bot = (\bot, \bot)$
⟨*proof*⟩

lemma *Pair-bottom-iff* [*simp*]: $(x, y) = \bot \longleftrightarrow x = \bot \wedge y = \bot$
⟨*proof*⟩

lemma *fst-strict* [*simp*]: *fst* $\bot = \bot$
⟨*proof*⟩

lemma *snd-strict* [*simp*]: *snd* $\bot = \bot$
⟨*proof*⟩

lemma *Pair-strict* [*simp*]: $(\bot, \bot) = \bot$
⟨*proof*⟩

lemma *split-strict* [*simp*]: *case-prod* $f \bot = f \bot \bot$
⟨*proof*⟩

6.6 Continuity of *Pair*, *fst*, *snd*

lemma *cont-pair1*: *cont* $(\lambda x.\ (x, y))$
⟨*proof*⟩

lemma *cont-pair2*: *cont* $(\lambda y.\ (x, y))$
⟨*proof*⟩

lemma *cont-fst*: *cont fst*
⟨*proof*⟩

lemma *cont-snd*: *cont snd*
⟨*proof*⟩

lemma *cont2cont-Pair* [*simp*, *cont2cont*]:
assumes *f*: *cont* $(\lambda x.\ f\ x)$
assumes *g*: *cont* $(\lambda x.\ g\ x)$
shows *cont* $(\lambda x.\ (f\ x, g\ x))$
⟨*proof*⟩

lemmas *cont2cont-fst* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-fst*]

lemmas *cont2cont-snd* [*simp*, *cont2cont*] = *cont-compose* [*OF cont-snd*]

lemma *cont2cont-case-prod*:
assumes *f1*: $\bigwedge a\ b.$ *cont* $(\lambda x.\ f\ x\ a\ b)$
assumes *f2*: $\bigwedge x\ b.$ *cont* $(\lambda a.\ f\ x\ a\ b)$
assumes *f3*: $\bigwedge x\ a.$ *cont* $(\lambda b.\ f\ x\ a\ b)$

assumes *g*: *cont* (λ*x*. *g x*)
shows *cont* (λ*x*. *case g x of* (*a*, *b*) ⇒ *f x a b*)
⟨*proof*⟩

lemma *prod-contI*:
assumes *f1*: ⋀*y*. *cont* (λ*x*. *f* (*x*, *y*))
assumes *f2*: ⋀*x*. *cont* (λ*y*. *f* (*x*, *y*))
shows *cont f*
⟨*proof*⟩

lemma *prod-cont-iff*: *cont f* ⟷ (∀ *y*. *cont* (λ*x*. *f* (*x*, *y*))) ∧ (∀ *x*. *cont* (λ*y*. *f* (*x*, *y*)))
⟨*proof*⟩

lemma *cont2cont-case-prod'* [*simp*, *cont2cont*]:
assumes *f*: *cont* (λ*p*. *f* (*fst p*) (*fst* (*snd p*)) (*snd* (*snd p*)))
assumes *g*: *cont* (λ*x*. *g x*)
shows *cont* (λ*x*. *case-prod* (*f x*) (*g x*))
⟨*proof*⟩

The simple version (due to Joachim Breitner) is needed if either element type of the pair is not a cpo.

lemma *cont2cont-split-simple* [*simp*, *cont2cont*]:
assumes ⋀*a b*. *cont* (λ*x*. *f x a b*)
shows *cont* (λ*x*. *case p of* (*a*, *b*) ⇒ *f x a b*)
⟨*proof*⟩

Admissibility of predicates on product types.

lemma *adm-case-prod* [*simp*]:
assumes *adm* (λ*x*. *P x* (*fst* (*f x*)) (*snd* (*f x*)))
shows *adm* (λ*x*. *case f x of* (*a*, *b*) ⇒ *P x a b*)
⟨*proof*⟩

6.7 Compactness and chain-finiteness

lemma *fst-below-iff*: *fst x* ⊑ *y* ⟷ *x* ⊑ (*y*, *snd x*) **for** *x* :: *'a* × *'b*
⟨*proof*⟩

lemma *snd-below-iff*: *snd x* ⊑ *y* ⟷ *x* ⊑ (*fst x*, *y*) **for** *x* :: *'a* × *'b*
⟨*proof*⟩

lemma *compact-fst*: *compact x* ⟹ *compact* (*fst x*)
⟨*proof*⟩

lemma *compact-snd*: *compact x* ⟹ *compact* (*snd x*)
⟨*proof*⟩

lemma *compact-Pair*: *compact x* ⟹ *compact y* ⟹ *compact* (*x*, *y*)
⟨*proof*⟩

lemma *compact-Pair-iff* [*simp*]: *compact* (*x*, *y*) ⟷ *compact x* ∧ *compact y*
⟨*proof*⟩

instance *prod* :: (*chfin*, *chfin*) *chfin*
⟨*proof*⟩

7 Discrete cpo types

datatype *'a discr* = *Discr 'a*::*type*

7.1 Discrete cpo class instance

instantiation *discr* :: (*type*) *discrete-cpo*
begin

definition ((⊑) :: *'a discr* ⇒ *'a discr* ⇒ *bool*) = (=)

instance
⟨*proof*⟩

end

7.2 *undiscr*

definition *undiscr* :: *'a*::*type discr* ⇒ *'a*
where *undiscr x* = (*case x of Discr y* ⇒ *y*)

lemma *undiscr-Discr* [*simp*]: *undiscr* (*Discr x*) = *x*
⟨*proof*⟩

lemma *Discr-undiscr* [*simp*]: *Discr* (*undiscr y*) = *y*
⟨*proof*⟩

end

8 Subtypes of pcpos

theory *Cpodef*
imports *Cpo*
keywords *pcpodef cpodef* :: *thy-goal-defn*
begin

8.1 Proving a subtype is a partial order

A subtype of a partial order is itself a partial order, if the ordering is defined in the standard way.

theorem (**in** *below*) *typedef-class-po*:

fixes *Abs* :: *'b*::*po* ⇒ *'a*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *class.po below*
⟨*proof*⟩

lemmas *typedef-po-class* = *below.typedef-class-po* [*THEN po.intro-of-class*]

8.2 Proving a subtype is finite

lemma *typedef-finite-UNIV*:
fixes *Abs* :: *'a*::*type* ⇒ *'b*::*type*
assumes *type*: *type-definition Rep Abs A*
shows *finite A* ⟹ *finite* (*UNIV* :: *'b set*)
⟨*proof*⟩

8.3 Proving a subtype is chain-finite

lemma *ch2ch-Rep*:
assumes *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *chain S* ⟹ *chain* (λ*i*. *Rep* (*S i*))
⟨*proof*⟩

theorem *typedef-chfin*:
fixes *Abs* :: *'a*::*chfin* ⇒ *'b*::*po*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *OFCLASS*(*'b*, *chfin-class*)
⟨*proof*⟩

8.4 Proving a subtype is complete

A subtype of a cpo is itself a cpo if the ordering is defined in the standard way, and the defining subset is closed with respect to limits of chains. A set is closed if and only if membership in the set is an admissible predicate.

lemma *typedef-is-lubI*:
assumes *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
shows *range* (λ*i*. *Rep* (*S i*)) <<| *Rep x* ⟹ *range S* <<| *x*
⟨*proof*⟩

lemma *Abs-inverse-lub-Rep*:
fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*po*
assumes *type*: *type-definition Rep Abs A*
and *below*: (⊑) ≡ λ*x y*. *Rep x* ⊑ *Rep y*
and *adm*: *adm* (λ*x*. *x* ∈ *A*)
shows *chain S* ⟹ *Rep* (*Abs* (⨆ *i*. *Rep* (*S i*))) = (⨆ *i*. *Rep* (*S i*))
⟨*proof*⟩

theorem *typedef-is-lub*:

fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*po*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: *adm* $(\lambda x.\ x \in A)$
assumes *S*: *chain S*
shows *range S* <<| *Abs* $(\bigsqcup i.\ Rep\ (S\ i))$
⟨*proof*⟩

lemmas *typedef-lub* = *typedef-is-lub* [*THEN lub-eqI*]

theorem *typedef-cpo*:
fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*po*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: *adm* $(\lambda x.\ x \in A)$
shows *OFCLASS*(*'b*, *cpo-class*)
⟨*proof*⟩

8.4.1 Continuity of *Rep* and *Abs*

For any sub-cpo, the *Rep* function is continuous.

theorem *typedef-cont-Rep*:
fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*cpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: *adm* $(\lambda x.\ x \in A)$
shows *cont* $(\lambda x.\ f\ x) \Longrightarrow$ *cont* $(\lambda x.\ Rep\ (f\ x))$
⟨*proof*⟩

For a sub-cpo, we can make the *Abs* function continuous only if we restrict its domain to the defining subset by composing it with another continuous function.

theorem *typedef-cont-Abs*:
fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*cpo*
fixes *f* :: *'c*::*cpo* ⇒ *'a*::*cpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *adm*: *adm* $(\lambda x.\ x \in A)$
and *f-in-A*: $\bigwedge x.\ f\ x \in A$
shows *cont f* $\Longrightarrow$ *cont* $(\lambda x.\ Abs\ (f\ x))$
⟨*proof*⟩

8.5 Proving subtype elements are compact

theorem *typedef-compact*:
fixes *Abs* :: *'a*::*cpo* ⇒ *'b*::*cpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$

and *adm*: *adm* $(\lambda x.\ x \in A)$
shows *compact* $(Rep\ k) \Longrightarrow$ *compact* k
⟨*proof*⟩

8.6 Proving a subtype is pointed

A subtype of a cpo has a least element if and only if the defining subset has a least element.

theorem *typedef-pcpo-generic*:
fixes *Abs* :: $'a$::*cpo* ⇒ $'b$::*cpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *z-in-A*: $z \in A$
and *z-least*: $\bigwedge x.\ x \in A \Longrightarrow z \sqsubseteq x$
shows *OFCLASS*($'b$, *pcpo-class*)
⟨*proof*⟩

As a special case, a subtype of a pcpo has a least element if the defining subset contains $\bot$.

theorem *typedef-pcpo*:
fixes *Abs* :: $'a$::*pcpo* ⇒ $'b$::*cpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows *OFCLASS*($'b$, *pcpo-class*)
⟨*proof*⟩

8.6.1 Strictness of *Rep* and *Abs*

For a sub-pcpo where $\bot$ is a member of the defining subset, *Rep* and *Abs* are both strict.

theorem *typedef-Abs-strict*:
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows $Abs\ \bot = \bot$
⟨*proof*⟩

theorem *typedef-Rep-strict*:
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows $Rep\ \bot = \bot$
⟨*proof*⟩

theorem *typedef-Abs-bottom-iff*:
assumes *type*: *type-definition Rep Abs A*

and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows $x \in A \Longrightarrow (Abs\ x = \bot) = (x = \bot)$
⟨*proof*⟩

theorem *typedef-Rep-bottom-iff*:
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows $(Rep\ x = \bot) = (x = \bot)$
⟨*proof*⟩

8.7 Proving a subtype is flat

theorem *typedef-flat*:
fixes *Abs* :: $'a$::*flat* ⇒ $'b$::*pcpo*
assumes *type*: *type-definition Rep Abs A*
and *below*: $(\sqsubseteq) \equiv \lambda x\ y.\ Rep\ x \sqsubseteq Rep\ y$
and *bottom-in-A*: $\bot \in A$
shows *OFCLASS*($'b$, *flat-class*)
⟨*proof*⟩

8.8 HOLCF type definition package

⟨*ML*⟩

end

9 The type of continuous functions

theory *Cfun*
imports *Cpodef*
begin

9.1 Definition of continuous function type

definition *cfun* = $\{f::'a \Rightarrow 'b.\ cont\ f\}$

cpodef ($'a$, $'b$) *cfun* (‹(‹*notation*=‹*infix* →››- →/ -)› [*1*, *0*] *0*) = *cfun* :: ($'a$ ⇒ $'b$) *set*
⟨*proof*⟩

type-notation (*ASCII*)
cfun (**infixr** ‹–>› *0*)

notation (*ASCII*)
Rep-cfun (‹(‹*notation*=‹*infix* $›-$/-)› [*999*,*1000*] *999*)

notation

Rep-cfun (‹(‹*notation*=‹*infix* ·››-·/-)› [*999*,*1000*] *999*)

9.2 Syntax for continuous lambda abstraction

syntax -*cabs* :: [*logic*, *logic*] ⇒ *logic*

⟨*ML*⟩

Syntax for nested abstractions

syntax (*ASCII*)
-*Lambda* :: [*cargs*, *logic*] ⇒ *logic* (‹(‹*indent*=*3* *notation*=‹*binder LAM*››*LAM* -./ -)› [*1000*, *10*] *10*)

syntax
-*Lambda* :: [*cargs*, *logic*] ⇒ *logic* (‹(‹*indent*=*3* *notation*=‹*binder* Λ››Λ -./ -)› [*1000*, *10*] *10*)

syntax-consts
-*Lambda* ⇌ *Abs-cfun*

⟨*ML*⟩

Dummy patterns for continuous abstraction

translations
Λ -. *t* ⇀ *CONST Abs-cfun* (λ-. *t*)

9.3 Continuous function space is pointed

lemma *bottom-cfun*: ⊥ ∈ *cfun*
⟨*proof*⟩

instance *cfun* :: (*cpo*, *discrete-cpo*) *discrete-cpo*
⟨*proof*⟩

instance *cfun* :: (*cpo*, *pcpo*) *pcpo*
⟨*proof*⟩

lemmas *Rep-cfun-strict* =
typedef-Rep-strict [*OF type-definition-cfun below-cfun-def bottom-cfun*]

lemmas *Abs-cfun-strict* =
typedef-Abs-strict [*OF type-definition-cfun below-cfun-def bottom-cfun*]

function application is strict in its first argument

lemma *Rep-cfun-strict1* [*simp*]: ⊥·*x* = ⊥
⟨*proof*⟩

lemma *LAM-strict* [*simp*]: (Λ *x*. ⊥) = ⊥
⟨*proof*⟩

for compatibility with old HOLCF-Version

lemma *inst-cfun-pcpo*: $\bot = (\Lambda\ x.\ \bot)$
⟨*proof*⟩

9.4 Basic properties of continuous functions

Beta-equality for continuous functions

lemma *Abs-cfun-inverse2*: *cont* $f \Longrightarrow$ *Rep-cfun* (*Abs-cfun* f) $= f$
⟨*proof*⟩

lemma *beta-cfun*: *cont* $f \Longrightarrow (\Lambda\ x.\ f\ x)\cdot u = f\ u$
⟨*proof*⟩

9.4.1 Beta-reduction simproc

Given the term $(\Lambda\ x.\ f\ x)\cdot y$, the procedure tries to construct the theorem $(\Lambda\ x.\ f\ x)\cdot y \equiv f\ y$. If this theorem cannot be completely solved by the cont2cont rules, then the procedure returns the ordinary conditional *beta-cfun* rule.

The simproc does not solve any more goals that would be solved by using *beta-cfun* as a simp rule. The advantage of the simproc is that it can avoid deeply-nested calls to the simplifier that would otherwise be caused by large continuity side conditions.

Update: The simproc now uses rule *Abs-cfun-inverse2* instead of *beta-cfun*, to avoid problems with eta-contraction.

⟨*ML*⟩

Eta-equality for continuous functions

lemma *eta-cfun*: $(\Lambda\ x.\ f{\cdot}x) = f$
⟨*proof*⟩

Extensionality for continuous functions

lemma *cfun-eq-iff*: $f = g \longleftrightarrow (\forall\, x.\ f{\cdot}x = g{\cdot}x)$
⟨*proof*⟩

lemma *cfun-eqI*: $(\bigwedge x.\ f{\cdot}x = g{\cdot}x) \Longrightarrow f = g$
⟨*proof*⟩

Extensionality wrt. ordering for continuous functions

lemma *cfun-below-iff*: $f \sqsubseteq g \longleftrightarrow (\forall\, x.\ f{\cdot}x \sqsubseteq g{\cdot}x)$
⟨*proof*⟩

lemma *cfun-belowI*: $(\bigwedge x.\ f{\cdot}x \sqsubseteq g{\cdot}x) \Longrightarrow f \sqsubseteq g$
⟨*proof*⟩

Congruence for continuous function application

lemma *cfun-cong*: $f = g \Longrightarrow x = y \Longrightarrow f{\cdot}x = g{\cdot}y$

⟨*proof*⟩

lemma *cfun-fun-cong*: $f = g \Longrightarrow f{\cdot}x = g{\cdot}x$
⟨*proof*⟩

lemma *cfun-arg-cong*: $x = y \Longrightarrow f{\cdot}x = f{\cdot}y$
⟨*proof*⟩

9.5 Continuity of application

lemma *cont-Rep-cfun1*: *cont* $(\lambda f.\ f{\cdot}x)$
⟨*proof*⟩

lemma *cont-Rep-cfun2*: *cont* $(\lambda x.\ f{\cdot}x)$
⟨*proof*⟩

lemmas *monofun-Rep-cfun* = *cont-Rep-cfun* [*THEN cont2mono*]

lemmas *monofun-Rep-cfun1* = *cont-Rep-cfun1* [*THEN cont2mono*]
lemmas *monofun-Rep-cfun2* = *cont-Rep-cfun2* [*THEN cont2mono*]

contlub, cont properties of *Rep-cfun* in each argument

lemma *contlub-cfun-arg*: *chain* $Y \Longrightarrow f{\cdot}(\bigsqcup i.\ Y\ i) = (\bigsqcup i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

lemma *contlub-cfun-fun*: *chain* $F \Longrightarrow (\bigsqcup i.\ F\ i){\cdot}x = (\bigsqcup i.\ F\ i{\cdot}x)$
⟨*proof*⟩

monotonicity of application

lemma *monofun-cfun-fun*: $f \sqsubseteq g \Longrightarrow f{\cdot}x \sqsubseteq g{\cdot}x$
⟨*proof*⟩

lemma *monofun-cfun-arg*: $x \sqsubseteq y \Longrightarrow f{\cdot}x \sqsubseteq f{\cdot}y$
⟨*proof*⟩

lemma *monofun-cfun*: $f \sqsubseteq g \Longrightarrow x \sqsubseteq y \Longrightarrow f{\cdot}x \sqsubseteq g{\cdot}y$
⟨*proof*⟩

ch2ch - rules for the type $'a \to 'b$

lemma *chain-monofun*: *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

lemma *ch2ch-Rep-cfunR*: *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ f{\cdot}(Y\ i))$
⟨*proof*⟩

lemma *ch2ch-Rep-cfunL*: *chain* $F \Longrightarrow$ *chain* $(\lambda i.\ (F\ i){\cdot}x)$
⟨*proof*⟩

lemma *ch2ch-Rep-cfun* [*simp*]: *chain* $F \Longrightarrow$ *chain* $Y \Longrightarrow$ *chain* $(\lambda i.\ (F\ i){\cdot}(Y\ i))$

⟨*proof*⟩

lemma *ch2ch-LAM* [*simp*]:
(⋀*x*. *chain* (λ*i*. *S i x*)) ⟹ (⋀*i*. *cont* (λ*x*. *S i x*)) ⟹ *chain* (λ*i*. Λ *x*. *S i x*)
⟨*proof*⟩

contlub, cont properties of *Rep-cfun* in both arguments

lemma *lub-APP*: *chain F* ⟹ *chain Y* ⟹ (⨆ *i*. *F i*·(*Y i*)) = (⨆ *i*. *F i*)·(⨆ *i*. *Y i*)
⟨*proof*⟩

lemma *lub-LAM*:
assumes ⋀*x*. *chain* (λ*i*. *F i x*)
and ⋀*i*. *cont* (λ*x*. *F i x*)
shows (⨆ *i*. Λ *x*. *F i x*) = (Λ *x*. ⨆ *i*. *F i x*)
⟨*proof*⟩

lemmas *lub-distribs* = *lub-APP lub-LAM*

strictness

lemma *strictI*: *f*·*x* = ⊥ ⟹ *f*·⊥ = ⊥
⟨*proof*⟩

type $'a \to 'b$ is chain complete

lemma *lub-cfun*: *chain F* ⟹ (⨆ *i*. *F i*) = (Λ *x*. ⨆ *i*. *F i*·*x*)
⟨*proof*⟩

9.6 Continuity simplification procedure

cont2cont lemma for *Rep-cfun*

lemma *cont2cont-APP* [*simp*, *cont2cont*]:
assumes *f*: *cont* (λ*x*. *f x*)
assumes *t*: *cont* (λ*x*. *t x*)
shows *cont* (λ*x*. (*f x*)·(*t x*))
⟨*proof*⟩

Two specific lemmas for the combination of LCF and HOL terms. These lemmas are needed in theories that use types like $'a \to 'b \Rightarrow 'c$.

lemma *cont-APP-app* [*simp*]: *cont f* ⟹ *cont g* ⟹ *cont* (λ*x*. ((*f x*)·(*g x*)) *s*)
⟨*proof*⟩

lemma *cont-APP-app-app* [*simp*]: *cont f* ⟹ *cont g* ⟹ *cont* (λ*x*. ((*f x*)·(*g x*)) *s t*)
⟨*proof*⟩

cont2mono Lemma for λ*x*. Λ *y*. *c1 x y*

lemma *cont2mono-LAM*:

⟦⋀x. cont (λy. f x y); ⋀y. monofun (λx. f x y)⟧
  ⟹ monofun (λx. Λ y. f x y)
⟨proof⟩

cont2cont Lemma for λx. Λ y. f x y

Not suitable as a cont2cont rule, because on nested lambdas it causes exponential blow-up in the number of subgoals.

lemma *cont2cont-LAM*:
  assumes *f1*: ⋀x. cont (λy. f x y)
  assumes *f2*: ⋀y. cont (λx. f x y)
  shows cont (λx. Λ y. f x y)
⟨proof⟩

This version does work as a cont2cont rule, since it has only a single subgoal.

lemma *cont2cont-LAM'* [*simp*, *cont2cont*]:
  fixes f :: 'a::cpo ⇒ 'b::cpo ⇒ 'c::cpo
  assumes f: cont (λp. f (fst p) (snd p))
  shows cont (λx. Λ y. f x y)
  ⟨proof⟩

lemma *cont2cont-LAM-discrete* [*simp*, *cont2cont*]:
  (⋀y::'a::discrete-cpo. cont (λx. f x y)) ⟹ cont (λx. Λ y. f x y)
  ⟨proof⟩

9.7 Miscellaneous

Monotonicity of *Abs-cfun*

lemma *monofun-LAM*: cont f ⟹ cont g ⟹ (⋀x. f x ⊑ g x) ⟹ (Λ x. f x) ⊑ (Λ x. g x)
  ⟨proof⟩

some lemmata for functions with flat/chfin domain/range types

lemma *chfin-Rep-cfunR*: chain Y ⟹ ∀ s. ∃ n. (LUB i. Y i)·s = Y n·s
  for Y :: nat ⇒ 'a::cpo → 'b::chfin
  ⟨proof⟩

lemma *adm-chfindom*: adm (λ(u::'a::cpo → 'b::chfin). P(u·s))
  ⟨proof⟩

9.8 Continuous injection-retraction pairs

Continuous retractions are strict.

lemma *retraction-strict*: ∀ x. f·(g·x) = x ⟹ f·⊥ = ⊥
  ⟨proof⟩

lemma *injection-eq*: ∀ x. f·(g·x) = x ⟹ (g·x = g·y) = (x = y)

⟨*proof*⟩

lemma *injection-below*: $\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow (g{\cdot}x \sqsubseteq g{\cdot}y) = (x \sqsubseteq y)$
⟨*proof*⟩

lemma *injection-defined-rev*: $\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow g{\cdot}z = \bot \Longrightarrow z = \bot$
⟨*proof*⟩

lemma *injection-defined*: $\forall x.\ f{\cdot}(g{\cdot}x) = x \Longrightarrow z \neq \bot \Longrightarrow g{\cdot}z \neq \bot$
⟨*proof*⟩

a result about functions with flat codomain

lemma *flat-eqI*: $x \sqsubseteq y \Longrightarrow x \neq \bot \Longrightarrow x = y$
for $x\ y :: 'a{::}flat$
⟨*proof*⟩

lemma *flat-codom*: $f{\cdot}x = c \Longrightarrow f{\cdot}\bot = \bot \vee (\forall z.\ f{\cdot}z = c)$
for $c :: 'b{::}flat$
⟨*proof*⟩

9.9 Identity and composition

definition $ID :: 'a \to 'a$
where $ID = (\Lambda\ x.\ x)$

definition *cfcomp* $:: ('b \to 'c) \to ('a \to 'b) \to 'a \to 'c$
where *oo-def*: $cfcomp = (\Lambda\ f\ g\ x.\ f{\cdot}(g{\cdot}x))$

abbreviation *cfcomp-syn* $:: ['b \to 'c,\ 'a \to 'b] \Rightarrow 'a \to 'c$ (**infixr** ‹*oo*› *100*)
where $f\ oo\ g == cfcomp{\cdot}f{\cdot}g$

lemma *ID1* [*simp*]: $ID{\cdot}x = x$
⟨*proof*⟩

lemma *cfcomp1*: $(f\ oo\ g) = (\Lambda\ x.\ f{\cdot}(g{\cdot}x))$
⟨*proof*⟩

lemma *cfcomp2* [*simp*]: $(f\ oo\ g){\cdot}x = f{\cdot}(g{\cdot}x)$
⟨*proof*⟩

lemma *cfcomp-LAM*: $cont\ g \Longrightarrow f\ oo\ (\Lambda\ x.\ g\ x) = (\Lambda\ x.\ f{\cdot}(g\ x))$
⟨*proof*⟩

lemma *cfcomp-strict* [*simp*]: $\bot\ oo\ f = \bot$
⟨*proof*⟩

Show that interpretation of (pcpo, -→-) is a category.

- The class of objects is interpretation of syntactical class pcpo.

- The class of arrows between objects $'a$ and $'b$ is interpret. of $'a \to 'b$.
- The identity arrow is interpretation of *ID*.
- The composition of f and g is interpretation of *oo*.

lemma *ID2* [*simp*]: $f \mathbin{oo} ID = f$
⟨*proof*⟩

lemma *ID3* [*simp*]: $ID \mathbin{oo} f = f$
⟨*proof*⟩

lemma *assoc-oo*: $f \mathbin{oo} (g \mathbin{oo} h) = (f \mathbin{oo} g) \mathbin{oo} h$
⟨*proof*⟩

9.10 Strictified functions

definition *seq* :: $'a{::}pcpo \to 'b{::}pcpo \to 'b$
where $seq = (\Lambda\ x.\ \mathit{if}\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ ID)$

lemma *cont2cont-if-bottom* [*cont2cont*, *simp*]:
assumes f: $\mathit{cont}\ (\lambda x.\ f\ x)$
and g: $\mathit{cont}\ (\lambda x.\ g\ x)$
shows $\mathit{cont}\ (\lambda x.\ \mathit{if}\ f\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ g\ x)$
⟨*proof*⟩

lemma *seq-conv-if*: $seq{\cdot}x = (\mathit{if}\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ ID)$
⟨*proof*⟩

lemma *seq-simps* [*simp*]:
$seq{\cdot}\bot = \bot$
$seq{\cdot}x{\cdot}\bot = \bot$
$x \neq \bot \Longrightarrow seq{\cdot}x = ID$
⟨*proof*⟩

definition *strictify* :: $('a{::}pcpo \to 'b{::}pcpo) \to 'a \to 'b$
where $strictify = (\Lambda\ f\ x.\ seq{\cdot}x{\cdot}(f{\cdot}x))$

lemma *strictify-conv-if*: $strictify{\cdot}f{\cdot}x = (\mathit{if}\ x = \bot\ \mathit{then}\ \bot\ \mathit{else}\ f{\cdot}x)$
⟨*proof*⟩

lemma *strictify1* [*simp*]: $strictify{\cdot}f{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *strictify2* [*simp*]: $x \neq \bot \Longrightarrow strictify{\cdot}f{\cdot}x = f{\cdot}x$
⟨*proof*⟩

9.11 Continuity of let-bindings

lemma *cont2cont-Let*:

assumes f: *cont* $(\lambda x.\ f\ x)$
assumes *g1*: $\bigwedge y.$ *cont* $(\lambda x.\ g\ x\ y)$
assumes *g2*: $\bigwedge x.$ *cont* $(\lambda y.\ g\ x\ y)$
shows *cont* $(\lambda x.$ *let* $y = f\ x$ *in* $g\ x\ y)$
⟨*proof*⟩

lemma *cont2cont-Let'* [*simp*, *cont2cont*]:
assumes f: *cont* $(\lambda x.\ f\ x)$
assumes g: *cont* $(\lambda p.\ g\ (fst\ p)\ (snd\ p))$
shows *cont* $(\lambda x.$ *let* $y = f\ x$ *in* $g\ x\ y)$
⟨*proof*⟩

The simple version (suggested by Joachim Breitner) is needed if the type of the defined term is not a cpo.

lemma *cont2cont-Let-simple* [*simp*, *cont2cont*]:
assumes $\bigwedge y.$ *cont* $(\lambda x.\ g\ x\ y)$
shows *cont* $(\lambda x.$ *let* $y = t$ *in* $g\ x\ y)$
⟨*proof*⟩

end

10 Continuous deflations and ep-pairs

theory *Deflation*
imports *Cfun*
begin

10.1 Continuous deflations

locale *deflation* =
fixes $d :: 'a \to 'a$
assumes *idem*: $\bigwedge x.\ d{\cdot}(d{\cdot}x) = d{\cdot}x$
assumes *below*: $\bigwedge x.\ d{\cdot}x \sqsubseteq x$
begin

lemma *below-ID*: $d \sqsubseteq ID$
⟨*proof*⟩

The set of fixed points is the same as the range.

lemma *fixes-eq-range*: $\{x.\ d{\cdot}x = x\} = range\ (\lambda x.\ d{\cdot}x)$
⟨*proof*⟩

lemma *range-eq-fixes*: $range\ (\lambda x.\ d{\cdot}x) = \{x.\ d{\cdot}x = x\}$
⟨*proof*⟩

The pointwise ordering on deflation functions coincides with the subset ordering of their sets of fixed-points.

lemma *belowI*:

assumes f: $\bigwedge x.\ d{\cdot}x = x \Longrightarrow f{\cdot}x = x$
shows $d \sqsubseteq f$
⟨*proof*⟩

lemma *belowD*: $[\![f \sqsubseteq d;\ f{\cdot}x = x]\!] \Longrightarrow d{\cdot}x = x$
⟨*proof*⟩

end

lemma *deflation-strict*: *deflation* $d \Longrightarrow d{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *adm-deflation*: *adm* $(\lambda d.\ \mathit{deflation}\ d)$
⟨*proof*⟩

lemma *deflation-ID*: *deflation ID*
⟨*proof*⟩

lemma *deflation-bottom*: *deflation* $\bot$
⟨*proof*⟩

lemma *deflation-below-iff*: *deflation* $p \Longrightarrow$ *deflation* $q \Longrightarrow p \sqsubseteq q \longleftrightarrow (\forall x.\ p{\cdot}x = x \longrightarrow q{\cdot}x = x)$
⟨*proof*⟩

The composition of two deflations is equal to the lesser of the two (if they are comparable).

lemma *deflation-below-comp1*:
assumes *deflation f*
assumes *deflation g*
shows $f \sqsubseteq g \Longrightarrow f{\cdot}(g{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

lemma *deflation-below-comp2*: *deflation* $f \Longrightarrow$ *deflation* $g \Longrightarrow f \sqsubseteq g \Longrightarrow g{\cdot}(f{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

10.2 Deflations with finite range

lemma *finite-range-imp-finite-fixes*:
assumes *finite* (*range f*)
shows *finite* $\{x.\ f\,x = x\}$
⟨*proof*⟩

locale *finite-deflation* = *deflation* +
assumes *finite-fixes*: *finite* $\{x.\ d{\cdot}x = x\}$
begin

lemma *finite-range*: *finite* (*range* $(\lambda x.\ d{\cdot}x)$)

⟨*proof*⟩

lemma *finite-image*: *finite* ((λ*x*. *d*·*x*) ‘ *A*)
⟨*proof*⟩

lemma *compact*: *compact* (*d*·*x*)
⟨*proof*⟩

end

lemma *finite-deflation-intro*: *deflation d* ⟹ *finite* {*x*. *d*·*x* = *x*} ⟹ *finite-deflation d*
⟨*proof*⟩

lemma *finite-deflation-imp-deflation*: *finite-deflation d* ⟹ *deflation d*
⟨*proof*⟩

lemma *finite-deflation-bottom*: *finite-deflation* ⊥
⟨*proof*⟩

10.3 Continuous embedding-projection pairs

locale *ep-pair* =
fixes *e* :: ′*a* → ′*b* **and** *p* :: ′*b* → ′*a*
assumes *e-inverse* [*simp*]: ⋀*x*. *p*·(*e*·*x*) = *x*
and *e-p-below*: ⋀*y*. *e*·(*p*·*y*) ⊑ *y*
begin

lemma *e-below-iff* [*simp*]: *e*·*x* ⊑ *e*·*y* ⟷ *x* ⊑ *y*
⟨*proof*⟩

lemma *e-eq-iff* [*simp*]: *e*·*x* = *e*·*y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *p-eq-iff*: *e*·(*p*·*x*) = *x* ⟹ *e*·(*p*·*y*) = *y* ⟹ *p*·*x* = *p*·*y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *p-inverse*: (∃ *x*. *y* = *e*·*x*) ⟷ *e*·(*p*·*y*) = *y*
⟨*proof*⟩

lemma *e-below-iff-below-p*: *e*·*x* ⊑ *y* ⟷ *x* ⊑ *p*·*y*
⟨*proof*⟩

lemma *compact-e-rev*: *compact* (*e*·*x*) ⟹ *compact x*
⟨*proof*⟩

lemma *compact-e*:
assumes *compact x*
shows *compact* (*e*·*x*)

⟨*proof*⟩

lemma *compact-e-iff*: *compact* (*e*·*x*) ⟷ *compact x*
⟨*proof*⟩

Deflations from ep-pairs

lemma *deflation-e-p*: *deflation* (*e oo p*)
⟨*proof*⟩

lemma *deflation-e-d-p*:
assumes *deflation d*
shows *deflation* (*e oo d oo p*)
⟨*proof*⟩

lemma *finite-deflation-e-d-p*:
assumes *finite-deflation d*
shows *finite-deflation* (*e oo d oo p*)
⟨*proof*⟩

lemma *deflation-p-d-e*:
assumes *deflation d*
assumes *d*: ⋀*x*. *d*·*x* ⊑ *e*·(*p*·*x*)
shows *deflation* (*p oo d oo e*)
⟨*proof*⟩

lemma *finite-deflation-p-d-e*:
assumes *finite-deflation d*
assumes *d*: ⋀*x*. *d*·*x* ⊑ *e*·(*p*·*x*)
shows *finite-deflation* (*p oo d oo e*)
⟨*proof*⟩

end

10.4 Uniqueness of ep-pairs

lemma *ep-pair-unique-e-lemma*:
assumes *1*: *ep-pair e1 p*
and *2*: *ep-pair e2 p*
shows *e1* ⊑ *e2*
⟨*proof*⟩

lemma *ep-pair-unique-e*: *ep-pair e1 p* ⟹ *ep-pair e2 p* ⟹ *e1* = *e2*
⟨*proof*⟩

lemma *ep-pair-unique-p-lemma*:
assumes *1*: *ep-pair e p1*
and *2*: *ep-pair e p2*
shows *p1* ⊑ *p2*
⟨*proof*⟩

lemma *ep-pair-unique-p*: *ep-pair e p1* $\Longrightarrow$ *ep-pair e p2* $\Longrightarrow$ *p1* = *p2*
⟨*proof*⟩

10.5 Composing ep-pairs

lemma *ep-pair-ID-ID*: *ep-pair ID ID*
⟨*proof*⟩

lemma *ep-pair-comp*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*e2 oo e1*) (*p1 oo p2*)
⟨*proof*⟩

locale *pcpo-ep-pair* = *ep-pair e p*
for *e* :: *'a*::*pcpo* $\rightarrow$ *'b*::*pcpo*
and *p* :: *'b*::*pcpo* $\rightarrow$ *'a*::*pcpo*
begin

lemma *e-strict* [*simp*]: *e*·$\bot$ = $\bot$
⟨*proof*⟩

lemma *e-bottom-iff* [*simp*]: *e*·*x* = $\bot$ $\longleftrightarrow$ *x* = $\bot$
⟨*proof*⟩

lemma *e-defined*: *x* ≠ $\bot$ $\Longrightarrow$ *e*·*x* ≠ $\bot$
⟨*proof*⟩

lemma *p-strict* [*simp*]: *p*·$\bot$ = $\bot$
⟨*proof*⟩

lemmas *stricts* = *e-strict p-strict*

end

end

11 The type of strict products

theory *Sprod*
imports *Cfun*
begin

11.1 Definition of strict product type

definition *sprod* = {*p*::*'a*::*pcpo* × *'b*::*pcpo*. *p* = $\bot$ ∨ (*fst p* ≠ $\bot$ ∧ *snd p* ≠ $\bot$)}

pcpodef (*'a*::*pcpo*, *'b*::*pcpo*) *sprod* (‹(‹*notation*=‹*infix strict product*››- ⊗/ -)› [*21*,*20*] *20*) =

sprod :: (*'a* × *'b*) *set*
⟨*proof*⟩

instance *sprod* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
⟨*proof*⟩

type-notation (*ASCII*)
sprod (**infixr** ‹**›› *20*)

11.2 Definitions of constants

definition *sfst* :: (*'a*::*pcpo* ** *'b*::*pcpo*) → *'a*
where *sfst* = (Λ *p*. *fst* (*Rep-sprod p*))

definition *ssnd* :: (*'a*::*pcpo* ** *'b*::*pcpo*) → *'b*
where *ssnd* = (Λ *p*. *snd* (*Rep-sprod p*))

definition *spair* :: *'a*::*pcpo* → *'b*::*pcpo* → (*'a* ** *'b*)
where *spair* = (Λ *a b*. *Abs-sprod* (*seq*·*b*·*a*, *seq*·*a*·*b*))

definition *ssplit* :: (*'a*::*pcpo* → *'b*::*pcpo* → *'c*::*pcpo*) → (*'a* ** *'b*) → *'c*
where *ssplit* = (Λ *f p*. *seq*·*p*·(*f*·(*sfst*·*p*)·(*ssnd*·*p*)))

syntax
-stuple :: [*logic*, *args*] ⇒ *logic* (‹(‹*indent=1 notation=‹mixfix strict tuple›*›'(:-,/ -:')›)›)
syntax-consts
-stuple ⇌ *spair*
translations
(:*x*, *y*, *z*:) ⇌ (:*x*, (:*y*, *z*:):)
(:*x*, *y*:) ⇌ *CONST spair*·*x*·*y*

translations
Λ(*CONST spair*·*x*·*y*). *t* ⇌ *CONST ssplit*·(Λ *x y*. *t*)

11.3 Case analysis

lemma *spair-sprod*: (*seq*·*b*·*a*, *seq*·*a*·*b*) ∈ *sprod*
⟨*proof*⟩

lemma *Rep-sprod-spair*: *Rep-sprod* (:*a*, *b*:) = (*seq*·*b*·*a*, *seq*·*a*·*b*)
⟨*proof*⟩

lemmas *Rep-sprod-simps* =
Rep-sprod-inject [*symmetric*] *below-sprod-def*
prod-eq-iff below-prod-def
Rep-sprod-strict Rep-sprod-spair

lemma *sprodE* [*case-names bottom spair*, *cases type*: *sprod*]:
obtains *p* = ⊥ | *x y* **where** *p* = (:*x*, *y*:) **and** *x* ≠ ⊥ **and** *y* ≠ ⊥

⟨*proof*⟩

lemma *sprod-induct* [*case-names bottom spair*, *induct type*: *sprod*]:
⟦$P\ \bot$; $\bigwedge x\ y.\ ⟦x \neq \bot; y \neq \bot⟧ \Longrightarrow P\ (:x, y:)⟧ \Longrightarrow P\ x$
⟨*proof*⟩

11.4 Properties of *spair*

lemma *spair-strict1* [*simp*]: $(:\bot, y:) = \bot$
⟨*proof*⟩

lemma *spair-strict2* [*simp*]: $(:x, \bot:) = \bot$
⟨*proof*⟩

lemma *spair-bottom-iff* [*simp*]: $(:x, y:) = \bot \longleftrightarrow x = \bot \vee y = \bot$
⟨*proof*⟩

lemma *spair-below-iff*: $(:a, b:) \sqsubseteq (:c, d:) \longleftrightarrow a = \bot \vee b = \bot \vee (a \sqsubseteq c \wedge b \sqsubseteq d)$
⟨*proof*⟩

lemma *spair-eq-iff*: $(:a, b:) = (:c, d:) \longleftrightarrow a = c \wedge b = d \vee (a = \bot \vee b = \bot) \wedge (c = \bot \vee d = \bot)$
⟨*proof*⟩

lemma *spair-strict*: $x = \bot \vee y = \bot \Longrightarrow (:x, y:) = \bot$
⟨*proof*⟩

lemma *spair-strict-rev*: $(:x, y:) \neq \bot \Longrightarrow x \neq \bot \wedge y \neq \bot$
⟨*proof*⟩

lemma *spair-defined*: $⟦x \neq \bot; y \neq \bot⟧ \Longrightarrow (:x, y:) \neq \bot$
⟨*proof*⟩

lemma *spair-defined-rev*: $(:x, y:) = \bot \Longrightarrow x = \bot \vee y = \bot$
⟨*proof*⟩

lemma *spair-below*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x, y:) \sqsubseteq (:a, b:) \longleftrightarrow x \sqsubseteq a \wedge y \sqsubseteq b$
⟨*proof*⟩

lemma *spair-eq*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x, y:) = (:a, b:) \longleftrightarrow x = a \wedge y = b$
⟨*proof*⟩

lemma *spair-inject*: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow (:x, y:) = (:a, b:) \Longrightarrow x = a \wedge y = b$
⟨*proof*⟩

lemma *inst-sprod-pcpo2*: $\bot = (:\bot, \bot:)$
⟨*proof*⟩

lemma *sprodE2*: $(\bigwedge x\ y.\ p = (:x, y:) \Longrightarrow Q) \Longrightarrow Q$

⟨*proof*⟩

11.5 Properties of *sfst* and *ssnd*

lemma *sfst-strict* [*simp*]: $sfst{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *ssnd-strict* [*simp*]: $ssnd{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *sfst-spair* [*simp*]: $y \neq \bot \Longrightarrow sfst{\cdot}(:x,\ y:) = x$
⟨*proof*⟩

lemma *ssnd-spair* [*simp*]: $x \neq \bot \Longrightarrow ssnd{\cdot}(:x,\ y:) = y$
⟨*proof*⟩

lemma *sfst-bottom-iff* [*simp*]: $sfst{\cdot}p = \bot \longleftrightarrow p = \bot$
⟨*proof*⟩

lemma *ssnd-bottom-iff* [*simp*]: $ssnd{\cdot}p = \bot \longleftrightarrow p = \bot$
⟨*proof*⟩

lemma *sfst-defined*: $p \neq \bot \Longrightarrow sfst{\cdot}p \neq \bot$
⟨*proof*⟩

lemma *ssnd-defined*: $p \neq \bot \Longrightarrow ssnd{\cdot}p \neq \bot$
⟨*proof*⟩

lemma *spair-sfst-ssnd*: $(:sfst{\cdot}p,\ ssnd{\cdot}p:) = p$
⟨*proof*⟩

lemma *below-sprod*: $x \sqsubseteq y \longleftrightarrow sfst{\cdot}x \sqsubseteq sfst{\cdot}y \wedge ssnd{\cdot}x \sqsubseteq ssnd{\cdot}y$
⟨*proof*⟩

lemma *eq-sprod*: $x = y \longleftrightarrow sfst{\cdot}x = sfst{\cdot}y \wedge ssnd{\cdot}x = ssnd{\cdot}y$
⟨*proof*⟩

lemma *sfst-below-iff*: $sfst{\cdot}x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:y,\ ssnd{\cdot}x:)$
⟨*proof*⟩

lemma *ssnd-below-iff*: $ssnd{\cdot}x \sqsubseteq y \longleftrightarrow x \sqsubseteq (:sfst{\cdot}x,\ y:)$
⟨*proof*⟩

11.6 Compactness

lemma *compact-sfst*: $compact\ x \Longrightarrow compact\ (sfst{\cdot}x)$
⟨*proof*⟩

lemma *compact-ssnd*: $compact\ x \Longrightarrow compact\ (ssnd{\cdot}x)$
⟨*proof*⟩

lemma *compact-spair*: *compact* $x \Longrightarrow$ *compact* $y \Longrightarrow$ *compact* $(:x,\ y:)$
⟨*proof*⟩

lemma *compact-spair-iff*: *compact* $(:x,\ y:) \longleftrightarrow x = \bot \vee y = \bot \vee (\textit{compact}\ x \wedge \textit{compact}\ y)$
⟨*proof*⟩

11.7 Properties of *ssplit*

lemma *ssplit1* [*simp*]: $\textit{ssplit}{\cdot}f{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *ssplit2* [*simp*]: $x \neq \bot \Longrightarrow y \neq \bot \Longrightarrow \textit{ssplit}{\cdot}f{\cdot}(:x,\ y:) = f{\cdot}x{\cdot}y$
⟨*proof*⟩

lemma *ssplit3* [*simp*]: $\textit{ssplit}{\cdot}\textit{spair}{\cdot}z = z$
⟨*proof*⟩

11.8 Strict product preserves flatness

instance *sprod* :: (*flat*, *flat*) *flat*
⟨*proof*⟩

end

12 The type of lifted values

theory *Up*
imports *Cfun*
begin

12.1 Definition of new type for lifting

datatype $'a\ u$ (‹(‹*notation*=‹*postfix lifting*››-$_\perp$)› [*1000*] *999*) = *Ibottom* | *Iup* $'a$

primrec *Ifup* :: $('a \to 'b{::}\textit{pcpo}) \Rightarrow 'a\ u \Rightarrow 'b$
where
Ifup f Ibottom = $\bot$
| *Ifup f* (*Iup x*) = $f{\cdot}x$

12.2 Ordering on lifted cpo

instantiation u :: (*cpo*) *below*
begin

definition *below-up-def*:
$(\sqsubseteq) \equiv$
$(\lambda x\ y.$

(*case x of*
Ibottom ⇒ *True*
| *Iup a* ⇒ (*case y of Ibottom* ⇒ *False* | *Iup b* ⇒ *a* ⊑ *b*)))

instance ⟨*proof*⟩

end

lemma *minimal-up* [*iff*]: *Ibottom* ⊑ *z*
⟨*proof*⟩

lemma *not-Iup-below* [*iff*]: *Iup x* ⋢ *Ibottom*
⟨*proof*⟩

lemma *Iup-below* [*iff*]: (*Iup x* ⊑ *Iup y*) = (*x* ⊑ *y*)
⟨*proof*⟩

12.3 Lifted cpo is a partial order

instance *u* :: (*cpo*) *po*
⟨*proof*⟩

12.4 Lifted cpo is a cpo

lemma *is-lub-Iup*: *range S* <<| *x* ⟹ *range* (λ*i*. *Iup* (*S i*)) <<| *Iup x*
⟨*proof*⟩

lemma *up-chain-lemma*:
assumes *Y*: *chain Y*
obtains ∀ *i*. *Y i* = *Ibottom*
| *A k* **where** ∀ *i*. *Iup* (*A i*) = *Y* (*i* + *k*) **and** *chain A* **and** *range Y* <<| *Iup* (⨆ *i*. *A i*)
⟨*proof*⟩

instance *u* :: (*cpo*) *cpo*
⟨*proof*⟩

12.5 Lifted cpo is pointed

instance *u* :: (*cpo*) *pcpo*
⟨*proof*⟩

for compatibility with old HOLCF-Version

lemma *inst-up-pcpo*: ⊥ = *Ibottom*
⟨*proof*⟩

12.6 Continuity of *Iup* and *Ifup*

continuity for *Iup*

lemma *cont-Iup*: *cont Iup*
⟨*proof*⟩

continuity for *Ifup*

lemma *cont-Ifup1*: *cont* (λ*f*. *Ifup f x*)
⟨*proof*⟩

lemma *monofun-Ifup2*: *monofun* (λ*x*. *Ifup f x*)
⟨*proof*⟩

lemma *cont-Ifup2*: *cont* (λ*x*. *Ifup f x*)
⟨*proof*⟩

12.7 Continuous versions of constants

definition *up* :: $'a \to 'a\ u$
where *up* = (Λ *x*. *Iup x*)

definition *fup* :: $('a \to 'b{::}pcpo) \to 'a\ u \to 'b$
where *fup* = (Λ *f p*. *Ifup f p*)

translations
case l of XCONST up·*x* ⇒ *t* ⇌ *CONST fup*·(Λ *x*. *t*)·*l*
case l of (*XCONST up* :: $'a$)·*x* ⇒ *t* ⇀ *CONST fup*·(Λ *x*. *t*)·*l*
Λ(*XCONST up*·*x*). *t* ⇌ *CONST fup*·(Λ *x*. *t*)

continuous versions of lemmas for $'a_\perp$

lemma *Exh-Up*: $z = \perp \vee (\exists x.\ z = up{\cdot}x)$
⟨*proof*⟩

lemma *up-eq* [*simp*]: $(up{\cdot}x = up{\cdot}y) = (x = y)$
⟨*proof*⟩

lemma *up-inject*: $up{\cdot}x = up{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

lemma *up-defined* [*simp*]: $up{\cdot}x \neq \perp$
⟨*proof*⟩

lemma *not-up-less-UU*: $up{\cdot}x \not\sqsubseteq \perp$
⟨*proof*⟩

lemma *up-below* [*simp*]: $up{\cdot}x \sqsubseteq up{\cdot}y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *upE* [*case-names bottom up*, *cases type*: *u*]: $[\![p = \perp \Longrightarrow Q;\ \bigwedge x.\ p = up{\cdot}x \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *up-induct* [*case-names bottom up*, *induct type*: *u*]: $P\ \bot \Longrightarrow (\bigwedge x.\ P\ (up{\cdot}x)) \Longrightarrow P\ x$
⟨*proof*⟩

lifting preserves chain-finiteness

lemma *up-chain-cases*:
assumes *Y*: *chain Y*
obtains $\forall i.\ Y\ i = \bot$
| *A k* **where** $\forall i.\ up{\cdot}(A\ i) = Y\ (i + k)$ **and** *chain A* **and** $(\bigsqcup i.\ Y\ i) = up{\cdot}(\bigsqcup i.\ A\ i)$
⟨*proof*⟩

lemma *compact-up*: *compact x* $\Longrightarrow$ *compact* $(up{\cdot}x)$
⟨*proof*⟩

lemma *compact-upD*: *compact* $(up{\cdot}x) \Longrightarrow$ *compact x*
⟨*proof*⟩

lemma *compact-up-iff* [*simp*]: *compact* $(up{\cdot}x) =$ *compact x*
⟨*proof*⟩

instance *u* :: (*chfin*) *chfin*
⟨*proof*⟩

properties of fup

lemma *fup1* [*simp*]: $fup{\cdot}f{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *fup2* [*simp*]: $fup{\cdot}f{\cdot}(up{\cdot}x) = f{\cdot}x$
⟨*proof*⟩

lemma *fup3* [*simp*]: $fup{\cdot}up{\cdot}x = x$
⟨*proof*⟩

end

13 Lifting types of class type to flat pcpo's

theory *Lift*
imports *Up*
begin

pcpodef *'a*::*type lift* = *UNIV* :: *'a discr u set*
⟨*proof*⟩

lemmas *inst-lift-pcpo* = *Abs-lift-strict* [*symmetric*]

definition

Def :: *'a*::*type* ⇒ *'a lift* **where**
Def x = *Abs-lift* (*up*·(*Discr x*))

13.1 Lift as a datatype

lemma *lift-induct*: ⟦*P* ⊥; ⋀*x*. *P* (*Def x*)⟧ ⟹ *P y*
⟨*proof*⟩

old-rep-datatype ⊥::*'a*::*type lift Def*
⟨*proof*⟩

⊥ and *Def*

lemma *not-Undef-is-Def*: (*x* ≠ ⊥) = (∃ *y*. *x* = *Def y*)
⟨*proof*⟩

lemma *lift-definedE*: ⟦*x* ≠ ⊥; ⋀*a*. *x* = *Def a* ⟹ *R*⟧ ⟹ *R*
⟨*proof*⟩

For *x* ≠ ⊥ in assumptions *defined* replaces *x* by *Def a* in conclusion.

⟨*ML*⟩

lemma *DefE*: *Def x* = ⊥ ⟹ *R*
⟨*proof*⟩

lemma *DefE2*: ⟦*x* = *Def s*; *x* = ⊥⟧ ⟹ *R*
⟨*proof*⟩

lemma *Def-below-Def*: *Def x* ⊑ *Def y* ⟷ *x* = *y*
⟨*proof*⟩

lemma *Def-below-iff* [*simp*]: *Def x* ⊑ *y* ⟷ *Def x* = *y*
⟨*proof*⟩

13.2 Lift is flat

instance *lift* :: (*type*) *flat*
⟨*proof*⟩

13.3 Continuity of *case-lift*

lemma *case-lift-eq*: *case-lift* ⊥ *f x* = *fup*·(Λ *y*. *f* (*undiscr y*))·(*Rep-lift x*)
⟨*proof*⟩

lemma *cont2cont-case-lift* [*simp*]:
⟦⋀*y*. *cont* (λ*x*. *f x y*); *cont g*⟧ ⟹ *cont* (λ*x*. *case-lift* ⊥ (*f x*) (*g x*))
⟨*proof*⟩

13.4 Further operations

definition

flift1 :: ('a::*type* ⇒ 'b::*pcpo*) ⇒ ('a *lift* → 'b) (**binder** ‹*FLIFT* › *10*) **where**
flift1 = (λ*f*. (Λ *x*. *case-lift* ⊥ *f x*))

translations
Λ(*XCONST Def x*). *t* => *CONST flift1* (λ*x*. *t*)
Λ(*CONST Def x*). *FLIFT y*. *t* <= *FLIFT x y*. *t*
Λ(*CONST Def x*). *t* <= *FLIFT x*. *t*

definition
flift2 :: ('a::*type* ⇒ 'b::*type*) ⇒ ('a *lift* → 'b *lift*) **where**
flift2 f = (*FLIFT x*. *Def* (*f x*))

lemma *flift1-Def* [*simp*]: *flift1 f*·(*Def x*) = (*f x*)
⟨*proof*⟩

lemma *flift2-Def* [*simp*]: *flift2 f*·(*Def x*) = *Def* (*f x*)
⟨*proof*⟩

lemma *flift1-strict* [*simp*]: *flift1 f*·⊥ = ⊥
⟨*proof*⟩

lemma *flift2-strict* [*simp*]: *flift2 f*·⊥ = ⊥
⟨*proof*⟩

lemma *flift2-defined* [*simp*]: *x* ≠ ⊥ ⟹ (*flift2 f*)·*x* ≠ ⊥
⟨*proof*⟩

lemma *flift2-bottom-iff* [*simp*]: (*flift2 f*·*x* = ⊥) = (*x* = ⊥)
⟨*proof*⟩

lemma *FLIFT-mono*:
(⋀*x*. *f x* ⊑ *g x*) ⟹ (*FLIFT x*. *f x*) ⊑ (*FLIFT x*. *g x*)
⟨*proof*⟩

lemma *cont2cont-flift1* [*simp*, *cont2cont*]:
⟦⋀*y*. *cont* (λ*x*. *f x y*)⟧ ⟹ *cont* (λ*x*. *FLIFT y*. *f x y*)
⟨*proof*⟩

end

14 The type of lifted booleans

theory *Tr*
imports *Lift*
begin

14.1 Type definition and constructors

type-synonym *tr* = *bool lift*

translations
(*type*) *tr* ↽ (*type*) *bool lift*

definition *TT* :: *tr*
where *TT* = *Def True*

definition *FF* :: *tr*
where *FF* = *Def False*

Exhaustion and Elimination for type *tr*

lemma *Exh-tr*: $t = \bot \vee t = TT \vee t = FF$
⟨*proof*⟩

lemma *trE* [*case-names bottom TT FF*, *cases type*: *tr*]:
$[\![p = \bot \Longrightarrow Q;\ p = TT \Longrightarrow Q;\ p = FF \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *tr-induct* [*case-names bottom TT FF*, *induct type*: *tr*]:
$P\ \bot \Longrightarrow P\ TT \Longrightarrow P\ FF \Longrightarrow P\ x$
⟨*proof*⟩

distinctness for type *tr*

lemma *dist-below-tr* [*simp*]:
$TT \not\sqsubseteq \bot\ FF \not\sqsubseteq \bot\ TT \not\sqsubseteq FF\ FF \not\sqsubseteq TT$
⟨*proof*⟩

lemma *dist-eq-tr* [*simp*]: $TT \neq \bot\ FF \neq \bot\ TT \neq FF\ \bot \neq TT\ \bot \neq FF\ FF \neq TT$
⟨*proof*⟩

lemma *TT-below-iff* [*simp*]: $TT \sqsubseteq x \longleftrightarrow x = TT$
⟨*proof*⟩

lemma *FF-below-iff* [*simp*]: $FF \sqsubseteq x \longleftrightarrow x = FF$
⟨*proof*⟩

lemma *not-below-TT-iff* [*simp*]: $x \not\sqsubseteq TT \longleftrightarrow x = FF$
⟨*proof*⟩

lemma *not-below-FF-iff* [*simp*]: $x \not\sqsubseteq FF \longleftrightarrow x = TT$
⟨*proof*⟩

14.2 Case analysis

definition *tr-case* :: *'a*::*pcpo* → *'a* → *tr* → *'a*
where *tr-case* = (Λ *t e* (*Def b*). *if b then t else e*)

abbreviation *cifte-syn* :: [*tr*, *'c*::*pcpo*, *'c*] ⇒ *'c* (‹(‹*notation*=‹*mixfix If expression*››*If* (-)/ *then* (-)/ *else* (-))› [*0*, *0*, *60*] *60*)

where *If b then e1 else e2* ≡ *tr-case·e1·e2·b*

translations
Λ (*XCONST TT*). *t* ⇌ *CONST tr-case·t·⊥*
Λ (*XCONST FF*). *t* ⇌ *CONST tr-case·⊥·t*

lemma *ifte-thms* [*simp*]:
If ⊥ then e1 else e2 = ⊥
If FF then e1 else e2 = *e2*
If TT then e1 else e2 = *e1*
⟨*proof*⟩

14.3 Boolean connectives

definition *trand* :: *tr* → *tr* → *tr*
where *andalso-def*: *trand* = (Λ *x y*. *If x then y else FF*)

abbreviation *andalso-syn* :: *tr* ⇒ *tr* ⇒ *tr* (‹- *andalso* -› [*36*,*35*] *35*)
where *x andalso y* ≡ *trand·x·y*

definition *tror* :: *tr* → *tr* → *tr*
where *orelse-def*: *tror* = (Λ *x y*. *If x then TT else y*)

abbreviation *orelse-syn* :: *tr* ⇒ *tr* ⇒ *tr* (‹- *orelse* -› [*31*,*30*] *30*)
where *x orelse y* ≡ *tror·x·y*

definition *neg* :: *tr* → *tr*
where *neg* = *flift2 Not*

definition *If2* :: *tr* ⇒ *'c*::*pcpo* ⇒ *'c* ⇒ *'c*
where *If2 Q x y* = (*If Q then x else y*)

tactic for tr-thms with case split

lemmas *tr-defs* = *andalso-def orelse-def neg-def tr-case-def TT-def FF-def*

lemmas about andalso, orelse, neg and if

lemma *andalso-thms* [*simp*]:
(*TT andalso y*) = *y*
(*FF andalso y*) = *FF*
(⊥ *andalso y*) = ⊥
(*y andalso TT*) = *y*
(*y andalso y*) = *y*
⟨*proof*⟩

lemma *orelse-thms* [*simp*]:
(*TT orelse y*) = *TT*
(*FF orelse y*) = *y*
(⊥ *orelse y*) = ⊥
(*y orelse FF*) = *y*

$(y\ orelse\ y) = y$
⟨*proof*⟩

lemma *neg-thms* [*simp*]:
$neg{\cdot}TT = FF$
$neg{\cdot}FF = TT$
$neg{\cdot}\bot = \bot$
⟨*proof*⟩

split-tac for If via If2 because the constant has to be a constant

lemma *split-If2*: $P\ (If2\ Q\ x\ y) \longleftrightarrow ((Q = \bot \longrightarrow P\ \bot) \wedge (Q = TT \longrightarrow P\ x) \wedge (Q = FF \longrightarrow P\ y))$
⟨*proof*⟩

⟨*ML*⟩

14.4 Rewriting of HOLCF operations to HOL functions

lemma *andalso-or*: $t \neq \bot \Longrightarrow (t\ andalso\ s) = FF \longleftrightarrow t = FF \vee s = FF$
⟨*proof*⟩

lemma *andalso-and*: $t \neq \bot \Longrightarrow ((t\ andalso\ s) \neq FF) \longleftrightarrow t \neq FF \wedge s \neq FF$
⟨*proof*⟩

lemma *Def-bool1* [*simp*]: $Def\ x \neq FF \longleftrightarrow x$
⟨*proof*⟩

lemma *Def-bool2* [*simp*]: $Def\ x = FF \longleftrightarrow \neg\ x$
⟨*proof*⟩

lemma *Def-bool3* [*simp*]: $Def\ x = TT \longleftrightarrow x$
⟨*proof*⟩

lemma *Def-bool4* [*simp*]: $Def\ x \neq TT \longleftrightarrow \neg\ x$
⟨*proof*⟩

lemma *If-and-if*: $(If\ Def\ P\ then\ A\ else\ B) = (if\ P\ then\ A\ else\ B)$
⟨*proof*⟩

14.5 Compactness

lemma *compact-TT*: *compact TT*
⟨*proof*⟩

lemma *compact-FF*: *compact FF*
⟨*proof*⟩

end

15 The type of strict sums

theory *Ssum*
imports *Tr*
begin

15.1 Definition of strict sum type

definition *ssum* =
$\{p :: tr \times ('a{::}pcpo \times 'b{::}pcpo).\ p = \bot \vee$
$(fst\ p = TT \wedge fst\ (snd\ p) \neq \bot \wedge snd\ (snd\ p) = \bot) \vee$
$(fst\ p = FF \wedge fst\ (snd\ p) = \bot \wedge snd\ (snd\ p) \neq \bot)\}$

pcpodef ('*a*::*pcpo*, '*b*::*pcpo*) *ssum* (‹(‹*notation*=‹*infix strict sum*››- ⊕/ -)› [*21*, *20*] *20*) =
ssum :: (*tr* × '*a* × '*b*) *set*
⟨*proof*⟩

instance *ssum* :: ({*chfin*,*pcpo*}, {*chfin*,*pcpo*}) *chfin*
⟨*proof*⟩

type-notation (*ASCII*)
ssum (**infixr** ‹++› *10*)

15.2 Definitions of constructors

definition *sinl* :: '*a*::*pcpo* → ('*a* ++ '*b*::*pcpo*)
where *sinl* = (Λ *a*. *Abs-ssum* (*seq*·*a*·*TT*, *a*, ⊥))

definition *sinr* :: '*b*::*pcpo* → ('*a*::*pcpo* ++ '*b*)
where *sinr* = (Λ *b*. *Abs-ssum* (*seq*·*b*·*FF*, ⊥, *b*))

lemma *sinl-ssum*: (*seq*·*a*·*TT*, *a*, ⊥) ∈ *ssum*
⟨*proof*⟩

lemma *sinr-ssum*: (*seq*·*b*·*FF*, ⊥, *b*) ∈ *ssum*
⟨*proof*⟩

lemma *Rep-ssum-sinl*: *Rep-ssum* (*sinl*·*a*) = (*seq*·*a*·*TT*, *a*, ⊥)
⟨*proof*⟩

lemma *Rep-ssum-sinr*: *Rep-ssum* (*sinr*·*b*) = (*seq*·*b*·*FF*, ⊥, *b*)
⟨*proof*⟩

lemmas *Rep-ssum-simps* =
Rep-ssum-inject [*symmetric*] *below-ssum-def*
prod-eq-iff below-prod-def
Rep-ssum-strict Rep-ssum-sinl Rep-ssum-sinr

15.3 Properties of *sinl* and *sinr*

Ordering

lemma *sinl-below* [*simp*]: $sinl{\cdot}x \sqsubseteq sinl{\cdot}y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *sinr-below* [*simp*]: $sinr{\cdot}x \sqsubseteq sinr{\cdot}y \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *sinl-below-sinr* [*simp*]: $sinl{\cdot}x \sqsubseteq sinr{\cdot}y \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *sinr-below-sinl* [*simp*]: $sinr{\cdot}x \sqsubseteq sinl{\cdot}y \longleftrightarrow x = \bot$
⟨*proof*⟩

Equality

lemma *sinl-eq* [*simp*]: $sinl{\cdot}x = sinl{\cdot}y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *sinr-eq* [*simp*]: $sinr{\cdot}x = sinr{\cdot}y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *sinl-eq-sinr* [*simp*]: $sinl{\cdot}x = sinr{\cdot}y \longleftrightarrow x = \bot \wedge y = \bot$
⟨*proof*⟩

lemma *sinr-eq-sinl* [*simp*]: $sinr{\cdot}x = sinl{\cdot}y \longleftrightarrow x = \bot \wedge y = \bot$
⟨*proof*⟩

lemma *sinl-inject*: $sinl{\cdot}x = sinl{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

lemma *sinr-inject*: $sinr{\cdot}x = sinr{\cdot}y \Longrightarrow x = y$
⟨*proof*⟩

Strictness

lemma *sinl-strict* [*simp*]: $sinl{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *sinr-strict* [*simp*]: $sinr{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *sinl-bottom-iff* [*simp*]: $sinl{\cdot}x = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *sinr-bottom-iff* [*simp*]: $sinr{\cdot}x = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *sinl-defined*: $x \neq \bot \Longrightarrow sinl{\cdot}x \neq \bot$
⟨*proof*⟩

lemma *sinr-defined*: $x \neq \bot \Longrightarrow sinr{\cdot}x \neq \bot$
⟨*proof*⟩

Compactness

lemma *compact-sinl*: $compact\ x \Longrightarrow compact\ (sinl{\cdot}x)$
⟨*proof*⟩

lemma *compact-sinr*: $compact\ x \Longrightarrow compact\ (sinr{\cdot}x)$
⟨*proof*⟩

lemma *compact-sinlD*: $compact\ (sinl{\cdot}x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-sinrD*: $compact\ (sinr{\cdot}x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-sinl-iff* [*simp*]: $compact\ (sinl{\cdot}x) = compact\ x$
⟨*proof*⟩

lemma *compact-sinr-iff* [*simp*]: $compact\ (sinr{\cdot}x) = compact\ x$
⟨*proof*⟩

15.4 Case analysis

lemma *ssumE* [*case-names bottom sinl sinr*, *cases type*: *ssum*]:
obtains $p = \bot$
| x **where** $p = sinl{\cdot}x$ **and** $x \neq \bot$
| y **where** $p = sinr{\cdot}y$ **and** $y \neq \bot$
⟨*proof*⟩

lemma *ssum-induct* [*case-names bottom sinl sinr*, *induct type*: *ssum*]:
$[\![P\ \bot;$
$\bigwedge x.\ x \neq \bot \Longrightarrow P\ (sinl{\cdot}x);$
$\bigwedge y.\ y \neq \bot \Longrightarrow P\ (sinr{\cdot}y)]\!] \Longrightarrow P\ x$
⟨*proof*⟩

lemma *ssumE2* [*case-names sinl sinr*]:
$[\![\bigwedge x.\ p = sinl{\cdot}x \Longrightarrow Q;\ \bigwedge y.\ p = sinr{\cdot}y \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *below-sinlD*: $p \sqsubseteq sinl{\cdot}x \Longrightarrow \exists\, y.\ p = sinl{\cdot}y \wedge y \sqsubseteq x$
⟨*proof*⟩

lemma *below-sinrD*: $p \sqsubseteq sinr{\cdot}x \Longrightarrow \exists\, y.\ p = sinr{\cdot}y \wedge y \sqsubseteq x$
⟨*proof*⟩

15.5 Case analysis combinator

definition *sscase* :: $('a{::}pcpo \to 'c{::}pcpo) \to ('b{::}pcpo \to 'c) \to ('a \mathbin{++} 'b) \to 'c$

where *sscase* = (Λ *f g s*. (λ(*t*, *x*, *y*). *If t then f·x else g·y*) (*Rep-ssum s*))

translations
case s of XCONST sinl·x ⇒ t1 | XCONST sinr·y ⇒ t2 ⇌ *CONST sscase·*(Λ *x*. *t1*)·(Λ *y*. *t2*)·*s*
case s of (*XCONST sinl* :: *'a*)·*x ⇒ t1 | XCONST sinr·y ⇒ t2* ⇀ *CONST sscase·*(Λ *x*. *t1*)·(Λ *y*. *t2*)·*s*

translations
Λ(*XCONST sinl·x*). *t* ⇌ *CONST sscase·*(Λ *x*. *t*)·⊥
Λ(*XCONST sinr·y*). *t* ⇌ *CONST sscase·*⊥·(Λ *y*. *t*)

lemma *beta-sscase*: *sscase·f·g·s* = (λ(*t*, *x*, *y*). *If t then f·x else g·y*) (*Rep-ssum s*)
⟨*proof*⟩

lemma *sscase1* [*simp*]: *sscase·f·g·*⊥ = ⊥
⟨*proof*⟩

lemma *sscase2* [*simp*]: *x* ≠ ⊥ ⟹ *sscase·f·g·*(*sinl·x*) = *f·x*
⟨*proof*⟩

lemma *sscase3* [*simp*]: *y* ≠ ⊥ ⟹ *sscase·f·g·*(*sinr·y*) = *g·y*
⟨*proof*⟩

lemma *sscase4* [*simp*]: *sscase·sinl·sinr·z* = *z*
⟨*proof*⟩

15.6 Strict sum preserves flatness

instance *ssum* :: (*flat*, *flat*) *flat*
⟨*proof*⟩

end

16 The Strict Function Type

theory *Sfun*
imports *Cfun*
begin

pcpodef ('*a*::*pcpo*, '*b*::*pcpo*) *sfun* (**infixr** ‹→!› *0*) = {*f* :: '*a* → '*b*. *f·*⊥ = ⊥}
⟨*proof*⟩

type-notation (*ASCII*)
sfun (**infixr** ‹->!› *0*)

TODO: Define nice syntax for abstraction, application.

definition *sfun-abs* :: ('*a*::*pcpo* → '*b*::*pcpo*) → ('*a* →! '*b*)
where *sfun-abs* = (Λ *f*. *Abs-sfun* (*strictify·f*))

definition *sfun-rep* :: ('a::pcpo →! 'b::pcpo) → 'a → 'b
where *sfun-rep* = (Λ *f*. *Rep-sfun f*)

lemma *sfun-rep-beta*: *sfun-rep*·*f* = *Rep-sfun f*
⟨*proof*⟩

lemma *sfun-rep-strict1* [*simp*]: *sfun-rep*·⊥ = ⊥
⟨*proof*⟩

lemma *sfun-rep-strict2* [*simp*]: *sfun-rep*·*f*·⊥ = ⊥
⟨*proof*⟩

lemma *strictify-cancel*: *f*·⊥ = ⊥ ⟹ *strictify*·*f* = *f*
⟨*proof*⟩

lemma *sfun-abs-sfun-rep* [*simp*]: *sfun-abs*·(*sfun-rep*·*f*) = *f*
⟨*proof*⟩

lemma *sfun-rep-sfun-abs* [*simp*]: *sfun-rep*·(*sfun-abs*·*f*) = *strictify*·*f*
⟨*proof*⟩

lemma *sfun-eq-iff*: *f* = *g* ⟷ *sfun-rep*·*f* = *sfun-rep*·*g*
⟨*proof*⟩

lemma *sfun-below-iff*: *f* ⊑ *g* ⟷ *sfun-rep*·*f* ⊑ *sfun-rep*·*g*
⟨*proof*⟩

end

17 Map functions for various types

theory *Map-Functions*
imports *Deflation Sprod Ssum Sfun Up*
begin

17.1 Map operator for continuous function space

definition *cfun-map* :: ('b → 'a) → ('c → 'd) → ('a → 'c) → ('b → 'd)
where *cfun-map* = (Λ *a b f x*. *b*·(*f*·(*a*·*x*)))

lemma *cfun-map-beta* [*simp*]: *cfun-map*·*a*·*b*·*f*·*x* = *b*·(*f*·(*a*·*x*))
⟨*proof*⟩

lemma *cfun-map-ID*: *cfun-map*·*ID*·*ID* = *ID*
⟨*proof*⟩

lemma *cfun-map-map*: *cfun-map*·*f1*·*g1*·(*cfun-map*·*f2*·*g2*·*p*) = *cfun-map*·(Λ *x*. *f2*·(*f1*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*

⟨*proof*⟩

lemma *ep-pair-cfun-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*cfun-map*·*p1*·*e2*) (*cfun-map*·*e1*·*p2*)
⟨*proof*⟩

lemma *deflation-cfun-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* (*cfun-map*·*d1*·*d2*)
⟨*proof*⟩

lemma *finite-range-cfun-map*:
assumes *a*: *finite* (*range* (λ*x*. *a*·*x*))
assumes *b*: *finite* (*range* (λ*y*. *b*·*y*))
shows *finite* (*range* (λ*f*. *cfun-map*·*a*·*b*·*f*)) (**is** *finite* (*range* ?*h*))
⟨*proof*⟩

lemma *finite-deflation-cfun-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* (*cfun-map*·*d1*·*d2*)
⟨*proof*⟩

Finite deflations are compact elements of the function space

lemma *finite-deflation-imp-compact*: *finite-deflation d* ⟹ *compact d*
⟨*proof*⟩

17.2 Map operator for product type

definition *prod-map* :: ('*a* → '*b*) → ('*c* → '*d*) → '*a* × '*c* → '*b* × '*d*
where *prod-map* = (Λ *f g p*. (*f*·(*fst p*), *g*·(*snd p*)))

lemma *prod-map-Pair* [*simp*]: *prod-map*·*f*·*g*·(*x*, *y*) = (*f*·*x*, *g*·*y*)
⟨*proof*⟩

lemma *prod-map-ID*: *prod-map*·*ID*·*ID* = *ID*
⟨*proof*⟩

lemma *prod-map-map*: *prod-map*·*f1*·*g1*·(*prod-map*·*f2*·*g2*·*p*) = *prod-map*·(Λ *x*. *f1*·(*f2*·*x*))·(Λ *x*. *g1*·(*g2*·*x*))·*p*
⟨*proof*⟩

lemma *ep-pair-prod-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*prod-map*·*e1*·*e2*) (*prod-map*·*p1*·*p2*)
⟨*proof*⟩

lemma *deflation-prod-map*:
assumes *deflation d1* **and** *deflation d2*

shows *deflation* (*prod-map*·*d1*·*d2*)
⟨*proof*⟩

lemma *finite-deflation-prod-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* (*prod-map*·*d1*·*d2*)
⟨*proof*⟩

17.3 Map function for lifted cpo

definition *u-map* :: ('*a* → '*b*) → '*a u* → '*b u*
where *u-map* = (Λ *f*. *fup*·(*up oo f*))

lemma *u-map-strict* [*simp*]: *u-map*·*f*·⊥ = ⊥
⟨*proof*⟩

lemma *u-map-up* [*simp*]: *u-map*·*f*·(*up*·*x*) = *up*·(*f*·*x*)
⟨*proof*⟩

lemma *u-map-ID*: *u-map*·*ID* = *ID*
⟨*proof*⟩

lemma *u-map-map*: *u-map*·*f*·(*u-map*·*g*·*p*) = *u-map*·(Λ *x*. *f*·(*g*·*x*))·*p*
⟨*proof*⟩

lemma *u-map-oo*: *u-map*·(*f oo g*) = *u-map*·*f oo u-map*·*g*
⟨*proof*⟩

lemma *ep-pair-u-map*: *ep-pair e p* ⟹ *ep-pair* (*u-map*·*e*) (*u-map*·*p*)
⟨*proof*⟩

lemma *deflation-u-map*: *deflation d* ⟹ *deflation* (*u-map*·*d*)
⟨*proof*⟩

lemma *finite-deflation-u-map*:
assumes *finite-deflation d*
shows *finite-deflation* (*u-map*·*d*)
⟨*proof*⟩

17.4 Map function for strict products

definition *sprod-map* :: ('*a*::*pcpo* → '*b*::*pcpo*) → ('*c*::*pcpo* → '*d*::*pcpo*) → '*a* ⊗ '*c* → '*b* ⊗ '*d*
where *sprod-map* = (Λ *f g*. *ssplit*·(Λ *x y*. (:*f*·*x*, *g*·*y*:)))

lemma *sprod-map-strict* [*simp*]: *sprod-map*·*a*·*b*·⊥ = ⊥
⟨*proof*⟩

lemma *sprod-map-spair* [*simp*]: *x* ≠ ⊥ ⟹ *y* ≠ ⊥ ⟹ *sprod-map*·*f*·*g*·(:*x*, *y*:) = (:*f*·*x*, *g*·*y*:)

⟨*proof*⟩

lemma *sprod-map-spair'*: $f{\cdot}\bot = \bot \Longrightarrow g{\cdot}\bot = \bot \Longrightarrow$ *sprod-map*$\cdot f{\cdot}g{\cdot}(:x, y:) = (:f{\cdot}x, g{\cdot}y:)$
⟨*proof*⟩

lemma *sprod-map-ID*: *sprod-map*$\cdot$*ID*$\cdot$*ID* = *ID*
⟨*proof*⟩

lemma *sprod-map-map*:
$[\![f1{\cdot}\bot = \bot;\ g1{\cdot}\bot = \bot]\!] \Longrightarrow$
sprod-map$\cdot f1{\cdot}g1{\cdot}$(*sprod-map*$\cdot f2{\cdot}g2{\cdot}p$) =
sprod-map$\cdot(\Lambda\ x.\ f1{\cdot}(f2{\cdot}x)){\cdot}(\Lambda\ x.\ g1{\cdot}(g2{\cdot}x)){\cdot}p$
⟨*proof*⟩

lemma *ep-pair-sprod-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*sprod-map*$\cdot e1{\cdot}e2$) (*sprod-map*$\cdot p1{\cdot}p2$)
⟨*proof*⟩

lemma *deflation-sprod-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* (*sprod-map*$\cdot d1{\cdot}d2$)
⟨*proof*⟩

lemma *finite-deflation-sprod-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* (*sprod-map*$\cdot d1{\cdot}d2$)
⟨*proof*⟩

17.5 Map function for strict sums

definition *ssum-map* :: ('*a*::*pcpo* → '*b*::*pcpo*) → ('*c*::*pcpo* → '*d*::*pcpo*) → '*a* ⊕ '*c* → '*b* ⊕ '*d*
where *ssum-map* = $(\Lambda\ f\ g.$ *sscase*$\cdot$(*sinl oo f*)$\cdot$(*sinr oo g*))

lemma *ssum-map-strict* [*simp*]: *ssum-map*$\cdot f{\cdot}g{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *ssum-map-sinl* [*simp*]: $x \neq \bot \Longrightarrow$ *ssum-map*$\cdot f{\cdot}g{\cdot}$(*sinl*$\cdot x$) = *sinl*$\cdot(f{\cdot}x)$
⟨*proof*⟩

lemma *ssum-map-sinr* [*simp*]: $x \neq \bot \Longrightarrow$ *ssum-map*$\cdot f{\cdot}g{\cdot}$(*sinr*$\cdot x$) = *sinr*$\cdot(g{\cdot}x)$
⟨*proof*⟩

lemma *ssum-map-sinl'*: $f{\cdot}\bot = \bot \Longrightarrow$ *ssum-map*$\cdot f{\cdot}g{\cdot}$(*sinl*$\cdot x$) = *sinl*$\cdot(f{\cdot}x)$
⟨*proof*⟩

lemma *ssum-map-sinr'*: $g{\cdot}\bot = \bot \Longrightarrow$ *ssum-map*$\cdot f{\cdot}g{\cdot}$(*sinr*$\cdot x$) = *sinr*$\cdot(g{\cdot}x)$

⟨*proof*⟩

lemma *ssum-map-ID*: *ssum-map·ID·ID = ID*
⟨*proof*⟩

lemma *ssum-map-map*:
⟦*f1·⊥ = ⊥*; *g1·⊥ = ⊥*⟧ ⟹
ssum-map·f1·g1·(*ssum-map·f2·g2·p*) =
ssum-map·(Λ *x*. *f1·*(*f2·x*))·(Λ *x*. *g1·*(*g2·x*))·*p*
⟨*proof*⟩

lemma *ep-pair-ssum-map*:
assumes *ep-pair e1 p1* **and** *ep-pair e2 p2*
shows *ep-pair* (*ssum-map·e1·e2*) (*ssum-map·p1·p2*)
⟨*proof*⟩

lemma *deflation-ssum-map*:
assumes *deflation d1* **and** *deflation d2*
shows *deflation* (*ssum-map·d1·d2*)
⟨*proof*⟩

lemma *finite-deflation-ssum-map*:
assumes *finite-deflation d1* **and** *finite-deflation d2*
shows *finite-deflation* (*ssum-map·d1·d2*)
⟨*proof*⟩

17.6 Map operator for strict function space

definition *sfun-map* :: ('*b*::*pcpo* → '*a*::*pcpo*) → ('*c*::*pcpo* → '*d*::*pcpo*) → ('*a* →! '*c*)
→ ('*b* →! '*d*)
where *sfun-map* = (Λ *a b*. *sfun-abs oo cfun-map·a·b oo sfun-rep*)

lemma *sfun-map-ID*: *sfun-map·ID·ID = ID*
⟨*proof*⟩

lemma *sfun-map-map*:
assumes *f2·⊥ = ⊥* **and** *g2·⊥ = ⊥*
shows *sfun-map·f1·g1·*(*sfun-map·f2·g2·p*) =
sfun-map·(Λ *x*. *f2·*(*f1·x*))·(Λ *x*. *g1·*(*g2·x*))·*p*
⟨*proof*⟩

lemma *ep-pair-sfun-map*:
assumes *1*: *ep-pair e1 p1*
assumes *2*: *ep-pair e2 p2*
shows *ep-pair* (*sfun-map·p1·e2*) (*sfun-map·e1·p2*)
⟨*proof*⟩

lemma *deflation-sfun-map*:
assumes *1*: *deflation d1*

assumes *2*: *deflation d2*
shows *deflation* (*sfun-map*·*d1*·*d2*)
⟨*proof*⟩

lemma *finite-deflation-sfun-map*:
assumes *finite-deflation d1*
and *finite-deflation d2*
shows *finite-deflation* (*sfun-map*·*d1*·*d2*)
⟨*proof*⟩

end

18 The cpo of cartesian products

theory *Cprod*
imports *Cfun*
begin

18.1 Continuous case function for unit type

definition *unit-when* :: $'a \to unit \to 'a$
where *unit-when* = (Λ *a* -. *a*)

translations
Λ(). *t* ⇌ *CONST unit-when*·*t*

lemma *unit-when* [*simp*]: *unit-when*·*a*·*u* = *a*
⟨*proof*⟩

18.2 Continuous version of split function

definition *csplit* :: $('a \to 'b \to 'c) \to ('a \times 'b) \to 'c$
where *csplit* = (Λ *f p*. *f*·(*fst p*)·(*snd p*))

translations
Λ(*CONST Pair x y*). *t* ⇌ *CONST csplit*·(Λ *x y*. *t*)

abbreviation *cfst* :: $'a \times 'b \to 'a$
where *cfst* ≡ *Abs-cfun fst*

abbreviation *csnd* :: $'a \times 'b \to 'b$
where *csnd* ≡ *Abs-cfun snd*

18.3 Convert all lemmas to the continuous versions

lemma *csplit1* [*simp*]: *csplit*·*f*·⊥ = *f*·⊥·⊥
⟨*proof*⟩

lemma *csplit-Pair* [*simp*]: *csplit*·*f*·(*x*, *y*) = *f*·*x*·*y*

⟨*proof*⟩

end

19 Profinite and bifinite cpos

theory *Bifinite*
imports *Map-Functions Cprod Sprod Sfun Up HOL−Library.Countable*
begin

19.1 Chains of finite deflations

locale *approx-chain* =
fixes *approx* :: *nat* ⇒ *'a* → *'a*
assumes *chain-approx* [*simp*]: *chain* (λ*i*. *approx i*)
assumes *lub-approx* [*simp*]: (⨆ *i*. *approx i*) = *ID*
assumes *finite-deflation-approx* [*simp*]: ⋀*i*. *finite-deflation* (*approx i*)
begin

lemma *deflation-approx*: *deflation* (*approx i*)
⟨*proof*⟩

lemma *approx-idem*: *approx i*·(*approx i*·*x*) = *approx i*·*x*
⟨*proof*⟩

lemma *approx-below*: *approx i*·*x* ⊑ *x*
⟨*proof*⟩

lemma *finite-range-approx*: *finite* (*range* (λ*x*. *approx i*·*x*))
⟨*proof*⟩

lemma *compact-approx* [*simp*]: *compact* (*approx n*·*x*)
⟨*proof*⟩

lemma *compact-eq-approx*: *compact x* ⟹ ∃ *i*. *approx i*·*x* = *x*
⟨*proof*⟩

end

19.2 Omega-profinite and bifinite domains

class *bifinite* = *pcpo* +
assumes *bifinite*: ∃ (*a*::*nat* ⇒ *'a* → *'a*). *approx-chain a*

class *profinite* = *cpo* +
assumes *profinite*: ∃ (*a*::*nat* ⇒ *'a*⊥ → *'a*⊥). *approx-chain a*

19.3 Building approx chains

lemma *approx-chain-iso*:
assumes *a*: *approx-chain a*
assumes [*simp*]: $\bigwedge x.\ f{\cdot}(g{\cdot}x) = x$
assumes [*simp*]: $\bigwedge y.\ g{\cdot}(f{\cdot}y) = y$
shows *approx-chain* ($\lambda i.$ *f oo a i oo g*)
⟨*proof*⟩

lemma *approx-chain-u-map*:
assumes *approx-chain a*
shows *approx-chain* ($\lambda i.$ *u-map*·(*a i*))
⟨*proof*⟩

lemma *approx-chain-sfun-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *sfun-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-sprod-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *sprod-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-ssum-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *ssum-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-cfun-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *cfun-map*·(*a i*)·(*b i*))
⟨*proof*⟩

lemma *approx-chain-prod-map*:
assumes *approx-chain a* **and** *approx-chain b*
shows *approx-chain* ($\lambda i.$ *prod-map*·(*a i*)·(*b i*))
⟨*proof*⟩

Approx chains for countable discrete types.

definition *discr-approx* :: *nat* ⇒ *'a*::*countable discr u* → *'a discr u*
where *discr-approx* = ($\lambda i.\ \Lambda(up{\cdot}x).$ *if to-nat* (*undiscr x*) < *i then up*·*x else* ⊥)

lemma *chain-discr-approx* [*simp*]: *chain discr-approx*
⟨*proof*⟩

lemma *lub-discr-approx* [*simp*]: ($\bigsqcup i.$ *discr-approx i*) = *ID*
⟨*proof*⟩

lemma *inj-on-undiscr* [*simp*]: *inj-on undiscr A*

⟨*proof*⟩

lemma *finite-deflation-discr-approx*: *finite-deflation* (*discr-approx i*)
⟨*proof*⟩

lemma *discr-approx*: *approx-chain discr-approx*
⟨*proof*⟩

19.4 Class instance proofs

instance *bifinite* ⊆ *profinite*
⟨*proof*⟩

instance *u* :: (*profinite*) *bifinite*
⟨*proof*⟩

Types $'a \to 'b$ and $'a_\perp \to! 'b$ are isomorphic.

definition *encode-cfun* = (Λ *f*. *sfun-abs*·(*fup*·*f*))

definition *decode-cfun* = (Λ *g x*. *sfun-rep*·*g*·(*up*·*x*))

lemma *decode-encode-cfun* [*simp*]: *decode-cfun*·(*encode-cfun*·*x*) = *x*
⟨*proof*⟩

lemma *encode-decode-cfun* [*simp*]: *encode-cfun*·(*decode-cfun*·*y*) = *y*
⟨*proof*⟩

instance *cfun* :: (*profinite*, *bifinite*) *bifinite*
⟨*proof*⟩

Types $('a \times 'b)_\perp$ and $'a_\perp \otimes 'b_\perp$ are isomorphic.

definition *encode-prod-u* = (Λ(*up*·(*x*, *y*)). (:*up*·*x*, *up*·*y*:))

definition *decode-prod-u* = (Λ(:*up*·*x*, *up*·*y*:). *up*·(*x*, *y*))

lemma *decode-encode-prod-u* [*simp*]: *decode-prod-u*·(*encode-prod-u*·*x*) = *x*
⟨*proof*⟩

lemma *encode-decode-prod-u* [*simp*]: *encode-prod-u*·(*decode-prod-u*·*y*) = *y*
⟨*proof*⟩

instance *prod* :: (*profinite*, *profinite*) *profinite*
⟨*proof*⟩

instance *prod* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

instance *sfun* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

instance *sprod* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

instance *ssum* :: (*bifinite*, *bifinite*) *bifinite*
⟨*proof*⟩

lemma *approx-chain-unit*: *approx-chain* (⊥ :: *nat* ⇒ *unit* → *unit*)
⟨*proof*⟩

instance *unit* :: *bifinite*
⟨*proof*⟩

instance *discr* :: (*countable*) *profinite*
⟨*proof*⟩

instance *lift* :: (*countable*) *bifinite*
⟨*proof*⟩

end

20 Defining algebraic domains by ideal completion

theory *Completion*
imports *Cfun*
begin

20.1 Ideals over a preorder

locale *preorder* =
fixes *r* :: *'a::type* ⇒ *'a* ⇒ *bool* (**infix** ‹⪯› *50*)
assumes *r-refl*: $x \preceq x$
assumes *r-trans*: $[\![x \preceq y; y \preceq z]\!] \Longrightarrow x \preceq z$
begin

definition
ideal :: *'a set* ⇒ *bool* **where**
$ideal\ A = ((\exists x.\ x \in A) \wedge (\forall x{\in}A.\ \forall y{\in}A.\ \exists z{\in}A.\ x \preceq z \wedge y \preceq z) \wedge (\forall x\ y.\ x \preceq y \longrightarrow y \in A \longrightarrow x \in A))$

lemma *idealI*:
assumes $\exists x.\ x \in A$
assumes $\bigwedge x\ y.\ [\![x \in A; y \in A]\!] \Longrightarrow \exists z{\in}A.\ x \preceq z \wedge y \preceq z$
assumes $\bigwedge x\ y.\ [\![x \preceq y; y \in A]\!] \Longrightarrow x \in A$
shows *ideal A*
⟨*proof*⟩

lemma *idealD1*:
$ideal\ A \Longrightarrow \exists x.\ x \in A$

⟨*proof*⟩

lemma *idealD2*:
  ⟦*ideal A*; $x \in A$; $y \in A$⟧ ⟹ $\exists\, z{\in}A.\ x \preceq z \wedge y \preceq z$
⟨*proof*⟩

lemma *idealD3*:
  ⟦*ideal A*; $x \preceq y$; $y \in A$⟧ ⟹ $x \in A$
⟨*proof*⟩

lemma *ideal-principal*: *ideal* $\{x.\ x \preceq z\}$
  ⟨*proof*⟩

lemma *ex-ideal*: $\exists\, A.\ A \in \{A.\ \mathit{ideal}\ A\}$
⟨*proof*⟩

The set of ideals is a cpo

lemma *ideal-UN*:
  fixes *A* :: *nat* ⇒ *'a set*
  assumes *ideal-A*: $\bigwedge i.\ \mathit{ideal}\ (A\ i)$
  assumes *chain-A*: $\bigwedge i\ j.\ i \leq j \Longrightarrow A\ i \subseteq A\ j$
  shows *ideal* $(\bigcup i.\ A\ i)$
  ⟨*proof*⟩

lemma *typedef-ideal-po*:
  fixes *Abs* :: *'a set* ⇒ *'b*::*below*
  assumes *type*: *type-definition Rep Abs* $\{S.\ \mathit{ideal}\ S\}$
  assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow \mathit{Rep}\ x \subseteq \mathit{Rep}\ y$
  shows *OFCLASS*(*'b*, *po-class*)
⟨*proof*⟩

lemma
  fixes *Abs* :: *'a set* ⇒ *'b*::*po*
  assumes *type*: *type-definition Rep Abs* $\{S.\ \mathit{ideal}\ S\}$
  assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow \mathit{Rep}\ x \subseteq \mathit{Rep}\ y$
  assumes *S*: *chain S*
  shows *typedef-ideal-lub*: *range S* <<| *Abs* $(\bigcup i.\ \mathit{Rep}\ (S\ i))$
    and *typedef-ideal-rep-lub*: *Rep* $(\bigsqcup i.\ S\ i) = (\bigcup i.\ \mathit{Rep}\ (S\ i))$
⟨*proof*⟩

lemma *typedef-ideal-cpo*:
  fixes *Abs* :: *'a set* ⇒ *'b*::*po*
  assumes *type*: *type-definition Rep Abs* $\{S.\ \mathit{ideal}\ S\}$
  assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow \mathit{Rep}\ x \subseteq \mathit{Rep}\ y$
  shows *OFCLASS*(*'b*, *cpo-class*)
  ⟨*proof*⟩

end

interpretation *below*: *preorder below* :: $'a$::*po* $\Rightarrow$ $'a \Rightarrow$ *bool*
⟨*proof*⟩

20.2 Lemmas about least upper bounds

lemma *is-ub-thelub-ex*: $[\![\exists\, u.\ S <<|\ u;\ x \in S]\!] \Longrightarrow x \sqsubseteq \mathit{lub}\ S$
⟨*proof*⟩

lemma *is-lub-thelub-ex*: $[\![\exists\, u.\ S <<|\ u;\ S <|\ x]\!] \Longrightarrow \mathit{lub}\ S \sqsubseteq x$
⟨*proof*⟩

20.3 Locale for ideal completion

hide-const (**open**) *Filter.principal*

locale *ideal-completion* = *preorder* +
  fixes *principal* :: $'a$::*type* $\Rightarrow$ $'b$
  fixes *rep* :: $'b \Rightarrow$ $'a$::*type set*
  assumes *ideal-rep*: $\bigwedge x.$ *ideal* (*rep x*)
  assumes *rep-lub*: $\bigwedge Y.$ *chain* $Y \Longrightarrow$ *rep* $(\bigsqcup i.\ Y\ i) = (\bigcup i.\ \mathit{rep}\ (Y\ i))$
  assumes *rep-principal*: $\bigwedge a.$ *rep* (*principal a*) $= \{b.\ b \preceq a\}$
  assumes *belowI*: $\bigwedge x\ y.$ *rep x* $\subseteq$ *rep y* $\Longrightarrow x \sqsubseteq y$
  assumes *countable*: $\exists f$::$'a \Rightarrow$ *nat. inj f*
begin

lemma *rep-mono*: $x \sqsubseteq y \Longrightarrow \mathit{rep}\ x \subseteq \mathit{rep}\ y$
⟨*proof*⟩

lemma *below-def*: $x \sqsubseteq y \longleftrightarrow \mathit{rep}\ x \subseteq \mathit{rep}\ y$
⟨*proof*⟩

lemma *principal-below-iff-mem-rep*: *principal a* $\sqsubseteq x \longleftrightarrow a \in \mathit{rep}\ x$
⟨*proof*⟩

lemma *principal-below-iff* [*simp*]: *principal a* $\sqsubseteq$ *principal b* $\longleftrightarrow a \preceq b$
⟨*proof*⟩

lemma *principal-eq-iff*: *principal a* = *principal b* $\longleftrightarrow a \preceq b \wedge b \preceq a$
⟨*proof*⟩

lemma *eq-iff*: $x = y \longleftrightarrow \mathit{rep}\ x = \mathit{rep}\ y$
⟨*proof*⟩

lemma *principal-mono*: $a \preceq b \Longrightarrow$ *principal a* $\sqsubseteq$ *principal b*
⟨*proof*⟩

lemma *ch2ch-principal* [*simp*]:
  $\forall i.\ Y\ i \preceq Y\ (\mathit{Suc}\ i) \Longrightarrow \mathit{chain}\ (\lambda i.\ \mathit{principal}\ (Y\ i))$
⟨*proof*⟩

20.3.1 Principal ideals approximate all elements

lemma *compact-principal* [*simp*]: *compact* (*principal a*)
⟨*proof*⟩

Construct a chain whose lub is the same as a given ideal

lemma *obtain-principal-chain*:
obtains *Y* **where** ∀ *i*. *Y i* ≼ *Y* (*Suc i*) **and** *x* = (⨆ *i*. *principal* (*Y i*))
⟨*proof*⟩

lemma *principal-induct*:
assumes *adm*: *adm P*
assumes *P*: ⋀*a*. *P* (*principal a*)
shows *P x*
⟨*proof*⟩

lemma *compact-imp-principal*: *compact x* ⟹ ∃ *a*. *x* = *principal a*
⟨*proof*⟩

20.4 Defining functions in terms of basis elements

definition
extension :: ('*a*::*type* ⇒ '*c*) ⇒ '*b* → '*c* **where**
extension = (λ*f*. (Λ *x*. *lub* (*f* ‘ *rep x*)))

lemma *extension-lemma*:
fixes *f* :: '*a*::*type* ⇒ '*c*
assumes *f-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *f a* ⊑ *f b*
shows ∃ *u*. *f* ‘ *rep x* <<| *u*
⟨*proof*⟩

lemma *extension-beta*:
fixes *f* :: '*a*::*type* ⇒ '*c*
assumes *f-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *f a* ⊑ *f b*
shows *extension f*·*x* = *lub* (*f* ‘ *rep x*)
⟨*proof*⟩

lemma *extension-principal*:
fixes *f* :: '*a*::*type* ⇒ '*c*
assumes *f-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *f a* ⊑ *f b*
shows *extension f*·(*principal a*) = *f a*
⟨*proof*⟩

lemma *extension-mono*:
assumes *f-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *f a* ⊑ *f b*
assumes *g-mono*: ⋀*a b*. *a* ≼ *b* ⟹ *g a* ⊑ *g b*
assumes *below*: ⋀*a*. *f a* ⊑ *g a*
shows *extension f* ⊑ *extension g*
⟨*proof*⟩

lemma *cont-extension*:
assumes *f-mono*: $\bigwedge a\ b\ x.\ a \preceq b \Longrightarrow f\ x\ a \sqsubseteq f\ x\ b$
assumes *f-cont*: $\bigwedge a.\ cont\ (\lambda x.\ f\ x\ a)$
shows *cont* $(\lambda x.\ extension\ (\lambda a.\ f\ x\ a))$
⟨*proof*⟩

end

lemma (**in** *preorder*) *typedef-ideal-completion*:
fixes *Abs* :: $'a\ set \Rightarrow 'b$
assumes *type*: *type-definition Rep Abs* $\{S.\ ideal\ S\}$
assumes *below*: $\bigwedge x\ y.\ x \sqsubseteq y \longleftrightarrow Rep\ x \subseteq Rep\ y$
assumes *principal*: $\bigwedge a.\ principal\ a = Abs\ \{b.\ b \preceq a\}$
assumes *countable*: $\exists f::'a \Rightarrow nat.\ inj\ f$
shows *ideal-completion r principal Rep*
⟨*proof*⟩

end

21 A universal bifinite domain

theory *Universal*
imports *Bifinite Completion HOL−Library.Nat-Bijection*
begin

unbundle *no binomial-syntax*

21.1 Basis for universal domain

21.1.1 Basis datatype

type-synonym *ubasis* = *nat*

definition
node :: *nat* ⇒ *ubasis* ⇒ *ubasis set* ⇒ *ubasis*
where
node i a S = *Suc* (*prod-encode* (*i*, *prod-encode* (*a*, *set-encode S*)))

lemma *node-not-0* [*simp*]: *node i a S* ≠ *0*
⟨*proof*⟩

lemma *node-gt-0* [*simp*]: *0* < *node i a S*
⟨*proof*⟩

lemma *node-inject* [*simp*]:
⟦*finite S*; *finite T*⟧
⟹ *node i a S* = *node j b T* ⟷ *i* = *j* ∧ *a* = *b* ∧ *S* = *T*
⟨*proof*⟩

lemma *node-gt0*: $i < node\ i\ a\ S$
⟨*proof*⟩

lemma *node-gt1*: $a < node\ i\ a\ S$
⟨*proof*⟩

lemma *nat-less-power2*: $n < 2\hat{}n$
⟨*proof*⟩

lemma *node-gt2*: $[\![finite\ S;\ b \in S]\!] \Longrightarrow b < node\ i\ a\ S$
⟨*proof*⟩

lemma *eq-prod-encode-pairI*:
$[\![fst\ (prod\text{-}decode\ x) = a;\ snd\ (prod\text{-}decode\ x) = b]\!] \Longrightarrow x = prod\text{-}encode\ (a,\ b)$
⟨*proof*⟩

lemma *node-cases*:
assumes *1*: $x = 0 \Longrightarrow P$
assumes *2*: $\bigwedge i\ a\ S.\ [\![finite\ S;\ x = node\ i\ a\ S]\!] \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *node-induct*:
assumes *1*: $P\ 0$
assumes *2*: $\bigwedge i\ a\ S.\ [\![P\ a;\ finite\ S;\ \forall b{\in}S.\ P\ b]\!] \Longrightarrow P\ (node\ i\ a\ S)$
shows $P\ x$
⟨*proof*⟩

21.1.2 Basis ordering

inductive
ubasis-le :: $nat \Rightarrow nat \Rightarrow bool$
where
ubasis-le-refl: $ubasis\text{-}le\ a\ a$
| *ubasis-le-trans*:
$[\![ubasis\text{-}le\ a\ b;\ ubasis\text{-}le\ b\ c]\!] \Longrightarrow ubasis\text{-}le\ a\ c$
| *ubasis-le-lower*:
$finite\ S \Longrightarrow ubasis\text{-}le\ a\ (node\ i\ a\ S)$
| *ubasis-le-upper*:
$[\![finite\ S;\ b \in S;\ ubasis\text{-}le\ a\ b]\!] \Longrightarrow ubasis\text{-}le\ (node\ i\ a\ S)\ b$

lemma *ubasis-le-minimal*: $ubasis\text{-}le\ 0\ x$
⟨*proof*⟩

interpretation *udom*: *preorder ubasis-le*
⟨*proof*⟩

21.1.3 Generic take function

function

ubasis-until :: (*ubasis* ⇒ *bool*) ⇒ *ubasis* ⇒ *ubasis*
where
ubasis-until P 0 = 0
| *finite S* ⟹ *ubasis-until P* (*node i a S*) =
(*if P* (*node i a S*) *then node i a S else ubasis-until P a*)
⟨*proof*⟩

termination *ubasis-until*
⟨*proof*⟩

lemma *ubasis-until*: *P 0* ⟹ *P* (*ubasis-until P x*)
⟨*proof*⟩

lemma *ubasis-until'*: *0 < ubasis-until P x* ⟹ *P* (*ubasis-until P x*)
⟨*proof*⟩

lemma *ubasis-until-same*: *P x* ⟹ *ubasis-until P x = x*
⟨*proof*⟩

lemma *ubasis-until-idem*:
P 0 ⟹ *ubasis-until P* (*ubasis-until P x*) = *ubasis-until P x*
⟨*proof*⟩

lemma *ubasis-until-0*:
∀ *x*. *x* ≠ *0* ⟶ ¬ *P x* ⟹ *ubasis-until P x = 0*
⟨*proof*⟩

lemma *ubasis-until-less*: *ubasis-le* (*ubasis-until P x*) *x*
⟨*proof*⟩

lemma *ubasis-until-chain*:
assumes *PQ*: ⋀*x*. *P x* ⟹ *Q x*
shows *ubasis-le* (*ubasis-until P x*) (*ubasis-until Q x*)
⟨*proof*⟩

lemma *ubasis-until-mono*:
assumes ⋀*i a S b*. ⟦*finite S*; *P* (*node i a S*); *b* ∈ *S*; *ubasis-le a b*⟧ ⟹ *P b*
shows *ubasis-le a b* ⟹ *ubasis-le* (*ubasis-until P a*) (*ubasis-until P b*)
⟨*proof*⟩

lemma *finite-range-ubasis-until*:
finite {*x*. *P x*} ⟹ *finite* (*range* (*ubasis-until P*))
⟨*proof*⟩

21.2 Defining the universal domain by ideal completion

typedef *udom* = {*S*. *udom.ideal S*}
⟨*proof*⟩

instantiation *udom* :: *below*
begin

definition
$x \sqsubseteq y \longleftrightarrow$ *Rep-udom* $x \subseteq$ *Rep-udom* y

instance ⟨*proof*⟩
end

instance *udom* :: *po*
⟨*proof*⟩

instance *udom* :: *cpo*
⟨*proof*⟩

definition
udom-principal :: *nat* ⇒ *udom* **where**
udom-principal t = *Abs-udom* $\{u.$ *ubasis-le* u $t\}$

lemma *ubasis-countable*: $\exists f$::*ubasis* ⇒ *nat*. *inj* f
⟨*proof*⟩

interpretation *udom*:
ideal-completion ubasis-le udom-principal Rep-udom
⟨*proof*⟩

Universal domain is pointed

lemma *udom-minimal*: *udom-principal 0* $\sqsubseteq x$
⟨*proof*⟩

instance *udom* :: *pcpo*
⟨*proof*⟩

lemma *inst-udom-pcpo*: $\bot$ = *udom-principal 0*
⟨*proof*⟩

21.3 Compact bases of domains

typedef *'a compact-basis* = $\{x$::*'a*::*pcpo*. *compact* $x\}$
⟨*proof*⟩

lemma *Rep-compact-basis'* [*simp*]: *compact* (*Rep-compact-basis a*)
⟨*proof*⟩

lemma *Abs-compact-basis-inverse'* [*simp*]:
compact $x \Longrightarrow$ *Rep-compact-basis* (*Abs-compact-basis* x) = x
⟨*proof*⟩

instantiation *compact-basis* :: (*pcpo*) *below*

begin

definition
compact-le-def:
$(\sqsubseteq) \equiv (\lambda x\ y.\ \textit{Rep-compact-basis}\ x \sqsubseteq \textit{Rep-compact-basis}\ y)$

instance ⟨*proof*⟩
end

instance *compact-basis* :: (*pcpo*) *po*
⟨*proof*⟩

definition
approximants :: $'a$::*pcpo* ⇒ $'a$ *compact-basis set* **where**
approximants = $(\lambda x.\ \{a.\ \textit{Rep-compact-basis}\ a \sqsubseteq x\})$

definition
compact-bot :: $'a$::*pcpo compact-basis* **where**
compact-bot = *Abs-compact-basis* ⊥

lemma *Rep-compact-bot* [*simp*]: *Rep-compact-basis compact-bot* = ⊥
⟨*proof*⟩

lemma *compact-bot-minimal* [*simp*]: *compact-bot* ⊑ *a*
⟨*proof*⟩

21.4 Universality of *udom*

We use a locale to parameterize the construction over a chain of approx functions on the type to be embedded.

locale *bifinite-approx-chain* =
approx-chain approx **for** *approx* :: *nat* ⇒ $'a$::*bifinite* → $'a$
begin

21.4.1 Choosing a maximal element from a finite set

lemma *finite-has-maximal*:
fixes A :: $'a$ *compact-basis set*
shows $[\![\textit{finite}\ A;\ A \neq \{\}]\!] \Longrightarrow \exists\, x{\in}A.\ \forall\, y{\in}A.\ x \sqsubseteq y \longrightarrow x = y$
⟨*proof*⟩

definition
choose :: $'a$ *compact-basis set* ⇒ $'a$ *compact-basis*
where
choose $A = (\textit{SOME}\ x.\ x \in \{x{\in}A.\ \forall\, y{\in}A.\ x \sqsubseteq y \longrightarrow x = y\})$

lemma *choose-lemma*:
$[\![\textit{finite}\ A;\ A \neq \{\}]\!] \Longrightarrow \textit{choose}\ A \in \{x{\in}A.\ \forall\, y{\in}A.\ x \sqsubseteq y \longrightarrow x = y\}$
⟨*proof*⟩

lemma *maximal-choose*:
⟦*finite A*; *y* ∈ *A*; *choose A* ⊑ *y*⟧ ⟹ *choose A* = *y*
⟨*proof*⟩

lemma *choose-in*: ⟦*finite A*; *A* ≠ {}⟧ ⟹ *choose A* ∈ *A*
⟨*proof*⟩

function
choose-pos :: *'a compact-basis set* ⇒ *'a compact-basis* ⇒ *nat*
where
choose-pos A x =
(*if finite A* ∧ *x* ∈ *A* ∧ *x* ≠ *choose A*
then Suc (*choose-pos* (*A* − {*choose A*}) *x*) *else 0*)
⟨*proof*⟩

termination *choose-pos*
⟨*proof*⟩

declare *choose-pos.simps* [*simp del*]

lemma *choose-pos-choose*: *finite A* ⟹ *choose-pos A* (*choose A*) = *0*
⟨*proof*⟩

lemma *inj-on-choose-pos* [*OF refl*]:
⟦*card A* = *n*; *finite A*⟧ ⟹ *inj-on* (*choose-pos A*) *A*
⟨*proof*⟩

lemma *choose-pos-bounded* [*OF refl*]:
⟦*card A* = *n*; *finite A*; *x* ∈ *A*⟧ ⟹ *choose-pos A x* < *n*
⟨*proof*⟩

lemma *choose-pos-lessD*:
⟦*choose-pos A x* < *choose-pos A y*; *finite A*; *x* ∈ *A*; *y* ∈ *A*⟧ ⟹ *x* ⋢ *y*
⟨*proof*⟩

21.4.2 Compact basis take function

primrec
cb-take :: *nat* ⇒ *'a compact-basis* ⇒ *'a compact-basis* **where**
cb-take 0 = (λ*x*. *compact-bot*)
| *cb-take* (*Suc n*) = (λ*a*. *Abs-compact-basis* (*approx n*·(*Rep-compact-basis a*)))

declare *cb-take.simps* [*simp del*]

lemma *cb-take-zero* [*simp*]: *cb-take 0 a* = *compact-bot*
⟨*proof*⟩

lemma *Rep-cb-take*:

Rep-compact-basis (*cb-take* (*Suc n*) *a*) = *approx n*·(*Rep-compact-basis a*)
⟨*proof*⟩

lemmas *approx-Rep-compact-basis* = *Rep-cb-take* [*symmetric*]

lemma *cb-take-covers*: ∃ *n*. *cb-take n x* = *x*
⟨*proof*⟩

lemma *cb-take-less*: *cb-take n x* ⊑ *x*
⟨*proof*⟩

lemma *cb-take-idem*: *cb-take n* (*cb-take n x*) = *cb-take n x*
⟨*proof*⟩

lemma *cb-take-mono*: *x* ⊑ *y* ⟹ *cb-take n x* ⊑ *cb-take n y*
⟨*proof*⟩

lemma *cb-take-chain-le*: *m* ≤ *n* ⟹ *cb-take m x* ⊑ *cb-take n x*
⟨*proof*⟩

lemma *finite-range-cb-take*: *finite* (*range* (*cb-take n*))
⟨*proof*⟩

21.4.3 Rank of basis elements

definition
rank :: *'a compact-basis* ⇒ *nat*
where
rank x = (*LEAST n*. *cb-take n x* = *x*)

lemma *compact-approx-rank*: *cb-take* (*rank x*) *x* = *x*
⟨*proof*⟩

lemma *rank-leD*: *rank x* ≤ *n* ⟹ *cb-take n x* = *x*
⟨*proof*⟩

lemma *rank-leI*: *cb-take n x* = *x* ⟹ *rank x* ≤ *n*
⟨*proof*⟩

lemma *rank-le-iff*: *rank x* ≤ *n* ⟷ *cb-take n x* = *x*
⟨*proof*⟩

lemma *rank-compact-bot* [*simp*]: *rank compact-bot* = *0*
⟨*proof*⟩

lemma *rank-eq-0-iff* [*simp*]: *rank x* = *0* ⟷ *x* = *compact-bot*
⟨*proof*⟩

definition

rank-le :: *'a compact-basis* ⇒ *'a compact-basis set*
where
rank-le x = {*y*. *rank y* ≤ *rank x*}

definition
rank-lt :: *'a compact-basis* ⇒ *'a compact-basis set*
where
rank-lt x = {*y*. *rank y* < *rank x*}

definition
rank-eq :: *'a compact-basis* ⇒ *'a compact-basis set*
where
rank-eq x = {*y*. *rank y* = *rank x*}

lemma *rank-eq-cong*: *rank x* = *rank y* ⟹ *rank-eq x* = *rank-eq y*
⟨*proof*⟩

lemma *rank-lt-cong*: *rank x* = *rank y* ⟹ *rank-lt x* = *rank-lt y*
⟨*proof*⟩

lemma *rank-eq-subset*: *rank-eq x* ⊆ *rank-le x*
⟨*proof*⟩

lemma *rank-lt-subset*: *rank-lt x* ⊆ *rank-le x*
⟨*proof*⟩

lemma *finite-rank-le*: *finite* (*rank-le x*)
⟨*proof*⟩

lemma *finite-rank-eq*: *finite* (*rank-eq x*)
⟨*proof*⟩

lemma *finite-rank-lt*: *finite* (*rank-lt x*)
⟨*proof*⟩

lemma *rank-lt-Int-rank-eq*: *rank-lt x* ∩ *rank-eq x* = {}
⟨*proof*⟩

lemma *rank-lt-Un-rank-eq*: *rank-lt x* ∪ *rank-eq x* = *rank-le x*
⟨*proof*⟩

21.4.4 Sequencing basis elements

definition
place :: *'a compact-basis* ⇒ *nat*
where
place x = *card* (*rank-lt x*) + *choose-pos* (*rank-eq x*) *x*

lemma *place-bounded*: *place x* < *card* (*rank-le x*)

⟨*proof*⟩

lemma *place-ge*: *card* (*rank-lt x*) ≤ *place x*
⟨*proof*⟩

lemma *place-rank-mono*:
fixes *x y* :: *'a compact-basis*
shows *rank x* < *rank y* ⟹ *place x* < *place y*
⟨*proof*⟩

lemma *place-eqD*: *place x* = *place y* ⟹ *x* = *y*
⟨*proof*⟩

lemma *inj-place*: *inj place*
⟨*proof*⟩

21.4.5 Embedding and projection on basis elements

definition
sub :: *'a compact-basis* ⇒ *'a compact-basis*
where
sub x = (*case rank x of 0* ⇒ *compact-bot* | *Suc k* ⇒ *cb-take k x*)

lemma *rank-sub-less*: *x* ≠ *compact-bot* ⟹ *rank* (*sub x*) < *rank x*
⟨*proof*⟩

lemma *place-sub-less*: *x* ≠ *compact-bot* ⟹ *place* (*sub x*) < *place x*
⟨*proof*⟩

lemma *sub-below*: *sub x* ⊑ *x*
⟨*proof*⟩

lemma *rank-less-imp-below-sub*: ⟦*x* ⊑ *y*; *rank x* < *rank y*⟧ ⟹ *x* ⊑ *sub y*
⟨*proof*⟩

function *basis-emb* :: *'a compact-basis* ⇒ *ubasis*
where *basis-emb x* = (*if x* = *compact-bot then 0 else*
node (*place x*) (*basis-emb* (*sub x*))
(*basis-emb* ‘ {*y*. *place y* < *place x* ∧ *x* ⊑ *y*}))
⟨*proof*⟩

termination *basis-emb*
⟨*proof*⟩

declare *basis-emb.simps* [*simp del*]

lemma *basis-emb-compact-bot* [*simp*]:
basis-emb compact-bot = *0*
⟨*proof*⟩

lemma *basis-emb-rec*:
basis-emb x = node (place x) (basis-emb (sub x)) (basis-emb ' {y. place y < place x ∧ x ⊑ y})
if *x ≠ compact-bot*
⟨*proof*⟩

lemma *basis-emb-eq-0-iff* [*simp*]:
basis-emb x = 0 ⟷ x = compact-bot
⟨*proof*⟩

lemma *fin1*: *finite {y. place y < place x ∧ x ⊑ y}*
⟨*proof*⟩

lemma *fin2*: *finite (basis-emb ' {y. place y < place x ∧ x ⊑ y})*
⟨*proof*⟩

lemma *rank-place-mono*:
⟦place x < place y; x ⊑ y⟧ ⟹ rank x < rank y
⟨*proof*⟩

lemma *basis-emb-mono*:
x ⊑ y ⟹ ubasis-le (basis-emb x) (basis-emb y)
⟨*proof*⟩

lemma *inj-basis-emb*: *inj basis-emb*
⟨*proof*⟩

definition
basis-prj :: ubasis ⇒ 'a compact-basis
where
basis-prj x = inv basis-emb
(ubasis-until (λx. x ∈ range (basis-emb :: 'a compact-basis ⇒ ubasis)) x)

lemma *basis-prj-basis-emb*: *⋀x. basis-prj (basis-emb x) = x*
⟨*proof*⟩

lemma *basis-prj-node*:
⟦finite S; node i a S ∉ range (basis-emb :: 'a compact-basis ⇒ nat)⟧
⟹ basis-prj (node i a S) = (basis-prj a :: 'a compact-basis)
⟨*proof*⟩

lemma *basis-prj-0*: *basis-prj 0 = compact-bot*
⟨*proof*⟩

lemma *node-eq-basis-emb-iff*:
finite S ⟹ node i a S = basis-emb x ⟷
x ≠ compact-bot ∧ i = place x ∧ a = basis-emb (sub x) ∧
S = basis-emb ' {y. place y < place x ∧ x ⊑ y}

⟨*proof*⟩

lemma *basis-prj-mono*: *ubasis-le a b* ⟹ *basis-prj a* ⊑ *basis-prj b*
⟨*proof*⟩

lemma *basis-emb-prj-less*: *ubasis-le* (*basis-emb* (*basis-prj x*)) *x*
⟨*proof*⟩

lemma *ideal-completion*:
ideal-completion below Rep-compact-basis (*approximants* :: *'a* ⇒ -)
⟨*proof*⟩

end

interpretation *compact-basis*:
ideal-completion below Rep-compact-basis
approximants :: *'a*::*bifinite* ⇒ *'a compact-basis set*
⟨*proof*⟩

21.4.6 EP-pair from any bifinite domain into *udom*

context *bifinite-approx-chain* **begin**

definition
udom-emb :: *'a* → *udom*
where
udom-emb = *compact-basis.extension* (λ*x*. *udom-principal* (*basis-emb x*))

definition
udom-prj :: *udom* → *'a*
where
udom-prj = *udom.extension* (λ*x*. *Rep-compact-basis* (*basis-prj x*))

lemma *udom-emb-principal*:
udom-emb·(*Rep-compact-basis x*) = *udom-principal* (*basis-emb x*)
⟨*proof*⟩

lemma *udom-prj-principal*:
udom-prj·(*udom-principal x*) = *Rep-compact-basis* (*basis-prj x*)
⟨*proof*⟩

lemma *ep-pair-udom*: *ep-pair udom-emb udom-prj*
⟨*proof*⟩

end

abbreviation *udom-emb* ≡ *bifinite-approx-chain.udom-emb*
abbreviation *udom-prj* ≡ *bifinite-approx-chain.udom-prj*

lemmas *ep-pair-udom* =
bifinite-approx-chain.ep-pair-udom [*unfolded bifinite-approx-chain-def*]

21.5 Chain of approx functions for type *udom*

definition
udom-approx :: *nat* ⇒ *udom* → *udom*
where
udom-approx i =
udom.extension (λ*x*. *udom-principal* (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*))

lemma *udom-approx-mono*:
ubasis-le a b ⟹
udom-principal (*ubasis-until* (λ*y*. *y* ≤ *i*) *a*) ⊑
udom-principal (*ubasis-until* (λ*y*. *y* ≤ *i*) *b*)
⟨*proof*⟩

lemma *adm-mem-finite*: ⟦*cont f*; *finite S*⟧ ⟹ *adm* (λ*x*. *f x* ∈ *S*)
⟨*proof*⟩

lemma *udom-approx-principal*:
udom-approx i·(*udom-principal x*) =
udom-principal (*ubasis-until* (λ*y*. *y* ≤ *i*) *x*)
⟨*proof*⟩

lemma *finite-deflation-udom-approx*: *finite-deflation* (*udom-approx i*)
⟨*proof*⟩

interpretation *udom-approx*: *finite-deflation udom-approx i*
⟨*proof*⟩

lemma *chain-udom-approx* [*simp*]: *chain* (λ*i*. *udom-approx i*)
⟨*proof*⟩

lemma *lub-udom-approx* [*simp*]: (⨆ *i*. *udom-approx i*) = *ID*
⟨*proof*⟩

lemma *udom-approx* [*simp*]: *approx-chain udom-approx*
⟨*proof*⟩

instance *udom* :: *bifinite*
⟨*proof*⟩

hide-const (**open**) *node*

unbundle *binomial-syntax*

end

22 Algebraic deflations

theory *Algebraic*
imports *Universal Map-Functions*
begin

22.1 Type constructor for finite deflations

typedef $'a$::*bifinite fin-defl* = $\{d::'a \to 'a.$ *finite-deflation* $d\}$
⟨*proof*⟩

instantiation *fin-defl* :: (*bifinite*) *below*
begin

definition *below-fin-defl-def*:
$below \equiv \lambda x\ y.$ *Rep-fin-defl* $x \sqsubseteq$ *Rep-fin-defl* y

instance ⟨*proof*⟩
end

instance *fin-defl* :: (*bifinite*) *po*
⟨*proof*⟩

lemma *finite-deflation-Rep-fin-defl*: *finite-deflation* (*Rep-fin-defl* d)
⟨*proof*⟩

lemma *deflation-Rep-fin-defl*: *deflation* (*Rep-fin-defl* d)
⟨*proof*⟩

interpretation *Rep-fin-defl*: *finite-deflation Rep-fin-defl d*
⟨*proof*⟩

lemma *fin-defl-belowI*:
$(\bigwedge x.$ *Rep-fin-defl* $a{\cdot}x = x \Longrightarrow$ *Rep-fin-defl* $b{\cdot}x = x) \Longrightarrow a \sqsubseteq b$
⟨*proof*⟩

lemma *fin-defl-belowD*:
$[\![a \sqsubseteq b;$ *Rep-fin-defl* $a{\cdot}x = x]\!] \Longrightarrow$ *Rep-fin-defl* $b{\cdot}x = x$
⟨*proof*⟩

lemma *fin-defl-eqI*:
$a = b$ **if** $(\bigwedge x.$ *Rep-fin-defl* $a{\cdot}x = x \longleftrightarrow$ *Rep-fin-defl* $b{\cdot}x = x)$
⟨*proof*⟩

lemma *Rep-fin-defl-mono*: $a \sqsubseteq b \Longrightarrow$ *Rep-fin-defl* $a \sqsubseteq$ *Rep-fin-defl* b
⟨*proof*⟩

lemma *Abs-fin-defl-mono*:
$[\![$*finite-deflation* $a;$ *finite-deflation* $b;$ $a \sqsubseteq b]\!]$
$\Longrightarrow$ *Abs-fin-defl* $a \sqsubseteq$ *Abs-fin-defl* b

⟨*proof*⟩

lemma (**in** *finite-deflation*) *compact-belowI*:
$d \sqsubseteq f$ **if** $\bigwedge x.\ \mathit{compact}\ x \Longrightarrow d{\cdot}x = x \Longrightarrow f{\cdot}x = x$
⟨*proof*⟩

lemma *compact-Rep-fin-defl* [*simp*]: *compact* (*Rep-fin-defl a*)
⟨*proof*⟩

22.2 Defining algebraic deflations by ideal completion

typedef $'a$::*bifinite defl* = $\{S::'a\ \mathit{fin\text{-}defl}\ \mathit{set}.\ \mathit{below.ideal}\ S\}$
⟨*proof*⟩

instantiation *defl* :: (*bifinite*) *below*
begin

definition $x \sqsubseteq y \longleftrightarrow \mathit{Rep\text{-}defl}\ x \subseteq \mathit{Rep\text{-}defl}\ y$

instance ⟨*proof*⟩

end

instance *defl* :: (*bifinite*) *po*
⟨*proof*⟩

instance *defl* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition *defl-principal* :: $'a$::*bifinite fin-defl* ⇒ $'a$ *defl*
where *defl-principal t* = *Abs-defl* $\{u.\ u \sqsubseteq t\}$

lemma *fin-defl-countable*: $\exists f::'a::\mathit{bifinite}\ \mathit{fin\text{-}defl} \Rightarrow \mathit{nat}.\ \mathit{inj}\ f$
⟨*proof*⟩

interpretation *defl*: *ideal-completion below defl-principal Rep-defl*
⟨*proof*⟩

Algebraic deflations are pointed

lemma *defl-minimal*: *defl-principal* (*Abs-fin-defl* ⊥) ⊑ *x*
⟨*proof*⟩

instance *defl* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-defl-pcpo*: ⊥ = *defl-principal* (*Abs-fin-defl* ⊥)
⟨*proof*⟩

22.3 Applying algebraic deflations

definition *cast* :: *'a*::*bifinite defl* → *'a* → *'a*
where *cast* = *defl.extension Rep-fin-defl*

lemma *cast-defl-principal*: *cast*·(*defl-principal a*) = *Rep-fin-defl a*
⟨*proof*⟩

lemma *deflation-cast*: *deflation* (*cast*·*d*)
⟨*proof*⟩

lemma *finite-deflation-cast*: *compact d* ⟹ *finite-deflation* (*cast*·*d*)
⟨*proof*⟩

interpretation *cast*: *deflation cast*·*d*
⟨*proof*⟩

declare *cast.idem* [*simp*]

lemma *compact-cast* [*simp*]: *compact* (*cast*·*d*) **if** *compact d*
⟨*proof*⟩

lemma *cast-below-cast*: *cast*·*A* ⊑ *cast*·*B* ⟷ *A* ⊑ *B*
⟨*proof*⟩

lemma *compact-cast-iff*: *compact* (*cast*·*d*) ⟷ *compact d*
⟨*proof*⟩

lemma *cast-below-imp-below*: *cast*·*A* ⊑ *cast*·*B* ⟹ *A* ⊑ *B*
⟨*proof*⟩

lemma *cast-eq-imp-eq*: *cast*·*A* = *cast*·*B* ⟹ *A* = *B*
⟨*proof*⟩

lemma *cast-strict1* [*simp*]: *cast*·⊥ = ⊥
⟨*proof*⟩

lemma *cast-strict2* [*simp*]: *cast*·*A*·⊥ = ⊥
⟨*proof*⟩

22.4 Deflation combinators

definition
defl-fun1 e p f =
defl.extension (λ*a*.
defl-principal (*Abs-fin-defl*
(*e oo f*·(*Rep-fin-defl a*) *oo p*)))

definition
defl-fun2 e p f =

```
    defl.extension (λa.
      defl.extension (λb.
        defl-principal (Abs-fin-defl
          (e oo f·(Rep-fin-defl a)·(Rep-fin-defl b) oo p))))

lemma cast-defl-fun1:
  assumes ep: ep-pair e p
  assumes f: ⋀a. finite-deflation a ⟹ finite-deflation (f·a)
  shows cast·(defl-fun1 e p f·A) = e oo f·(cast·A) oo p
⟨proof⟩

lemma cast-defl-fun2:
  assumes ep: ep-pair e p
  assumes f: ⋀a b. finite-deflation a ⟹ finite-deflation b ⟹
              finite-deflation (f·a·b)
  shows cast·(defl-fun2 e p f·A·B) = e oo f·(cast·A)·(cast·B) oo p
⟨proof⟩

end
```

23 Representable domains

```
theory Representable
imports Algebraic Map-Functions HOL−Library.Countable
begin
```

23.1 Class of representable domains

We define a "domain" as a pcpo that is isomorphic to some algebraic deflation over the universal domain; this is equivalent to being omega-bifinite.

A predomain is a cpo that, when lifted, becomes a domain. Predomains are represented by deflations over a lifted universal domain type.

```
class predomain-syn = cpo +
  fixes liftemb :: 'a⊥ → udom⊥
  fixes liftprj :: udom⊥ → 'a⊥
  fixes liftdefl :: 'a itself ⇒ udom u defl

class predomain = predomain-syn +
  assumes predomain-ep: ep-pair liftemb liftprj
  assumes cast-liftdefl: cast·(liftdefl TYPE('a)) = liftemb oo liftprj

syntax -LIFTDEFL :: type ⇒ logic  (‹(1LIFTDEFL/(1'(-')))›)
syntax-consts -LIFTDEFL ⇌ liftdefl
translations LIFTDEFL('t) ⇌ CONST liftdefl TYPE('t)

definition liftdefl-of :: udom defl → udom u defl
  where liftdefl-of = defl-fun1 ID ID u-map
```

lemma *cast-liftdefl-of*: *cast·(liftdefl-of·t) = u-map·(cast·t)*
⟨*proof*⟩

class *domain = predomain-syn + pcpo +*
  fixes *emb* :: *'a → udom*
  fixes *prj* :: *udom → 'a*
  fixes *defl* :: *'a itself ⇒ udom defl*
  assumes *ep-pair-emb-prj*: *ep-pair emb prj*
  assumes *cast-DEFL*: *cast·(defl TYPE('a)) = emb oo prj*
  assumes *liftemb-eq*: *liftemb = u-map·emb*
  assumes *liftprj-eq*: *liftprj = u-map·prj*
  assumes *liftdefl-eq*: *liftdefl TYPE('a) = liftdefl-of·(defl TYPE('a))*

syntax *-DEFL* :: *type ⇒ logic* (‹(*1DEFL/(1'(-'))*)›)
syntax-consts *-DEFL* ⇌ *defl*
translations *DEFL('t)* ⇌ *CONST defl TYPE('t)*

instance *domain* ⊆ *predomain*
⟨*proof*⟩

Constants *liftemb* and *liftprj* imply class predomain.

⟨*ML*⟩

interpretation *predomain*: *pcpo-ep-pair liftemb liftprj*
  ⟨*proof*⟩

interpretation *domain*: *pcpo-ep-pair emb prj*
  ⟨*proof*⟩

lemmas *emb-inverse = domain.e-inverse*
lemmas *emb-prj-below = domain.e-p-below*
lemmas *emb-eq-iff = domain.e-eq-iff*
lemmas *emb-strict = domain.e-strict*
lemmas *prj-strict = domain.p-strict*

23.2 Domains are bifinite

lemma *approx-chain-ep-cast*:
  assumes *ep*: *ep-pair* (*e*::*'a*::*pcpo → 'b*::*bifinite*) (*p*::*'b → 'a*)
  assumes *cast-t*: *cast·t = e oo p*
  shows ∃(*a*::*nat ⇒ 'a*::*pcpo → 'a*). *approx-chain a*
⟨*proof*⟩

instance *domain* ⊆ *bifinite*
⟨*proof*⟩

instance *predomain* ⊆ *profinite*
⟨*proof*⟩

23.3 Universal domain ep-pairs

definition *u-emb* = *udom-emb* (λ*i*. *u-map*·(*udom-approx i*))
definition *u-prj* = *udom-prj* (λ*i*. *u-map*·(*udom-approx i*))

definition *prod-emb* = *udom-emb* (λ*i*. *prod-map*·(*udom-approx i*)·(*udom-approx i*))
definition *prod-prj* = *udom-prj* (λ*i*. *prod-map*·(*udom-approx i*)·(*udom-approx i*))

definition *sprod-emb* = *udom-emb* (λ*i*. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))
definition *sprod-prj* = *udom-prj* (λ*i*. *sprod-map*·(*udom-approx i*)·(*udom-approx i*))

definition *ssum-emb* = *udom-emb* (λ*i*. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))
definition *ssum-prj* = *udom-prj* (λ*i*. *ssum-map*·(*udom-approx i*)·(*udom-approx i*))

definition *sfun-emb* = *udom-emb* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))
definition *sfun-prj* = *udom-prj* (λ*i*. *sfun-map*·(*udom-approx i*)·(*udom-approx i*))

lemma *ep-pair-u*: *ep-pair u-emb u-prj*
⟨*proof*⟩

lemma *ep-pair-prod*: *ep-pair prod-emb prod-prj*
⟨*proof*⟩

lemma *ep-pair-sprod*: *ep-pair sprod-emb sprod-prj*
⟨*proof*⟩

lemma *ep-pair-ssum*: *ep-pair ssum-emb ssum-prj*
⟨*proof*⟩

lemma *ep-pair-sfun*: *ep-pair sfun-emb sfun-prj*
⟨*proof*⟩

23.4 Type combinators

definition *u-defl* :: *udom defl* → *udom defl*
where *u-defl* = *defl-fun1 u-emb u-prj u-map*

definition *prod-defl* :: *udom defl* → *udom defl* → *udom defl*
where *prod-defl* = *defl-fun2 prod-emb prod-prj prod-map*

definition *sprod-defl* :: *udom defl* → *udom defl* → *udom defl*
where *sprod-defl* = *defl-fun2 sprod-emb sprod-prj sprod-map*

definition *ssum-defl* :: *udom defl* → *udom defl* → *udom defl*
where *ssum-defl* = *defl-fun2 ssum-emb ssum-prj ssum-map*

definition *sfun-defl* :: *udom defl* → *udom defl* → *udom defl*

where *sfun-defl* = *defl-fun2 sfun-emb sfun-prj sfun-map*

lemma *cast-u-defl*:
cast·(*u-defl*·*A*) = *u-emb oo u-map*·(*cast*·*A*) *oo u-prj*
⟨*proof*⟩

lemma *cast-prod-defl*:
cast·(*prod-defl*·*A*·*B*) =
prod-emb oo prod-map·(*cast*·*A*)·(*cast*·*B*) *oo prod-prj*
⟨*proof*⟩

lemma *cast-sprod-defl*:
cast·(*sprod-defl*·*A*·*B*) =
sprod-emb oo sprod-map·(*cast*·*A*)·(*cast*·*B*) *oo sprod-prj*
⟨*proof*⟩

lemma *cast-ssum-defl*:
cast·(*ssum-defl*·*A*·*B*) =
ssum-emb oo ssum-map·(*cast*·*A*)·(*cast*·*B*) *oo ssum-prj*
⟨*proof*⟩

lemma *cast-sfun-defl*:
cast·(*sfun-defl*·*A*·*B*) =
sfun-emb oo sfun-map·(*cast*·*A*)·(*cast*·*B*) *oo sfun-prj*
⟨*proof*⟩

Special deflation combinator for unpointed types.

definition *u-liftdefl* :: *udom u defl* → *udom defl*
where *u-liftdefl* = *defl-fun1 u-emb u-prj ID*

lemma *cast-u-liftdefl*:
cast·(*u-liftdefl*·*A*) = *u-emb oo cast*·*A oo u-prj*
⟨*proof*⟩

lemma *u-liftdefl-liftdefl-of*:
u-liftdefl·(*liftdefl-of*·*A*) = *u-defl*·*A*
⟨*proof*⟩

23.5 Class instance proofs

23.5.1 Universal domain

instantiation *udom* :: *domain*
begin

definition [*simp*]:
emb = (*ID* :: *udom* → *udom*)

definition [*simp*]:
prj = (*ID* :: *udom* → *udom*)

definition
defl (*t*::*udom itself*) = (⨆ *i*. *defl-principal* (*Abs-fin-defl* (*udom-approx i*)))

definition
(*liftemb* :: *udom u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → *udom u*) = *u-map*·*prj*

definition
liftdefl (*t*::*udom itself*) = *liftdefl-of*·*DEFL*(*udom*)

instance ⟨*proof*⟩

end

23.5.2 Lifted cpo

instantiation *u* :: (*predomain*) *domain*
begin

definition
emb = *u-emb oo liftemb*

definition
prj = *liftprj oo u-prj*

definition
defl (*t*::'*a u itself*) = *u-liftdefl*·*LIFTDEFL*('*a*)

definition
(*liftemb* :: '*a u u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a u u*) = *u-map*·*prj*

definition
liftdefl (*t*::'*a u itself*) = *liftdefl-of*·*DEFL*('*a u*)

instance ⟨*proof*⟩

end

lemma *DEFL-u*: *DEFL*('*a*::*predomain u*) = *u-liftdefl*·*LIFTDEFL*('*a*)
⟨*proof*⟩

23.5.3 Strict function space

instantiation *sfun* :: (*domain*, *domain*) *domain*

begin

definition
emb = *sfun-emb oo sfun-map·prj·emb*

definition
prj = *sfun-map·emb·prj oo sfun-prj*

definition
defl (*t*::('*a* →! '*b*) *itself*) = *sfun-defl·DEFL*('*a*)·*DEFL*('*b*)

definition
(*liftemb* :: ('*a* →! '*b*) *u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → ('*a* →! '*b*) *u*) = *u-map·prj*

definition
liftdefl (*t*::('*a* →! '*b*) *itself*) = *liftdefl-of·DEFL*('*a* →! '*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-sfun*:
DEFL('*a*::*domain* →! '*b*::*domain*) = *sfun-defl·DEFL*('*a*)·*DEFL*('*b*)
⟨*proof*⟩

23.5.4 Continuous function space

instantiation *cfun* :: (*predomain*, *domain*) *domain*
begin

definition
emb = *emb oo encode-cfun*

definition
prj = *decode-cfun oo prj*

definition
defl (*t*::('*a* → '*b*) *itself*) = *DEFL*('*a u* →! '*b*)

definition
(*liftemb* :: ('*a* → '*b*) *u* → *udom u*) = *u-map·emb*

definition
(*liftprj* :: *udom u* → ('*a* → '*b*) *u*) = *u-map·prj*

definition

liftdefl (*t*::('*a* → '*b*) *itself*) = *liftdefl-of*·*DEFL*('*a* → '*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-cfun*:
DEFL('*a*::*predomain* → '*b*::*domain*) = *DEFL*('*a* *u* →! '*b*)
⟨*proof*⟩

23.5.5 Strict product

instantiation *sprod* :: (*domain*, *domain*) *domain*
begin

definition
emb = *sprod-emb oo sprod-map*·*emb*·*emb*

definition
prj = *sprod-map*·*prj*·*prj oo sprod-prj*

definition
defl (*t*::('*a* ⊗ '*b*) *itself*) = *sprod-defl*·*DEFL*('*a*)·*DEFL*('*b*)

definition
(*liftemb* :: ('*a* ⊗ '*b*) *u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → ('*a* ⊗ '*b*) *u*) = *u-map*·*prj*

definition
liftdefl (*t*::('*a* ⊗ '*b*) *itself*) = *liftdefl-of*·*DEFL*('*a* ⊗ '*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-sprod*:
DEFL('*a*::*domain* ⊗ '*b*::*domain*) = *sprod-defl*·*DEFL*('*a*)·*DEFL*('*b*)
⟨*proof*⟩

23.5.6 Cartesian product

definition *prod-liftdefl* :: *udom u defl* → *udom u defl* → *udom u defl*
where *prod-liftdefl* = *defl-fun2* (*u-map*·*prod-emb oo decode-prod-u*)
(*encode-prod-u oo u-map*·*prod-prj*) *sprod-map*

lemma *cast-prod-liftdefl*:
cast·(*prod-liftdefl*·*a*·*b*) =
(*u-map*·*prod-emb oo decode-prod-u*) *oo sprod-map*·(*cast*·*a*)·(*cast*·*b*) *oo*

(*encode-prod-u oo u-map·prod-prj*)
⟨*proof*⟩

instantiation *prod* :: (*predomain*, *predomain*) *predomain*
begin

definition
liftemb = (*u-map·prod-emb oo decode-prod-u*) *oo*
(*sprod-map·liftemb·liftemb oo encode-prod-u*)

definition
liftprj = (*decode-prod-u oo sprod-map·liftprj·liftprj*) *oo*
(*encode-prod-u oo u-map·prod-prj*)

definition
liftdefl (*t*::('*a* × '*b*) *itself*) = *prod-liftdefl·LIFTDEFL*('*a*)·*LIFTDEFL*('*b*)

instance ⟨*proof*⟩

end

instantiation *prod* :: (*domain*, *domain*) *domain*
begin

definition
emb = *prod-emb oo prod-map·emb·emb*

definition
prj = *prod-map·prj·prj oo prod-prj*

definition
defl (*t*::('*a* × '*b*) *itself*) = *prod-defl·DEFL*('*a*)·*DEFL*('*b*)

instance ⟨*proof*⟩

end

lemma *DEFL-prod*:
DEFL('*a*::*domain* × '*b*::*domain*) = *prod-defl·DEFL*('*a*)·*DEFL*('*b*)
⟨*proof*⟩

lemma *LIFTDEFL-prod*:
LIFTDEFL('*a*::*predomain* × '*b*::*predomain*) =
prod-liftdefl·LIFTDEFL('*a*)·*LIFTDEFL*('*b*)
⟨*proof*⟩

23.5.7 Unit type

instantiation *unit* :: *domain*

begin

definition
emb = (⊥ :: *unit* → *udom*)

definition
prj = (⊥ :: *udom* → *unit*)

definition
defl (*t*::*unit itself*) = ⊥

definition
(*liftemb* :: *unit u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → *unit u*) = *u-map*·*prj*

definition
liftdefl (*t*::*unit itself*) = *liftdefl-of*·*DEFL*(*unit*)

instance ⟨*proof*⟩

end

23.5.8 Discrete cpo

instantiation *discr* :: (*countable*) *predomain*
begin

definition
(*liftemb* :: *'a discr u* → *udom u*) = *strictify*·*up oo udom-emb discr-approx*

definition
(*liftprj* :: *udom u* → *'a discr u*) = *udom-prj discr-approx oo fup*·*ID*

definition
liftdefl (*t*::*'a discr itself*) =
(⨆ *i*. *defl-principal* (*Abs-fin-defl* (*liftemb oo discr-approx i oo* (*liftprj*::*udom u* → *'a discr u*))))

instance ⟨*proof*⟩

end

23.5.9 Strict sum

instantiation *ssum* :: (*domain*, *domain*) *domain*
begin

definition

emb = ssum-emb oo ssum-map·emb·emb

definition
prj = ssum-map·prj·prj oo ssum-prj

definition
defl (t::('a ⊕ 'b) itself) = ssum-defl·DEFL('a)·DEFL('b)

definition
(liftemb :: ('a ⊕ 'b) u → udom u) = u-map·emb

definition
(liftprj :: udom u → ('a ⊕ 'b) u) = u-map·prj

definition
liftdefl (t::('a ⊕ 'b) itself) = liftdefl-of·DEFL('a ⊕ 'b)

instance ⟨*proof*⟩

end

lemma *DEFL-ssum*:
DEFL('a::domain ⊕ 'b::domain) = ssum-defl·DEFL('a)·DEFL('b)
⟨*proof*⟩

23.5.10 Lifted HOL type

instantiation *lift* :: (*countable*) *domain*
begin

definition
emb = emb oo (Λ x. Rep-lift x)

definition
prj = (Λ y. Abs-lift y) oo prj

definition
defl (t::'a lift itself) = DEFL('a discr u)

definition
(liftemb :: 'a lift u → udom u) = u-map·emb

definition
(liftprj :: udom u → 'a lift u) = u-map·prj

definition
liftdefl (t::'a lift itself) = liftdefl-of·DEFL('a lift)

instance ⟨*proof*⟩

end

end

24 The unit domain

theory *One*
imports *Lift*
begin

type-synonym *one* = *unit lift*

translations
(*type*) *one* ↽ (*type*) *unit lift*

definition *ONE* :: *one*
where *ONE* ≡ *Def* ()

Exhaustion and Elimination for type *one*

lemma *Exh-one*: $t = \bot \vee t = ONE$
⟨*proof*⟩

lemma *oneE* [*case-names bottom ONE*]: $[\![p = \bot \Longrightarrow Q;\ p = ONE \Longrightarrow Q]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma *one-induct* [*case-names bottom ONE*]: $P\ \bot \Longrightarrow P\ ONE \Longrightarrow P\ x$
⟨*proof*⟩

lemma *dist-below-one* [*simp*]: $ONE \not\sqsubseteq \bot$
⟨*proof*⟩

lemma *below-ONE* [*simp*]: $x \sqsubseteq ONE$
⟨*proof*⟩

lemma *ONE-below-iff* [*simp*]: $ONE \sqsubseteq x \longleftrightarrow x = ONE$
⟨*proof*⟩

lemma *ONE-defined* [*simp*]: $ONE \neq \bot$
⟨*proof*⟩

lemma *one-neq-iffs* [*simp*]:
$x \neq ONE \longleftrightarrow x = \bot$
$ONE \neq x \longleftrightarrow x = \bot$
$x \neq \bot \longleftrightarrow x = ONE$
$\bot \neq x \longleftrightarrow x = ONE$
⟨*proof*⟩

lemma *compact-ONE*: *compact ONE*

⟨*proof*⟩

Case analysis function for type *one*

definition *one-case* :: *'a::pcpo → one → 'a*
where *one-case* = (Λ *a x*. *seq·x·a*)

translations
case x of XCONST ONE ⇒ t ⇌ *CONST one-case·t·x*
case x of XCONST ONE :: 'a ⇒ t ⇀ *CONST one-case·t·x*
Λ (*XCONST ONE*). *t* ⇌ *CONST one-case·t*

lemma *one-case1* [*simp*]: (*case ⊥ of ONE ⇒ t*) = *⊥*
⟨*proof*⟩

lemma *one-case2* [*simp*]: (*case ONE of ONE ⇒ t*) = *t*
⟨*proof*⟩

lemma *one-case3* [*simp*]: (*case x of ONE ⇒ ONE*) = *x*
⟨*proof*⟩

end

theory *Fixrec*
imports *Cprod Sprod Ssum Up One Tr Cfun*
keywords *fixrec* :: *thy-defn*
begin

25 Fixed point operator and admissibility

25.1 Iteration

primrec *iterate* :: *nat ⇒ ('a → 'a) → ('a → 'a)*
where
iterate 0 = (Λ *F x*. *x*)
| *iterate* (*Suc n*) = (Λ *F x*. *F·*(*iterate n·F·x*))

Derive inductive properties of iterate from primitive recursion

lemma *iterate-0* [*simp*]: *iterate 0·F·x* = *x*
⟨*proof*⟩

lemma *iterate-Suc* [*simp*]: *iterate* (*Suc n*)*·F·x* = *F·*(*iterate n·F·x*)
⟨*proof*⟩

declare *iterate.simps* [*simp del*]

lemma *iterate-Suc2*: *iterate* (*Suc n*)*·F·x* = *iterate n·F·*(*F·x*)
⟨*proof*⟩

lemma *iterate-iterate*: $iterate\ m{\cdot}F{\cdot}(iterate\ n{\cdot}F{\cdot}x) = iterate\ (m + n){\cdot}F{\cdot}x$
⟨*proof*⟩

The sequence of function iterations is a chain.

lemma *chain-iterate* [*simp*]: $chain\ (\lambda i.\ iterate\ i{\cdot}F{\cdot}\bot)$
⟨*proof*⟩

25.2 Least fixed point operator

definition $fix :: ('a{::}pcpo \to 'a) \to 'a$
where $fix = (\Lambda\ F.\ \bigsqcup i.\ iterate\ i{\cdot}F{\cdot}\bot)$

Binder syntax for *fix*

abbreviation $fix\text{-}syn :: ('a{::}pcpo \Rightarrow 'a) \Rightarrow 'a$ (**binder** ‹μ › *10*)
where $fix\text{-}syn\ (\lambda x.\ f\ x) \equiv fix{\cdot}(\Lambda\ x.\ f\ x)$

notation (*ASCII*)
fix-syn (**binder** ‹*FIX* › *10*)

Properties of *fix*

direct connection between *fix* and iteration

lemma *fix-def2*: $fix{\cdot}F = (\bigsqcup i.\ iterate\ i{\cdot}F{\cdot}\bot)$
⟨*proof*⟩

lemma *iterate-below-fix*: $iterate\ n{\cdot}f{\cdot}\bot \sqsubseteq fix{\cdot}f$
⟨*proof*⟩

Kleene's fixed point theorems for continuous functions in pointed omega cpo's

lemma *fix-eq*: $fix{\cdot}F = F{\cdot}(fix{\cdot}F)$
⟨*proof*⟩

lemma *fix-least-below*: $F{\cdot}x \sqsubseteq x \Longrightarrow fix{\cdot}F \sqsubseteq x$
⟨*proof*⟩

lemma *fix-least*: $F{\cdot}x = x \Longrightarrow fix{\cdot}F \sqsubseteq x$
⟨*proof*⟩

lemma *fix-eqI*:
assumes *fixed*: $F{\cdot}x = x$
and *least*: $\bigwedge z.\ F{\cdot}z = z \Longrightarrow x \sqsubseteq z$
shows $fix{\cdot}F = x$
⟨*proof*⟩

lemma *fix-eq2*: $f \equiv fix{\cdot}F \Longrightarrow f = F{\cdot}f$
⟨*proof*⟩

lemma *fix-eq3*: $f \equiv fix{\cdot}F \Longrightarrow f{\cdot}x = F{\cdot}f{\cdot}x$

⟨*proof*⟩

lemma *fix-eq4*: $f = \mathit{fix}{\cdot}F \Longrightarrow f = F{\cdot}f$
⟨*proof*⟩

lemma *fix-eq5*: $f = \mathit{fix}{\cdot}F \Longrightarrow f{\cdot}x = F{\cdot}f{\cdot}x$
⟨*proof*⟩

strictness of *fix*

lemma *fix-bottom-iff*: $\mathit{fix}{\cdot}F = \bot \longleftrightarrow F{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *fix-strict*: $F{\cdot}\bot = \bot \Longrightarrow \mathit{fix}{\cdot}F = \bot$
⟨*proof*⟩

lemma *fix-defined*: $F{\cdot}\bot \neq \bot \Longrightarrow \mathit{fix}{\cdot}F \neq \bot$
⟨*proof*⟩

fix applied to identity and constant functions

lemma *fix-id*: $(\mu\ x.\ x) = \bot$
⟨*proof*⟩

lemma *fix-const*: $(\mu\ x.\ c) = c$
⟨*proof*⟩

25.3 Fixed point induction

lemma *fix-ind*: $\mathit{adm}\ P \Longrightarrow P\ \bot \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow P\ (F{\cdot}x)) \Longrightarrow P\ (\mathit{fix}{\cdot}F)$
⟨*proof*⟩

lemma *cont-fix-ind*: $\mathit{cont}\ F \Longrightarrow \mathit{adm}\ P \Longrightarrow P\ \bot \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow P\ (F\ x)) \Longrightarrow P\ (\mathit{fix}{\cdot}(\mathit{Abs\text{-}cfun}\ F))$
⟨*proof*⟩

lemma *def-fix-ind*: $[\![f \equiv \mathit{fix}{\cdot}F;\ \mathit{adm}\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F{\cdot}x)]\!] \Longrightarrow P\ f$
⟨*proof*⟩

lemma *fix-ind2*:
assumes *adm*: $\mathit{adm}\ P$
assumes *0*: $P\ \bot$ **and** *1*: $P\ (F{\cdot}\bot)$
assumes *step*: $\bigwedge x.\ [\![P\ x;\ P\ (F{\cdot}x)]\!] \Longrightarrow P\ (F{\cdot}(F{\cdot}x))$
shows $P\ (\mathit{fix}{\cdot}F)$
⟨*proof*⟩

lemma *parallel-fix-ind*:
assumes *adm*: $\mathit{adm}\ (\lambda x.\ P\ (\mathit{fst}\ x)\ (\mathit{snd}\ x))$
assumes *base*: $P\ \bot\ \bot$
assumes *step*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow P\ (F{\cdot}x)\ (G{\cdot}y)$
shows $P\ (\mathit{fix}{\cdot}F)\ (\mathit{fix}{\cdot}G)$

⟨*proof*⟩

lemma *cont-parallel-fix-ind*:
assumes *cont F* **and** *cont G*
assumes *adm* (λ*x*. *P* (*fst x*) (*snd x*))
assumes *P* ⊥ ⊥
assumes ⋀*x y*. *P x y* ⟹ *P* (*F x*) (*G y*)
shows *P* (*fix*·(*Abs-cfun F*)) (*fix*·(*Abs-cfun G*))
⟨*proof*⟩

25.4 Fixed-points on product types

Bekic's Theorem: Simultaneous fixed points over pairs can be written in terms of separate fixed points.

lemma *fix-cprod*:
fixes *F* :: *'a*::*pcpo* × *'b*::*pcpo* → *'a* × *'b*
shows
fix·*F* =
(μ *x*. *fst* (*F*·(*x*, μ *y*. *snd* (*F*·(*x*, *y*)))),
μ *y*. *snd* (*F*·(μ *x*. *fst* (*F*·(*x*, μ *y*. *snd* (*F*·(*x*, *y*)))), *y*)))
(**is** *fix*·*F* = (*?x*, *?y*))
⟨*proof*⟩

26 Package for defining recursive functions in HOLCF

26.1 Pattern-match monad

pcpodef *'a match* = *UNIV*::(*one* ++ *'a u*) *set*
⟨*proof*⟩

definition
fail :: *'a match* **where**
fail = *Abs-match* (*sinl*·*ONE*)

definition
succeed :: *'a* → *'a match* **where**
succeed = (Λ *x*. *Abs-match* (*sinr*·(*up*·*x*)))

lemma *matchE* [*case-names bottom fail succeed*, *cases type*: *match*]:
⟦*p* = ⊥ ⟹ *Q*; *p* = *fail* ⟹ *Q*; ⋀*x*. *p* = *succeed*·*x* ⟹ *Q*⟧ ⟹ *Q*
⟨*proof*⟩

lemma *succeed-defined* [*simp*]: *succeed*·*x* ≠ ⊥
⟨*proof*⟩

lemma *fail-defined* [*simp*]: *fail* ≠ ⊥
⟨*proof*⟩

lemma *succeed-eq* [*simp*]: $(succeed{\cdot}x = succeed{\cdot}y) = (x = y)$
⟨*proof*⟩

lemma *succeed-neq-fail* [*simp*]:
$succeed{\cdot}x \neq fail\ fail \neq succeed{\cdot}x$
⟨*proof*⟩

26.1.1 Run operator

definition
run :: *'a match* → *'a*::*pcpo* **where**
$run = (\Lambda\ m.\ sscase{\cdot}\bot{\cdot}(fup{\cdot}ID){\cdot}(Rep\text{-}match\ m))$

rewrite rules for run

lemma *run-strict* [*simp*]: $run{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *run-fail* [*simp*]: $run{\cdot}fail = \bot$
⟨*proof*⟩

lemma *run-succeed* [*simp*]: $run{\cdot}(succeed{\cdot}x) = x$
⟨*proof*⟩

26.1.2 Monad plus operator

definition
mplus :: *'a match* → *'a match* → *'a match* **where**
$mplus = (\Lambda\ m1\ m2.\ sscase{\cdot}(\Lambda\ \text{-}.\ m2){\cdot}(\Lambda\ \text{-}.\ m1){\cdot}(Rep\text{-}match\ m1))$

abbreviation
mplus-syn :: [*'a match*, *'a match*] ⇒ *'a match* (**infixr** ‹+++› *65*) **where**
$m1 +\!\!+\!\!+ m2 == mplus{\cdot}m1{\cdot}m2$

rewrite rules for mplus

lemma *mplus-strict* [*simp*]: $\bot +\!\!+\!\!+ m = \bot$
⟨*proof*⟩

lemma *mplus-fail* [*simp*]: $fail +\!\!+\!\!+ m = m$
⟨*proof*⟩

lemma *mplus-succeed* [*simp*]: $succeed{\cdot}x +\!\!+\!\!+ m = succeed{\cdot}x$
⟨*proof*⟩

lemma *mplus-fail2* [*simp*]: $m +\!\!+\!\!+ fail = m$
⟨*proof*⟩

lemma *mplus-assoc*: $(x +\!\!+\!\!+ y) +\!\!+\!\!+ z = x +\!\!+\!\!+ (y +\!\!+\!\!+ z)$
⟨*proof*⟩

26.2 Match functions for built-in types

definition
match-bottom :: *'a*::*pcpo* → *'c match* → *'c match*
where
match-bottom = (Λ *x k*. *seq*·*x*·*fail*)

definition
match-Pair :: *'a* × *'b* → (*'a* → *'b* → *'c match*) → *'c match*
where
match-Pair = (Λ *x k*. *csplit*·*k*·*x*)

definition
match-spair :: *'a*::*pcpo* ⊗ *'b*::*pcpo* → (*'a* → *'b* → *'c match*) → *'c*::*pcpo match*
where
match-spair = (Λ *x k*. *ssplit*·*k*·*x*)

definition
match-sinl :: *'a*::*pcpo* ⊕ *'b*::*pcpo* → (*'a* → *'c*::*pcpo match*) → *'c match*
where
match-sinl = (Λ *x k*. *sscase*·*k*·(Λ *b*. *fail*)·*x*)

definition
match-sinr :: *'a*::*pcpo* ⊕ *'b*::*pcpo* → (*'b* → *'c*::*pcpo match*) → *'c match*
where
match-sinr = (Λ *x k*. *sscase*·(Λ *a*. *fail*)·*k*·*x*)

definition
match-up :: *'a u* → (*'a* → *'c*::*pcpo match*) → *'c match*
where
match-up = (Λ *x k*. *fup*·*k*·*x*)

definition
match-ONE :: *one* → *'c*::*pcpo match* → *'c match*
where
match-ONE = (Λ *ONE k*. *k*)

definition
match-TT :: *tr* → *'c*::*pcpo match* → *'c match*
where
match-TT = (Λ *x k*. *If x then k else fail*)

definition
match-FF :: *tr* → *'c*::*pcpo match* → *'c match*
where
match-FF = (Λ *x k*. *If x then fail else k*)

lemma *match-bottom-simps* [*simp*]:
match-bottom·*x*·*k* = (*if x* = ⊥ *then* ⊥ *else fail*)
⟨*proof*⟩

lemma *match-Pair-simps* [*simp*]:
match-Pair·(*x*, *y*)·*k* = *k*·*x*·*y*
⟨*proof*⟩

lemma *match-spair-simps* [*simp*]:
⟦*x* ≠ ⊥; *y* ≠ ⊥⟧ ⟹ *match-spair*·(:*x*, *y*:)·*k* = *k*·*x*·*y*
match-spair·⊥·*k* = ⊥
⟨*proof*⟩

lemma *match-sinl-simps* [*simp*]:
x ≠ ⊥ ⟹ *match-sinl*·(*sinl*·*x*)·*k* = *k*·*x*
y ≠ ⊥ ⟹ *match-sinl*·(*sinr*·*y*)·*k* = *fail*
match-sinl·⊥·*k* = ⊥
⟨*proof*⟩

lemma *match-sinr-simps* [*simp*]:
x ≠ ⊥ ⟹ *match-sinr*·(*sinl*·*x*)·*k* = *fail*
y ≠ ⊥ ⟹ *match-sinr*·(*sinr*·*y*)·*k* = *k*·*y*
match-sinr·⊥·*k* = ⊥
⟨*proof*⟩

lemma *match-up-simps* [*simp*]:
match-up·(*up*·*x*)·*k* = *k*·*x*
match-up·⊥·*k* = ⊥
⟨*proof*⟩

lemma *match-ONE-simps* [*simp*]:
match-ONE·*ONE*·*k* = *k*
match-ONE·⊥·*k* = ⊥
⟨*proof*⟩

lemma *match-TT-simps* [*simp*]:
match-TT·*TT*·*k* = *k*
match-TT·*FF*·*k* = *fail*
match-TT·⊥·*k* = ⊥
⟨*proof*⟩

lemma *match-FF-simps* [*simp*]:
match-FF·*FF*·*k* = *k*
match-FF·*TT*·*k* = *fail*
match-FF·⊥·*k* = ⊥
⟨*proof*⟩

26.3 Mutual recursion

The following rules are used to prove unfolding theorems from fixed-point definitions of mutually recursive functions.

lemma *Pair-equalI*: ⟦*x* ≡ *fst p*; *y* ≡ *snd p*⟧ ⟹ (*x*, *y*) ≡ *p*

⟨*proof*⟩

lemma *Pair-eqD1*: $(x, y) = (x', y') \Longrightarrow x = x'$
⟨*proof*⟩

lemma *Pair-eqD2*: $(x, y) = (x', y') \Longrightarrow y = y'$
⟨*proof*⟩

lemma *def-cont-fix-eq*:
$[\![f \equiv fix{\cdot}(Abs\text{-}cfun\ F);\ cont\ F]\!] \Longrightarrow f = F\ f$
⟨*proof*⟩

lemma *def-cont-fix-ind*:
$[\![f \equiv fix{\cdot}(Abs\text{-}cfun\ F);\ cont\ F;\ adm\ P;\ P\ \bot;\ \bigwedge x.\ P\ x \Longrightarrow P\ (F\ x)]\!] \Longrightarrow P\ f$
⟨*proof*⟩

lemma for proving rewrite rules

lemma *ssubst-lhs*: $[\![t = s;\ P\ s = Q]\!] \Longrightarrow P\ t = Q$
⟨*proof*⟩

26.4 Initializing the fixrec package

⟨*ML*⟩

hide-const (**open**) *succeed fail run*

end

27 Domain package

theory *Domain*
imports *Representable Map-Functions Fixrec*
keywords
lazy unsafe **and**
domaindef domain :: *thy-defn* **and**
domain-isomorphism :: *thy-decl*
begin

27.1 Continuous isomorphisms

A locale for continuous isomorphisms

locale *iso* =
fixes *abs* :: *'a*::*pcpo* → *'b*::*pcpo*
fixes *rep* :: *'b* → *'a*
assumes *abs-iso* [*simp*]: $rep{\cdot}(abs{\cdot}x) = x$
assumes *rep-iso* [*simp*]: $abs{\cdot}(rep{\cdot}y) = y$
begin

lemma *swap*: *iso rep abs*
⟨*proof*⟩

lemma *abs-below*: $(abs{\cdot}x \sqsubseteq abs{\cdot}y) = (x \sqsubseteq y)$
⟨*proof*⟩

lemma *rep-below*: $(rep{\cdot}x \sqsubseteq rep{\cdot}y) = (x \sqsubseteq y)$
⟨*proof*⟩

lemma *abs-eq*: $(abs{\cdot}x = abs{\cdot}y) = (x = y)$
⟨*proof*⟩

lemma *rep-eq*: $(rep{\cdot}x = rep{\cdot}y) = (x = y)$
⟨*proof*⟩

lemma *abs-strict*: $abs{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *rep-strict*: $rep{\cdot}\bot = \bot$
⟨*proof*⟩

lemma *abs-defin'*: $abs{\cdot}x = \bot \Longrightarrow x = \bot$
⟨*proof*⟩

lemma *rep-defin'*: $rep{\cdot}z = \bot \Longrightarrow z = \bot$
⟨*proof*⟩

lemma *abs-defined*: $z \neq \bot \Longrightarrow abs{\cdot}z \neq \bot$
⟨*proof*⟩

lemma *rep-defined*: $z \neq \bot \Longrightarrow rep{\cdot}z \neq \bot$
⟨*proof*⟩

lemma *abs-bottom-iff*: $(abs{\cdot}x = \bot) = (x = \bot)$
⟨*proof*⟩

lemma *rep-bottom-iff*: $(rep{\cdot}x = \bot) = (x = \bot)$
⟨*proof*⟩

lemma *casedist-rule*: $rep{\cdot}x = \bot \vee P \Longrightarrow x = \bot \vee P$
⟨*proof*⟩

lemma *compact-abs-rev*: $compact\ (abs{\cdot}x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-rep-rev*: $compact\ (rep{\cdot}x) \Longrightarrow compact\ x$
⟨*proof*⟩

lemma *compact-abs*: $compact\ x \Longrightarrow compact\ (abs{\cdot}x)$

⟨*proof*⟩

lemma *compact-rep*: *compact x* ⟹ *compact* (*rep*·*x*)
⟨*proof*⟩

lemma *iso-swap*: (*x* = *abs*·*y*) = (*rep*·*x* = *y*)
⟨*proof*⟩

end

27.2 Proofs about take functions

This section contains lemmas that are used in a module that supports the domain isomorphism package; the module contains proofs related to take functions and the finiteness predicate.

lemma *deflation-abs-rep*:
fixes *abs* **and** *rep* **and** *d*
assumes *abs-iso*: ⋀*x*. *rep*·(*abs*·*x*) = *x*
assumes *rep-iso*: ⋀*y*. *abs*·(*rep*·*y*) = *y*
shows *deflation d* ⟹ *deflation* (*abs oo d oo rep*)
⟨*proof*⟩

lemma *deflation-chain-min*:
assumes *chain*: *chain d*
assumes *defl*: ⋀*n*. *deflation* (*d n*)
shows *d m*·(*d n*·*x*) = *d* (*min m n*)·*x*
⟨*proof*⟩

lemma *lub-ID-take-lemma*:
assumes *chain t* **and** (⨆ *n*. *t n*) = *ID*
assumes ⋀*n*. *t n*·*x* = *t n*·*y* **shows** *x* = *y*
⟨*proof*⟩

lemma *lub-ID-reach*:
assumes *chain t* **and** (⨆ *n*. *t n*) = *ID*
shows (⨆ *n*. *t n*·*x*) = *x*
⟨*proof*⟩

lemma *lub-ID-take-induct*:
assumes *chain t* **and** (⨆ *n*. *t n*) = *ID*
assumes *adm P* **and** ⋀*n*. *P* (*t n*·*x*) **shows** *P x*
⟨*proof*⟩

27.3 Finiteness

Let a "decisive" function be a deflation that maps every input to either itself or bottom. Then if a domain's take functions are all decisive, then all values in the domain are finite.

definition
  decisive :: (*'a*::*pcpo* → *'a*) ⇒ *bool*
where
  decisive d ⟷ (∀ *x*. *d*·*x* = *x* ∨ *d*·*x* = ⊥)

lemma *decisiveI*: (⋀*x*. *d*·*x* = *x* ∨ *d*·*x* = ⊥) ⟹ *decisive d*
  ⟨*proof*⟩

lemma *decisive-cases*:
  assumes *decisive d* **obtains** *d*·*x* = *x* | *d*·*x* = ⊥
⟨*proof*⟩

lemma *decisive-bottom*: *decisive* ⊥
  ⟨*proof*⟩

lemma *decisive-ID*: *decisive ID*
  ⟨*proof*⟩

lemma *decisive-ssum-map*:
  assumes *f*: *decisive f*
  assumes *g*: *decisive g*
  shows *decisive* (*ssum-map*·*f*·*g*)
  ⟨*proof*⟩

lemma *decisive-sprod-map*:
  assumes *f*: *decisive f*
  assumes *g*: *decisive g*
  shows *decisive* (*sprod-map*·*f*·*g*)
  ⟨*proof*⟩

lemma *decisive-abs-rep*:
  fixes *abs rep*
  assumes *iso*: *iso abs rep*
  assumes *d*: *decisive d*
  shows *decisive* (*abs oo d oo rep*)
  ⟨*proof*⟩

lemma *lub-ID-finite*:
  assumes *chain*: *chain d*
  assumes *lub*: (⨆ *n*. *d n*) = *ID*
  assumes *decisive*: ⋀*n*. *decisive* (*d n*)
  shows ∃ *n*. *d n*·*x* = *x*
⟨*proof*⟩

lemma *lub-ID-finite-take-induct*:
  assumes *chain d* **and** (⨆ *n*. *d n*) = *ID* **and** ⋀*n*. *decisive* (*d n*)
  shows (⋀*n*. *P* (*d n*·*x*)) ⟹ *P x*
⟨*proof*⟩

27.4 Proofs about constructor functions

Lemmas for proving nchotomy rule:

lemma *ex-one-bottom-iff*:
$(\exists x.\ P\ x \wedge x \neq \bot) = P\ ONE$
⟨*proof*⟩

lemma *ex-up-bottom-iff*:
$(\exists x.\ P\ x \wedge x \neq \bot) = (\exists x.\ P\ (up{\cdot}x))$
⟨*proof*⟩

lemma *ex-sprod-bottom-iff*:
$(\exists y.\ P\ y \wedge y \neq \bot) =$
$(\exists x\ y.\ (P\ (:x,\ y:) \wedge x \neq \bot) \wedge y \neq \bot)$
⟨*proof*⟩

lemma *ex-sprod-up-bottom-iff*:
$(\exists y.\ P\ y \wedge y \neq \bot) =$
$(\exists x\ y.\ P\ (:up{\cdot}x,\ y:) \wedge y \neq \bot)$
⟨*proof*⟩

lemma *ex-ssum-bottom-iff*:
$(\exists x.\ P\ x \wedge x \neq \bot) =$
$((\exists x.\ P\ (sinl{\cdot}x) \wedge x \neq \bot) \vee$
$(\exists x.\ P\ (sinr{\cdot}x) \wedge x \neq \bot))$
⟨*proof*⟩

lemma *exh-start*: $p = \bot \vee (\exists x.\ p = x \wedge x \neq \bot)$
⟨*proof*⟩

lemmas *ex-bottom-iffs* =
ex-ssum-bottom-iff
ex-sprod-up-bottom-iff
ex-sprod-bottom-iff
ex-up-bottom-iff
ex-one-bottom-iff

Rules for turning nchotomy into exhaust:

lemma *exh-casedist0*: $[\![R;\ R \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma *exh-casedist1*: $((P \vee Q \Longrightarrow R) \Longrightarrow S) \equiv ([\![P \Longrightarrow R;\ Q \Longrightarrow R]\!] \Longrightarrow S)$
⟨*proof*⟩

lemma *exh-casedist2*: $(\exists x.\ P\ x \Longrightarrow Q) \equiv (\bigwedge x.\ P\ x \Longrightarrow Q)$
⟨*proof*⟩

lemma *exh-casedist3*: $(P \wedge Q \Longrightarrow R) \equiv (P \Longrightarrow Q \Longrightarrow R)$
⟨*proof*⟩

lemmas *exh-casedists = exh-casedist1 exh-casedist2 exh-casedist3*

Rules for proving constructor properties

lemmas *con-strict-rules =*
sinl-strict sinr-strict spair-strict1 spair-strict2

lemmas *con-bottom-iff-rules =*
sinl-bottom-iff sinr-bottom-iff spair-bottom-iff up-defined ONE-defined

lemmas *con-below-iff-rules =*
sinl-below sinr-below sinl-below-sinr sinr-below-sinl con-bottom-iff-rules

lemmas *con-eq-iff-rules =*
sinl-eq sinr-eq sinl-eq-sinr sinr-eq-sinl con-bottom-iff-rules

lemmas *sel-strict-rules =*
cfcomp2 sscase1 sfst-strict ssnd-strict fup1

lemma *sel-app-extra-rules*:
sscase·ID·⊥·(sinr·x) = ⊥
sscase·ID·⊥·(sinl·x) = x
sscase·⊥·ID·(sinl·x) = ⊥
sscase·⊥·ID·(sinr·x) = x
fup·ID·(up·x) = x
⟨*proof*⟩

lemmas *sel-app-rules =*
sel-strict-rules sel-app-extra-rules
ssnd-spair sfst-spair up-defined spair-defined

lemmas *sel-bottom-iff-rules =*
cfcomp2 sfst-bottom-iff ssnd-bottom-iff

lemmas *take-con-rules =*
ssum-map-sinl' ssum-map-sinr' sprod-map-spair' u-map-up
deflation-strict deflation-ID ID1 cfcomp2

27.5 ML setup

named-theorems *domain-deflation theorems like deflation a ==> deflation (foo-map$a)*
and *domain-map-ID theorems like foo-map$ID = ID*

⟨*ML*⟩

27.6 Representations of types

lemma *emb-prj*: *emb·((prj·x)::'a::domain) = cast·DEFL('a)·x*
⟨*proof*⟩

lemma *emb-prj-emb*:
fixes $x :: 'a::domain$
assumes $DEFL('a) \sqsubseteq DEFL('b)$
shows $emb{\cdot}(prj{\cdot}(emb{\cdot}x) :: 'b::domain) = emb{\cdot}x$
⟨*proof*⟩

lemma *prj-emb-prj*:
assumes $DEFL('a::domain) \sqsubseteq DEFL('b::domain)$
shows $prj{\cdot}(emb{\cdot}(prj{\cdot}x :: 'b)) = (prj{\cdot}x :: 'a)$
⟨*proof*⟩

Isomorphism lemmas used internally by the domain package:

lemma *domain-abs-iso*:
fixes *abs* **and** *rep*
assumes *DEFL*: $DEFL('b::domain) = DEFL('a::domain)$
assumes *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
assumes *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
shows $rep{\cdot}(abs{\cdot}x) = x$
⟨*proof*⟩

lemma *domain-rep-iso*:
fixes *abs* **and** *rep*
assumes *DEFL*: $DEFL('b::domain) = DEFL('a::domain)$
assumes *abs-def*: $(abs :: 'a \to 'b) \equiv prj\ oo\ emb$
assumes *rep-def*: $(rep :: 'b \to 'a) \equiv prj\ oo\ emb$
shows $abs{\cdot}(rep{\cdot}x) = x$
⟨*proof*⟩

27.7 Deflations as sets

definition *defl-set* :: $'a::bifinite\ defl \Rightarrow 'a\ set$
where $defl\text{-}set\ A = \{x.\ cast{\cdot}A{\cdot}x = x\}$

lemma *adm-defl-set*: $adm\ (\lambda x.\ x \in defl\text{-}set\ A)$
⟨*proof*⟩

lemma *defl-set-bottom*: $\bot \in defl\text{-}set\ A$
⟨*proof*⟩

lemma *defl-set-cast* [*simp*]: $cast{\cdot}A{\cdot}x \in defl\text{-}set\ A$
⟨*proof*⟩

lemma *defl-set-subset-iff*: $defl\text{-}set\ A \subseteq defl\text{-}set\ B \longleftrightarrow A \sqsubseteq B$
⟨*proof*⟩

27.8 Proving a subtype is representable

Temporarily relax type constraints.

⟨*ML*⟩

lemma *typedef-domain-class*:
fixes *Rep* :: *'a::pcpo ⇒ udom*
fixes *Abs* :: *udom ⇒ 'a::pcpo*
fixes *t* :: *udom defl*
assumes *type*: *type-definition Rep Abs* (*defl-set t*)
assumes *below*: (⊑) ≡ *λx y. Rep x ⊑ Rep y*
assumes *emb*: *emb* ≡ (*Λ x. Rep x*)
assumes *prj*: *prj* ≡ (*Λ x. Abs* (*cast·t·x*))
assumes *defl*: *defl* ≡ (*λ a::'a itself. t*)
assumes *liftemb*: (*liftemb* :: *'a u → udom u*) ≡ *u-map·emb*
assumes *liftprj*: (*liftprj* :: *udom u → 'a u*) ≡ *u-map·prj*
assumes *liftdefl*: (*liftdefl* :: *'a itself ⇒ -*) ≡ (*λt. liftdefl-of·DEFL*('*a*))
shows *OFCLASS*('*a*, *domain-class*)
⟨*proof*⟩

lemma *typedef-DEFL*:
assumes *defl* ≡ (*λa::'a::pcpo itself. t*)
shows *DEFL*('*a::pcpo*) = *t*
⟨*proof*⟩

Restore original typing constraints.

⟨*ML*⟩

27.9 Isomorphic deflations

definition *isodefl* :: (*'a::domain → 'a*) ⇒ *udom defl ⇒ bool*
where *isodefl d t* ⟷ *cast·t = emb oo d oo prj*

definition *isodefl'* :: (*'a::predomain → 'a*) ⇒ *udom u defl ⇒ bool*
where *isodefl' d t* ⟷ *cast·t = liftemb oo u-map·d oo liftprj*

lemma *isodeflI*: (⋀*x. cast·t·x = emb·*(*d·*(*prj·x*))) ⟹ *isodefl d t*
⟨*proof*⟩

lemma *cast-isodefl*: *isodefl d t* ⟹ *cast·t* = (*Λ x. emb·*(*d·*(*prj·x*)))
⟨*proof*⟩

lemma *isodefl-strict*: *isodefl d t* ⟹ *d·⊥* = *⊥*
⟨*proof*⟩

lemma *isodefl-imp-deflation*:
fixes *d* :: *'a::domain → 'a*
assumes *isodefl d t* **shows** *deflation d*
⟨*proof*⟩

lemma *isodefl-ID-DEFL*: *isodefl* (*ID* :: *'a → 'a*) *DEFL*('*a::domain*)
⟨*proof*⟩

lemma *isodefl-LIFTDEFL*:
isodefl' (*ID* :: *'a* → *'a*) *LIFTDEFL*(*'a*::*predomain*)
⟨*proof*⟩

lemma *isodefl-DEFL-imp-ID*: *isodefl* (*d* :: *'a* → *'a*) *DEFL*(*'a*::*domain*) ⟹ *d* = *ID*
⟨*proof*⟩

lemma *isodefl-bottom*: *isodefl* ⊥ ⊥
⟨*proof*⟩

lemma *adm-isodefl*:
cont f ⟹ *cont g* ⟹ *adm* (λ*x*. *isodefl* (*f x*) (*g x*))
⟨*proof*⟩

lemma *isodefl-lub*:
assumes *chain d* **and** *chain t*
assumes ⋀*i*. *isodefl* (*d i*) (*t i*)
shows *isodefl* (⨆ *i*. *d i*) (⨆ *i*. *t i*)
⟨*proof*⟩

lemma *isodefl-fix*:
assumes ⋀*d t*. *isodefl d t* ⟹ *isodefl* (*f*·*d*) (*g*·*t*)
shows *isodefl* (*fix*·*f*) (*fix*·*g*)
⟨*proof*⟩

lemma *isodefl-abs-rep*:
fixes *abs* **and** *rep* **and** *d*
assumes *DEFL*: *DEFL*(*'b*::*domain*) = *DEFL*(*'a*::*domain*)
assumes *abs-def*: (*abs* :: *'a* → *'b*) ≡ *prj oo emb*
assumes *rep-def*: (*rep* :: *'b* → *'a*) ≡ *prj oo emb*
shows *isodefl d t* ⟹ *isodefl* (*abs oo d oo rep*) *t*
⟨*proof*⟩

lemma *isodefl'-liftdefl-of*: *isodefl d t* ⟹ *isodefl' d* (*liftdefl-of*·*t*)
⟨*proof*⟩

lemma *isodefl-sfun*:
isodefl d1 t1 ⟹ *isodefl d2 t2* ⟹
isodefl (*sfun-map*·*d1*·*d2*) (*sfun-defl*·*t1*·*t2*)
⟨*proof*⟩

lemma *isodefl-ssum*:
isodefl d1 t1 ⟹ *isodefl d2 t2* ⟹
isodefl (*ssum-map*·*d1*·*d2*) (*ssum-defl*·*t1*·*t2*)
⟨*proof*⟩

lemma *isodefl-sprod*:

isodefl d1 t1 ⟹ isodefl d2 t2 ⟹
isodefl (sprod-map·d1·d2) (sprod-defl·t1·t2)
⟨*proof*⟩

lemma *isodefl-prod*:
isodefl d1 t1 ⟹ isodefl d2 t2 ⟹
isodefl (prod-map·d1·d2) (prod-defl·t1·t2)
⟨*proof*⟩

lemma *isodefl-u*:
isodefl d t ⟹ isodefl (u-map·d) (u-defl·t)
⟨*proof*⟩

lemma *isodefl-u-liftdefl*:
isodefl' d t ⟹ isodefl (u-map·d) (u-liftdefl·t)
⟨*proof*⟩

lemma *encode-prod-u-map*:
encode-prod-u·(u-map·(prod-map·f·g)·(decode-prod-u·x))
= sprod-map·(u-map·f)·(u-map·g)·x
⟨*proof*⟩

lemma *isodefl-prod-u*:
assumes *isodefl' d1 t1* **and** *isodefl' d2 t2*
shows *isodefl' (prod-map·d1·d2) (prod-liftdefl·t1·t2)*
⟨*proof*⟩

lemma *encode-cfun-map*:
encode-cfun·(cfun-map·f·g·(decode-cfun·x))
= sfun-map·(u-map·f)·g·x
⟨*proof*⟩

lemma *isodefl-cfun*:
assumes *isodefl (u-map·d1) t1* **and** *isodefl d2 t2*
shows *isodefl (cfun-map·d1·d2) (sfun-defl·t1·t2)*
⟨*proof*⟩

27.10 Setting up the domain package

named-theorems *domain-defl-simps theorems like DEFL('a t) = t-defl$DEFL('a)*
and *domain-isodefl theorems like isodefl d t ==> isodefl (foo-map$d) (foo-defl$t)*

⟨*ML*⟩

lemmas [*domain-defl-simps*] =
DEFL-cfun DEFL-sfun DEFL-ssum DEFL-sprod DEFL-prod DEFL-u
liftdefl-eq LIFTDEFL-prod u-liftdefl-liftdefl-of

lemmas [*domain-map-ID*] =

cfun-map-ID sfun-map-ID ssum-map-ID sprod-map-ID prod-map-ID u-map-ID

lemmas [*domain-isodefl*] =
isodefl-u isodefl-sfun isodefl-ssum isodefl-sprod
isodefl-cfun isodefl-prod isodefl-prod-u isodefl'-liftdefl-of
isodefl-u-liftdefl

lemmas [*domain-deflation*] =
deflation-cfun-map deflation-sfun-map deflation-ssum-map
deflation-sprod-map deflation-prod-map deflation-u-map

⟨*ML*⟩

end

28 A compact basis for powerdomains

theory *Compact-Basis*
imports *Universal*
begin

28.1 A compact basis for powerdomains

definition *pd-basis* = {*S*::'*a*::*bifinite compact-basis set*. *finite S* ∧ *S* ≠ {}}

typedef '*a*::*bifinite pd-basis* = *pd-basis* :: '*a compact-basis set set*
⟨*proof*⟩

lemma *finite-Rep-pd-basis* [*simp*]: *finite* (*Rep-pd-basis u*)
⟨*proof*⟩

lemma *Rep-pd-basis-nonempty* [*simp*]: *Rep-pd-basis u* ≠ {}
⟨*proof*⟩

The powerdomain basis type is countable.

lemma *pd-basis-countable*: ∃*f*::'*a*::*bifinite pd-basis* ⇒ *nat*. *inj f* (**is** *Ex ?P*)
⟨*proof*⟩

28.2 Unit and plus constructors

definition
PDUnit :: '*a*::*bifinite compact-basis* ⇒ '*a pd-basis* **where**
PDUnit = (λ*x*. *Abs-pd-basis* {*x*})

definition
PDPlus :: '*a*::*bifinite pd-basis* ⇒ '*a pd-basis* ⇒ '*a pd-basis* **where**
PDPlus t u = *Abs-pd-basis* (*Rep-pd-basis t* ∪ *Rep-pd-basis u*)

lemma *Rep-PDUnit*:

Rep-pd-basis (*PDUnit x*) = {*x*}
⟨*proof*⟩

lemma *Rep-PDPlus*:
Rep-pd-basis (*PDPlus u v*) = *Rep-pd-basis u* ∪ *Rep-pd-basis v*
⟨*proof*⟩

lemma *PDUnit-inject* [*simp*]: (*PDUnit a* = *PDUnit b*) = (*a* = *b*)
⟨*proof*⟩

lemma *PDPlus-assoc*: *PDPlus* (*PDPlus t u*) *v* = *PDPlus t* (*PDPlus u v*)
⟨*proof*⟩

lemma *PDPlus-commute*: *PDPlus t u* = *PDPlus u t*
⟨*proof*⟩

lemma *PDPlus-absorb*: *PDPlus t t* = *t*
⟨*proof*⟩

lemma *pd-basis-induct1* [*case-names PDUnit PDPlus*]:
assumes *PDUnit*: ⋀*a*. *P* (*PDUnit a*)
assumes *PDPlus*: ⋀*a t*. *P t* ⟹ *P* (*PDPlus* (*PDUnit a*) *t*)
shows *P x*
⟨*proof*⟩

lemma *pd-basis-induct* [*case-names PDUnit PDPlus*]:
assumes *PDUnit*: ⋀*a*. *P* (*PDUnit a*)
assumes *PDPlus*: ⋀*t u*. ⟦*P t*; *P u*⟧ ⟹ *P* (*PDPlus t u*)
shows *P x*
⟨*proof*⟩

28.3 Fold operator

definition
fold-pd ::
('*a*::*bifinite compact-basis* ⇒ '*b*::*type*) ⇒ ('*b* ⇒ '*b* ⇒ '*b*) ⇒ '*a pd-basis* ⇒ '*b*
where *fold-pd g f t* = *semilattice-set.F f* (*g* ‘ *Rep-pd-basis t*)

lemma *fold-pd-PDUnit*:
assumes *semilattice f*
shows *fold-pd g f* (*PDUnit x*) = *g x*
⟨*proof*⟩

lemma *fold-pd-PDPlus*:
assumes *semilattice f*
shows *fold-pd g f* (*PDPlus t u*) = *f* (*fold-pd g f t*) (*fold-pd g f u*)
⟨*proof*⟩

end

29 Upper powerdomain

theory *UpperPD*
imports *Compact-Basis*
begin

29.1 Basis preorder

definition
upper-le :: *'a::bifinite pd-basis* ⇒ *'a pd-basis* ⇒ *bool* (**infix** ‹≤♯› *50*) **where**
upper-le = (λ*u v*. ∀ *y*∈*Rep-pd-basis v*. ∃ *x*∈*Rep-pd-basis u*. *x* ⊑ *y*)

lemma *upper-le-refl* [*simp*]: *t* ≤♯ *t*
⟨*proof*⟩

lemma *upper-le-trans*: ⟦*t* ≤♯ *u*; *u* ≤♯ *v*⟧ ⟹ *t* ≤♯ *v*
⟨*proof*⟩

interpretation *upper-le*: *preorder upper-le*
⟨*proof*⟩

lemma *upper-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♯ *t*
⟨*proof*⟩

lemma *PDUnit-upper-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♯ *PDUnit y*
⟨*proof*⟩

lemma *PDPlus-upper-mono*: ⟦*s* ≤♯ *t*; *u* ≤♯ *v*⟧ ⟹ *PDPlus s u* ≤♯ *PDPlus t v*
⟨*proof*⟩

lemma *PDPlus-upper-le*: *PDPlus t u* ≤♯ *t*
⟨*proof*⟩

lemma *upper-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* ≤♯ *PDUnit b*) = (*a* ⊑ *b*)
⟨*proof*⟩

lemma *upper-le-PDPlus-PDUnit-iff*:
(*PDPlus t u* ≤♯ *PDUnit a*) = (*t* ≤♯ *PDUnit a* ∨ *u* ≤♯ *PDUnit a*)
⟨*proof*⟩

lemma *upper-le-PDPlus-iff*: (*t* ≤♯ *PDPlus u v*) = (*t* ≤♯ *u* ∧ *t* ≤♯ *v*)
⟨*proof*⟩

lemma *upper-le-induct* [*induct set*: *upper-le*]:
assumes *le*: *t* ≤♯ *u*
assumes *1*: ⋀*a b*. *a* ⊑ *b* ⟹ *P* (*PDUnit a*) (*PDUnit b*)
assumes *2*: ⋀*t u a*. *P t* (*PDUnit a*) ⟹ *P* (*PDPlus t u*) (*PDUnit a*)
assumes *3*: ⋀*t u v*. ⟦*P t u*; *P t v*⟧ ⟹ *P t* (*PDPlus u v*)
shows *P t u*

⟨*proof*⟩

29.2 Type definition

typedef *'a*::*bifinite upper-pd* (‹(‹*notation*=‹*postfix upper-pd*›› *'*(-*'*)♯)›) =
{*S*::*'a pd-basis set*. *upper-le.ideal S*}
⟨*proof*⟩

instantiation *upper-pd* :: (*bifinite*) *below*
begin

definition
x ⊑ *y* ⟷ *Rep-upper-pd x* ⊆ *Rep-upper-pd y*

instance ⟨*proof*⟩
end

instance *upper-pd* :: (*bifinite*) *po*
⟨*proof*⟩

instance *upper-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
upper-principal :: *'a*::*bifinite pd-basis* ⇒ *'a upper-pd* **where**
upper-principal t = *Abs-upper-pd* {*u*. *u* ≤♯ *t*}

interpretation *upper-pd*:
ideal-completion upper-le upper-principal Rep-upper-pd
⟨*proof*⟩

Upper powerdomain is pointed

lemma *upper-pd-minimal*: *upper-principal* (*PDUnit compact-bot*) ⊑ *ys*
⟨*proof*⟩

instance *upper-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-upper-pd-pcpo*: ⊥ = *upper-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

29.3 Monadic unit and plus

definition
upper-unit :: *'a*::*bifinite* → *'a upper-pd* **where**
upper-unit = *compact-basis.extension* (λ*a*. *upper-principal* (*PDUnit a*))

definition
upper-plus :: *'a*::*bifinite upper-pd* → *'a upper-pd* → *'a upper-pd* **where**

upper-plus = *upper-pd.extension* (λ*t*. *upper-pd.extension* (λ*u*. *upper-principal* (*PDPlus t u*)))

abbreviation
upper-add :: '*a*::*bifinite upper-pd* ⇒ '*a upper-pd* ⇒ '*a upper-pd*
(**infixl** ‹∪♯› *65*) **where**
xs ∪♯ *ys* == *upper-plus*·*xs*·*ys*

syntax
-upper-pd :: *args* ⇒ *logic* (‹(‹*indent=1 notation=‹mixfix upper-pd enumeration›*›{-}♯)›)
translations
{*x*,*xs*}♯ == {*x*}♯ ∪♯ {*xs*}♯
{*x*}♯ == *CONST upper-unit*·*x*

lemma *upper-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♯ = *upper-principal* (*PDUnit a*)
⟨*proof*⟩

lemma *upper-plus-principal* [*simp*]:
upper-principal t ∪♯ *upper-principal u* = *upper-principal* (*PDPlus t u*)
⟨*proof*⟩

interpretation *upper-add*: *semilattice upper-add* ⟨*proof*⟩

lemmas *upper-plus-assoc* = *upper-add.assoc*
lemmas *upper-plus-commute* = *upper-add.commute*
lemmas *upper-plus-absorb* = *upper-add.idem*
lemmas *upper-plus-left-commute* = *upper-add.left-commute*
lemmas *upper-plus-left-absorb* = *upper-add.left-idem*

Useful for *simp add*: *upper-plus-ac*

lemmas *upper-plus-ac* =
upper-plus-assoc upper-plus-commute upper-plus-left-commute

Useful for *simp only*: *upper-plus-aci*

lemmas *upper-plus-aci* =
upper-plus-ac upper-plus-absorb upper-plus-left-absorb

lemma *upper-plus-below1*: *xs* ∪♯ *ys* ⊑ *xs*
⟨*proof*⟩

lemma *upper-plus-below2*: *xs* ∪♯ *ys* ⊑ *ys*
⟨*proof*⟩

lemma *upper-plus-greatest*: ⟦*xs* ⊑ *ys*; *xs* ⊑ *zs*⟧ ⟹ *xs* ⊑ *ys* ∪♯ *zs*
⟨*proof*⟩

lemma *upper-below-plus-iff* [*simp*]:

$xs \sqsubseteq ys \mathbin{\natural\!\cup} zs \longleftrightarrow xs \sqsubseteq ys \wedge xs \sqsubseteq zs$
⟨*proof*⟩

lemma *upper-plus-below-unit-iff* [*simp*]:
$xs \mathbin{\natural\!\cup} ys \sqsubseteq \{z\}\natural \longleftrightarrow xs \sqsubseteq \{z\}\natural \vee ys \sqsubseteq \{z\}\natural$
⟨*proof*⟩

lemma *upper-unit-below-iff* [*simp*]: $\{x\}\natural \sqsubseteq \{y\}\natural \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemmas *upper-pd-below-simps* =
upper-unit-below-iff
upper-below-plus-iff
upper-plus-below-unit-iff

lemma *upper-unit-eq-iff* [*simp*]: $\{x\}\natural = \{y\}\natural \longleftrightarrow x = y$
⟨*proof*⟩

lemma *upper-unit-strict* [*simp*]: $\{\bot\}\natural = \bot$
⟨*proof*⟩

lemma *upper-plus-strict1* [*simp*]: $\bot \mathbin{\natural\!\cup} ys = \bot$
⟨*proof*⟩

lemma *upper-plus-strict2* [*simp*]: $xs \mathbin{\natural\!\cup} \bot = \bot$
⟨*proof*⟩

lemma *upper-unit-bottom-iff* [*simp*]: $\{x\}\natural = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *upper-plus-bottom-iff* [*simp*]:
$xs \mathbin{\natural\!\cup} ys = \bot \longleftrightarrow xs = \bot \vee ys = \bot$
⟨*proof*⟩

lemma *compact-upper-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\natural$
⟨*proof*⟩

lemma *compact-upper-unit-iff* [*simp*]: *compact* $\{x\}\natural \longleftrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-upper-plus* [*simp*]:
$[\![$*compact* xs; *compact* $ys]\!] \Longrightarrow$ *compact* $(xs \mathbin{\natural\!\cup} ys)$
⟨*proof*⟩

29.4 Induction rules

lemma *upper-pd-induct1*:
assumes P: *adm* P
assumes *unit*: $\bigwedge x.\ P\ \{x\}\natural$

assumes *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\sharp;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\sharp \cup\sharp\ ys)$
shows $P\ (xs::'a::bifinite\ upper\text{-}pd)$
⟨*proof*⟩

lemma *upper-pd-induct* [*case-names adm upper-unit upper-plus*, *induct type*: *upper-pd*]:
assumes P: *adm P*
assumes *unit*: $\bigwedge x.\ P\ \{x\}\sharp$
assumes *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\sharp\ ys)$
shows $P\ (xs::'a::bifinite\ upper\text{-}pd)$
⟨*proof*⟩

29.5 Monadic bind

definition
upper-bind-basis ::
$'a::bifinite\ pd\text{-}basis \Rightarrow ('a \to 'b\ upper\text{-}pd) \to 'b::bifinite\ upper\text{-}pd$ **where**
upper-bind-basis = *fold-pd*
$(\lambda a.\ \Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a))$
$(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\sharp\ y{\cdot}f)$

lemma *ACI-upper-bind*:
semilattice $(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\sharp\ y{\cdot}f)$
⟨*proof*⟩

lemma *upper-bind-basis-simps* [*simp*]:
upper-bind-basis (*PDUnit a*) =
$(\Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a))$
upper-bind-basis (*PDPlus t u*) =
$(\Lambda\ f.\ upper\text{-}bind\text{-}basis\ t{\cdot}f \cup\sharp\ upper\text{-}bind\text{-}basis\ u{\cdot}f)$
⟨*proof*⟩

lemma *upper-bind-basis-mono*:
$t \leq\sharp\ u \Longrightarrow upper\text{-}bind\text{-}basis\ t \sqsubseteq upper\text{-}bind\text{-}basis\ u$
⟨*proof*⟩

definition
upper-bind :: $'a::bifinite\ upper\text{-}pd \to ('a \to 'b\ upper\text{-}pd) \to 'b::bifinite\ upper\text{-}pd$
where
upper-bind = *upper-pd.extension upper-bind-basis*

syntax
-upper-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
(‹(‹*indent=3 notation=‹binder upper-bind›*›⋃♯-∈-./ -)› [*0*, *0*, *10*] *10*)

translations
$\bigcup\sharp x{\in}xs.\ e == CONST\ upper\text{-}bind{\cdot}xs{\cdot}(\Lambda\ x.\ e)$

lemma *upper-bind-principal* [*simp*]:

upper-bind·(*upper-principal t*) = *upper-bind-basis t*
⟨*proof*⟩

lemma *upper-bind-unit* [*simp*]:
upper-bind·{*x*}♯·*f* = *f*·*x*
⟨*proof*⟩

lemma *upper-bind-plus* [*simp*]:
upper-bind·(*xs* ∪♯ *ys*)·*f* = *upper-bind*·*xs*·*f* ∪♯ *upper-bind*·*ys*·*f*
⟨*proof*⟩

lemma *upper-bind-strict* [*simp*]: *upper-bind*·⊥·*f* = *f*·⊥
⟨*proof*⟩

lemma *upper-bind-bind*:
upper-bind·(*upper-bind*·*xs*·*f*)·*g* = *upper-bind*·*xs*·(Λ *x*. *upper-bind*·(*f*·*x*)·*g*)
⟨*proof*⟩

29.6 Map

definition
upper-map :: ('*a*::*bifinite* → '*b*::*bifinite*) → '*a upper-pd* → '*b upper-pd* **where**
upper-map = (Λ *f xs*. *upper-bind*·*xs*·(Λ *x*. {*f*·*x*}♯))

lemma *upper-map-unit* [*simp*]:
upper-map·*f*·{*x*}♯ = {*f*·*x*}♯
⟨*proof*⟩

lemma *upper-map-plus* [*simp*]:
upper-map·*f*·(*xs* ∪♯ *ys*) = *upper-map*·*f*·*xs* ∪♯ *upper-map*·*f*·*ys*
⟨*proof*⟩

lemma *upper-map-bottom* [*simp*]: *upper-map*·*f*·⊥ = {*f*·⊥}♯
⟨*proof*⟩

lemma *upper-map-ident*: *upper-map*·(Λ *x*. *x*)·*xs* = *xs*
⟨*proof*⟩

lemma *upper-map-ID*: *upper-map*·*ID* = *ID*
⟨*proof*⟩

lemma *upper-map-map*:
upper-map·*f*·(*upper-map*·*g*·*xs*) = *upper-map*·(Λ *x*. *f*·(*g*·*x*))·*xs*
⟨*proof*⟩

lemma *upper-bind-map*:
upper-bind·(*upper-map*·*f*·*xs*)·*g* = *upper-bind*·*xs*·(Λ *x*. *g*·(*f*·*x*))
⟨*proof*⟩

lemma *upper-map-bind*:
upper-map·f·(upper-bind·xs·g) = upper-bind·xs·(Λ x. upper-map·f·(g·x))
⟨*proof*⟩

lemma *ep-pair-upper-map*: *ep-pair e p ⟹ ep-pair (upper-map·e) (upper-map·p)*
⟨*proof*⟩

lemma *deflation-upper-map*: *deflation d ⟹ deflation (upper-map·d)*
⟨*proof*⟩

lemma *finite-deflation-upper-map*:
assumes *finite-deflation d* **shows** *finite-deflation (upper-map·d)*
⟨*proof*⟩

29.7 Upper powerdomain is bifinite

lemma *approx-chain-upper-map*:
assumes *approx-chain a*
shows *approx-chain (λi. upper-map·(a i))*
⟨*proof*⟩

instance *upper-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

29.8 Join

definition
upper-join :: *'a*::*bifinite upper-pd upper-pd → 'a upper-pd* **where**
upper-join = (Λ xss. upper-bind·xss·(Λ xs. xs))

lemma *upper-join-unit* [*simp*]:
upper-join·{xs}♯ = xs
⟨*proof*⟩

lemma *upper-join-plus* [*simp*]:
upper-join·(xss ∪♯ yss) = upper-join·xss ∪♯ upper-join·yss
⟨*proof*⟩

lemma *upper-join-bottom* [*simp*]: *upper-join·⊥ = ⊥*
⟨*proof*⟩

lemma *upper-join-map-unit*:
upper-join·(upper-map·upper-unit·xs) = xs
⟨*proof*⟩

lemma *upper-join-map-join*:
upper-join·(upper-map·upper-join·xsss) = upper-join·(upper-join·xsss)
⟨*proof*⟩

lemma *upper-join-map-map*:
upper-join·(*upper-map*·(*upper-map*·*f*)·*xss*) =
upper-map·*f*·(*upper-join*·*xss*)
⟨*proof*⟩

end

30 Lower powerdomain

theory *LowerPD*
imports *Compact-Basis*
begin

30.1 Basis preorder

definition
lower-le :: 'a::*bifinite pd-basis* ⇒ 'a *pd-basis* ⇒ *bool* (**infix** ‹≤♭› *50*) **where**
lower-le = (λ*u v*. ∀ *x*∈*Rep-pd-basis u*. ∃ *y*∈*Rep-pd-basis v*. *x* ⊑ *y*)

lemma *lower-le-refl* [*simp*]: *t* ≤♭ *t*
⟨*proof*⟩

lemma *lower-le-trans*: ⟦*t* ≤♭ *u*; *u* ≤♭ *v*⟧ ⟹ *t* ≤♭ *v*
⟨*proof*⟩

interpretation *lower-le*: *preorder lower-le*
⟨*proof*⟩

lemma *lower-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♭ *t*
⟨*proof*⟩

lemma *PDUnit-lower-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♭ *PDUnit y*
⟨*proof*⟩

lemma *PDPlus-lower-mono*: ⟦*s* ≤♭ *t*; *u* ≤♭ *v*⟧ ⟹ *PDPlus s u* ≤♭ *PDPlus t v*
⟨*proof*⟩

lemma *PDPlus-lower-le*: *t* ≤♭ *PDPlus t u*
⟨*proof*⟩

lemma *lower-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* ≤♭ *PDUnit b*) = (*a* ⊑ *b*)
⟨*proof*⟩

lemma *lower-le-PDUnit-PDPlus-iff*:
(*PDUnit a* ≤♭ *PDPlus t u*) = (*PDUnit a* ≤♭ *t* ∨ *PDUnit a* ≤♭ *u*)
⟨*proof*⟩

lemma *lower-le-PDPlus-iff*: (*PDPlus t u* ≤♭ *v*) = (*t* ≤♭ *v* ∧ *u* ≤♭ *v*)

⟨*proof*⟩

lemma *lower-le-induct* [*induct set*: *lower-le*]:
assumes *le*: $t \leq\flat\ u$
assumes *1*: $\bigwedge a\ b.\ a \sqsubseteq b \Longrightarrow P\ (PDUnit\ a)\ (PDUnit\ b)$
assumes *2*: $\bigwedge t\ u\ a.\ P\ (PDUnit\ a)\ t \Longrightarrow P\ (PDUnit\ a)\ (PDPlus\ t\ u)$
assumes *3*: $\bigwedge t\ u\ v.\ [\![P\ t\ v;\ P\ u\ v]\!] \Longrightarrow P\ (PDPlus\ t\ u)\ v$
shows $P\ t\ u$
⟨*proof*⟩

30.2 Type definition

typedef *'a*::*bifinite lower-pd* (‹(‹*notation*=‹*postfix lower-pd*››'(-')♭)›) =
$\{S::'a\ pd\text{-}basis\ set.\ lower\text{-}le.ideal\ S\}$
⟨*proof*⟩

instantiation *lower-pd* :: (*bifinite*) *below*
begin

definition
$x \sqsubseteq y \longleftrightarrow Rep\text{-}lower\text{-}pd\ x \subseteq Rep\text{-}lower\text{-}pd\ y$

instance ⟨*proof*⟩
end

instance *lower-pd* :: (*bifinite*) *po*
⟨*proof*⟩

instance *lower-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
lower-principal :: *'a*::*bifinite pd-basis* ⇒ *'a lower-pd* **where**
lower-principal t = *Abs-lower-pd* $\{u.\ u \leq\flat\ t\}$

interpretation *lower-pd*:
ideal-completion lower-le lower-principal Rep-lower-pd
⟨*proof*⟩

Lower powerdomain is pointed

lemma *lower-pd-minimal*: *lower-principal* (*PDUnit compact-bot*) $\sqsubseteq$ *ys*
⟨*proof*⟩

instance *lower-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-lower-pd-pcpo*: $\bot$ = *lower-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

30.3 Monadic unit and plus

definition
lower-unit :: *'a::bifinite → 'a lower-pd* **where**
lower-unit = compact-basis.extension (*λa. lower-principal* (*PDUnit a*))

definition
lower-plus :: *'a::bifinite lower-pd → 'a lower-pd → 'a lower-pd* **where**
lower-plus = lower-pd.extension (*λt. lower-pd.extension* (*λu.*
lower-principal (*PDPlus t u*)))

abbreviation
lower-add :: *'a::bifinite lower-pd ⇒ 'a lower-pd ⇒ 'a lower-pd*
(**infixl** ‹∪♭› *65*) **where**
xs ∪♭ ys == lower-plus·xs·ys

syntax
-lower-pd :: *args ⇒ logic* (‹(‹*indent=1 notation=‹mixfix lower-pd enumeration›*›{-}♭)›)
translations
{x,xs}♭ == {x}♭ ∪♭ {xs}♭
{x}♭ == CONST lower-unit·x

lemma *lower-unit-Rep-compact-basis* [*simp*]:
{Rep-compact-basis a}♭ = lower-principal (*PDUnit a*)
⟨*proof*⟩

lemma *lower-plus-principal* [*simp*]:
lower-principal t ∪♭ lower-principal u = lower-principal (*PDPlus t u*)
⟨*proof*⟩

interpretation *lower-add*: *semilattice lower-add* ⟨*proof*⟩

lemmas *lower-plus-assoc = lower-add.assoc*
lemmas *lower-plus-commute = lower-add.commute*
lemmas *lower-plus-absorb = lower-add.idem*
lemmas *lower-plus-left-commute = lower-add.left-commute*
lemmas *lower-plus-left-absorb = lower-add.left-idem*

Useful for *simp add*: *lower-plus-ac*

lemmas *lower-plus-ac =*
lower-plus-assoc lower-plus-commute lower-plus-left-commute

Useful for *simp only*: *lower-plus-aci*

lemmas *lower-plus-aci =*
lower-plus-ac lower-plus-absorb lower-plus-left-absorb

lemma *lower-plus-below1*: *xs ⊑ xs ∪♭ ys*
⟨*proof*⟩

lemma *lower-plus-below2*: $ys \sqsubseteq xs \mathbin{\cup\!\flat} ys$
⟨*proof*⟩

lemma *lower-plus-least*: $[\![xs \sqsubseteq zs;\ ys \sqsubseteq zs]\!] \Longrightarrow xs \mathbin{\cup\!\flat} ys \sqsubseteq zs$
⟨*proof*⟩

lemma *lower-plus-below-iff* [*simp*]:
$xs \mathbin{\cup\!\flat} ys \sqsubseteq zs \longleftrightarrow xs \sqsubseteq zs \wedge ys \sqsubseteq zs$
⟨*proof*⟩

lemma *lower-unit-below-plus-iff* [*simp*]:
$\{x\}\flat \sqsubseteq ys \mathbin{\cup\!\flat} zs \longleftrightarrow \{x\}\flat \sqsubseteq ys \vee \{x\}\flat \sqsubseteq zs$
⟨*proof*⟩

lemma *lower-unit-below-iff* [*simp*]: $\{x\}\flat \sqsubseteq \{y\}\flat \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemmas *lower-pd-below-simps* =
lower-unit-below-iff
lower-plus-below-iff
lower-unit-below-plus-iff

lemma *lower-unit-eq-iff* [*simp*]: $\{x\}\flat = \{y\}\flat \longleftrightarrow x = y$
⟨*proof*⟩

lemma *lower-unit-strict* [*simp*]: $\{\bot\}\flat = \bot$
⟨*proof*⟩

lemma *lower-unit-bottom-iff* [*simp*]: $\{x\}\flat = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *lower-plus-bottom-iff* [*simp*]:
$xs \mathbin{\cup\!\flat} ys = \bot \longleftrightarrow xs = \bot \wedge ys = \bot$
⟨*proof*⟩

lemma *lower-plus-strict1* [*simp*]: $\bot \mathbin{\cup\!\flat} ys = ys$
⟨*proof*⟩

lemma *lower-plus-strict2* [*simp*]: $xs \mathbin{\cup\!\flat} \bot = xs$
⟨*proof*⟩

lemma *compact-lower-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\flat$
⟨*proof*⟩

lemma *compact-lower-unit-iff* [*simp*]: *compact* $\{x\}\flat \longleftrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-lower-plus* [*simp*]:
$[\![$*compact* xs; *compact* $ys]\!] \Longrightarrow$ *compact* $(xs \mathbin{\cup\!\flat} ys)$

⟨*proof*⟩

30.4 Induction rules

lemma *lower-pd-induct1*:
assumes *P*: *adm P*
assumes *unit*: $\bigwedge x.\ P\ \{x\}\flat$
assumes *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\flat;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\flat \cup\flat\ ys)$
shows *P* (*xs*::'*a*::*bifinite lower-pd*)
⟨*proof*⟩

lemma *lower-pd-induct* [*case-names adm lower-unit lower-plus*, *induct type*: *lower-pd*]:
assumes *P*: *adm P*
assumes *unit*: $\bigwedge x.\ P\ \{x\}\flat$
assumes *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\flat\ ys)$
shows *P* (*xs*::'*a*::*bifinite lower-pd*)
⟨*proof*⟩

30.5 Monadic bind

definition
lower-bind-basis ::
'*a*::*bifinite pd-basis* ⇒ ('*a* → '*b lower-pd*) → '*b*::*bifinite lower-pd* **where**
lower-bind-basis = *fold-pd*
($\lambda a.\ \Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a)$)
($\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\flat\ y{\cdot}f$)

lemma *ACI-lower-bind*:
semilattice ($\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\flat\ y{\cdot}f$)
⟨*proof*⟩

lemma *lower-bind-basis-simps* [*simp*]:
lower-bind-basis (*PDUnit a*) =
($\Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a)$)
lower-bind-basis (*PDPlus t u*) =
($\Lambda\ f.$ *lower-bind-basis* $t{\cdot}f \cup\flat$ *lower-bind-basis* $u{\cdot}f$)
⟨*proof*⟩

lemma *lower-bind-basis-mono*:
$t \leq\flat\ u \Longrightarrow$ *lower-bind-basis* $t \sqsubseteq$ *lower-bind-basis* u
⟨*proof*⟩

definition
lower-bind :: '*a*::*bifinite lower-pd* → ('*a* → '*b lower-pd*) → '*b*::*bifinite lower-pd*
where
lower-bind = *lower-pd.extension lower-bind-basis*

syntax
-*lower-bind* :: [*logic*, *logic*, *logic*] ⇒ *logic*
(‹(‹*indent=3 notation=‹binder lower-bind›*›⋃♭-∈-./ -)› [*0*, *0*, *10*] *10*)

translations
$\bigcup\flat x{\in}xs.\ e == CONST\ lower\text{-}bind{\cdot}xs{\cdot}(\Lambda\ x.\ e)$

lemma *lower-bind-principal* [*simp*]:
$lower\text{-}bind{\cdot}(lower\text{-}principal\ t) = lower\text{-}bind\text{-}basis\ t$
⟨*proof*⟩

lemma *lower-bind-unit* [*simp*]:
$lower\text{-}bind{\cdot}\{x\}\flat{\cdot}f = f{\cdot}x$
⟨*proof*⟩

lemma *lower-bind-plus* [*simp*]:
$lower\text{-}bind{\cdot}(xs \cup\flat\ ys){\cdot}f = lower\text{-}bind{\cdot}xs{\cdot}f \cup\flat\ lower\text{-}bind{\cdot}ys{\cdot}f$
⟨*proof*⟩

lemma *lower-bind-strict* [*simp*]: $lower\text{-}bind{\cdot}\bot{\cdot}f = f{\cdot}\bot$
⟨*proof*⟩

lemma *lower-bind-bind*:
$lower\text{-}bind{\cdot}(lower\text{-}bind{\cdot}xs{\cdot}f){\cdot}g = lower\text{-}bind{\cdot}xs{\cdot}(\Lambda\ x.\ lower\text{-}bind{\cdot}(f{\cdot}x){\cdot}g)$
⟨*proof*⟩

30.6 Map

definition
$lower\text{-}map :: ('a{::}bifinite \to 'b{::}bifinite) \to 'a\ lower\text{-}pd \to 'b\ lower\text{-}pd$ **where**
$lower\text{-}map = (\Lambda\ f\ xs.\ lower\text{-}bind{\cdot}xs{\cdot}(\Lambda\ x.\ \{f{\cdot}x\}\flat))$

lemma *lower-map-unit* [*simp*]:
$lower\text{-}map{\cdot}f{\cdot}\{x\}\flat = \{f{\cdot}x\}\flat$
⟨*proof*⟩

lemma *lower-map-plus* [*simp*]:
$lower\text{-}map{\cdot}f{\cdot}(xs \cup\flat\ ys) = lower\text{-}map{\cdot}f{\cdot}xs \cup\flat\ lower\text{-}map{\cdot}f{\cdot}ys$
⟨*proof*⟩

lemma *lower-map-bottom* [*simp*]: $lower\text{-}map{\cdot}f{\cdot}\bot = \{f{\cdot}\bot\}\flat$
⟨*proof*⟩

lemma *lower-map-ident*: $lower\text{-}map{\cdot}(\Lambda\ x.\ x){\cdot}xs = xs$
⟨*proof*⟩

lemma *lower-map-ID*: $lower\text{-}map{\cdot}ID = ID$
⟨*proof*⟩

lemma *lower-map-map*:
$lower\text{-}map{\cdot}f{\cdot}(lower\text{-}map{\cdot}g{\cdot}xs) = lower\text{-}map{\cdot}(\Lambda\ x.\ f{\cdot}(g{\cdot}x)){\cdot}xs$
⟨*proof*⟩

lemma *lower-bind-map*:
lower-bind·(*lower-map*·*f*·*xs*)·*g* = *lower-bind*·*xs*·(Λ *x*. *g*·(*f*·*x*))
⟨*proof*⟩

lemma *lower-map-bind*:
lower-map·*f*·(*lower-bind*·*xs*·*g*) = *lower-bind*·*xs*·(Λ *x*. *lower-map*·*f*·(*g*·*x*))
⟨*proof*⟩

lemma *ep-pair-lower-map*: *ep-pair e p* ⟹ *ep-pair* (*lower-map*·*e*) (*lower-map*·*p*)
⟨*proof*⟩

lemma *deflation-lower-map*: *deflation d* ⟹ *deflation* (*lower-map*·*d*)
⟨*proof*⟩

lemma *finite-deflation-lower-map*:
assumes *finite-deflation d* **shows** *finite-deflation* (*lower-map*·*d*)
⟨*proof*⟩

30.7 Lower powerdomain is bifinite

lemma *approx-chain-lower-map*:
assumes *approx-chain a*
shows *approx-chain* (λ*i*. *lower-map*·(*a i*))
⟨*proof*⟩

instance *lower-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

30.8 Join

definition
lower-join :: *'a*::*bifinite lower-pd lower-pd* → *'a lower-pd* **where**
lower-join = (Λ *xss*. *lower-bind*·*xss*·(Λ *xs*. *xs*))

lemma *lower-join-unit* [*simp*]:
lower-join·{*xs*}♭ = *xs*
⟨*proof*⟩

lemma *lower-join-plus* [*simp*]:
lower-join·(*xss* ∪♭ *yss*) = *lower-join*·*xss* ∪♭ *lower-join*·*yss*
⟨*proof*⟩

lemma *lower-join-bottom* [*simp*]: *lower-join*·⊥ = ⊥
⟨*proof*⟩

lemma *lower-join-map-unit*:
lower-join·(*lower-map*·*lower-unit*·*xs*) = *xs*
⟨*proof*⟩

lemma *lower-join-map-join*:
lower-join·(*lower-map*·*lower-join*·*xsss*) = *lower-join*·(*lower-join*·*xsss*)
⟨*proof*⟩

lemma *lower-join-map-map*:
lower-join·(*lower-map*·(*lower-map*·*f*)·*xss*) =
lower-map·*f*·(*lower-join*·*xss*)
⟨*proof*⟩

end

31 Convex powerdomain

theory *ConvexPD*
imports *UpperPD LowerPD*
begin

31.1 Basis preorder

definition
convex-le :: 'a::*bifinite pd-basis* ⇒ 'a *pd-basis* ⇒ *bool* (**infix** ‹≤♮› *50*) **where**
convex-le = (λ*u v*. *u* ≤♯ *v* ∧ *u* ≤♭ *v*)

lemma *convex-le-refl* [*simp*]: *t* ≤♮ *t*
⟨*proof*⟩

lemma *convex-le-trans*: ⟦*t* ≤♮ *u*; *u* ≤♮ *v*⟧ ⟹ *t* ≤♮ *v*
⟨*proof*⟩

interpretation *convex-le*: *preorder convex-le*
⟨*proof*⟩

lemma *upper-le-minimal* [*simp*]: *PDUnit compact-bot* ≤♮ *t*
⟨*proof*⟩

lemma *PDUnit-convex-mono*: *x* ⊑ *y* ⟹ *PDUnit x* ≤♮ *PDUnit y*
⟨*proof*⟩

lemma *PDPlus-convex-mono*: ⟦*s* ≤♮ *t*; *u* ≤♮ *v*⟧ ⟹ *PDPlus s u* ≤♮ *PDPlus t v*
⟨*proof*⟩

lemma *convex-le-PDUnit-PDUnit-iff* [*simp*]:
(*PDUnit a* ≤♮ *PDUnit b*) = (*a* ⊑ *b*)
⟨*proof*⟩

lemma *convex-le-PDUnit-lemma1*:
(*PDUnit a* ≤♮ *t*) = (∀ *b*∈*Rep-pd-basis t*. *a* ⊑ *b*)
⟨*proof*⟩

lemma *convex-le-PDUnit-PDPlus-iff* [*simp*]:
(*PDUnit a* $\leq\natural$ *PDPlus t u*) = (*PDUnit a* $\leq\natural$ *t* $\wedge$ *PDUnit a* $\leq\natural$ *u*)
⟨*proof*⟩

lemma *convex-le-PDUnit-lemma2*:
(*t* $\leq\natural$ *PDUnit b*) = ($\forall$ *a*$\in$*Rep-pd-basis t*. *a* $\sqsubseteq$ *b*)
⟨*proof*⟩

lemma *convex-le-PDPlus-PDUnit-iff* [*simp*]:
(*PDPlus t u* $\leq\natural$ *PDUnit a*) = (*t* $\leq\natural$ *PDUnit a* $\wedge$ *u* $\leq\natural$ *PDUnit a*)
⟨*proof*⟩

lemma *convex-le-PDPlus-lemma*:
assumes *z*: *PDPlus t u* $\leq\natural$ *z*
shows $\exists$ *v w*. *z* = *PDPlus v w* $\wedge$ *t* $\leq\natural$ *v* $\wedge$ *u* $\leq\natural$ *w*
⟨*proof*⟩

lemma *convex-le-induct* [*induct set*: *convex-le*]:
assumes *le*: *t* $\leq\natural$ *u*
assumes *2*: $\bigwedge$*t u v*. $[\![$*P t u*; *P u v*$]\!] \Longrightarrow$ *P t v*
assumes *3*: $\bigwedge$*a b*. *a* $\sqsubseteq$ *b* $\Longrightarrow$ *P* (*PDUnit a*) (*PDUnit b*)
assumes *4*: $\bigwedge$*t u v w*. $[\![$*P t v*; *P u w*$]\!] \Longrightarrow$ *P* (*PDPlus t u*) (*PDPlus v w*)
shows *P t u*
⟨*proof*⟩

31.2 Type definition

typedef *'a*::*bifinite convex-pd* (‹(‹*notation*=‹*postfix convex-pd*››'(-')$\natural$)›) =
{*S*::*'a pd-basis set*. *convex-le.ideal S*}
⟨*proof*⟩

instantiation *convex-pd* :: (*bifinite*) *below*
begin

definition
x $\sqsubseteq$ *y* $\longleftrightarrow$ *Rep-convex-pd x* $\subseteq$ *Rep-convex-pd y*

instance ⟨*proof*⟩
end

instance *convex-pd* :: (*bifinite*) *po*
⟨*proof*⟩

instance *convex-pd* :: (*bifinite*) *cpo*
⟨*proof*⟩

definition
convex-principal :: *'a*::*bifinite pd-basis* $\Rightarrow$ *'a convex-pd* **where**

convex-principal t = Abs-convex-pd {u. u ≤♮ t}

interpretation *convex-pd*:
ideal-completion convex-le convex-principal Rep-convex-pd
⟨*proof*⟩

Convex powerdomain is pointed

lemma *convex-pd-minimal*: *convex-principal* (*PDUnit compact-bot*) ⊑ *ys*
⟨*proof*⟩

instance *convex-pd* :: (*bifinite*) *pcpo*
⟨*proof*⟩

lemma *inst-convex-pd-pcpo*: ⊥ = *convex-principal* (*PDUnit compact-bot*)
⟨*proof*⟩

31.3 Monadic unit and plus

definition
convex-unit :: *'a*::*bifinite* → *'a convex-pd* **where**
convex-unit = *compact-basis.extension* (λ*a*. *convex-principal* (*PDUnit a*))

definition
convex-plus :: *'a*::*bifinite convex-pd* → *'a convex-pd* → *'a convex-pd* **where**
convex-plus = *convex-pd.extension* (λ*t*. *convex-pd.extension* (λ*u*.
convex-principal (*PDPlus t u*)))

abbreviation
convex-add :: *'a*::*bifinite convex-pd* ⇒ *'a convex-pd* ⇒ *'a convex-pd*
(**infixl** ‹∪♮› *65*) **where**
xs ∪♮ *ys* == *convex-plus*·*xs*·*ys*

syntax
-convex-pd :: *args* ⇒ *logic* (‹(‹*indent=1 notation=‹mixfix convex-pd enumeration*››{-}♮)›)
translations
{*x*,*xs*}♮ == {*x*}♮ ∪♮ {*xs*}♮
{*x*}♮ == *CONST convex-unit*·*x*

lemma *convex-unit-Rep-compact-basis* [*simp*]:
{*Rep-compact-basis a*}♮ = *convex-principal* (*PDUnit a*)
⟨*proof*⟩

lemma *convex-plus-principal* [*simp*]:
convex-principal t ∪♮ *convex-principal u* = *convex-principal* (*PDPlus t u*)
⟨*proof*⟩

interpretation *convex-add*: *semilattice convex-add* ⟨*proof*⟩

lemmas *convex-plus-assoc* = *convex-add.assoc*
lemmas *convex-plus-commute* = *convex-add.commute*
lemmas *convex-plus-absorb* = *convex-add.idem*
lemmas *convex-plus-left-commute* = *convex-add.left-commute*
lemmas *convex-plus-left-absorb* = *convex-add.left-idem*

Useful for *simp add*: *convex-plus-ac*

lemmas *convex-plus-ac* =
convex-plus-assoc convex-plus-commute convex-plus-left-commute

Useful for *simp only*: *convex-plus-aci*

lemmas *convex-plus-aci* =
convex-plus-ac convex-plus-absorb convex-plus-left-absorb

lemma *convex-unit-below-plus-iff* [*simp*]:
$\{x\}\natural \sqsubseteq ys \cup\natural zs \longleftrightarrow \{x\}\natural \sqsubseteq ys \wedge \{x\}\natural \sqsubseteq zs$
⟨*proof*⟩

lemma *convex-plus-below-unit-iff* [*simp*]:
$xs \cup\natural ys \sqsubseteq \{z\}\natural \longleftrightarrow xs \sqsubseteq \{z\}\natural \wedge ys \sqsubseteq \{z\}\natural$
⟨*proof*⟩

lemma *convex-unit-below-iff* [*simp*]: $\{x\}\natural \sqsubseteq \{y\}\natural \longleftrightarrow x \sqsubseteq y$
⟨*proof*⟩

lemma *convex-unit-eq-iff* [*simp*]: $\{x\}\natural = \{y\}\natural \longleftrightarrow x = y$
⟨*proof*⟩

lemma *convex-unit-strict* [*simp*]: $\{\bot\}\natural = \bot$
⟨*proof*⟩

lemma *convex-unit-bottom-iff* [*simp*]: $\{x\}\natural = \bot \longleftrightarrow x = \bot$
⟨*proof*⟩

lemma *compact-convex-unit*: *compact* $x \Longrightarrow$ *compact* $\{x\}\natural$
⟨*proof*⟩

lemma *compact-convex-unit-iff* [*simp*]: *compact* $\{x\}\natural \longleftrightarrow$ *compact* x
⟨*proof*⟩

lemma *compact-convex-plus* [*simp*]:
$[\![$*compact* xs; *compact* $ys]\!] \Longrightarrow$ *compact* $(xs \cup\natural ys)$
⟨*proof*⟩

31.4 Induction rules

lemma *convex-pd-induct1*:
assumes P: *adm* P
assumes *unit*: $\bigwedge x.\ P\ \{x\}\natural$

assumes *insert*: $\bigwedge x\ ys.\ [\![P\ \{x\}\natural;\ P\ ys]\!] \Longrightarrow P\ (\{x\}\natural \cup\natural\ ys)$
shows $P\ (xs{::}'a{::}bifinite\ convex\text{-}pd)$
⟨*proof*⟩

lemma *convex-pd-induct* [*case-names adm convex-unit convex-plus*, *induct type*: *convex-pd*]:
assumes P: *adm P*
assumes *unit*: $\bigwedge x.\ P\ \{x\}\natural$
assumes *plus*: $\bigwedge xs\ ys.\ [\![P\ xs;\ P\ ys]\!] \Longrightarrow P\ (xs \cup\natural\ ys)$
shows $P\ (xs{::}'a{::}bifinite\ convex\text{-}pd)$
⟨*proof*⟩

31.5 Monadic bind

definition
convex-bind-basis ::
$'a{::}bifinite\ pd\text{-}basis \Rightarrow ('a \to 'b\ convex\text{-}pd) \to 'b{::}bifinite\ convex\text{-}pd$ **where**
convex-bind-basis = *fold-pd*
$(\lambda a.\ \Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a))$
$(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\natural\ y{\cdot}f)$

lemma *ACI-convex-bind*:
semilattice $(\lambda x\ y.\ \Lambda\ f.\ x{\cdot}f \cup\natural\ y{\cdot}f)$
⟨*proof*⟩

lemma *convex-bind-basis-simps* [*simp*]:
convex-bind-basis $(PDUnit\ a) =$
$(\Lambda\ f.\ f{\cdot}(Rep\text{-}compact\text{-}basis\ a))$
convex-bind-basis $(PDPlus\ t\ u) =$
$(\Lambda\ f.\ convex\text{-}bind\text{-}basis\ t{\cdot}f \cup\natural\ convex\text{-}bind\text{-}basis\ u{\cdot}f)$
⟨*proof*⟩

lemma *convex-bind-basis-mono*:
$t \leq\natural\ u \Longrightarrow convex\text{-}bind\text{-}basis\ t \sqsubseteq convex\text{-}bind\text{-}basis\ u$
⟨*proof*⟩

definition
convex-bind :: $'a{::}bifinite\ convex\text{-}pd \to ('a \to 'b\ convex\text{-}pd) \to 'b{::}bifinite\ convex\text{-}pd$
where
convex-bind = *convex-pd.extension convex-bind-basis*

syntax
-convex-bind :: [*logic*, *logic*, *logic*] ⇒ *logic*
(‹(‹*indent=3 notation=‹binder convex-bind›*›⋃♮-∈-./ -)› [*0*, *0*, *10*] *10*)

translations
$\bigcup\natural x{\in}xs.\ e == CONST\ convex\text{-}bind{\cdot}xs{\cdot}(\Lambda\ x.\ e)$

lemma *convex-bind-principal* [*simp*]:

convex-bind·(convex-principal t) = convex-bind-basis t
⟨*proof*⟩

lemma *convex-bind-unit* [*simp*]:
convex-bind·{x}♮·f = f·x
⟨*proof*⟩

lemma *convex-bind-plus* [*simp*]:
convex-bind·(xs ∪♮ ys)·f = convex-bind·xs·f ∪♮ convex-bind·ys·f
⟨*proof*⟩

lemma *convex-bind-strict* [*simp*]: *convex-bind·⊥·f = f·⊥*
⟨*proof*⟩

lemma *convex-bind-bind*:
convex-bind·(convex-bind·xs·f)·g =
convex-bind·xs·(Λ x. convex-bind·(f·x)·g)
⟨*proof*⟩

31.6 Map

definition
convex-map :: (*'a*::*bifinite* → *'b*) → *'a convex-pd* → *'b*::*bifinite convex-pd* **where**
convex-map = (Λ f xs. convex-bind·xs·(Λ x. {f·x}♮))

lemma *convex-map-unit* [*simp*]:
convex-map·f·{x}♮ = {f·x}♮
⟨*proof*⟩

lemma *convex-map-plus* [*simp*]:
convex-map·f·(xs ∪♮ ys) = convex-map·f·xs ∪♮ convex-map·f·ys
⟨*proof*⟩

lemma *convex-map-bottom* [*simp*]: *convex-map·f·⊥ = {f·⊥}♮*
⟨*proof*⟩

lemma *convex-map-ident*: *convex-map·(Λ x. x)·xs = xs*
⟨*proof*⟩

lemma *convex-map-ID*: *convex-map·ID = ID*
⟨*proof*⟩

lemma *convex-map-map*:
convex-map·f·(convex-map·g·xs) = convex-map·(Λ x. f·(g·x))·xs
⟨*proof*⟩

lemma *convex-bind-map*:
convex-bind·(convex-map·f·xs)·g = convex-bind·xs·(Λ x. g·(f·x))
⟨*proof*⟩

lemma *convex-map-bind*:
convex-map·f·(convex-bind·xs·g) = convex-bind·xs·(Λ x. convex-map·f·(g·x))
⟨*proof*⟩

lemma *ep-pair-convex-map*: *ep-pair e p ⟹ ep-pair (convex-map·e) (convex-map·p)*
⟨*proof*⟩

lemma *deflation-convex-map*: *deflation d ⟹ deflation (convex-map·d)*
⟨*proof*⟩

lemma *finite-deflation-convex-map*:
assumes *finite-deflation d* **shows** *finite-deflation (convex-map·d)*
⟨*proof*⟩

31.7 Convex powerdomain is bifinite

lemma *approx-chain-convex-map*:
assumes *approx-chain a*
shows *approx-chain (λi. convex-map·(a i))*
⟨*proof*⟩

instance *convex-pd* :: (*bifinite*) *bifinite*
⟨*proof*⟩

31.8 Join

definition
convex-join :: *'a::bifinite convex-pd convex-pd → 'a convex-pd* **where**
convex-join = (Λ xss. convex-bind·xss·(Λ xs. xs))

lemma *convex-join-unit* [*simp*]:
convex-join·{xs}♮ = xs
⟨*proof*⟩

lemma *convex-join-plus* [*simp*]:
convex-join·(xss ∪♮ yss) = convex-join·xss ∪♮ convex-join·yss
⟨*proof*⟩

lemma *convex-join-bottom* [*simp*]: *convex-join·⊥ = ⊥*
⟨*proof*⟩

lemma *convex-join-map-unit*:
convex-join·(convex-map·convex-unit·xs) = xs
⟨*proof*⟩

lemma *convex-join-map-join*:
convex-join·(convex-map·convex-join·xsss) = convex-join·(convex-join·xsss)
⟨*proof*⟩

lemma *convex-join-map-map*:
convex-join·(*convex-map*·(*convex-map*·*f*)·*xss*) =
convex-map·*f*·(*convex-join*·*xss*)
⟨*proof*⟩

31.9 Conversions to other powerdomains

Convex to upper

lemma *convex-le-imp-upper-le*: $t \leq\natural\ u \Longrightarrow t \leq\sharp\ u$
⟨*proof*⟩

definition
convex-to-upper :: *'a*::*bifinite convex-pd* → *'a upper-pd* **where**
convex-to-upper = *convex-pd.extension upper-principal*

lemma *convex-to-upper-principal* [*simp*]:
convex-to-upper·(*convex-principal t*) = *upper-principal t*
⟨*proof*⟩

lemma *convex-to-upper-unit* [*simp*]:
convex-to-upper·{*x*}$\natural$ = {*x*}$\sharp$
⟨*proof*⟩

lemma *convex-to-upper-plus* [*simp*]:
convex-to-upper·(*xs* ∪$\natural$ *ys*) = *convex-to-upper*·*xs* ∪$\sharp$ *convex-to-upper*·*ys*
⟨*proof*⟩

lemma *convex-to-upper-bind* [*simp*]:
convex-to-upper·(*convex-bind*·*xs*·*f*) =
upper-bind·(*convex-to-upper*·*xs*)·(*convex-to-upper oo f*)
⟨*proof*⟩

lemma *convex-to-upper-map* [*simp*]:
convex-to-upper·(*convex-map*·*f*·*xs*) = *upper-map*·*f*·(*convex-to-upper*·*xs*)
⟨*proof*⟩

lemma *convex-to-upper-join* [*simp*]:
convex-to-upper·(*convex-join*·*xss*) =
upper-bind·(*convex-to-upper*·*xss*)·*convex-to-upper*
⟨*proof*⟩

Convex to lower

lemma *convex-le-imp-lower-le*: $t \leq\natural\ u \Longrightarrow t \leq\flat\ u$
⟨*proof*⟩

definition
convex-to-lower :: *'a*::*bifinite convex-pd* → *'a lower-pd* **where**
convex-to-lower = *convex-pd.extension lower-principal*

lemma *convex-to-lower-principal* [*simp*]:
convex-to-lower·(*convex-principal t*) = *lower-principal t*
⟨*proof*⟩

lemma *convex-to-lower-unit* [*simp*]:
convex-to-lower·{*x*}♮ = {*x*}♭
⟨*proof*⟩

lemma *convex-to-lower-plus* [*simp*]:
convex-to-lower·(*xs* ∪♮ *ys*) = *convex-to-lower*·*xs* ∪♭ *convex-to-lower*·*ys*
⟨*proof*⟩

lemma *convex-to-lower-bind* [*simp*]:
convex-to-lower·(*convex-bind*·*xs*·*f*) =
lower-bind·(*convex-to-lower*·*xs*)·(*convex-to-lower oo f*)
⟨*proof*⟩

lemma *convex-to-lower-map* [*simp*]:
convex-to-lower·(*convex-map*·*f*·*xs*) = *lower-map*·*f*·(*convex-to-lower*·*xs*)
⟨*proof*⟩

lemma *convex-to-lower-join* [*simp*]:
convex-to-lower·(*convex-join*·*xss*) =
lower-bind·(*convex-to-lower*·*xss*)·*convex-to-lower*
⟨*proof*⟩

Ordering property

lemma *convex-pd-below-iff*:
(*xs* ⊑ *ys*) =
(*convex-to-upper*·*xs* ⊑ *convex-to-upper*·*ys* ∧
convex-to-lower·*xs* ⊑ *convex-to-lower*·*ys*)
⟨*proof*⟩

lemmas *convex-plus-below-plus-iff* =
convex-pd-below-iff [**where** *xs*=*xs* ∪♮ *ys* **and** *ys*=*zs* ∪♮ *ws*]
for *xs ys zs ws*

lemmas *convex-pd-below-simps* =
convex-unit-below-plus-iff
convex-plus-below-unit-iff
convex-plus-below-plus-iff
convex-unit-below-iff
convex-to-upper-unit
convex-to-upper-plus
convex-to-lower-unit
convex-to-lower-plus
upper-pd-below-simps
lower-pd-below-simps

end

32 Powerdomains

theory *Powerdomains*
imports *ConvexPD Domain*
begin

32.1 Universal domain embeddings

definition *upper-emb* = *udom-emb* (λ*i*. *upper-map*·(*udom-approx i*))
definition *upper-prj* = *udom-prj* (λ*i*. *upper-map*·(*udom-approx i*))

definition *lower-emb* = *udom-emb* (λ*i*. *lower-map*·(*udom-approx i*))
definition *lower-prj* = *udom-prj* (λ*i*. *lower-map*·(*udom-approx i*))

definition *convex-emb* = *udom-emb* (λ*i*. *convex-map*·(*udom-approx i*))
definition *convex-prj* = *udom-prj* (λ*i*. *convex-map*·(*udom-approx i*))

lemma *ep-pair-upper*: *ep-pair upper-emb upper-prj*
⟨*proof*⟩

lemma *ep-pair-lower*: *ep-pair lower-emb lower-prj*
⟨*proof*⟩

lemma *ep-pair-convex*: *ep-pair convex-emb convex-prj*
⟨*proof*⟩

32.2 Deflation combinators

definition *upper-defl* :: *udom defl* → *udom defl*
where *upper-defl* = *defl-fun1 upper-emb upper-prj upper-map*

definition *lower-defl* :: *udom defl* → *udom defl*
where *lower-defl* = *defl-fun1 lower-emb lower-prj lower-map*

definition *convex-defl* :: *udom defl* → *udom defl*
where *convex-defl* = *defl-fun1 convex-emb convex-prj convex-map*

lemma *cast-upper-defl*:
cast·(*upper-defl*·*A*) = *upper-emb oo upper-map*·(*cast*·*A*) *oo upper-prj*
⟨*proof*⟩

lemma *cast-lower-defl*:
cast·(*lower-defl*·*A*) = *lower-emb oo lower-map*·(*cast*·*A*) *oo lower-prj*
⟨*proof*⟩

lemma *cast-convex-defl*:

cast·(*convex-defl*·*A*) = *convex-emb oo convex-map*·(*cast*·*A*) *oo convex-prj*
⟨*proof*⟩

32.3 Domain class instances

instantiation *upper-pd* :: (*domain*) *domain*
begin

definition
emb = *upper-emb oo upper-map*·*emb*

definition
prj = *upper-map*·*prj oo upper-prj*

definition
defl (*t*::'*a upper-pd itself*) = *upper-defl*·*DEFL*('*a*)

definition
(*liftemb* :: '*a upper-pd u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a upper-pd u*) = *u-map*·*prj*

definition
liftdefl (*t*::'*a upper-pd itself*) = *liftdefl-of*·*DEFL*('*a upper-pd*)

instance ⟨*proof*⟩

end

instantiation *lower-pd* :: (*domain*) *domain*
begin

definition
emb = *lower-emb oo lower-map*·*emb*

definition
prj = *lower-map*·*prj oo lower-prj*

definition
defl (*t*::'*a lower-pd itself*) = *lower-defl*·*DEFL*('*a*)

definition
(*liftemb* :: '*a lower-pd u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a lower-pd u*) = *u-map*·*prj*

definition

liftdefl (*t*::'*a lower-pd itself*) = *liftdefl-of*·*DEFL*('*a lower-pd*)

instance ⟨*proof*⟩

end

instantiation *convex-pd* :: (*domain*) *domain*
begin

definition
emb = *convex-emb oo convex-map*·*emb*

definition
prj = *convex-map*·*prj oo convex-prj*

definition
defl (*t*::'*a convex-pd itself*) = *convex-defl*·*DEFL*('*a*)

definition
(*liftemb* :: '*a convex-pd u* → *udom u*) = *u-map*·*emb*

definition
(*liftprj* :: *udom u* → '*a convex-pd u*) = *u-map*·*prj*

definition
liftdefl (*t*::'*a convex-pd itself*) = *liftdefl-of*·*DEFL*('*a convex-pd*)

instance ⟨*proof*⟩

end

lemma *DEFL-upper*: *DEFL*('*a*::*domain upper-pd*) = *upper-defl*·*DEFL*('*a*)
⟨*proof*⟩

lemma *DEFL-lower*: *DEFL*('*a*::*domain lower-pd*) = *lower-defl*·*DEFL*('*a*)
⟨*proof*⟩

lemma *DEFL-convex*: *DEFL*('*a*::*domain convex-pd*) = *convex-defl*·*DEFL*('*a*)
⟨*proof*⟩

32.4 Isomorphic deflations

lemma *isodefl-upper*:
isodefl d t ⟹ *isodefl* (*upper-map*·*d*) (*upper-defl*·*t*)
⟨*proof*⟩

lemma *isodefl-lower*:
isodefl d t ⟹ *isodefl* (*lower-map*·*d*) (*lower-defl*·*t*)
⟨*proof*⟩

lemma *isodefl-convex*:
isodefl d t ⟹ *isodefl* (*convex-map*·*d*) (*convex-defl*·*t*)
⟨*proof*⟩

32.5 Domain package setup for powerdomains

lemmas [*domain-defl-simps*] = *DEFL-upper DEFL-lower DEFL-convex*
lemmas [*domain-map-ID*] = *upper-map-ID lower-map-ID convex-map-ID*
lemmas [*domain-isodefl*] = *isodefl-upper isodefl-lower isodefl-convex*

lemmas [*domain-deflation*] =
deflation-upper-map deflation-lower-map deflation-convex-map

⟨*ML*⟩

end

theory *HOLCF*
imports
Main
Domain
Powerdomains
begin

default-sort *domain*

end