Fundamental Properties of Lambda-calculus

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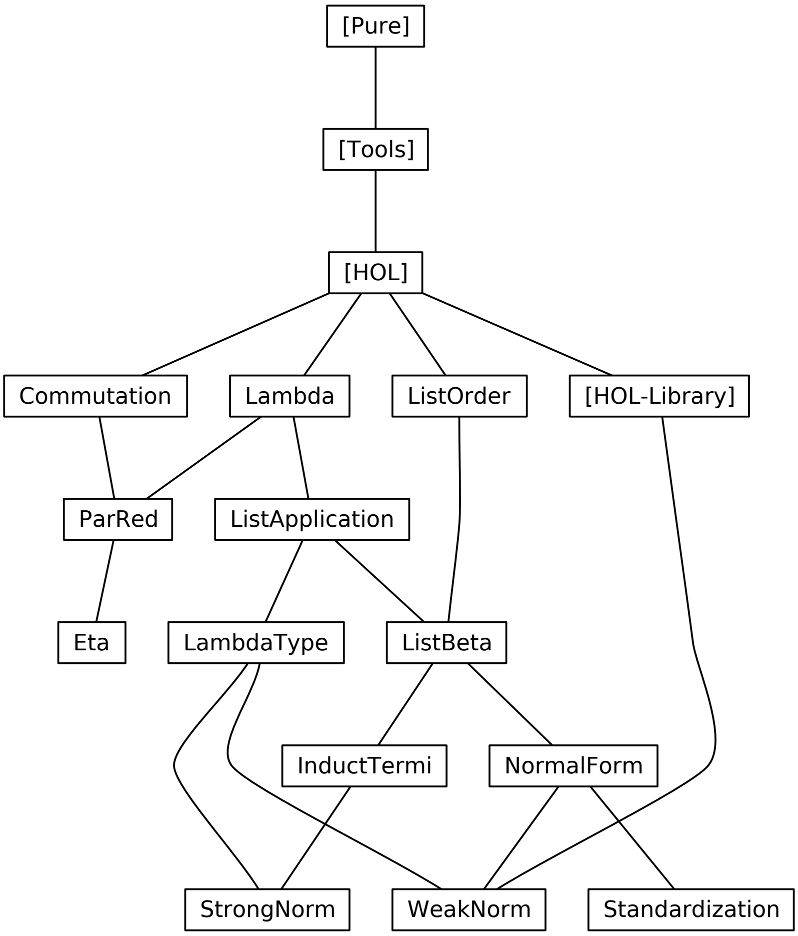
January 18, 2026

Contents


[Pure]
[Tools]
[HOL]
Commutation
Lambda
ListOrder
[HOL-Library]
ParRed
ListApplication
Eta
LambdaType
ListBeta
InductTermi
NormalForm
StrongNorm
WeakNorm
Standardization

1 Basic definitions of Lambda-calculus

theory *Lambda*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

1.1 Lambda-terms in de Bruijn notation and substitution

```
datatype dB =
    Var nat
  | App dB dB (infixl ‹°› 200)
  | Abs dB

primrec
  lift :: [dB, nat] => dB
where
    lift (Var i) k = (if i < k then Var i else Var (i + 1))
  | lift (s ° t) k = lift s k ° lift t k
  | lift (Abs s) k = Abs (lift s (k + 1))

primrec
  subst :: [dB, dB, nat] => dB  (‹-[-'/-]› [300, 0, 0] 300)
where
    subst-Var: (Var i)[s/k] =
      (if k < i then Var (i - 1) else if i = k then s else Var i)
  | subst-App: (t ° u)[s/k] = t[s/k] ° u[s/k]
  | subst-Abs: (Abs t)[s/k] = Abs (t[lift s 0 / k+1])

declare subst-Var [simp del]
```

Optimized versions of *subst* and *lift*.

```
primrec
  liftn :: [nat, dB, nat] => dB
where
    liftn n (Var i) k = (if i < k then Var i else Var (i + n))
  | liftn n (s ° t) k = liftn n s k ° liftn n t k
  | liftn n (Abs s) k = Abs (liftn n s (k + 1))

primrec
  substn :: [dB, dB, nat] => dB
where
    substn (Var i) s k =
      (if k < i then Var (i - 1) else if i = k then liftn k s 0 else Var i)
  | substn (t ° u) s k = substn t s k ° substn u s k
  | substn (Abs t) s k = Abs (substn t s (k + 1))
```

1.2 Beta-reduction

inductive *beta* :: [*dB*, *dB*] => *bool* (**infixl** ‹$\rightarrow_\beta$› *50*)
where
beta [*simp*, *intro*!]: $Abs\ s \circ t \rightarrow_\beta s[t/0]$
| *appL* [*simp*, *intro*!]: $s \rightarrow_\beta t \Longrightarrow s \circ u \rightarrow_\beta t \circ u$
| *appR* [*simp*, *intro*!]: $s \rightarrow_\beta t \Longrightarrow u \circ s \rightarrow_\beta u \circ t$
| *abs* [*simp*, *intro*!]: $s \rightarrow_\beta t \Longrightarrow Abs\ s \rightarrow_\beta Abs\ t$

abbreviation
beta-reds :: [*dB*, *dB*] => *bool* (**infixl** ‹$\rightarrow_\beta{}^*$› *50*) **where**
$s \rightarrow_\beta{}^* t == beta^{**}\ s\ t$

inductive-cases *beta-cases* [*elim*!]:
$Var\ i \rightarrow_\beta t$
$Abs\ r \rightarrow_\beta s$
$s \circ t \rightarrow_\beta u$

declare *if-not-P* [*simp*] *not-less-eq* [*simp*]
— don't add *r-into-rtrancl*[*intro*!]

1.3 Congruence rules

lemma *rtrancl-beta-Abs* [*intro*!]:
$s \rightarrow_\beta{}^* s' \Longrightarrow Abs\ s \rightarrow_\beta{}^* Abs\ s'$
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-beta-AppL*:
$s \rightarrow_\beta{}^* s' \Longrightarrow s \circ t \rightarrow_\beta{}^* s' \circ t$
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-beta-AppR*:
$t \rightarrow_\beta{}^* t' \Longrightarrow s \circ t \rightarrow_\beta{}^* s \circ t'$
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-beta-App* [*intro*]:
$[\![s \rightarrow_\beta{}^* s'; t \rightarrow_\beta{}^* t']\!] \Longrightarrow s \circ t \rightarrow_\beta{}^* s' \circ t'$
by (*blast intro*!: *rtrancl-beta-AppL rtrancl-beta-AppR intro*: *rtranclp-trans*)

1.4 Substitution-lemmas

lemma *subst-eq* [*simp*]: $(Var\ k)[u/k] = u$
by (*simp add*: *subst-Var*)

lemma *subst-gt* [*simp*]: $i < j \Longrightarrow (Var\ j)[u/i] = Var\ (j - 1)$
by (*simp add*: *subst-Var*)

lemma *subst-lt* [*simp*]: $j < i \Longrightarrow (Var\ j)[u/i] = Var\ j$
by (*simp add*: *subst-Var*)

lemma *lift-lift*:
$i < k + 1 \Longrightarrow lift\ (lift\ t\ i)\ (Suc\ k) = lift\ (lift\ t\ k)\ i$
by (*induct t arbitrary*: *i k*) *auto*

lemma *lift-subst* [*simp*]:
$j < i + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ (i + 1))\ [lift\ s\ i\ /\ j]$
by (*induct t arbitrary*: *i j s*)
(*simp-all add*: *diff-Suc subst-Var lift-lift split*: *nat.split*)

lemma *lift-subst-lt*:
$i < j + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ i)\ [lift\ s\ i\ /\ j + 1]$
by (*induct t arbitrary*: *i j s*) (*simp-all add*: *subst-Var lift-lift*)

lemma *subst-lift* [*simp*]:
$(lift\ t\ k)[s/k] = t$
by (*induct t arbitrary*: *k s*) *simp-all*

lemma *subst-subst*:
$i < j + 1 \Longrightarrow t[lift\ v\ i\ /\ Suc\ j][u[v/j]/i] = t[u/i][v/j]$
by (*induct t arbitrary*: *i j u v*)
(*simp-all add*: *diff-Suc subst-Var lift-lift* [*symmetric*] *lift-subst-lt*
split: *nat.split*)

1.5 Equivalence proof for optimized substitution

lemma *liftn-0* [*simp*]: $liftn\ 0\ t\ k = t$
by (*induct t arbitrary*: *k*) (*simp-all add*: *subst-Var*)

lemma *liftn-lift* [*simp*]: $liftn\ (Suc\ n)\ t\ k = lift\ (liftn\ n\ t\ k)\ k$
by (*induct t arbitrary*: *k*) (*simp-all add*: *subst-Var*)

lemma *substn-subst-n* [*simp*]: $substn\ t\ s\ n = t[liftn\ n\ s\ 0\ /\ n]$
by (*induct t arbitrary*: *n*) (*simp-all add*: *subst-Var*)

theorem *substn-subst-0*: $substn\ t\ s\ 0 = t[s/0]$
by *simp*

1.6 Preservation theorems

Not used in Church-Rosser proof, but in Strong Normalization.

theorem *subst-preserves-beta* [*simp*]:
$r \rightarrow_\beta s \Longrightarrow r[t/i] \rightarrow_\beta s[t/i]$
by (*induct arbitrary*: *t i set*: *beta*) (*simp-all add*: *subst-subst* [*symmetric*])

theorem *subst-preserves-beta'*: $r \rightarrow_\beta{}^* s \Longrightarrow r[t/i] \rightarrow_\beta{}^* s[t/i]$
proof (*induct set*: *rtranclp*)
case *base*
then show *?case*

by (*iprover intro*: *rtrancl-refl*)
next
case (*step y z*)
then show *?case*
by (*iprover intro*: *rtranclp.simps subst-preserves-beta*)
qed

theorem *lift-preserves-beta* [*simp*]:
$r \to_\beta s \Longrightarrow \mathit{lift}\ r\ i \to_\beta \mathit{lift}\ s\ i$
by (*induct arbitrary*: *i set*: *beta*) *auto*

theorem *lift-preserves-beta'*: $r \to_\beta{}^* s \Longrightarrow \mathit{lift}\ r\ i \to_\beta{}^* \mathit{lift}\ s\ i$
proof (*induct set*: *rtranclp*)
case *base*
then show *?case*
by (*iprover intro*: *rtrancl-refl*)
next
case (*step y z*)
then show *?case*
by (*iprover intro*: *lift-preserves-beta rtranclp.simps*)
qed

theorem *subst-preserves-beta2* [*simp*]: $r \to_\beta s \Longrightarrow t[r/i] \to_\beta{}^* t[s/i]$
proof (*induct t arbitrary*: *r s i*)
case (*Var x*)
then show *?case*
by (*simp add*: *subst-Var r-into-rtranclp*)
next
case (*App t1 t2*)
then show *?case*
by (*simp add*: *rtrancl-beta-App*)
next
case (*Abs t*)
then show *?case* **by** (*simp add*: *rtrancl-beta-Abs*)
qed

theorem *subst-preserves-beta2'*: $r \to_\beta{}^* s \Longrightarrow t[r/i] \to_\beta{}^* t[s/i]$
proof (*induct set*: *rtranclp*)
case *base*
then show *?case* **by** (*iprover intro*: *rtrancl-refl*)
next
case (*step y z*)
then show *?case*
by (*iprover intro*: *rtranclp-trans subst-preserves-beta2*)
qed

end

2 Abstract commutation and confluence notions

theory *Commutation*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

2.1 Basic definitions

definition
square :: ['*a* => '*a* => *bool*, '*a* => '*a* => *bool*, '*a* => '*a* => *bool*, '*a* => '*a* => *bool*] => *bool* **where**
square R S T U =
($\forall$ *x y*. *R x y* --> ($\forall$ *z*. *S x z* --> ($\exists$ *u*. *T y u* $\wedge$ *U z u*)))

definition
commute :: ['*a* => '*a* => *bool*, '*a* => '*a* => *bool*] => *bool* **where**
commute R S = *square R S S R*

definition
diamond :: ('*a* => '*a* => *bool*) => *bool* **where**
diamond R = *commute R R*

definition
Church-Rosser :: ('*a* => '*a* => *bool*) => *bool* **where**
Church-Rosser R =
($\forall$ *x y*. (*sup R* (R^{-1-1}))** *x y* $\longrightarrow$ ($\exists$ *z*. R^{**} *x z* $\wedge$ R^{**} *y z*))

abbreviation
confluent :: ('*a* => '*a* => *bool*) => *bool* **where**
confluent R == *diamond* (R^{**})

2.2 Basic lemmas

square

lemma *square-sym*: *square R S T U* ==> *square S R U T*
apply (*unfold square-def*)
apply *blast*
done

lemma *square-subset*:
[| *square R S T U*; $T \leq T'$ |] ==> *square R S T' U*
apply (*unfold square-def*)
apply (*blast dest*: *predicate2D*)
done

lemma *square-reflcl*:
[| *square R S T* ($R^{==}$); $S \leq T$ |] ==> *square* ($R^{==}$) *S T* ($R^{==}$)

apply (*unfold square-def*)
apply (*blast dest*: *predicate2D*)
done

lemma *square-rtrancl*:
square R S S T ==> square (R^{**}) *S S* (T^{**})
apply (*unfold square-def*)
apply (*intro strip*)
apply (*erule rtranclp-induct*)
apply *blast*
apply (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)
done

lemma *square-rtrancl-reflcl-commute*:
square R S (S^{**}) $(R^{==})$ *==> commute* (R^{**}) (S^{**})
apply (*unfold commute-def*)
apply (*fastforce dest*: *square-reflcl square-sym* [*THEN square-rtrancl*])
done

commute

lemma *commute-sym*: *commute R S ==> commute S R*
apply (*unfold commute-def*)
apply (*blast intro*: *square-sym*)
done

lemma *commute-rtrancl*: *commute R S ==> commute* (R^{**}) (S^{**})
apply (*unfold commute-def*)
apply (*blast intro*: *square-rtrancl square-sym*)
done

lemma *commute-Un*:
[| *commute R T*; *commute S T* |] *==> commute* (*sup R S*) *T*
apply (*unfold commute-def square-def*)
apply *blast*
done

diamond, *confluence*, **and** *union*

lemma *diamond-Un*:
[| *diamond R*; *diamond S*; *commute R S* |] *==> diamond* (*sup R S*)
apply (*unfold diamond-def*)
apply (*blast intro*: *commute-Un commute-sym*)
done

lemma *diamond-confluent*: *diamond R ==> confluent R*
apply (*unfold diamond-def*)
apply (*erule commute-rtrancl*)
done

lemma *square-reflcl-confluent*:
  square R R $(R^{==})$ $(R^{==})$ ==> *confluent R*
  apply (*unfold diamond-def*)
  apply (*fast intro*: *square-rtrancl-reflcl-commute elim*: *square-subset*)
  done

lemma *confluent-Un*:
  [| *confluent R*; *confluent S*; *commute* (R^{**}) (S^{**}) |] ==> *confluent* (*sup R S*)
  apply (*rule rtranclp-sup-rtranclp* [*THEN subst*])
  apply (*blast dest*: *diamond-Un intro*: *diamond-confluent*)
  done

lemma *diamond-to-confluence*:
  [| *diamond R*; $T \leq R$; $R \leq T^{**}$ |] ==> *confluent T*
  apply (*force intro*: *diamond-confluent*
  dest: *rtranclp-subset* [*symmetric*])
  done

2.3 Church-Rosser

lemma *Church-Rosser-confluent*: *Church-Rosser R* = *confluent R*
  apply (*unfold square-def commute-def diamond-def Church-Rosser-def*)
  apply (*tactic* ‹*safe-tac* (*put-claset HOL-cs* ***context***)›)
  apply (*tactic* ‹
  blast-tac (*put-claset HOL-cs* ***context*** *addIs*
  [@{*thm sup-ge2*} *RS* @{*thm rtranclp-mono*} *RS* @{*thm predicate2D*} *RS*
@{*thm rtranclp-trans*},
  @{*thm rtranclp-converseI*}, @{*thm conversepI*},
  @{*thm sup-ge1*} *RS* @{*thm rtranclp-mono*} *RS* @{*thm predicate2D*}]) *1*›)
  apply (*erule rtranclp-induct*)
  apply *blast*
  apply (*blast del*: *rtranclp.rtrancl-refl intro*: *rtranclp-trans*)
  done

2.4 Newman's lemma

Proof by Stefan Berghofer

theorem *newman*:
  assumes *wf*: *wfP* (R^{-1-1})
  and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
  $\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
  shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
  $\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
  using *wf*
proof *induct*
  case (*less x b c*)
  have *xc*: $R^{**}\ x\ c$ **by** *fact*
  have *xb*: $R^{**}\ x\ b$ **by** *fact* **thus** *?case*
  proof (*rule converse-rtranclpE*)

    assume $x = b$
    with *xc* **have** $R^{**}\ b\ c$ **by** *simp*
    thus *?thesis* **by** *iprover*
  next
    fix y
    assume *xy*: $R\ x\ y$
    assume *yb*: $R^{**}\ y\ b$
    from *xc* **show** *?thesis*
    proof (*rule converse-rtranclpE*)
      assume $x = c$
      with *xb* **have** $R^{**}\ c\ b$ **by** *simp*
      thus *?thesis* **by** *iprover*
    next
      fix y'
      assume *y'c*: $R^{**}\ y'\ c$
      assume *xy'*: $R\ x\ y'$
      with *xy* **have** $\exists\, u.\ R^{**}\ y\ u \wedge R^{**}\ y'\ u$ **by** (*rule lc*)
      then obtain u **where** *yu*: $R^{**}\ y\ u$ **and** *y'u*: $R^{**}\ y'\ u$ **by** *iprover*
      from *xy* **have** $R^{-1-1}\ y\ x$ **..**
      from *this* **and** *yb yu* **have** $\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ u\ d$ **by** (*rule less*)
      then obtain v **where** *bv*: $R^{**}\ b\ v$ **and** *uv*: $R^{**}\ u\ v$ **by** *iprover*
      from *xy'* **have** $R^{-1-1}\ y'\ x$ **..**
      moreover from *y'u* **and** *uv* **have** $R^{**}\ y'\ v$ **by** (*rule rtranclp-trans*)
      moreover note *y'c*
      ultimately have $\exists\, d.\ R^{**}\ v\ d \wedge R^{**}\ c\ d$ **by** (*rule less*)
      then obtain w **where** *vw*: $R^{**}\ v\ w$ **and** *cw*: $R^{**}\ c\ w$ **by** *iprover*
      from *bv vw* **have** $R^{**}\ b\ w$ **by** (*rule rtranclp-trans*)
      with *cw* **show** *?thesis* **by** *iprover*
    qed
  qed
qed

Alternative version. Partly automated by Tobias Nipkow. Takes 2 minutes (2002).

This is the maximal amount of automation possible using *blast*.

theorem *newman'*:
  assumes *wf*: $wfP\ (R^{-1-1})$
  and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
    $\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
  shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
    $\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
  using *wf*
proof *induct*
  case (*less x b c*)
  note $IH = \langle \bigwedge y\ b\ c.\ [\![R^{-1-1}\ y\ x;\ R^{**}\ y\ b;\ R^{**}\ y\ c]\!]$
            $\Longrightarrow \exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d\rangle$
  have *xc*: $R^{**}\ x\ c$ **by** *fact*
  have *xb*: $R^{**}\ x\ b$ **by** *fact*
  thus *?case*

```
  proof (rule converse-rtranclpE)
    assume x = b
    with xc have R** b c by simp
    thus ?thesis by iprover
  next
    fix y
    assume xy: R x y
    assume yb: R** y b
    from xc show ?thesis
    proof (rule converse-rtranclpE)
      assume x = c
      with xb have R** c b by simp
      thus ?thesis by iprover
    next
      fix y'
      assume y'c: R** y' c
      assume xy': R x y'
      with xy obtain u where u: R** y u R** y' u
        by (blast dest: lc)
      from yb u y'c show ?thesis
        by (blast del: rtranclp.rtrancl-refl
            intro: rtranclp-trans
            dest: IH [OF conversepI, OF xy] IH [OF conversepI, OF xy'])
    qed
  qed
qed
```

Using the coherent logic prover, the proof of the induction step is completely automatic.

```
lemma eq-imp-rtranclp: x = y ⟹ r** x y
  by simp

theorem newman'':
  assumes wf: wfP (R⁻¹⁻¹)
  and lc: ⋀a b c. R a b ⟹ R a c ⟹
    ∃ d. R** b d ∧ R** c d
  shows ⋀b c. R** a b ⟹ R** a c ⟹
    ∃ d. R** b d ∧ R** c d
  using wf
proof induct
  case (less x b c)
  note IH = ‹⋀y b c. ⟦R⁻¹⁻¹ y x; R** y b; R** y c⟧
                    ⟹ ∃ d. R** b d ∧ R** c d›
  show ?case
    by (coherent
      ‹R** x c› ‹R** x b›
      refl [where 'a='a] sym
      eq-imp-rtranclp
      r-into-rtranclp [of R]
```

rtranclp-trans
lc IH [*OF conversepI*]
converse-rtranclpE)
qed

end

3 Parallel reduction and a complete developments

theory *ParRed* **imports** *Lambda Commutation* **begin**

3.1 Parallel reduction

```
inductive par-beta :: [dB, dB] => bool  (infixl ‹=>› 50)
  where
    var [simp, intro!]: Var n => Var n
  | abs [simp, intro!]: s => t ==> Abs s => Abs t
  | app [simp, intro!]: [| s => s'; t => t' |] ==> s ° t => s' ° t'
  | beta [simp, intro!]: [| s => s'; t => t' |] ==> (Abs s) ° t => s'[t'/0]
```

```
inductive-cases par-beta-cases [elim!]:
  Var n => t
  Abs s => Abs t
  (Abs s) ° t => u
  s ° t => u
  Abs s => t
```

3.2 Inclusions

beta ⊆ *par-beta* ⊆ $beta^*$

```
lemma par-beta-varL [simp]:
    (Var n => t) = (t = Var n)
  by blast

lemma par-beta-refl [simp]: t => t
  by (induct t) simp-all

lemma beta-subset-par-beta: beta <= par-beta
  apply (rule predicate2I)
  apply (erule beta.induct)
     apply (blast intro!: par-beta-refl)+
  done

lemma par-beta-subset-beta: par-beta ≤ beta**
  apply (rule predicate2I)
  apply (erule par-beta.induct)
     apply blast
    apply (blast del: rtranclp.rtrancl-refl intro: rtranclp.rtrancl-into-rtrancl)+
```

— *rtrancl-refl* complicates the proof by increasing the branching factor
done

3.3 Misc properties of *par-beta*

lemma *par-beta-lift* [*simp*]:
$t => t' \Longrightarrow lift\ t\ n => lift\ t'\ n$
by (*induct t arbitrary*: *t' n*) *fastforce*+

lemma *par-beta-subst*:
$s => s' \Longrightarrow t => t' \Longrightarrow t[s/n] => t'[s'/n]$
apply (*induct t arbitrary*: *s s' t' n*)
apply (*simp add*: *subst-Var*)
apply (*erule par-beta-cases*)
apply *simp*
apply (*simp add*: *subst-subst* [*symmetric*])
apply (*fastforce intro*!: *par-beta-lift*)
apply *fastforce*
done

3.4 Confluence (directly)

lemma *diamond-par-beta*: *diamond par-beta*
apply (*unfold diamond-def commute-def square-def*)
apply (*rule impI* [*THEN allI* [*THEN allI*]])
apply (*erule par-beta.induct*)
apply (*blast intro*!: *par-beta-subst*)+
done

3.5 Complete developments

fun
cd :: *dB* => *dB*
where
cd (*Var n*) = *Var n*
| *cd* (*Var n* ° *t*) = *Var n* ° *cd t*
| *cd* ((*s1* ° *s2*) ° *t*) = *cd* (*s1* ° *s2*) ° *cd t*
| *cd* (*Abs u* ° *t*) = (*cd u*)[*cd t*/*0*]
| *cd* (*Abs s*) = *Abs* (*cd s*)

lemma *par-beta-cd*: $s => t \Longrightarrow t => cd\ s$
apply (*induct s arbitrary*: *t rule*: *cd.induct*)
apply *auto*
apply (*fast intro*!: *par-beta-subst*)
done

3.6 Confluence (via complete developments)

lemma *diamond-par-beta2*: *diamond par-beta*
unfolding *diamond-def commute-def square-def*

by (*blast intro*: *par-beta-cd*)

theorem *beta-confluent*: *confluent beta*
by (*rule diamond-par-beta2 diamond-to-confluence par-beta-subset-beta beta-subset-par-beta*)+

end

4 Eta-reduction

theory *Eta* **imports** *ParRed* **begin**

4.1 Definition of eta-reduction and relatives

primrec
free :: *dB* => *nat* => *bool*
where
free (*Var j*) *i* = (*j* = *i*)
| *free* (*s* ° *t*) *i* = (*free s i* ∨ *free t i*)
| *free* (*Abs s*) *i* = *free s* (*i* + *1*)

inductive
eta :: [*dB*, *dB*] => *bool* (**infixl** ‹$\to_\eta$› *50*)
where
eta [*simp*, *intro*]: ¬ *free s 0* ==> *Abs* (*s* ° *Var 0*) $\to_\eta$ *s*[*dummy*/*0*]
| *appL* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *s* ° *u* $\to_\eta$ *t* ° *u*
| *appR* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *u* ° *s* $\to_\eta$ *u* ° *t*
| *abs* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *Abs s* $\to_\eta$ *Abs t*

abbreviation
eta-reds :: [*dB*, *dB*] => *bool* (**infixl** ‹$\to_\eta{}^*$› *50*) **where**
s $\to_\eta{}^*$ *t* == *eta*** *s t*

abbreviation
eta-red0 :: [*dB*, *dB*] => *bool* (**infixl** ‹$\to_\eta{}^=$› *50*) **where**
s $\to_\eta{}^=$ *t* == *eta*$^{==}$ *s t*

inductive-cases *eta-cases* [*elim*!]:
Abs s $\to_\eta$ *z*
s ° *t* $\to_\eta$ *u*
Var i $\to_\eta$ *t*

4.2 Properties of *eta*, *subst* and *free*

lemma *subst-not-free* [*simp*]: ¬ *free s i* ⟹ *s*[*t*/*i*] = *s*[*u*/*i*]
by (*induct s arbitrary*: *i t u*) (*simp-all add*: *subst-Var*)

lemma *free-lift* [*simp*]:
free (*lift t k*) *i* = (*i* < *k* ∧ *free t i* ∨ *k* < *i* ∧ *free t* (*i* − *1*))

apply (*induct t arbitrary*: *i k*)
apply (*auto cong*: *conj-cong*)
done

lemma *free-subst* [*simp*]:
free (*s*[*t*/*k*]) *i* =
(*free s k* ∧ *free t i* ∨ *free s* (*if i* < *k then i else i* + *1*))
apply (*induct s arbitrary*: *i k t*)
prefer *2*
apply *simp*
apply *blast*
prefer *2*
apply *simp*
apply (*simp add*: *diff-Suc subst-Var split*: *nat.split*)
done

lemma *free-eta*: $s \rightarrow_\eta t$ ==> *free t i* = *free s i*
by (*induct arbitrary*: *i set*: *eta*) (*simp-all cong*: *conj-cong*)

lemma *not-free-eta*:
[| $s \rightarrow_\eta t$; ¬ *free s i* |] ==> ¬ *free t i*
by (*simp add*: *free-eta*)

lemma *eta-subst* [*simp*]:
$s \rightarrow_\eta t$ ==> $s[u/i] \rightarrow_\eta t[u/i]$
by (*induct arbitrary*: *u i set*: *eta*) (*simp-all add*: *subst-subst* [*symmetric*])

theorem *lift-subst-dummy*: ¬ *free s i* ⟹ *lift* (*s*[*dummy*/*i*]) *i* = *s*
by (*induct s arbitrary*: *i dummy*) *simp-all*

4.3 Confluence of *eta*

lemma *square-eta*: *square eta eta* ($eta^{==}$) ($eta^{==}$)
apply (*unfold square-def id-def*)
apply (*rule impI* [*THEN allI* [*THEN allI*]])
apply (*erule eta.induct*)
apply (*slowsimp intro*: *subst-not-free eta-subst free-eta* [*THEN iffD1*])
apply *safe*
prefer *5*
apply (*blast intro*!: *eta-subst intro*: *free-eta* [*THEN iffD1*])
apply *blast*+
done

theorem *eta-confluent*: *confluent eta*
apply (*rule square-eta* [*THEN square-reflcl-confluent*])
done

4.4 Congruence rules for eta^*

lemma *rtrancl-eta-Abs*: $s \rightarrow_\eta{}^* s'$ ==> *Abs s* $\rightarrow_\eta{}^*$ *Abs s'*

by (*induct set*: *rtranclp*)
(*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-eta-AppL*: $s \rightarrow_\eta{}^* s' ==> s \circ t \rightarrow_\eta{}^* s' \circ t$
by (*induct set*: *rtranclp*)
(*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-eta-AppR*: $t \rightarrow_\eta{}^* t' ==> s \circ t \rightarrow_\eta{}^* s \circ t'$
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-eta-App*:
$[| s \rightarrow_\eta{}^* s'; t \rightarrow_\eta{}^* t' |] ==> s \circ t \rightarrow_\eta{}^* s' \circ t'$
by (*blast intro*!: *rtrancl-eta-AppL rtrancl-eta-AppR intro*: *rtranclp-trans*)

4.5 Commutation of *beta* and *eta*

lemma *free-beta*:
$s \rightarrow_\beta t ==> free\ t\ i \Longrightarrow free\ s\ i$
by (*induct arbitrary*: *i set*: *beta*) *auto*

lemma *beta-subst* [*intro*]: $s \rightarrow_\beta t ==> s[u/i] \rightarrow_\beta t[u/i]$
by (*induct arbitrary*: *u i set*: *beta*) (*simp-all add*: *subst-subst* [*symmetric*])

lemma *subst-Var-Suc* [*simp*]: $t[Var\ i/i] = t[Var(i)/i + 1]$
by (*induct t arbitrary*: *i*) (*auto elim*!: *linorder-neqE simp*: *subst-Var*)

lemma *eta-lift* [*simp*]: $s \rightarrow_\eta t ==> lift\ s\ i \rightarrow_\eta lift\ t\ i$
by (*induct arbitrary*: *i set*: *eta*) *simp-all*

lemma *rtrancl-eta-subst*: $s \rightarrow_\eta t \Longrightarrow u[s/i] \rightarrow_\eta{}^* u[t/i]$
apply (*induct u arbitrary*: *s t i*)
apply (*simp-all add*: *subst-Var*)
apply *blast*
apply (*blast intro*: *rtrancl-eta-App*)
apply (*blast intro*!: *rtrancl-eta-Abs eta-lift*)
done

lemma *rtrancl-eta-subst'*:
fixes *s t* :: *dB*
assumes *eta*: $s \rightarrow_\eta{}^* t$
shows $s[u/i] \rightarrow_\eta{}^* t[u/i]$ **using** *eta*
by *induct* (*iprover intro*: *eta-subst*)+

lemma *rtrancl-eta-subst''*:
fixes *s t* :: *dB*
assumes *eta*: $s \rightarrow_\eta{}^* t$
shows $u[s/i] \rightarrow_\eta{}^* u[t/i]$ **using** *eta*
by *induct* (*iprover intro*: *rtrancl-eta-subst rtranclp-trans*)+

lemma *square-beta-eta*: *square beta eta* (*eta***) (*beta*$^{==}$)
  apply (*unfold square-def*)
  apply (*rule impI* [*THEN allI* [*THEN allI*]])
  apply (*erule beta.induct*)
     apply (*slowsimp intro*: *rtrancl-eta-subst eta-subst*)
    apply (*blast intro*: *rtrancl-eta-AppL*)
   apply (*blast intro*: *rtrancl-eta-AppR*)
  apply *simp*
  apply (*slowsimp intro*: *rtrancl-eta-Abs free-beta*
    iff del: *dB.distinct simp*: *dB.distinct*)
  done

lemma *confluent-beta-eta*: *confluent* (*sup beta eta*)
  apply (*assumption* |
    rule square-rtrancl-reflcl-commute confluent-Un
      beta-confluent eta-confluent square-beta-eta)+
  done

4.6 Implicit definition of *eta*

Abs (*lift s 0* ° *Var 0*) $\to_\eta$ *s*

lemma *not-free-iff-lifted*:
    (¬ *free s i*) = (∃ *t*. *s* = *lift t i*)
  apply (*induct s arbitrary*: *i*)
    apply *simp*
    apply (*rule iffI*)
     apply (*erule linorder-neqE*)
      apply (*rename-tac nat a*, *rule-tac x* = *Var nat* **in** *exI*)
      apply *simp*
     apply (*rename-tac nat a*, *rule-tac x* = *Var* (*nat* − *1*) **in** *exI*)
     apply *simp*
    apply *clarify*
    apply (*rule notE*)
     prefer *2*
     apply *assumption*
    apply (*erule thin-rl*)
    apply (*case-tac t*)
      apply *simp*
     apply *simp*
    apply *simp*
   apply *simp*
   apply (*erule thin-rl*)
   apply (*erule thin-rl*)
   apply (*rule iffI*)
    apply (*elim conjE exE*)
    apply (*rename-tac u1 u2*)
    apply (*rule-tac x* = *u1* ° *u2* **in** *exI*)
    apply *simp*
   apply (*erule exE*)

apply (*erule rev-mp*)
apply (*case-tac t*)
apply *simp*
apply *simp*
apply *blast*
apply *simp*
apply *simp*
apply (*erule thin-rl*)
apply (*rule iffI*)
apply (*erule exE*)
apply (*rule-tac x = Abs t* **in** *exI*)
apply *simp*
apply (*erule exE*)
apply (*erule rev-mp*)
apply (*case-tac t*)
apply *simp*
apply *simp*
apply *simp*
apply *blast*
done

theorem *explicit-is-implicit*:
(∀ *s u*. (¬ *free s 0*) −−> *R* (*Abs* (*s* ° *Var 0*)) (*s*[*u*/*0*])) =
(∀ *s*. *R* (*Abs* (*lift s 0* ° *Var 0*)) *s*)
by (*auto simp add*: *not-free-iff-lifted*)

4.7 Eta-postponement theorem

Based on a paper proof due to Andreas Abel. Unlike the proof by Masako Takahashi [4], it does not use parallel eta reduction, which only seems to complicate matters unnecessarily.

theorem *eta-case*:
fixes *s* :: *dB*
assumes *free*: ¬ *free s 0*
and *s*: *s*[*dummy*/*0*] => *u*
shows ∃ *t'*. *Abs* (*s* ° *Var 0*) => *t'* ∧ $t' \rightarrow_\eta{}^*$ *u*
proof −
from *s* **have** *lift* (*s*[*dummy*/*0*]) *0* => *lift u 0* **by** (*simp del*: *lift-subst*)
with *free* **have** *s* => *lift u 0* **by** (*simp add*: *lift-subst-dummy del*: *lift-subst*)
hence *Abs* (*s* ° *Var 0*) => *Abs* (*lift u 0* ° *Var 0*) **by** *simp*
moreover have ¬ *free* (*lift u 0*) *0* **by** *simp*
hence *Abs* (*lift u 0* ° *Var 0*) $\rightarrow_\eta$ *lift u 0*[*dummy*/*0*]
by (*rule eta.eta*)
hence *Abs* (*lift u 0* ° *Var 0*) $\rightarrow_\eta{}^*$ *u* **by** *simp*
ultimately show *?thesis* **by** *iprover*
qed

theorem *eta-par-beta*:
assumes *st*: $s \rightarrow_\eta t$

and *tu*: *t* => *u*
shows ∃ *t'*. *s* => *t'* ∧ *t'* $\to_\eta{}^*$ *u* **using** *tu st*
proof (*induct arbitrary*: *s*)
case (*var n*)
thus *?case* **by** (*iprover intro*: *par-beta-refl*)
next
case (*abs s' t*)
note *abs'* = *this*
from ‹*s* $\to_\eta$ *Abs s'*› **show** *?case*
proof *cases*
case (*eta s'' dummy*)
from *abs* **have** *Abs s'* => *Abs t* **by** *simp*
with *eta* **have** *s''*[*dummy*/*0*] => *Abs t* **by** *simp*
with ‹¬ *free s'' 0*› **have** ∃ *t'*. *Abs* (*s''* ° *Var 0*) => *t'* ∧ *t'* $\to_\eta{}^*$ *Abs t*
by (*rule eta-case*)
with *eta* **show** *?thesis* **by** *simp*
next
case (*abs r*)
from ‹*r* $\to_\eta$ *s'*›
obtain *t'* **where** *r*: *r* => *t'* **and** *t'*: *t'* $\to_\eta{}^*$ *t* **by** (*iprover dest*: *abs'*)
from *r* **have** *Abs r* => *Abs t'* **..**
moreover from *t'* **have** *Abs t'* $\to_\eta{}^*$ *Abs t* **by** (*rule rtrancl-eta-Abs*)
ultimately show *?thesis* **using** *abs* **by** *simp iprover*
qed
next
case (*app u u' t t'*)
from ‹*s* $\to_\eta$ *u* ° *t*› **show** *?case*
proof *cases*
case (*eta s' dummy*)
from *app* **have** *u* ° *t* => *u'* ° *t'* **by** *simp*
with *eta* **have** *s'*[*dummy*/*0*] => *u'* ° *t'* **by** *simp*
with ‹¬ *free s' 0*› **have** ∃ *r*. *Abs* (*s'* ° *Var 0*) => *r* ∧ *r* $\to_\eta{}^*$ *u'* ° *t'*
by (*rule eta-case*)
with *eta* **show** *?thesis* **by** *simp*
next
case (*appL s'*)
from ‹*s'* $\to_\eta$ *u*›
obtain *r* **where** *s'*: *s'* => *r* **and** *r*: *r* $\to_\eta{}^*$ *u'* **by** (*iprover dest*: *app*)
from *s'* **and** *app* **have** *s'* ° *t* => *r* ° *t'* **by** *simp*
moreover from *r* **have** *r* ° *t'* $\to_\eta{}^*$ *u'* ° *t'* **by** (*simp add*: *rtrancl-eta-AppL*)
ultimately show *?thesis* **using** *appL* **by** *simp iprover*
next
case (*appR s'*)
from ‹*s'* $\to_\eta$ *t*›
obtain *r* **where** *s'*: *s'* => *r* **and** *r*: *r* $\to_\eta{}^*$ *t'* **by** (*iprover dest*: *app*)
from *s'* **and** *app* **have** *u* ° *s'* => *u'* ° *r* **by** *simp*
moreover from *r* **have** *u'* ° *r* $\to_\eta{}^*$ *u'* ° *t'* **by** (*simp add*: *rtrancl-eta-AppR*)
ultimately show *?thesis* **using** *appR* **by** *simp iprover*
qed

```
next
  case (beta u u' t t')
  from ‹s →η Abs u ° t› show ?case
  proof cases
    case (eta s' dummy)
    from beta have Abs u ° t => u'[t'/0] by simp
    with eta have s'[dummy/0] => u'[t'/0] by simp
    with ‹¬ free s' 0› have ∃ r. Abs (s' ° Var 0) => r ∧ r →η* u'[t'/0]
      by (rule eta-case)
    with eta show ?thesis by simp
  next
    case (appL s')
    from ‹s' →η Abs u› show ?thesis
    proof cases
      case (eta s'' dummy)
      have Abs (lift u 1) = lift (Abs u) 0 by simp
      also from eta have ... = s'' by (simp add: lift-subst-dummy del: lift-subst)
      finally have s: s = Abs (Abs (lift u 1) ° Var 0) ° t using appL and eta by
simp
      from beta have lift u 1 => lift u' 1 by simp
      hence Abs (lift u 1) ° Var 0 => lift u' 1[Var 0/0]
        using par-beta.var ..
      hence Abs (Abs (lift u 1) ° Var 0) ° t => lift u' 1[Var 0/0][t'/0]
        using ‹t => t'› ..
      with s have s => u'[t'/0] by simp
      thus ?thesis by iprover
    next
      case (abs r)
      from ‹r →η u›
      obtain r'' where r: r => r'' and r'': r'' →η* u' by (iprover dest: beta)
      from r and beta have Abs r ° t => r''[t'/0] by simp
      moreover from r'' have r''[t'/0] →η* u'[t'/0]
        by (rule rtrancl-eta-subst')
      ultimately show ?thesis using abs and appL by simp iprover
    qed
  next
    case (appR s')
    from ‹s' →η t›
    obtain r where s': s' => r and r: r →η* t' by (iprover dest: beta)
    from s' and beta have Abs u ° s' => u'[r/0] by simp
    moreover from r have u'[r/0] →η* u'[t'/0]
      by (rule rtrancl-eta-subst'')
    ultimately show ?thesis using appR by simp iprover
  qed
qed

theorem eta-postponement':
  assumes eta: s →η* t and beta: t => u
  shows ∃ t'. s => t' ∧ t' →η* u using eta beta
```

proof (*induct arbitrary*: *u*)
  case *base*
  thus *?case* **by** *blast*
next
  case (*step* s' s'' s''')
  then obtain t' **where** s': $s' => t'$ **and** t': $t' \rightarrow_\eta{}^* s'''$
    by (*auto dest*: *eta-par-beta*)
  from s' **obtain** t'' **where** s: $s => t''$ **and** t'': $t'' \rightarrow_\eta{}^* t'$ **using** *step*
    by *blast*
  from t'' **and** t' **have** $t'' \rightarrow_\eta{}^* s'''$ **by** (*rule rtranclp-trans*)
  with *s* **show** *?case* **by** *iprover*
qed

theorem *eta-postponement*:
  assumes (*sup beta eta*)** *s t*
  shows (*beta*** *OO eta***) *s t* **using** *assms*
proof *induct*
  case *base*
  show *?case* **by** *blast*
next
  case (*step* s' s'')
  from *step*(*3*) **obtain** t' **where** s: $s \rightarrow_\beta{}^* t'$ **and** t': $t' \rightarrow_\eta{}^* s'$ **by** *blast*
  from *step*(*2*) **show** *?case*
  proof
    assume $s' \rightarrow_\beta s''$
    with *beta-subset-par-beta* **have** $s' => s''$ **..**
    with t' **obtain** t'' **where** *st*: $t' => t''$ **and** *tu*: $t'' \rightarrow_\eta{}^* s''$
      by (*auto dest*: *eta-postponement'*)
    from *par-beta-subset-beta st* **have** $t' \rightarrow_\beta{}^* t''$ **..**
    with *s* **have** $s \rightarrow_\beta{}^* t''$ **by** (*rule rtranclp-trans*)
    thus *?thesis* **using** *tu* **..**
  next
    assume $s' \rightarrow_\eta s''$
    with t' **have** $t' \rightarrow_\eta{}^* s''$ **..**
    with *s* **show** *?thesis* **..**
  qed
qed

end

5 Application of a term to a list of terms

theory *ListApplication* **imports** *Lambda* **begin**

abbreviation
  list-application :: *dB => dB list => dB* (**infixl** ‹°°› *150*) **where**
  t °° *ts* == *foldl* (°) *t ts*

lemma *apps-eq-tail-conv* [*iff*]: (*r* °° *ts* = *s* °° *ts*) = (*r* = *s*)

by (*induct ts rule*: *rev-induct*) *auto*

lemma *Var-eq-apps-conv* [*iff*]: (*Var m* = *s* °° *ss*) = (*Var m* = *s* ∧ *ss* = [])
by (*induct ss arbitrary*: *s*) *auto*

lemma *Var-apps-eq-Var-apps-conv* [*iff*]:
(*Var m* °° *rs* = *Var n* °° *ss*) = (*m* = *n* ∧ *rs* = *ss*)
apply (*induct rs arbitrary*: *ss rule*: *rev-induct*)
apply *simp*
apply *blast*
apply (*induct-tac ss rule*: *rev-induct*)
apply *auto*
done

lemma *App-eq-foldl-conv*:
(*r* ° *s* = *t* °° *ts*) =
(*if ts* = [] *then r* ° *s* = *t*
else (∃ *ss*. *ts* = *ss* @ [*s*] ∧ *r* = *t* °° *ss*))
apply (*rule-tac xs* = *ts* **in** *rev-exhaust*)
apply *auto*
done

lemma *Abs-eq-apps-conv* [*iff*]:
(*Abs r* = *s* °° *ss*) = (*Abs r* = *s* ∧ *ss* = [])
by (*induct ss rule*: *rev-induct*) *auto*

lemma *apps-eq-Abs-conv* [*iff*]: (*s* °° *ss* = *Abs r*) = (*s* = *Abs r* ∧ *ss* = [])
by (*induct ss rule*: *rev-induct*) *auto*

lemma *Abs-apps-eq-Abs-apps-conv* [*iff*]:
(*Abs r* °° *rs* = *Abs s* °° *ss*) = (*r* = *s* ∧ *rs* = *ss*)
apply (*induct rs arbitrary*: *ss rule*: *rev-induct*)
apply *simp*
apply *blast*
apply (*induct-tac ss rule*: *rev-induct*)
apply *auto*
done

lemma *Abs-App-neq-Var-apps* [*iff*]:
Abs s ° *t* ≠ *Var n* °° *ss*
by (*induct ss arbitrary*: *s t rule*: *rev-induct*) *auto*

lemma *Var-apps-neq-Abs-apps* [*iff*]:
Var n °° *ts* ≠ *Abs r* °° *ss*
apply (*induct ss arbitrary*: *ts rule*: *rev-induct*)
apply *simp*
apply (*induct-tac ts rule*: *rev-induct*)
apply *auto*
done

```
lemma ex-head-tail:
  ∃ts h. t = h °° ts ∧ ((∃n. h = Var n) ∨ (∃u. h = Abs u))
  apply (induct t)
    apply (rule-tac x = [] in exI)
    apply simp
   apply clarify
   apply (rename-tac ts1 ts2 h1 h2)
   apply (rule-tac x = ts1 @ [h2 °° ts2] in exI)
   apply simp
  apply simp
  done

lemma size-apps [simp]:
  size (r °° rs) = size r + foldl (+) 0 (map size rs) + length rs
  by (induct rs rule: rev-induct) auto

lemma lem0: [| (0::nat) < k; m <= n |] ==> m < n + k
  by simp

lemma lift-map [simp]:
    lift (t °° ts) i = lift t i °° map (λt. lift t i) ts
  by (induct ts arbitrary: t) simp-all

lemma subst-map [simp]:
    subst (t °° ts) u i = subst t u i °° map (λt. subst t u i) ts
  by (induct ts arbitrary: t) simp-all

lemma app-last: (t °° ts) ° u = t °° (ts @ [u])
  by simp
```

A customized induction schema for °°.

```
lemma lem:
  assumes !!n ts. ∀t ∈ set ts. P t ==> P (Var n °° ts)
    and !!u ts. [| P u; ∀t ∈ set ts. P t |] ==> P (Abs u °° ts)
  shows size t = n ⟹ P t
  apply (induct n arbitrary: t rule: nat-less-induct)
  apply (cut-tac t = t in ex-head-tail)
  apply clarify
  apply (erule disjE)
   apply clarify
   apply (rule assms)
   apply clarify
   apply (erule allE, erule impE)
    prefer 2
    apply (erule allE, erule mp, rule refl)
   apply simp
   apply (simp only: foldl-conv-fold add.commute fold-plus-sum-list-rev)
   apply (fastforce simp add: sum-list-map-remove1)
```

```
  apply clarify
  apply (rule assms)
   apply (erule allE, erule impE)
    prefer 2
    apply (erule allE, erule mp, rule refl)
   apply simp
  apply clarify
  apply (erule allE, erule impE)
   prefer 2
   apply (erule allE, erule mp, rule refl)
  apply simp
  apply (rule le-imp-less-Suc)
  apply (rule trans-le-add1)
  apply (rule trans-le-add2)
  apply (simp only: foldl-conv-fold add.commute fold-plus-sum-list-rev)
  apply (simp add: member-le-sum-list)
  done

theorem Apps-dB-induct:
  assumes !!n ts. ∀ t ∈ set ts. P t ==> P (Var n °° ts)
    and !!u ts. [| P u; ∀ t ∈ set ts. P t |] ==> P (Abs u °° ts)
  shows P t
  apply (rule-tac t = t in lem)
    prefer 3
    apply (rule refl)
    using assms apply iprover+
  done

end
```

6 Simply-typed lambda terms

```
theory LambdaType imports ListApplication begin
```

6.1 Environments

```
definition
  shift :: (nat ⇒ 'a) ⇒ nat ⇒ 'a ⇒ nat ⇒ 'a  (‹-⟨-:-⟩› [90, 0, 0] 91) where
  e⟨i:a⟩ = (λj. if j < i then e j else if j = i then a else e (j − 1))

lemma shift-eq [simp]: i = j ⟹ (e⟨i:T⟩) j = T
  by (simp add: shift-def)

lemma shift-gt [simp]: j < i ⟹ (e⟨i:T⟩) j = e j
  by (simp add: shift-def)

lemma shift-lt [simp]: i < j ⟹ (e⟨i:T⟩) j = e (j − 1)
  by (simp add: shift-def)
```

lemma *shift-commute* [*simp*]: $e\langle i{:}U\rangle\langle 0{:}T\rangle = e\langle 0{:}T\rangle\langle Suc\ i{:}U\rangle$
by (*rule ext*) (*simp-all add*: *shift-def split*: *nat.split*)

6.2 Types and typing rules

datatype *type* =
Atom nat
| *Fun type type* (**infixr** ‹⇒› *200*)

inductive *typing* :: (*nat* ⇒ *type*) ⇒ *dB* ⇒ *type* ⇒ *bool* (‹- ⊢ - : -› [*50*, *50*, *50*] *50*)
where
Var [*intro*!]: *env x* = *T* ⟹ *env* ⊢ *Var x* : *T*
| *Abs* [*intro*!]: *env*⟨*0*:*T*⟩ ⊢ *t* : *U* ⟹ *env* ⊢ *Abs t* : (*T* ⇒ *U*)
| *App* [*intro*!]: *env* ⊢ *s* : *T* ⇒ *U* ⟹ *env* ⊢ *t* : *T* ⟹ *env* ⊢ (*s* ° *t*) : *U*

inductive-cases *typing-elims* [*elim*!]:
e ⊢ *Var i* : *T*
e ⊢ *t* ° *u* : *T*
e ⊢ *Abs t* : *T*

primrec
typings :: (*nat* ⇒ *type*) ⇒ *dB list* ⇒ *type list* ⇒ *bool*
where
typings e [] *Ts* = (*Ts* = [])
| *typings e* (*t* # *ts*) *Ts* =
(*case Ts of*
[] ⇒ *False*
| *T* # *Ts* ⇒ *e* ⊢ *t* : *T* ∧ *typings e ts Ts*)

abbreviation
typings-rel :: (*nat* ⇒ *type*) ⇒ *dB list* ⇒ *type list* ⇒ *bool*
(‹- ⊩ - : -› [*50*, *50*, *50*] *50*) **where**
env ⊩ *ts* : *Ts* == *typings env ts Ts*

abbreviation
funs :: *type list* ⇒ *type* ⇒ *type* (**infixr** ‹⇛› *200*) **where**
Ts ⇛ *T* == *foldr Fun Ts T*

6.3 Some examples

schematic-goal *e* ⊢ *Abs* (*Abs* (*Abs* (*Var 1* ° (*Var 2* ° *Var 1* ° *Var 0*)))) : *?T*
by *force*

schematic-goal *e* ⊢ *Abs* (*Abs* (*Abs* (*Var 2* ° *Var 0* ° (*Var 1* ° *Var 0*)))) : *?T*
by *force*

6.4 Lists of types

lemma *lists-typings*:

$e \Vdash ts : Ts \Longrightarrow listsp\ (\lambda t.\ \exists\, T.\ e \vdash t : T)\ ts$

proof (*induct ts arbitrary*: *Ts*)
case *Nil*
then show *?case*
by *simp*
next
case *c*: (*Cons a ts*)
show *?case*
proof (*cases Ts*)
case *Nil*
with *c* **show** *?thesis*
by *simp*
next
case (*Cons T list*)
with *c* **show** *?thesis* **by** *force*
qed
qed

lemma *types-snoc*: $e \Vdash ts : Ts \Longrightarrow e \vdash t : T \Longrightarrow e \Vdash ts\ @\ [t] : Ts\ @\ [T]$
by (*induct ts arbitrary*: *Ts*) (*auto split*: *list.split-asm*)

lemma *types-snoc-eq*: $e \Vdash ts\ @\ [t] : Ts\ @\ [T] =$
$(e \Vdash ts : Ts \land e \vdash t : T)$
proof (*induct ts arbitrary*: *Ts*)
case *Nil*
then show *?case*
by (*auto split*: *list.split*)
next
case (*Cons a ts*)
have $\neg\ e \Vdash ts\ @\ [t] : []$
by (*cases ts* @ [*t*]; *simp*)
with *Cons* **show** *?case*
by (*auto split*: *list.split*)
qed

Cannot use *rev-exhaust* from the *List* theory, since it is not constructive

lemma *rev-exhaust2* [*extraction-expand*]:
obtains (*Nil*) $xs = []$ | (*snoc*) *ys y* **where** $xs = ys\ @\ [y]$
proof −
have §: $xs = rev\ ys \Longrightarrow thesis$ **for** *ys*
by (*cases ys*) (*simp-all add*: *local.Nil snoc*)
show *thesis*
using § [*of rev xs*] **by** *simp*
qed

lemma *types-snocE*:
assumes ‹$e \Vdash ts\ @\ [t] : Ts$›
obtains *Us* **and** *U* **where** ‹$Ts = Us\ @\ [U]$› ‹$e \Vdash ts : Us$› ‹$e \vdash t : U$›
proof (*cases Ts rule*: *rev-exhaust2*)

case *Nil*
with *assms* **show** *?thesis*
by (*cases ts* @ [*t*]) *simp-all*
next
case (*snoc Us U*)
with *assms* **have** $e \Vdash ts$ @ $[t] : Us$ @ $[U]$ **by** *simp*
then have $e \Vdash ts : Us$ $e \vdash t : U$ **by** (*simp-all add*: *types-snoc-eq*)
with *snoc* **show** *?thesis* **by** (*rule that*)
qed

6.5 n-ary function types

lemma *list-app-typeD*:
$e \vdash t \circ\circ ts : T \Longrightarrow \exists Ts.\ e \vdash t : Ts \Rrightarrow T \wedge e \Vdash ts : Ts$
proof (*induct ts arbitrary*: *t T*)
case *Nil*
then show *?case* **by** *auto*
next
case (*Cons a b t T*)
then show *?case*
by (*auto simp*: *split*: *list.split*)
qed

lemma *list-app-typeE*:
$e \vdash t \circ\circ ts : T \Longrightarrow (\bigwedge Ts.\ e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow C) \Longrightarrow C$
using *list-app-typeD* **by** *iprover*

lemma *list-app-typeI*:
$e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow e \vdash t \circ\circ ts : T$
by (*induct ts arbitrary*: *t Ts*) (*auto simp add*: *split*: *list.split-asm*)

For the specific case where the head of the term is a variable, the following theorems allow to infer the types of the arguments without analyzing the typing derivation. This is crucial for program extraction.

theorem *var-app-type-eq*:
$e \vdash Var\ i \circ\circ ts : T \Longrightarrow e \vdash Var\ i \circ\circ ts : U \Longrightarrow T = U$
by (*induct ts arbitrary*: *T U rule*: *rev-induct*) *auto*

lemma *var-app-types*: $e \vdash Var\ i \circ\circ ts \circ\circ us : T \Longrightarrow e \Vdash ts : Ts \Longrightarrow$
$e \vdash Var\ i \circ\circ ts : U \Longrightarrow \exists Us.\ U = Us \Rrightarrow T \wedge e \Vdash us : Us$
proof (*induct us arbitrary*: *ts Ts U*)
case *Nil*
then show *?case*
by (*simp add*: *var-app-type-eq*)
next
case (*Cons a b ts Ts U*)
then show *?case*
apply *atomize*
apply (*case-tac U*)

apply (*rule FalseE*)
apply *simp*
apply (*erule list-app-typeE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
apply (*rename-tac nat Ts′ T′*)
apply (*drule-tac* T=*Atom nat* **and** $U = T' \Rightarrow Ts' \Rrightarrow T$ **in** *var-app-type-eq*)
apply *assumption*
apply *simp*
apply (*rename-tac type1 type2*)
apply (*erule-tac x=ts* @ [*a*] **in** *allE*)
apply (*erule-tac x=Ts* @ [*type1*] **in** *allE*)
apply (*erule-tac x=type2* **in** *allE*)
apply *simp*
apply (*erule impE*)
apply (*rule types-snoc*)
apply *assumption*
apply (*erule list-app-typeE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
using *var-app-type-eq* **apply** *fastforce*
apply (*erule impE*)
apply (*rule typing.App*)
apply *assumption*
apply (*erule list-app-typeE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
using *var-app-type-eq* **apply** *fastforce*
apply (*erule exE*)
apply (*rule-tac x=type1 # Us* **in** *exI*)
apply *simp*
apply (*erule list-app-typeE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
using *var-app-type-eq* **by** *fastforce*
qed

lemma *var-app-typesE*: $e \vdash Var\ i \circ\circ ts : T \Longrightarrow$
$(\bigwedge Ts.\ e \vdash Var\ i : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow P) \Longrightarrow P$
by (*iprover intro*: *typing.Var dest*: *var-app-types* [*of* - - [], *simplified*])

lemma *abs-typeE*:
assumes $e \vdash Abs\ t : T \bigwedge U\ V.\ e\langle 0{:}U\rangle \vdash t : V \Longrightarrow P$
shows P
proof (*cases T*)
case (*Atom x1*)
with *assms*(*1*) **show** *?thesis*
apply−
apply (*rule FalseE*)
apply (*erule typing.cases*)
apply *simp-all*
done
next

case (*Fun type1 type2*)
with *assms* **show** *?thesis*
apply *atomize*
apply (*erule-tac x=type1* **in** *allE*)
apply (*erule-tac x=type2* **in** *allE*)
apply (*erule mp*)
apply (*erule typing.cases*)
apply *simp-all*
done
qed

6.6 Lifting preserves well-typedness

lemma *lift-type* [*intro*!]: $e \vdash t : T \Longrightarrow e\langle i{:}U\rangle \vdash \mathit{lift}\ t\ i : T$
by (*induct arbitrary*: *i U set*: *typing*) *auto*

lemma *lift-types*:
$e \Vdash ts : Ts \Longrightarrow e\langle i{:}U\rangle \Vdash (\mathit{map}\ (\lambda t.\ \mathit{lift}\ t\ i)\ ts) : Ts$
by (*induct ts arbitrary*: *Ts*) (*auto split*: *list.split*)

6.7 Substitution lemmas

lemma *subst-lemma*:
$e \vdash t : T \Longrightarrow e' \vdash u : U \Longrightarrow e = e'\langle i{:}U\rangle \Longrightarrow e' \vdash t[u/i] : T$
proof (*induct arbitrary*: *e' i U u set*: *typing*)
case (*Var env x T*)
then show *?case*
by (*force simp add*: *shift-def*)
next
case (*Abs env T t U*)
then show *?case* **by** *force*
qed *auto*

lemma *substs-lemma*:
$e \vdash u : T \Longrightarrow e\langle i{:}T\rangle \Vdash ts : Ts \Longrightarrow$
$e \Vdash (\mathit{map}\ (\lambda t.\ t[u/i])\ ts) : Ts$
proof (*induct ts arbitrary*: *Ts*)
case *Nil*
then show *?case*
by *auto*
next
case (*Cons a ts*)
with *subst-lemma* **show** *?case*
by (*auto split*: *list.split*)
qed

6.8 Subject reduction

lemma *subject-reduction*: $e \vdash t : T \Longrightarrow t \to_\beta t' \Longrightarrow e \vdash t' : T$
proof (*induct arbitrary*: *t' set*: *typing*)

case (*App env s T U t*)
with *subst-lemma* **show** *?case*
by *auto*
qed *auto*

theorem *subject-reduction'*: $t \to_\beta{}^* t' \Longrightarrow e \vdash t : T \Longrightarrow e \vdash t' : T$
by (*induct set*: *rtranclp*) (*iprover intro*: *subject-reduction*)+

6.9 Alternative induction rule for types

lemma *type-induct* [*induct type*]:
assumes
$(\bigwedge T.\ (\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T1) \Longrightarrow$
$(\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T2) \Longrightarrow P\ T)$
shows *P T*
proof (*induct T*)
case *Atom*
show *?case* **by** (*rule assms*) *simp-all*
next
case *Fun*
show *?case* **by** (*rule assms*) (*insert Fun*, *simp-all*)
qed

end

7 Lifting an order to lists of elements

theory *ListOrder*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

Lifting an order to lists of elements, relating exactly one element.

definition
step1 :: (*'a* => *'a* => *bool*) => *'a list* => *'a list* => *bool* **where**
step1 r =
$(\lambda ys\ xs.\ \exists us\ z\ z'\ vs.\ xs = us\ @\ z\ \#\ vs \wedge r\ z'\ z \wedge ys =$
$us\ @\ z'\ \#\ vs)$

lemma *step1-converse* [*simp*]: $step1\ (r^{-1-1}) = (step1\ r)^{-1-1}$
apply (*unfold step1-def*)
apply (*blast intro*!: *order-antisym*)
done

lemma *in-step1-converse* [*iff*]: $(step1\ (r^{-1-1})\ x\ y) = ((step1\ r)^{-1-1}\ x\ y)$
apply *auto*
done

```
lemma not-Nil-step1 [iff]: ¬ step1 r [] xs
  apply (unfold step1-def)
  apply blast
  done

lemma not-step1-Nil [iff]: ¬ step1 r xs []
  apply (unfold step1-def)
  apply blast
  done

lemma Cons-step1-Cons [iff]:
    (step1 r (y # ys) (x # xs)) =
      (r y x ∧ xs = ys ∨ x = y ∧ step1 r ys xs)
  apply (unfold step1-def)
  apply (rule iffI)
   apply (erule exE)
   apply (rename-tac ts)
   apply (case-tac ts)
    apply fastforce
   apply force
  apply (erule disjE)
   apply blast
  apply (blast intro: Cons-eq-appendI)
  done

lemma append-step1I:
  step1 r ys xs ∧ vs = us ∨ ys = xs ∧ step1 r vs us
    ==> step1 r (ys @ vs) (xs @ us)
  apply (unfold step1-def)
  apply auto
   apply blast
  apply (blast intro: append-eq-appendI)
  done

lemma Cons-step1E [elim!]:
  assumes step1 r ys (x # xs)
    and !!y. ys = y # xs ⟹ r y x ⟹ R
    and !!zs. ys = x # zs ⟹ step1 r zs xs ⟹ R
  shows R
  using assms
  apply (cases ys)
   apply (simp add: step1-def)
  apply blast
  done

lemma Snoc-step1-SnocD:
  step1 r (ys @ [y]) (xs @ [x])
    ==> (step1 r ys xs ∧ y = x ∨ ys = xs ∧ r y x)
```

apply (*unfold step1-def*)
apply (*clarify del*: *disjCI*)
apply (*rename-tac vs*)
apply (*rule-tac xs* = *vs* **in** *rev-exhaust*)
apply *force*
apply *simp*
apply *blast*
done

lemma *Cons-acc-step1I* [*intro*!]:
Wellfounded.accp r x ==> *Wellfounded.accp* (*step1 r*) *xs* $\Longrightarrow$ *Wellfounded.accp* (*step1 r*) (*x* # *xs*)
apply (*induct arbitrary*: *xs set*: *Wellfounded.accp*)
apply (*erule thin-rl*)
apply (*erule accp-induct*)
apply (*rule accp.accI*)
apply *blast*
done

lemma *lists-accD*: *listsp* (*Wellfounded.accp r*) *xs* ==> *Wellfounded.accp* (*step1 r*) *xs*
apply (*induct set*: *listsp*)
apply (*rule accp.accI*)
apply *simp*
apply (*rule accp.accI*)
apply (*fast dest*: *accp-downward*)
done

lemma *ex-step1I*:
[| $x \in$ *set xs*; *r y x* |]
==> $\exists$ *ys*. *step1 r ys xs* $\wedge$ $y \in$ *set ys*
apply (*unfold step1-def*)
apply (*drule in-set-conv-decomp* [*THEN iffD1*])
apply *force*
done

lemma *lists-accI*: *Wellfounded.accp* (*step1 r*) *xs* ==> *listsp* (*Wellfounded.accp r*) *xs*
apply (*induct set*: *Wellfounded.accp*)
apply *clarify*
apply (*rule accp.accI*)
apply (*drule-tac r*=*r* **in** *ex-step1I*, *assumption*)
apply *blast*
done

end

8 Lifting beta-reduction to lists

theory *ListBeta* **imports** *ListApplication ListOrder* **begin**

Lifting beta-reduction to lists of terms, reducing exactly one element.

abbreviation
  list-beta :: *dB list => dB list => bool* (**infixl** ‹=>› *50*) **where**
  rs => ss == step1 beta rs ss

lemma *head-Var-reduction*:
  $Var\ n\ ^{\circ\circ}\ rs \rightarrow_\beta v \Longrightarrow \exists ss.\ rs => ss \wedge v = Var\ n\ ^{\circ\circ}\ ss$
  apply (*induct u == Var n °° rs v arbitrary*: *rs set*: *beta*)
     apply *simp*
    apply (*rule-tac xs = rs* **in** *rev-exhaust*)
     apply *simp*
    apply (*atomize*, *force intro*: *append-step1I*)
   apply (*rule-tac xs = rs* **in** *rev-exhaust*)
    apply *simp*
    apply (*auto 0 3 intro*: *disjI2* [*THEN append-step1I*])
  done

lemma *apps-betasE* [*elim*!]:
  assumes *major*: $r\ ^{\circ\circ}\ rs \rightarrow_\beta s$
    and *cases*: !!*r′*. [| $r \rightarrow_\beta r'$; *s = r′ °° rs* |] ==> *R*
      !!*rs′*. [| *rs => rs′*; *s = r °° rs′* |] ==> *R*
      !!*t u us*. [| *r = Abs t*; *rs = u # us*; *s = t*[*u/0*] °° *us* |] ==> *R*
  shows *R*
proof −
  from *major* **have**
   (∃ *r′*. $r \rightarrow_\beta r'$ ∧ *s = r′ °° rs*) ∨
    (∃ *rs′*. *rs => rs′* ∧ *s = r °° rs′*) ∨
    (∃ *t u us*. *r = Abs t* ∧ *rs = u # us* ∧ *s = t*[*u/0*] °° *us*)
    apply (*induct u == r °° rs s arbitrary*: *r rs set*: *beta*)
       apply (*case-tac r*)
         apply *simp*
        apply (*simp add*: *App-eq-foldl-conv*)
        apply (*split if-split-asm*)
         apply *simp*
         apply *blast*
        apply *simp*
       apply (*simp add*: *App-eq-foldl-conv*)
       apply (*split if-split-asm*)
        apply *simp*
       apply *simp*
      apply (*drule App-eq-foldl-conv* [*THEN iffD1*])
      apply (*split if-split-asm*)
       apply *simp*
       apply *blast*
      apply (*force intro*!: *disjI1* [*THEN append-step1I*])

```
    apply (drule App-eq-foldl-conv [THEN iffD1])
    apply (split if-split-asm)
     apply simp
     apply blast
    apply (clarify, auto 0 3 intro!: exI intro: append-step1I)
   done
  with cases show ?thesis by blast
qed

lemma apps-preserves-beta [simp]:
    r →β s ==> r °° ss →β s °° ss
  by (induct ss rule: rev-induct) auto

lemma apps-preserves-beta2 [simp]:
    r →β* s ==> r °° ss →β* s °° ss
  apply (induct set: rtranclp)
   apply blast
  apply (blast intro: apps-preserves-beta rtranclp.rtrancl-into-rtrancl)
  done

lemma apps-preserves-betas [simp]:
    rs => ss ⟹ r °° rs →β r °° ss
  apply (induct rs arbitrary: ss rule: rev-induct)
   apply simp
  apply simp
  apply (rule-tac xs = ss in rev-exhaust)
   apply simp
  apply simp
  apply (drule Snoc-step1-SnocD)
  apply blast
  done

end
```

9 Inductive characterization of terminating lambda terms

```
theory InductTermi imports ListBeta begin
```

9.1 Terminating lambda terms

```
inductive IT :: dB => bool
  where
    Var [intro]: listsp IT rs ==> IT (Var n °° rs)
  | Lambda [intro]: IT r ==> IT (Abs r)
  | Beta [intro]: IT ((r[s/0]) °° ss) ==> IT s ==> IT ((Abs r ° s) °° ss)
```

9.2 Every term in *IT* terminates

lemma *double-induction-lemma* [*rule-format*]:
termip beta s ==> ∀ *t*. *termip beta t* −−>
(∀ *r ss*. *t* = *r*[*s*/*0*] °° *ss* −−> *termip beta* (*Abs r* ° *s* °° *ss*))
apply (*erule accp-induct*)
apply (*rule allI*)
apply (*rule impI*)
apply (*erule thin-rl*)
apply (*erule accp-induct*)
apply *clarify*
apply (*rule accp.accI*)
apply (*safe elim*!: *apps-betasE*)
apply (*blast intro*: *subst-preserves-beta apps-preserves-beta*)
apply (*blast intro*: *apps-preserves-beta2 subst-preserves-beta2 rtranclp-converseI*
dest: *accp-downwards*)
apply (*blast dest*: *apps-preserves-betas*)
done

lemma *IT-implies-termi*: *IT t* ==> *termip beta t*
apply (*induct set*: *IT*)
apply (*drule rev-predicate1D* [*OF - listsp-mono* [**where** *B*=*termip beta*]])
apply (*fast intro*!: *predicate1I*)
apply (*drule lists-accD*)
apply (*erule accp-induct*)
apply (*rule accp.accI*)
apply (*blast dest*: *head-Var-reduction*)
apply (*erule accp-induct*)
apply (*rule accp.accI*)
apply *blast*
apply (*blast intro*: *double-induction-lemma*)
done

9.3 Every terminating term is in *IT*

declare *Var-apps-neq-Abs-apps* [*symmetric*, *simp*]

lemma [*simp*, *THEN not-sym*, *simp*]: *Var n* °° *ss* ≠ *Abs r* ° *s* °° *ts*
by (*simp add*: *foldl-Cons* [*symmetric*] *del*: *foldl-Cons*)

lemma [*simp*]:
(*Abs r* ° *s* °° *ss* = *Abs r*′ ° *s*′ °° *ss*′) = (*r* = *r*′ ∧ *s* = *s*′ ∧ *ss* = *ss*′)
by (*simp add*: *foldl-Cons* [*symmetric*] *del*: *foldl-Cons*)

inductive-cases [*elim*!]:
IT (*Var n* °° *ss*)
IT (*Abs t*)
IT (*Abs r* ° *s* °° *ts*)

theorem *termi-implies-IT*: *termip beta r* ==> *IT r*

apply (*erule accp-induct*)
apply (*rename-tac r*)
apply (*erule thin-rl*)
apply (*erule rev-mp*)
apply *simp*
apply (*rule-tac t = r* **in** *Apps-dB-induct*)
apply *clarify*
apply (*rule IT.intros*)
apply *clarify*
apply (*drule bspec*, *assumption*)
apply (*erule mp*)
apply *clarify*
apply (*drule-tac r=beta* **in** *conversepI*)
apply (*drule-tac r=beta*$^{-1-1}$ **in** *ex-step1I*, *assumption*)
apply *clarify*
apply (*rename-tac us*)
apply (*erule-tac x = Var n °° us* **in** *allE*)
apply *force*
apply (*rename-tac u ts*)
apply (*case-tac ts*)
apply *simp*
apply *blast*
apply (*rename-tac s ss*)
apply *simp*
apply *clarify*
apply (*rule IT.intros*)
apply (*blast intro*: *apps-preserves-beta*)
apply (*erule mp*)
apply *clarify*
apply (*rename-tac t*)
apply (*erule-tac x = Abs u ° t °° ss* **in** *allE*)
apply *force*
done

end

10 Strong normalization for simply-typed lambda calculus

theory *StrongNorm* **imports** *LambdaType InductTermi* **begin**

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

10.1 Properties of *IT*

lemma *lift-IT* [*intro*!]: *IT t* $\Longrightarrow$ *IT* (*lift t i*)
apply (*induct arbitrary*: *i set*: *IT*)

apply (*simp* (*no-asm*))
apply (*rule conjI*)
apply
(*rule impI*,
rule IT.Var,
erule listsp.induct,
simp (*no-asm*),
simp (*no-asm*),
rule listsp.Cons,
blast,
assumption)+
apply *auto*
done

lemma *lifts-IT*: *listsp IT ts* $\Longrightarrow$ *listsp IT* (*map* (λt. *lift t 0*) *ts*)
by (*induct ts*) *auto*

lemma *subst-Var-IT*: *IT r* $\Longrightarrow$ *IT* (*r*[*Var i*/*j*])
apply (*induct arbitrary*: *i j set*: *IT*)

Case *Var*:

apply (*simp* (*no-asm*) *add*: *subst-Var*)
apply
((*rule conjI impI*)+,
rule IT.Var,
erule listsp.induct,
simp (*no-asm*),
simp (*no-asm*),
rule listsp.Cons,
fast,
assumption)+

Case *Lambda*:

apply *atomize*
apply *simp*
apply (*rule IT.Lambda*)
apply *fast*

Case *Beta*:

apply *atomize*
apply (*simp* (*no-asm-use*) *add*: *subst-subst* [*symmetric*])
apply (*rule IT.Beta*)
apply *auto*
done

lemma *Var-IT*: *IT* (*Var n*)
apply (*subgoal-tac IT* (*Var n* °° []))
apply *simp*
apply (*rule IT.Var*)

  apply (*rule listsp.Nil*)
  done

lemma *app-Var-IT*: *IT t* $\Longrightarrow$ *IT* (*t* ° *Var i*)
  apply (*induct set*: *IT*)
    apply (*subst app-last*)
    apply (*rule IT.Var*)
    apply *simp*
    apply (*rule listsp.Cons*)
     apply (*rule Var-IT*)
    apply (*rule listsp.Nil*)
   apply (*rule IT.Beta* [**where** *?ss* = [], *unfolded foldl-Nil* [*THEN eq-reflection*]])
    apply (*erule subst-Var-IT*)
   apply (*rule Var-IT*)
  apply (*subst app-last*)
  apply (*rule IT.Beta*)
   apply (*subst app-last* [*symmetric*])
   apply *assumption*
  apply *assumption*
  done

10.2 Well-typed substitution preserves termination

lemma *subst-type-IT*:
  $\bigwedge t\ e\ T\ u\ i.\ IT\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow$
    $IT\ u \Longrightarrow e \vdash u : U \Longrightarrow IT\ (t[u/i])$
  (**is** *PROP ?P U* **is** $\bigwedge t\ e\ T\ u\ i.$ - $\Longrightarrow$ *PROP ?Q t e T u i U*)
proof (*induct U*)
  fix *T t*
  assume *MI1*: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow$ *PROP ?P T1*
  assume *MI2*: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow$ *PROP ?P T2*
  assume *IT t*
  thus $\bigwedge e\ T'\ u\ i.$ *PROP ?Q t e T′ u i T*
  proof *induct*
    fix *e T′ u i*
    assume *uIT*: *IT u*
    assume *uT*: $e \vdash u : T$
    {
      case (*Var rs n e1 T′1 u1 i1*)
      assume *nT*: $e\langle i{:}T\rangle \vdash Var\ n\ {}^{\circ\circ}\ rs : T'$
      let $?ty = \lambda t.\ \exists T'.\ e\langle i{:}T\rangle \vdash t : T'$
      let $?R = \lambda t.\ \forall e\ T'\ u\ i.$
        $e\langle i{:}T\rangle \vdash t : T' \longrightarrow IT\ u \longrightarrow e \vdash u : T \longrightarrow IT\ (t[u/i])$
      show *IT* $((Var\ n\ {}^{\circ\circ}\ rs)[u/i])$
      proof (*cases n* = *i*)
        case *True*
        show *?thesis*
        proof (*cases rs*)
          case *Nil*

```
  with uIT True show ?thesis by simp
next
  case (Cons a as)
  with nT have e⟨i:T⟩ ⊢ Var n ° a °° as : T′ by simp
  then obtain Ts
      where headT: e⟨i:T⟩ ⊢ Var n ° a : Ts ⇛ T′
      and argsT: e⟨i:T⟩ ⊩ as : Ts
    by (rule list-app-typeE)
  from headT obtain T′′
      where varT: e⟨i:T⟩ ⊢ Var n : T′′ ⇒ Ts ⇛ T′
      and argT: e⟨i:T⟩ ⊢ a : T′′
    by cases simp-all
  from varT True have T: T = T′′ ⇒ Ts ⇛ T′
    by cases auto
  with uT have uT′: e ⊢ u : T′′ ⇒ Ts ⇛ T′ by simp
  from T have IT ((Var 0 °° map (λt. lift t 0)
    (map (λt. t[u/i]) as))[(u ° a[u/i])/0])
  proof (rule MI2)
    from T have IT ((lift u 0 ° Var 0)[a[u/i]/0])
    proof (rule MI1)
      have IT (lift u 0) by (rule lift-IT [OF uIT])
      thus IT (lift u 0 ° Var 0) by (rule app-Var-IT)
      show e⟨0:T′′⟩ ⊢ lift u 0 ° Var 0 : Ts ⇛ T′
      proof (rule typing.App)
        show e⟨0:T′′⟩ ⊢ lift u 0 : T′′ ⇒ Ts ⇛ T′
          by (rule lift-type) (rule uT′)
        show e⟨0:T′′⟩ ⊢ Var 0 : T′′
          by (rule typing.Var) simp
      qed
      from Var have ?R a by cases (simp-all add: Cons)
      with argT uIT uT show IT (a[u/i]) by simp
      from argT uT show e ⊢ a[u/i] : T′′
        by (rule subst-lemma) simp
    qed
    thus IT (u ° a[u/i]) by simp
    from Var have listsp ?R as
      by cases (simp-all add: Cons)
    moreover from argsT have listsp ?ty as
      by (rule lists-typings)
    ultimately have listsp (λt. ?R t ∧ ?ty t) as
      by simp
    hence listsp IT (map (λt. lift t 0) (map (λt. t[u/i]) as))
      (is listsp IT (?ls as))
    proof induct
      case Nil
      show ?case by fastforce
    next
      case (Cons b bs)
      hence I: ?R b by simp
```

```
        from Cons obtain U where e⟨i:T⟩ ⊢ b : U by fast
        with uT uIT I have IT (b[u/i]) by simp
        hence IT (lift (b[u/i]) 0) by (rule lift-IT)
        hence listsp IT (lift (b[u/i]) 0 # ?ls bs)
          by (rule listsp.Cons) (rule Cons)
        thus ?case by simp
      qed
      thus IT (Var 0 °° ?ls as) by (rule IT.Var)
      have e⟨0:Ts ⇛ T'⟩ ⊢ Var 0 : Ts ⇛ T'
        by (rule typing.Var) simp
      moreover from uT argsT have e ⊩ map (λt. t[u/i]) as : Ts
        by (rule substs-lemma)
      hence e⟨0:Ts ⇛ T'⟩ ⊩ ?ls as : Ts
        by (rule lift-types)
      ultimately show e⟨0:Ts ⇛ T'⟩ ⊢ Var 0 °° ?ls as : T'
        by (rule list-app-typeI)
      from argT uT have e ⊢ a[u/i] : T''
        by (rule subst-lemma) (rule refl)
      with uT' show e ⊢ u ° a[u/i] : Ts ⇛ T'
        by (rule typing.App)
    qed
    with Cons True show ?thesis
      by (simp add: comp-def)
  qed
next
  case False
  from Var have listsp ?R rs by simp
  moreover from nT obtain Ts where e⟨i:T⟩ ⊩ rs : Ts
    by (rule list-app-typeE)
  hence listsp ?ty rs by (rule lists-typings)
  ultimately have listsp (λt. ?R t ∧ ?ty t) rs
    by simp
  hence listsp IT (map (λx. x[u/i]) rs)
  proof induct
    case Nil
    show ?case by fastforce
  next
    case (Cons a as)
    hence I: ?R a by simp
    from Cons obtain U where e⟨i:T⟩ ⊢ a : U by fast
    with uT uIT I have IT (a[u/i]) by simp
    hence listsp IT (a[u/i] # map (λt. t[u/i]) as)
      by (rule listsp.Cons) (rule Cons)
    thus ?case by simp
  qed
  with False show ?thesis by (auto simp add: subst-Var)
qed
next
  case (Lambda r e1 T'1 u1 i1)
```

assume $e\langle i{:}T\rangle \vdash Abs\ r : T'$
and $\bigwedge e\ T'\ u\ i.\ PROP\ ?Q\ r\ e\ T'\ u\ i\ T$
with *uIT uT* **show** $IT\ (Abs\ r[u/i])$
by *fastforce*
next
case $(Beta\ r\ a\ as\ e1\ T'1\ u1\ i1)$
assume T: $e\langle i{:}T\rangle \vdash Abs\ r \circ a \circ\circ as : T'$
assume $SI1$: $\bigwedge e\ T'\ u\ i.\ PROP\ ?Q\ (r[a/0] \circ\circ as)\ e\ T'\ u\ i\ T$
assume $SI2$: $\bigwedge e\ T'\ u\ i.\ PROP\ ?Q\ a\ e\ T'\ u\ i\ T$
have $IT\ (Abs\ (r[lift\ u\ 0/Suc\ i]) \circ a[u/i] \circ\circ map\ (\lambda t.\ t[u/i])\ as)$
proof (*rule IT.Beta*)
have $Abs\ r \circ a \circ\circ as \rightarrow_\beta r[a/0] \circ\circ as$
by (*rule apps-preserves-beta*) (*rule beta.beta*)
with T **have** $e\langle i{:}T\rangle \vdash r[a/0] \circ\circ as : T'$
by (*rule subject-reduction*)
hence $IT\ ((r[a/0] \circ\circ as)[u/i])$
using *uIT uT* **by** (*rule SI1*)
thus $IT\ (r[lift\ u\ 0/Suc\ i][a[u/i]/0] \circ\circ map\ (\lambda t.\ t[u/i])\ as)$
by (*simp del*: *subst-map add*: *subst-subst subst-map* [*symmetric*])
from T **obtain** U **where** $e\langle i{:}T\rangle \vdash Abs\ r \circ a : U$
by (*rule list-app-typeE*) *fast*
then obtain T'' **where** $e\langle i{:}T\rangle \vdash a : T''$ **by** *cases simp-all*
thus $IT\ (a[u/i])$ **using** *uIT uT* **by** (*rule SI2*)
qed
thus $IT\ ((Abs\ r \circ a \circ\circ as)[u/i])$ **by** *simp*
}
qed
qed

10.3 Well-typed terms are strongly normalizing

lemma *type-implies-IT*:
assumes $e \vdash t : T$
shows $IT\ t$
using *assms*
proof *induct*
case *Var*
show *?case* **by** (*rule Var-IT*)
next
case *Abs*
show *?case* **by** (*rule IT.Lambda*) (*rule Abs*)
next
case $(App\ e\ s\ T\ U\ t)$
have $IT\ ((Var\ 0 \circ lift\ t\ 0)[s/0])$
proof (*rule subst-type-IT*)
have $IT\ (lift\ t\ 0)$ **using** ‹$IT\ t$› **by** (*rule lift-IT*)
hence $listsp\ IT\ [lift\ t\ 0]$ **by** (*rule listsp.Cons*) (*rule listsp.Nil*)
hence $IT\ (Var\ 0 \circ\circ [lift\ t\ 0])$ **by** (*rule IT.Var*)
also have $Var\ 0 \circ\circ [lift\ t\ 0] = Var\ 0 \circ lift\ t\ 0$ **by** *simp*

    finally show *IT* . . . .
    have $e\langle 0{:}T \Rightarrow U\rangle \vdash Var\ 0 : T \Rightarrow U$
      by (*rule typing.Var*) *simp*
    moreover have $e\langle 0{:}T \Rightarrow U\rangle \vdash lift\ t\ 0 : T$
      by (*rule lift-type*) (*rule App.hyps*)
    ultimately show $e\langle 0{:}T \Rightarrow U\rangle \vdash Var\ 0 \circ lift\ t\ 0 : U$
      by (*rule typing.App*)
    show *IT s* **by** *fact*
    show $e \vdash s : T \Rightarrow U$ **by** *fact*
  qed
  thus *?case* **by** *simp*
qed

theorem *type-implies-termi*: $e \vdash t : T \Longrightarrow termip\ beta\ t$
proof −
  assume $e \vdash t : T$
  hence *IT t* **by** (*rule type-implies-IT*)
  thus *?thesis* **by** (*rule IT-implies-termi*)
qed

end

11 Inductive characterization of lambda terms in normal form

theory *NormalForm*
imports *ListBeta*
begin

11.1 Terms in normal form

definition
  $listall :: ('a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow bool$ **where**
  $listall\ P\ xs \equiv (\forall i.\ i < length\ xs \longrightarrow P\ (xs\ !\ i))$

declare *listall-def* [*extraction-expand-def*]

theorem *listall-nil*: *listall P* []
  by (*simp add*: *listall-def*)

theorem *listall-nil-eq* [*simp*]: *listall P* [] = *True*
  by (*iprover intro*: *listall-nil*)

theorem *listall-cons*: $P\ x \Longrightarrow listall\ P\ xs \Longrightarrow listall\ P\ (x\ \#\ xs)$
  apply (*simp add*: *listall-def*)
  apply (*rule allI impI*)+
  apply (*case-tac i*)
  apply *simp*+

done

theorem *listall-cons-eq* [*simp*]: *listall P* (*x* # *xs*) = (*P x* ∧ *listall P xs*)
apply (*rule iffI*)
prefer *2*
apply (*erule conjE*)
apply (*erule listall-cons*)
apply *assumption*
apply (*unfold listall-def*)
apply (*rule conjI*)
apply (*erule-tac x=0* **in** *allE*)
apply *simp*
apply *simp*
apply (*rule allI*)
apply (*erule-tac x=Suc i* **in** *allE*)
apply *simp*
done

lemma *listall-conj1*: *listall* (λ*x*. *P x* ∧ *Q x*) *xs* ⟹ *listall P xs*
by (*induct xs*) *simp-all*

lemma *listall-conj2*: *listall* (λ*x*. *P x* ∧ *Q x*) *xs* ⟹ *listall Q xs*
by (*induct xs*) *simp-all*

lemma *listall-app*: *listall P* (*xs* @ *ys*) = (*listall P xs* ∧ *listall P ys*)
by (*induct xs*; *intro iffI*; *simp*)

lemma *listall-snoc* [*simp*]: *listall P* (*xs* @ [*x*]) = (*listall P xs* ∧ *P x*)
by (*rule iffI*; *simp add*: *listall-app*)

lemma *listall-cong* [*cong*, *extraction-expand*]:
xs = *ys* ⟹ *listall P xs* = *listall P ys*
— Currently needed for strange technical reasons
by (*unfold listall-def*) *simp*

listsp is equivalent to *listall*, but cannot be used for program extraction.

lemma *listall-listsp-eq*: *listall P xs* = *listsp P xs*
by (*induct xs*) (*auto intro*: *listsp.intros*)

inductive *NF* :: *dB* ⇒ *bool*
where
App: *listall NF ts* ⟹ *NF* (*Var x* °° *ts*)
| *Abs*: *NF t* ⟹ *NF* (*Abs t*)
monos *listall-def*

lemma *nat-eq-dec*: ⋀*n*::*nat*. *m* = *n* ∨ *m* ≠ *n*
proof (*induction m*)
case *0*
then show *?case*

    by (*cases n*; *simp only*: *nat.simps*; *iprover*)
next
  case (*Suc m*)
  then show *?case*
    by (*cases n*; *simp only*: *nat.simps*; *iprover*)
qed

lemma *nat-le-dec*: $\bigwedge n::nat.\ m < n \vee \neg\ (m < n)$
proof (*induction m*)
  case *0*
  then show *?case*
    by (*cases n*; *simp only*: *order.irrefl zero-less-Suc*; *iprover*)
next
  case (*Suc m*)
  then show *?case*
    by (*cases n*; *simp only*: *not-less-zero Suc-less-eq*; *iprover*)
qed

lemma *App-NF-D*: **assumes** *NF*: *NF* (*Var n* °° *ts*)
  shows *listall NF ts* **using** *NF*
  by *cases simp-all*

11.2 Properties of *NF*

lemma *Var-NF*: *NF* (*Var n*)
proof −
  have *NF* (*Var n* °° [])
    by (*rule NF.App*) *simp*
  then show *?thesis* **by** *simp*
qed

lemma *Abs-NF*:
  assumes *NF*: *NF* (*Abs t* °° *ts*)
  shows *ts* = [] **using** *NF*
proof *cases*
  case (*App us i*)
  thus *?thesis* **by** (*simp add*: *Var-apps-neq-Abs-apps* [*THEN not-sym*])
next
  case (*Abs u*)
  thus *?thesis* **by** *simp*
qed

lemma *subst-terms-NF*: *listall NF ts* $\Longrightarrow$
    listall ($\lambda t.\ \forall\, i\ j.\ NF\ (t[Var\ i/j])$) *ts* $\Longrightarrow$
    listall NF (*map* ($\lambda t.\ t[Var\ i/j]$) *ts*)
  by (*induct ts*) *simp-all*

lemma *subst-Var-NF*: *NF t* $\Longrightarrow$ *NF* ($t[Var\ i/j]$)
  apply (*induct arbitrary*: *i j set*: *NF*)

apply *simp*
apply (*frule listall-conj1*)
apply (*drule listall-conj2*)
apply (*drule-tac i=i* **and** *j=j* **in** *subst-terms-NF*)
apply *assumption*
apply (*rule-tac m1=x* **and** *n1=j* **in** *nat-eq-dec* [*THEN disjE*])
apply *simp*
apply (*erule NF.App*)
apply (*rule-tac m1=j* **and** *n1=x* **in** *nat-le-dec* [*THEN disjE*])
apply (*simp-all add*: *NF.App NF.Abs*)
done

lemma *app-Var-NF*: $NF\ t \Longrightarrow \exists\, t'.\ t \circ Var\ i \to_\beta{}^* t' \wedge NF\ t'$
apply (*induct set*: *NF*)
apply (*simplesubst app-last*) — Using *subst* makes extraction fail
apply (*rule exI*)
apply (*rule conjI*)
apply (*rule rtranclp.rtrancl-refl*)
apply (*rule NF.App*)
apply (*drule listall-conj1*)
apply (*simp add*: *listall-app*)
apply (*rule Var-NF*)
apply (*iprover intro*: *rtranclp.rtrancl-into-rtrancl rtranclp.rtrancl-refl beta subst-Var-NF*)
done

lemma *lift-terms-NF*: *listall NF ts* $\Longrightarrow$
listall ($\lambda t.\ \forall\, i.\ NF\ (lift\ t\ i)$) *ts* $\Longrightarrow$
listall NF (*map* ($\lambda t.\ lift\ t\ i$) *ts*)
by (*induct ts*) *simp-all*

lemma *lift-NF*: $NF\ t \Longrightarrow NF\ (lift\ t\ i)$
apply (*induct arbitrary*: *i set*: *NF*)
apply (*frule listall-conj1*)
apply (*drule listall-conj2*)
apply (*drule-tac i=i* **in** *lift-terms-NF*)
apply *assumption*
apply (*rule-tac m1=x* **and** *n1=i* **in** *nat-le-dec* [*THEN disjE*])
apply (*simp-all add*: *NF.App NF.Abs*)
done

NF characterizes exactly the terms that are in normal form.

lemma *NF-eq*: $NF\ t = (\forall\, t'.\ \neg\ t \to_\beta t')$
proof
assume *NF t*
then have $\bigwedge t'.\ \neg\ t \to_\beta t'$
proof *induct*
case (*App ts t*)
show *?case*
proof

```
      assume Var t °° ts →β t'
      then obtain rs where ts => rs
        by (iprover dest: head-Var-reduction)
      with App show False
        by (induct rs arbitrary: ts) auto
    qed
  next
    case (Abs t)
    show ?case
    proof
      assume Abs t →β t'
      then show False using Abs by cases simp-all
    qed
  qed
  then show ∀ t'. ¬ t →β t' ..
next
  assume H: ∀ t'. ¬ t →β t'
  then show NF t
  proof (induct t rule: Apps-dB-induct)
    case (1 n ts)
    then have ∀ ts'. ¬ ts => ts'
      by (iprover intro: apps-preserves-betas)
    with 1(1) have listall NF ts
      by (induct ts) auto
    then show ?case by (rule NF.App)
  next
    case (2 u ts)
    show ?case
    proof (cases ts)
      case Nil
      from 2 have ∀ u'. ¬ u →β u'
        by (auto intro: apps-preserves-beta)
      then have NF u by (rule 2)
      then have NF (Abs u) by (rule NF.Abs)
      with Nil show ?thesis by simp
    next
      case (Cons r rs)
      have Abs u ° r →β u[r/0] ..
      then have Abs u ° r °° rs →β u[r/0] °° rs
        by (rule apps-preserves-beta)
      with Cons have Abs u °° ts →β u[r/0] °° rs
        by simp
      with 2 show ?thesis by iprover
    qed
  qed
qed

end
```

12 Standardization

theory *Standardization*
imports *NormalForm*
begin

Based on lecture notes by Ralph Matthes [3], original proof idea due to Ralph Loader [2].

12.1 Standard reduction relation

declare *listrel-mono* [*mono-set*]

inductive
  sred :: *dB* ⇒ *dB* ⇒ *bool* (**infixl** ‹$\rightarrow_s$› *50*)
  and *sredlist* :: *dB list* ⇒ *dB list* ⇒ *bool* (**infixl** ‹$[\rightarrow_s]$› *50*)
where
  $s\ [\rightarrow_s]\ t \equiv$ *listrelp* $(\rightarrow_s)\ s\ t$
| *Var*: $rs\ [\rightarrow_s]\ rs' \Longrightarrow Var\ x\ {}^{\circ\circ}\ rs \rightarrow_s Var\ x\ {}^{\circ\circ}\ rs'$
| *Abs*: $r \rightarrow_s r' \Longrightarrow ss\ [\rightarrow_s]\ ss' \Longrightarrow Abs\ r\ {}^{\circ\circ}\ ss \rightarrow_s Abs\ r'\ {}^{\circ\circ}\ ss'$
| *Beta*: $r[s/0]\ {}^{\circ\circ}\ ss \rightarrow_s t \Longrightarrow Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ss \rightarrow_s t$

lemma *refl-listrelp*: $\forall x{\in}set\ xs.\ R\ x\ x \Longrightarrow$ *listrelp* $R\ xs\ xs$
  by (*induct xs*) (*auto intro*: *listrelp.intros*)

lemma *refl-sred*: $t \rightarrow_s t$
  by (*induct t rule*: *Apps-dB-induct*) (*auto intro*: *refl-listrelp sred.intros*)

lemma *refl-sreds*: $ts\ [\rightarrow_s]\ ts$
  by (*simp add*: *refl-sred refl-listrelp*)

lemma *listrelp-conj1*: *listrelp* $(\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow$ *listrelp* $R\ x\ y$
  by (*erule listrelp.induct*) (*auto intro*: *listrelp.intros*)

lemma *listrelp-conj2*: *listrelp* $(\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow$ *listrelp* $S\ x\ y$
  by (*erule listrelp.induct*) (*auto intro*: *listrelp.intros*)

lemma *listrelp-app*:
  assumes *xsys*: *listrelp* $R\ xs\ ys$
  shows *listrelp* $R\ xs'\ ys' \Longrightarrow$ *listrelp* $R\ (xs\ @\ xs')\ (ys\ @\ ys')$ **using** *xsys*
  by (*induct arbitrary*: $xs'\ ys'$) (*auto intro*: *listrelp.intros*)

lemma *lemma1*:
  assumes *r*: $r \rightarrow_s r'$ **and** *s*: $s \rightarrow_s s'$
  shows $r\ {}^{\circ}\ s \rightarrow_s r'\ {}^{\circ}\ s'$ **using** *r*
proof *induct*
  case (*Var rs rs' x*)
  then have $rs\ [\rightarrow_s]\ rs'$ **by** (*rule listrelp-conj1*)
  moreover have $[s]\ [\rightarrow_s]\ [s']$ **by** (*iprover intro*: *s listrelp.intros*)

ultimately have rs @ $[s]$ $[\rightarrow_s]$ rs' @ $[s']$ **by** (*rule listrelp-app*)
hence $Var\ x$ °° (rs @ $[s]$) $\rightarrow_s$ $Var\ x$ °° (rs' @ $[s']$) **by** (*rule sred.Var*)
thus *?case* **by** (*simp only*: *app-last*)
next
case (*Abs r r' ss ss'*)
from *Abs*(*3*) **have** ss $[\rightarrow_s]$ ss' **by** (*rule listrelp-conj1*)
moreover have $[s]$ $[\rightarrow_s]$ $[s']$ **by** (*iprover intro*: *s listrelp.intros*)
ultimately have ss @ $[s]$ $[\rightarrow_s]$ ss' @ $[s']$ **by** (*rule listrelp-app*)
with ‹$r \rightarrow_s r'$› **have** $Abs\ r$ °° (ss @ $[s]$) $\rightarrow_s$ $Abs\ r'$ °° (ss' @ $[s']$)
by (*rule sred.Abs*)
thus *?case* **by** (*simp only*: *app-last*)
next
case (*Beta r u ss t*)
hence $r[u/0]$ °° (ss @ $[s]$) $\rightarrow_s$ t ° s' **by** (*simp only*: *app-last*)
hence $Abs\ r$ ° u °° (ss @ $[s]$) $\rightarrow_s$ t ° s' **by** (*rule sred.Beta*)
thus *?case* **by** (*simp only*: *app-last*)
qed

lemma *lemma1'*:
assumes *ts*: ts $[\rightarrow_s]$ ts'
shows $r \rightarrow_s r' \Longrightarrow r$ °° $ts \rightarrow_s r'$ °° ts' **using** *ts*
by (*induct arbitrary*: *r r'*) (*auto intro*: *lemma1*)

lemma *lemma2-1*:
assumes *beta*: $t \rightarrow_\beta u$
shows $t \rightarrow_s u$ **using** *beta*
proof *induct*
case (*beta s t*)
have $Abs\ s$ ° t °° [] $\rightarrow_s$ $s[t/0]$ °° [] **by** (*iprover intro*: *sred.Beta refl-sred*)
thus *?case* **by** *simp*
next
case (*appL s t u*)
thus *?case* **by** (*iprover intro*: *lemma1 refl-sred*)
next
case (*appR s t u*)
thus *?case* **by** (*iprover intro*: *lemma1 refl-sred*)
next
case (*abs s t*)
hence $Abs\ s$ °° [] $\rightarrow_s$ $Abs\ t$ °° [] **by** (*iprover intro*: *sred.Abs listrelp.Nil*)
thus *?case* **by** *simp*
qed

lemma *listrelp-betas*:
assumes *ts*: *listrelp* ($\rightarrow_\beta{}^*$) *ts ts'*
shows $\bigwedge t\ t'.\ t \rightarrow_\beta{}^* t' \Longrightarrow t$ °° $ts \rightarrow_\beta{}^* t'$ °° ts' **using** *ts*
by *induct auto*

lemma *lemma2-2*:
assumes *t*: $t \rightarrow_s u$

shows $t \rightarrow_\beta{}^* u$ **using** *t*
by *induct* (*auto dest*: *listrelp-conj2*
intro: *listrelp-betas apps-preserves-beta converse-rtranclp-into-rtranclp*)

lemma *sred-lift*:
assumes *s*: $s \rightarrow_s t$
shows $lift\ s\ i \rightarrow_s lift\ t\ i$ **using** *s*
proof (*induct arbitrary*: *i*)
case (*Var rs rs′ x*)
hence $map\ (\lambda t.\ lift\ t\ i)\ rs\ [\rightarrow_s]\ map\ (\lambda t.\ lift\ t\ i)\ rs'$
by *induct* (*auto intro*: *listrelp.intros*)
thus *?case* **by** (*cases* $x < i$) (*auto intro*: *sred.Var*)
next
case (*Abs r r′ ss ss′*)
from *Abs(3)* **have** $map\ (\lambda t.\ lift\ t\ i)\ ss\ [\rightarrow_s]\ map\ (\lambda t.\ lift\ t\ i)\ ss'$
by *induct* (*auto intro*: *listrelp.intros*)
thus *?case* **by** (*auto intro*: *sred.Abs Abs*)
next
case (*Beta r s ss t*)
thus *?case* **by** (*auto intro*: *sred.Beta*)
qed

lemma *lemma3*:
assumes *r*: $r \rightarrow_s r'$
shows $s \rightarrow_s s' \Longrightarrow r[s/x] \rightarrow_s r'[s'/x]$ **using** *r*
proof (*induct arbitrary*: *s s′ x*)
case (*Var rs rs′ y*)
hence $map\ (\lambda t.\ t[s/x])\ rs\ [\rightarrow_s]\ map\ (\lambda t.\ t[s'/x])\ rs'$
by *induct* (*auto intro*: *listrelp.intros Var*)
moreover have $Var\ y[s/x] \rightarrow_s Var\ y[s'/x]$
proof (*cases* $y < x$)
case *True* **thus** *?thesis* **by** *simp* (*rule refl-sred*)
next
case *False*
thus *?thesis*
by (*cases* $y = x$) (*auto simp add*: *Var intro*: *refl-sred*)
qed
ultimately show *?case* **by** *simp* (*rule lemma1′*)
next
case (*Abs r r′ ss ss′*)
from *Abs(4)* **have** $lift\ s\ 0 \rightarrow_s lift\ s'\ 0$ **by** (*rule sred-lift*)
hence $r[lift\ s\ 0/Suc\ x] \rightarrow_s r'[lift\ s'\ 0/Suc\ x]$ **by** (*fast intro*: *Abs.hyps*)
moreover from *Abs(3)* **have** $map\ (\lambda t.\ t[s/x])\ ss\ [\rightarrow_s]\ map\ (\lambda t.\ t[s'/x])\ ss'$
by *induct* (*auto intro*: *listrelp.intros Abs*)
ultimately show *?case* **by** *simp* (*rule sred.Abs*)
next
case (*Beta r u ss t*)
thus *?case* **by** (*auto simp add*: *subst-subst intro*: *sred.Beta*)
qed

```
lemma lemma4-aux:
  assumes rs: listrelp (λt u. t →s u ∧ (∀r. u →β r ⟶ t →s r)) rs rs'
  shows rs' => ss ⟹ rs [→s] ss using rs
proof (induct arbitrary: ss)
  case Nil
  thus ?case by cases (auto intro: listrelp.Nil)
next
  case (Cons x y xs ys)
  note Cons' = Cons
  show ?case
  proof (cases ss)
    case Nil with Cons show ?thesis by simp
  next
    case (Cons y' ys')
    hence ss: ss = y' # ys' by simp
    from Cons Cons' have y →β y' ∧ ys' = ys ∨ y' = y ∧ ys => ys' by simp
    hence x # xs [→s] y' # ys'
    proof
      assume H: y →β y' ∧ ys' = ys
      with Cons' have x →s y' by blast
      moreover from Cons' have xs [→s] ys by (iprover dest: listrelp-conj1)
      ultimately have x # xs [→s] y' # ys by (rule listrelp.Cons)
      with H show ?thesis by simp
    next
      assume H: y' = y ∧ ys => ys'
      with Cons' have x →s y' by blast
      moreover from H have xs [→s] ys' by (blast intro: Cons')
      ultimately show ?thesis by (rule listrelp.Cons)
    qed
    with ss show ?thesis by simp
  qed
qed

lemma lemma4:
  assumes r: r →s r'
  shows r' →β r'' ⟹ r →s r'' using r
proof (induct arbitrary: r'')
  case (Var rs rs' x)
  then obtain ss where rs: rs' => ss and r'': r'' = Var x °° ss
    by (blast dest: head-Var-reduction)
  from Var(1) rs have rs [→s] ss by (rule lemma4-aux)
  hence Var x °° rs →s Var x °° ss by (rule sred.Var)
  with r'' show ?case by simp
next
  case (Abs r r' ss ss')
  from ‹Abs r' °° ss' →β r''› show ?case
  proof
    fix s
```

assume r'': $r'' = s \circ\circ ss'$
assume $Abs\ r' \to_\beta s$
then obtain r''' **where** s: $s = Abs\ r'''$ **and** r''': $r' \to_\beta r'''$ **by** *cases auto*
from r''' **have** $r \to_s r'''$ **by** (*blast intro*: *Abs*)
moreover from *Abs* **have** $ss\ [\to_s]\ ss'$ **by** (*iprover dest*: *listrelp-conj1*)
ultimately have $Abs\ r \circ\circ ss \to_s Abs\ r''' \circ\circ ss'$ **by** (*rule sred.Abs*)
with r'' s **show** $Abs\ r \circ\circ ss \to_s r''$ **by** *simp*
next
fix rs'
assume $ss' => rs'$
with *Abs*(*3*) **have** $ss\ [\to_s]\ rs'$ **by** (*rule lemma4-aux*)
with ‹$r \to_s r'$› **have** $Abs\ r \circ\circ ss \to_s Abs\ r' \circ\circ rs'$ **by** (*rule sred.Abs*)
moreover assume $r'' = Abs\ r' \circ\circ rs'$
ultimately show $Abs\ r \circ\circ ss \to_s r''$ **by** *simp*
next
fix $t\ u'\ us'$
assume $ss' = u' \#\ us'$
with *Abs*(*3*) **obtain** u us **where**
ss: $ss = u \#\ us$ **and** u: $u \to_s u'$ **and** us: $us\ [\to_s]\ us'$
by *cases* (*auto dest*!: *listrelp-conj1*)
have $r[u/0] \to_s r'[u'/0]$ **using** *Abs*(*1*) **and** u **by** (*rule lemma3*)
with us **have** $r[u/0] \circ\circ us \to_s r'[u'/0] \circ\circ us'$ **by** (*rule lemma1'*)
hence $Abs\ r \circ u \circ\circ us \to_s r'[u'/0] \circ\circ us'$ **by** (*rule sred.Beta*)
moreover assume $Abs\ r' = Abs\ t$ **and** $r'' = t[u'/0] \circ\circ us'$
ultimately show $Abs\ r \circ\circ ss \to_s r''$ **using** ss **by** *simp*
qed
next
case (*Beta r s ss t*)
show *?case*
by (*rule sred.Beta*) (*rule Beta*)+
qed

lemma *rtrancl-beta-sred*:
assumes r: $r \to_\beta^* r'$
shows $r \to_s r'$ **using** r
by *induct* (*iprover intro*: *refl-sred lemma4*)+

12.2 Leftmost reduction and weakly normalizing terms

inductive
lred :: $dB \Rightarrow dB \Rightarrow bool$ (**infixl** ‹$\to_l$› *50*)
and *lredlist* :: $dB\ list \Rightarrow dB\ list \Rightarrow bool$ (**infixl** ‹$[\to_l]$› *50*)
where
$s\ [\to_l]\ t \equiv listrelp\ (\to_l)\ s\ t$
| *Var*: $rs\ [\to_l]\ rs' \Longrightarrow Var\ x \circ\circ rs \to_l Var\ x \circ\circ rs'$
| *Abs*: $r \to_l r' \Longrightarrow Abs\ r \to_l Abs\ r'$
| *Beta*: $r[s/0] \circ\circ ss \to_l t \Longrightarrow Abs\ r \circ s \circ\circ ss \to_l t$

lemma *lred-imp-sred*:

```
  assumes lred: s →l t
  shows s →s t using lred
proof induct
  case (Var rs rs' x)
  then have rs [→s] rs'
    by induct (iprover intro: listrelp.intros)+
  then show ?case by (rule sred.Var)
next
  case (Abs r r')
  from ‹r →s r'›
  have Abs r °° [] →s Abs r' °° [] using listrelp.Nil
    by (rule sred.Abs)
  then show ?case by simp
next
  case (Beta r s ss t)
  from ‹r[s/0] °° ss →s t›
  show ?case by (rule sred.Beta)
qed

inductive WN :: dB => bool
  where
    Var: listsp WN rs ⟹ WN (Var n °° rs)
  | Lambda: WN r ⟹ WN (Abs r)
  | Beta: WN ((r[s/0]) °° ss) ⟹ WN ((Abs r ° s) °° ss)

lemma listrelp-imp-listsp1:
  assumes H: listrelp (λx y. P x) xs ys
  shows listsp P xs using H
  by induct auto

lemma listrelp-imp-listsp2:
  assumes H: listrelp (λx y. P y) xs ys
  shows listsp P ys using H
  by induct auto

lemma lemma5:
  assumes lred: r →l r'
  shows WN r and NF r' using lred
  by induct
    (iprover dest: listrelp-conj1 listrelp-conj2
     listrelp-imp-listsp1 listrelp-imp-listsp2 intro: WN.intros
     NF.intros [simplified listall-listsp-eq])+

lemma lemma6:
  assumes wn: WN r
  shows ∃r'. r →l r' using wn
proof induct
  case (Var rs n)
  then have ∃rs'. rs [→l] rs'
```

```
    by induct (iprover intro: listrelp.intros)+
  then show ?case by (iprover intro: lred.Var)
qed (iprover intro: lred.intros)+

lemma lemma7:
  assumes r: r →_s r'
  shows NF r' ⟹ r →_l r' using r
proof induct
  case (Var rs rs' x)
  from ‹NF (Var x °° rs')› have listall NF rs'
    by cases simp-all
  with Var(1) have rs [→_l] rs'
  proof induct
    case Nil
    show ?case by (rule listrelp.Nil)
  next
    case (Cons x y xs ys)
    hence x →_l y and xs [→_l] ys by simp-all
    thus ?case by (rule listrelp.Cons)
  qed
  thus ?case by (rule lred.Var)
next
  case (Abs r r' ss ss')
  from ‹NF (Abs r' °° ss')›
  have ss': ss' = [] by (rule Abs-NF)
  from Abs(3) have ss: ss = [] using ss'
    by cases simp-all
  from ss' Abs have NF (Abs r') by simp
  hence NF r' by cases simp-all
  with Abs have r →_l r' by simp
  hence Abs r →_l Abs r' by (rule lred.Abs)
  with ss ss' show ?case by simp
next
  case (Beta r s ss t)
  hence r[s/0] °° ss →_l t by simp
  thus ?case by (rule lred.Beta)
qed

lemma WN-eq: WN t = (∃ t'. t →_β* t' ∧ NF t')
proof
  assume WN t
  then have ∃ t'. t →_l t' by (rule lemma6)
  then obtain t' where t': t →_l t' ..
  then have NF: NF t' by (rule lemma5)
  from t' have t →_s t' by (rule lred-imp-sred)
  then have t →_β* t' by (rule lemma2-2)
  with NF show ∃ t'. t →_β* t' ∧ NF t' by iprover
next
  assume ∃ t'. t →_β* t' ∧ NF t'
```

then obtain t' **where** t': $t \rightarrow_\beta{}^* t'$ **and** *NF*: *NF* t'
  by *iprover*
from t' **have** $t \rightarrow_s t'$ **by** (*rule rtrancl-beta-sred*)
then have $t \rightarrow_l t'$ **using** *NF* **by** (*rule lemma7*)
then show *WN t* **by** (*rule lemma5*)
qed

end

13 Weak normalization for simply-typed lambda calculus

theory *WeakNorm*
imports *LambdaType NormalForm HOL−Library.Realizers HOL−Library.Code-Target-Int*
begin

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

13.1 Main theorems

lemma *norm-list*:
  assumes *f-compat*: $\bigwedge t\ t'.\ t \rightarrow_\beta{}^* t' \Longrightarrow f\ t \rightarrow_\beta{}^* f\ t'$
  and *f-NF*: $\bigwedge t.\ NF\ t \Longrightarrow NF\ (f\ t)$
  and *uNF*: *NF u* **and** *uT*: $e \vdash u : T$
  shows $\bigwedge Us.\ e\langle i{:}T\rangle \Vdash as : Us \Longrightarrow$
    $listall\ (\lambda t.\ \forall e\ T'\ u\ i.\ e\langle i{:}T\rangle \vdash t : T' \longrightarrow$
      $NF\ u \longrightarrow e \vdash u : T \longrightarrow (\exists t'.\ t[u/i] \rightarrow_\beta{}^* t' \land NF\ t'))\ as \Longrightarrow$
    $\exists as'.\ \forall j.\ Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ as \rightarrow_\beta{}^*$
      $Var\ j\ {}^{\circ\circ}\ map\ f\ as' \land NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ as')$
  (**is** $\bigwedge Us.\ \text{-} \Longrightarrow listall\ ?R\ as \Longrightarrow \exists as'.\ ?ex\ Us\ as\ as'$)
proof (*induct as rule*: *rev-induct*)
  case (*Nil Us*)
  with *Var-NF* **have** *?ex Us* [] [] **by** *simp*
  thus *?case* **..**
next
  case (*snoc b bs Us*)
  have $e\langle i{:}T\rangle \Vdash bs\ @\ [b] : Us$ **by** *fact*
  then obtain *Vs W* **where** *Us*: $Us = Vs\ @\ [W]$
    and *bs*: $e\langle i{:}T\rangle \Vdash bs : Vs$ **and** *bT*: $e\langle i{:}T\rangle \vdash b : W$
    by (*rule types-snocE*)
  from *snoc* **have** *listall ?R bs* **by** *simp*
  with *bs* **have** $\exists bs'.\ ?ex\ Vs\ bs\ bs'$ **by** (*rule snoc*)
  then obtain bs' **where** *bsred*: $Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ bs \rightarrow_\beta{}^* Var\ j\ {}^{\circ\circ}\ map$
$f\ bs'$
    and *bsNF*: $NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ bs')$ **for** *j*
    by *iprover*
  from *snoc* **have** *?R b* **by** *simp*

with *bT* **and** *uNF* **and** *uT* **have** $\exists\, b'.\ b[u/i] \to_\beta{}^* b' \wedge NF\ b'$
  by *iprover*
then obtain b' **where** *bred*: $b[u/i] \to_\beta{}^* b'$ **and** *bNF*: $NF\ b'$
  by *iprover*
from *bsNF* [*of 0*] **have** *listall NF* (*map f bs'*)
  by (*rule App-NF-D*)
moreover have *NF* (*f b'*) **using** *bNF* **by** (*rule f-NF*)
ultimately have *listall NF* (*map f* (*bs'* @ [*b'*]))
  by *simp*
hence $\bigwedge j.$ *NF* (*Var j* °° *map f* (*bs'* @ [*b'*])) **by** (*rule NF.App*)
moreover from *bred* **have** $f\ (b[u/i]) \to_\beta{}^* f\ b'$
  by (*rule f-compat*)
with *bsred* **have**
  $\bigwedge j.$ (*Var j* °° *map* ($\lambda t.\ f\ (t[u/i])$) *bs*) ° $f\ (b[u/i]) \to_\beta{}^*$
   (*Var j* °° *map f bs'*) ° *f b'* **by** (*rule rtrancl-beta-App*)
ultimately have *?ex Us* (*bs* @ [*b*]) (*bs'* @ [*b'*]) **by** *simp*
thus *?case* **..**
qed

lemma *subst-type-NF*:
$\bigwedge t\ e\ T\ u\ i.\ NF\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow NF\ u \Longrightarrow e \vdash u : U \Longrightarrow \exists\, t'.\ t[u/i]$
$\to_\beta{}^* t' \wedge NF\ t'$
(**is** *PROP ?P U* **is** $\bigwedge t\ e\ T\ u\ i.$ - $\Longrightarrow$ *PROP ?Q t e T u i U*)
proof (*induct U*)
fix *T t*
let $?R = \lambda t.\ \forall\, e\ T'\ u\ i.$
  $e\langle i{:}T\rangle \vdash t : T' \longrightarrow NF\ u \longrightarrow e \vdash u : T \longrightarrow (\exists\, t'.\ t[u/i] \to_\beta{}^* t' \wedge NF\ t')$
assume *MI1*: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow$ *PROP ?P T1*
assume *MI2*: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow$ *PROP ?P T2*
assume *NF t*
thus $\bigwedge e\ T'\ u\ i.$ *PROP ?Q t e T' u i T*
proof *induct*
  fix *e T' u i* **assume** *uNF*: *NF u* **and** *uT*: $e \vdash u : T$
  {
    case (*App ts x e1 T'1 u1 i1*)
    assume $e\langle i{:}T\rangle \vdash$ *Var x* °° $ts : T'$
    then obtain *Us*
      where *varT*: $e\langle i{:}T\rangle \vdash Var\ x : Us \Rrightarrow T'$
      and *argsT*: $e\langle i{:}T\rangle \Vdash ts : Us$
      by (*rule var-app-typesE*)
    from *nat-eq-dec* **show** $\exists\, t'.$ (*Var x* °° *ts*)$[u/i] \to_\beta{}^* t' \wedge NF\ t'$
    proof
      assume *eq*: $x = i$
      show *?thesis*
      proof (*cases ts*)
        case *Nil*
        with *eq* **have** (*Var x* °° [])$[u/i] \to_\beta{}^* u$ **by** *simp*
        with *Nil* **and** *uNF* **show** *?thesis* **by** *simp iprover*
      next

```
    case (Cons a as)
    with argsT obtain T'' Ts where Us: Us = T'' # Ts
      by (cases Us) (rule FalseE, simp)
    from varT and Us have varT: e⟨i:T⟩ ⊢ Var x : T'' ⇒ Ts ⇛ T'
      by simp
    from varT eq have T: T = T'' ⇒ Ts ⇛ T' by cases auto
    with uT have uT': e ⊢ u : T'' ⇒ Ts ⇛ T' by simp
    from argsT Us Cons have argsT': e⟨i:T⟩ ⊩ as : Ts by simp
    from argsT Us Cons have argT: e⟨i:T⟩ ⊢ a : T'' by simp
    from argT uT refl have aT: e ⊢ a[u/i] : T'' by (rule subst-lemma)
  from App and Cons have listall ?R as by simp (iprover dest: listall-conj2)
    with lift-preserves-beta' lift-NF uNF uT argsT'
    have ∃ as'. ∀ j. Var j °° map (λt. lift (t[u/i]) 0) as →β*
      Var j °° map (λt. lift t 0) as' ∧
      NF (Var j °° map (λt. lift t 0) as') by (rule norm-list)
    then obtain as' where
      asred: Var 0 °° map (λt. lift (t[u/i]) 0) as →β*
        Var 0 °° map (λt. lift t 0) as'
      and asNF: NF (Var 0 °° map (λt. lift t 0) as') by iprover
    from App and Cons have ?R a by simp
    with argT and uNF and uT have ∃ a'. a[u/i] →β* a' ∧ NF a'
      by iprover
    then obtain a' where ared: a[u/i] →β* a' and aNF: NF a' by iprover
    from uNF have NF (lift u 0) by (rule lift-NF)
    hence ∃ u'. lift u 0 ° Var 0 →β* u' ∧ NF u' by (rule app-Var-NF)
    then obtain u' where ured: lift u 0 ° Var 0 →β* u' and u'NF: NF u'
      by iprover
    from T and u'NF have ∃ ua. u'[a'/0] →β* ua ∧ NF ua
    proof (rule MI1)
      have e⟨0:T''⟩ ⊢ lift u 0 ° Var 0 : Ts ⇛ T'
      proof (rule typing.App)
       from uT' show e⟨0:T''⟩ ⊢ lift u 0 : T'' ⇒ Ts ⇛ T' by (rule lift-type)
        show e⟨0:T''⟩ ⊢ Var 0 : T'' by (rule typing.Var) simp
      qed
      with ured show e⟨0:T''⟩ ⊢ u' : Ts ⇛ T' by (rule subject-reduction')
      from ared aT show e ⊢ a' : T'' by (rule subject-reduction')
      show NF a' by fact
    qed
    then obtain ua where uared: u'[a'/0] →β* ua and uaNF: NF ua
      by iprover
    from ared have (lift u 0 ° Var 0)[a[u/i]/0] →β* (lift u 0 ° Var 0)[a'/0]
      by (rule subst-preserves-beta2')
    also from ured have (lift u 0 ° Var 0)[a'/0] →β* u'[a'/0]
      by (rule subst-preserves-beta')
    also note uared
    finally have (lift u 0 ° Var 0)[a[u/i]/0] →β* ua .
    hence uared': u ° a[u/i] →β* ua by simp
     from T asNF - uaNF have ∃ r. (Var 0 °° map (λt. lift t 0) as')[ua/0]
→β* r ∧ NF r
```

```
    proof (rule MI2)
      have e⟨0:Ts ⇛ T′⟩ ⊢ Var 0 °° map (λt. lift (t[u/i]) 0) as : T′
      proof (rule list-app-typeI)
        show e⟨0:Ts ⇛ T′⟩ ⊢ Var 0 : Ts ⇛ T′ by (rule typing.Var) simp
        from uT argsT′ have e ⊩ map (λt. t[u/i]) as : Ts
          by (rule substs-lemma)
        hence e⟨0:Ts ⇛ T′⟩ ⊩ map (λt. lift t 0) (map (λt. t[u/i]) as) : Ts
          by (rule lift-types)
        thus e⟨0:Ts ⇛ T′⟩ ⊩ map (λt. lift (t[u/i]) 0) as : Ts
          by (simp-all add: o-def)
      qed
      with asred show e⟨0:Ts ⇛ T′⟩ ⊢ Var 0 °° map (λt. lift t 0) as′ : T′
        by (rule subject-reduction′)
      from argT uT refl have e ⊢ a[u/i] : T′′ by (rule subst-lemma)
      with uT′ have e ⊢ u ° a[u/i] : Ts ⇛ T′ by (rule typing.App)
      with uared′ show e ⊢ ua : Ts ⇛ T′ by (rule subject-reduction′)
    qed
    then obtain r where rred: (Var 0 °° map (λt. lift t 0) as′)[ua/0] →β* r
      and rnf: NF r by iprover
    from asred have
      (Var 0 °° map (λt. lift (t[u/i]) 0) as)[u ° a[u/i]/0] →β*
      (Var 0 °° map (λt. lift t 0) as′)[u ° a[u/i]/0]
      by (rule subst-preserves-beta′)
    also from uared′ have (Var 0 °° map (λt. lift t 0) as′)[u ° a[u/i]/0] →β*
      (Var 0 °° map (λt. lift t 0) as′)[ua/0] by (rule subst-preserves-beta2′)
    also note rred
    finally have (Var 0 °° map (λt. lift (t[u/i]) 0) as)[u ° a[u/i]/0] →β* r .
    with rnf Cons eq show ?thesis
      by (simp add: o-def) iprover
  qed
next
  assume neq: x ≠ i
  from App have listall ?R ts by (iprover dest: listall-conj2)
  with uNF uT argsT
  have ∃ts′. ∀j. Var j °° map (λt. t[u/i]) ts →β* Var j °° ts′ ∧
    NF (Var j °° ts′) (is ∃ts′. ?ex ts′)
    by (rule norm-list [of λt. t, simplified])
  then obtain ts′ where NF: ?ex ts′ ..
  from nat-le-dec show ?thesis
  proof
    assume i < x
    with NF show ?thesis by simp iprover
  next
    assume ¬ (i < x)
    with NF neq show ?thesis by (simp add: subst-Var) iprover
  qed
qed
next
  case (Abs r e1 T′1 u1 i1)
```

```
      assume absT: e⟨i:T⟩ ⊢ Abs r : T'
      then obtain R S where e⟨0:R⟩⟨Suc i:T⟩ ⊢ r : S by (rule abs-typeE) simp
      moreover have NF (lift u 0) using ‹NF u› by (rule lift-NF)
      moreover have e⟨0:R⟩ ⊢ lift u 0 : T using uT by (rule lift-type)
      ultimately have ∃t'. r[lift u 0/Suc i] →β* t' ∧ NF t' by (rule Abs)
      thus ∃t'. Abs r[u/i] →β* t' ∧ NF t'
        by simp (iprover intro: rtrancl-beta-Abs NF.Abs)
    }
  qed
qed
```

— A computationally relevant copy of $e \vdash t : T$

```
inductive rtyping :: (nat ⇒ type) ⇒ dB ⇒ type ⇒ bool (‹- ⊢R - : -› [50, 50, 50]
50)
  where
    Var: e x = T ⟹ e ⊢R Var x : T
  | Abs: e⟨0:T⟩ ⊢R t : U ⟹ e ⊢R Abs t : (T ⇒ U)
  | App: e ⊢R s : T ⇒ U ⟹ e ⊢R t : T ⟹ e ⊢R (s ° t) : U

lemma rtyping-imp-typing: e ⊢R t : T ⟹ e ⊢ t : T
  apply (induct set: rtyping)
  apply (erule typing.Var)
  apply (erule typing.Abs)
  apply (erule typing.App)
  apply assumption
  done


theorem type-NF:
  assumes e ⊢R t : T
  shows ∃t'. t →β* t' ∧ NF t' using assms
proof induct
  case Var
  show ?case by (iprover intro: Var-NF)
next
  case Abs
  thus ?case by (iprover intro: rtrancl-beta-Abs NF.Abs)
next
  case (App e s T U t)
  from App obtain s' t' where
    sred: s →β* s' and NF s'
    and tred: t →β* t' and tNF: NF t' by iprover
  have ∃u. (Var 0 ° lift t' 0)[s'/0] →β* u ∧ NF u
  proof (rule subst-type-NF)
    have NF (lift t' 0) using tNF by (rule lift-NF)
    hence listall NF [lift t' 0] by (rule listall-cons) (rule listall-nil)
    hence NF (Var 0 °° [lift t' 0]) by (rule NF.App)
    thus NF (Var 0 ° lift t' 0) by simp
```

```
    show e⟨0:T ⇒ U⟩ ⊢ Var 0 ° lift t′ 0 : U
    proof (rule typing.App)
      show e⟨0:T ⇒ U⟩ ⊢ Var 0 : T ⇒ U
        by (rule typing.Var) simp
      from tred have e ⊢ t′ : T
        by (rule subject-reduction′) (rule rtyping-imp-typing, rule App.hyps)
      thus e⟨0:T ⇒ U⟩ ⊢ lift t′ 0 : T
        by (rule lift-type)
    qed
    from sred show e ⊢ s′ : T ⇒ U
      by (rule subject-reduction′) (rule rtyping-imp-typing, rule App.hyps)
    show NF s′ by fact
  qed
  then obtain u where ured: s′ ° t′ →β* u and unf: NF u by simp iprover
  from sred tred have s ° t →β* s′ ° t′ by (rule rtrancl-beta-App)
  hence s ° t →β* u using ured by (rule rtranclp-trans)
  with unf show ?case by iprover
qed
```

13.2 Extracting the program

```
declare NF.induct [ind-realizer]
declare rtranclp.induct [ind-realizer irrelevant]
declare rtyping.induct [ind-realizer]
lemmas [extraction-expand] = conj-assoc listall-cons-eq subst-all equal-allI

extract type-NF

lemma rtranclR-rtrancl-eq: rtranclpR r a b = r** a b
proof
  show rtranclpR r a b ⟹ r** a b
    apply (erule rtranclpR.induct)
     apply (rule rtranclp.rtrancl-refl)
    apply (metis rtranclp.rtrancl-into-rtrancl)
    done
  show r** a b ⟹ rtranclpR r a b
    apply (erule rtranclp.induct)
     apply (rule rtranclpR.rtrancl-refl)
    apply (metis rtranclpR.rtrancl-into-rtrancl)
    done
qed

lemma NFR-imp-NF: NFR nf t ⟹ NF t
  apply (erule NFR.induct)
  apply (rule NF.intros)
  apply (simp add: listall-def)
  apply (erule NF.intros)
  done
```

The program corresponding to the proof of the central lemma, which per-

```
subst-type-NF ≡
λx xa xb xc xd xe H Ha.
  type-induct-P xc
   (λx H2 H2a xa xaa xb xc xd H.
      compat-NFT.rec-split-NFT default
       (λts xa xaa r xb xc xd xe H.
          var-app-typesE-P (xb⟨xe:x⟩) xa ts
           (λUs--. case nat-eq-dec xa xe of
                 Left ⇒ case ts of [] ⇒ (xd, H)
                        | a # list ⇒
                           case Us-- of [] ⇒ default
                           | T''-- # Ts-- ⇒
                              let (x, y) =
                                    norm-list (λt. lift t 0) xd xb xe list Ts--
                                     (λt. lift-NF 0) H
                                     (listall-conj2-P-Q list (λi. (xaa (Suc i), r (Suc i))));
                                  (xa, ya) = snd (xaa 0, r 0) xb T''-- xd xe H;
                                  (xd, yb) = app-Var-NF 0 (lift-NF 0 H);
                                  (xa, ya) =
                                    H2 T''-- (Ts-- ⇛ xc) xd xb (Ts-- ⇛ xc) xa 0 yb ya;
                                  (x, y) =
                                    H2a T''-- (Ts-- ⇛ xc) (dB.Var 0 °° map (λt. lift t 0) x)
                                     xb xc xa 0 (y 0) ya
                              in (x, y)
                 | Right ⇒
                    let (x, y) =
                          let (x, y) =
                                norm-list (λt. t) xd xb xe ts Us-- (λx H. H) H
                                 (listall-conj2-P-Q ts (λz. (xaa z, r z)))
                          in (x, λx. y x)
                    in case nat-le-dec xe xa of
                         Left ⇒ (dB.Var (xa − Suc 0) °° x, y (xa − Suc 0))
                         | Right ⇒ (dB.Var xa °° x, y xa)))
       (λt x r xa xaa xb xc H.
          abs-typeE-P xaa
           (λU V. let (x, y) =
                        let (x, y) = r (λa. (xa⟨0:U⟩) a) V (lift xb 0) (Suc xc) (lift-NF 0 H)
                        in (dB.Abs x, NFT.Abs x y)
                 in (x, y)))
       H (λa. xaa a) xb xc xd)
   x xa xd xe xb H Ha
```

Figure 1: Program extracted from *subst-type-NF*

```
subst-Var-NF ≡
λx xa H.
   compat-NFT.rec-split-NFT default
    (λts x xa r xb xc.
        case nat-eq-dec x xc of
        Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) xb
                  (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                    (listall-conj2-P-Q ts (λz. (xa z, r z))))
        | Right ⇒
            case nat-le-dec xc x of
            Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) (x − Suc 0)
                      (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                        (listall-conj2-P-Q ts (λz. (xa z, r z))))
            | Right ⇒
                NFT.App (map (λt. t[dB.Var xb/xc]) ts) x
                 (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                   (listall-conj2-P-Q ts (λz. (xa z, r z)))))
    (λt x r xa xaa. NFT.Abs (t[dB.Var (Suc xa)/Suc xaa]) (r (Suc xa) (Suc xaa))) H x xa

app-Var-NF ≡
λx. compat-NFT.rec-split-NFT default
      (λts xa xaa r.
          (dB.Var xa °° (ts @ [dB.Var x]),
           NFT.App (ts @ [dB.Var x]) xa
            (snd (listall-app-P ts)
              (listall-conj1-P-Q ts (λz. (xaa z, r z)),
               listall-cons-P (Var-NF x) listall-nil-eq-P))))
      (λt xa r. let (xb, y) = r in (t[dB.Var x/0], subst-Var-NF x 0 xa))

lift-NF ≡
λx H. compat-NFT.rec-split-NFT default
        (λts x xa r xb.
            case nat-le-dec x xb of
            Left ⇒ NFT.App (map (λt. lift t xb) ts) x
                      (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                        (listall-conj2-P-Q ts (λz. (xa z, r z))))
            | Right ⇒
                NFT.App (map (λt. lift t xb) ts) (Suc x)
                 (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                   (listall-conj2-P-Q ts (λz. (xa z, r z)))))
        (λt x r xa. NFT.Abs (lift t (Suc xa)) (r (Suc xa))) H x

type-NF ≡
λH. rec-rtypingT (λe x T. (dB.Var x, Var-NF x))
      (λe T t U x r. let (x, y) = r in (dB.Abs x, NFT.Abs x y))
      (λe s T U t x xa r ra.
          let (x, y) = r; (xa, ya) = ra;
              (x, y) =
                let (x, y) =
                      subst-type-NF (dB.Var 0 ° lift xa 0) e 0 (T ⇒ U) U x
                       (NFT.App [lift xa 0] 0 (listall-cons-P (lift-NF 0 ya) listall-nil-P)) y
                in (x, y)
          in (x, y))
      H
```

Figure 2: Program extracted from lemmas and main theorem

forms substitution and normalization, is shown in Figure 1. The correctness theorem corresponding to the program *subst-type-NF* is

$$\bigwedge x.\ NFR\ x\ t \Longrightarrow$$
$$e\langle i{:}U\rangle \vdash t : T \Longrightarrow$$
$$(\bigwedge xa.\ NFR\ xa\ u \Longrightarrow$$
$$e \vdash u : U \Longrightarrow$$
$$t[u/i] \rightarrow_{\beta}{}^{*}\ fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa) \wedge$$
$$NFR\ (snd\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa))\ (fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa)))$$

where *NFR* is the realizability predicate corresponding to the datatype NFT, which is inductively defined by the rules

$\forall\, i{<}length\ ts.\ NFR\ (nfs\ i)\ (ts\ !\ i) \Longrightarrow NFR\ (NFT.App\ ts\ x\ nfs)\ (dB.Var\ x\ {}^{\circ\circ}\ ts)$
$NFR\ nf\ t \Longrightarrow NFR\ (NFT.Abs\ t\ nf)\ (dB.Abs\ t)$

The programs corresponding to the main theorem *type-NF*, as well as to some lemmas, are shown in Figure 2. The correctness statement for the main function *type-NF* is

$\bigwedge x.\ rtypingR\ x\ e\ t\ T \Longrightarrow t \rightarrow_\beta{}^*\ fst\ (type\text{-}NF\ x) \wedge NFR\ (snd\ (type\text{-}NF\ x))\ (fst\ (type\text{-}NF\ x))$

where the realizability predicate *rtypingR* corresponding to the computationally relevant version of the typing judgement is inductively defined by the rules

$e\ x = T \Longrightarrow rtypingR\ (rtypingT.Var\ e\ x\ T)\ e\ (dB.Var\ x)\ T$
$rtypingR\ ty\ (e\langle 0{:}T\rangle)\ t\ U \Longrightarrow rtypingR\ (rtypingT.Abs\ e\ T\ t\ U\ ty)\ e\ (dB.Abs\ t)\ (T \Rightarrow U)$
$rtypingR\ ty\ e\ s\ (T \Rightarrow U) \Longrightarrow$
$rtypingR\ ty'\ e\ t\ T \Longrightarrow rtypingR\ (rtypingT.App\ e\ s\ T\ U\ t\ ty\ ty')\ e\ (s\ {}^\circ\ t)\ U$

13.3 Generating executable code

instantiation *NFT* :: *default*
begin

definition *default* = *Dummy* ()

instance ..

end

instantiation *dB* :: *default*
begin

definition *default* = *dB.Var 0*

instance ..

end

instantiation *prod* :: (*default*, *default*) *default*
begin

definition *default* = (*default*, *default*)

instance ..

end

instantiation *list* :: (*type*) *default*
begin

definition *default* = []

instance ..

end

instantiation *fun* :: (*type*, *default*) *default*
begin

definition *default* = (λx. *default*)

instance ..

end

definition *int-of-nat* :: *nat* ⇒ *int* **where**
int-of-nat = *of-nat*

The following functions convert between Isabelle's built-in `term` datatype and the generated `dB` datatype. This allows to generate example terms using Isabelle's parser and inspect normalized terms using Isabelle's pretty printer.

```
ML ‹
val nat-of-integer = @{code nat} o @{code int-of-integer};

fun dBtype-of-typ (Type (fun, [T, U])) =
      @{code Fun} (dBtype-of-typ T, dBtype-of-typ U)
  | dBtype-of-typ (TFree (s, -)) = (case raw-explode s of
        [', a] => @{code Atom} (nat-of-integer (ord a − 97))
      | - => error dBtype-of-typ: variable name)
  | dBtype-of-typ - = error dBtype-of-typ: bad type;

fun dB-of-term (Bound i) = @{code dB.Var} (nat-of-integer i)
  | dB-of-term (t $ u) = @{code dB.App} (dB-of-term t, dB-of-term u)
  | dB-of-term (Abs (-, -, t)) = @{code dB.Abs} (dB-of-term t)
  | dB-of-term - = error dB-of-term: bad term;

fun term-of-dB Ts (Type (fun, [T, U])) (@{code dB.Abs} dBt) =
      Abs (x, T, term-of-dB (T :: Ts) U dBt)
  | term-of-dB Ts - dBt = term-of-dB′ Ts dBt
and term-of-dB′ Ts (@{code dB.Var} n) = Bound (@{code integer-of-nat} n)
  | term-of-dB′ Ts (@{code dB.App} (dBt, dBu)) =
      let val t = term-of-dB′ Ts dBt
      in case fastype-of1 (Ts, t) of
          Type (fun, [T, -]) => t $ term-of-dB Ts T dBu
```

```
    | - => error term-of-dB: function type expected
    end
  | term-of-dB' - - = error term-of-dB: term not in normal form;

fun typing-of-term Ts e (Bound i) =
      @{code Var} (e, nat-of-integer i, dBtype-of-typ (nth Ts i))
  | typing-of-term Ts e (t $ u) = (case fastype-of1 (Ts, t) of
        Type (fun, [T, U]) => @{code App} (e, dB-of-term t,
          dBtype-of-typ T, dBtype-of-typ U, dB-of-term u,
          typing-of-term Ts e t, typing-of-term Ts e u)
      | - => error typing-of-term: function type expected)
  | typing-of-term Ts e (Abs (-, T, t)) =
      let val dBT = dBtype-of-typ T
      in @{code Abs} (e, dBT, dB-of-term t,
        dBtype-of-typ (fastype-of1 (T :: Ts, t)),
        typing-of-term (T :: Ts) (@{code shift} e @{code 0::nat} dBT) t)
      end
  | typing-of-term - - - = error typing-of-term: bad term;

fun dummyf - = error dummy;

val ct1 = @{cterm %f. ((%f x. f (f (f x))) ((%f x. f (f (f (f x)))) f))};
val (dB1, -) = @{code type-NF} (typing-of-term [] dummyf (Thm.term-of ct1));
val ct1' = Thm.cterm-of @{context} (term-of-dB [] (Thm.typ-of-cterm ct1) dB1);

val ct2 = @{cterm %f x. (%x. f x x) ((%x. f x x) ((%x. f x x) ((%x. f x x) ((%x. f
x x) ((%x. f x x) x)))))};
val (dB2, -) = @{code type-NF} (typing-of-term [] dummyf (Thm.term-of ct2));
val ct2' = Thm.cterm-of @{context} (term-of-dB [] (Thm.typ-of-cterm ct2) dB2);
›
```

end

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