NanoJava

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Abstract

These theories define *NanoJava*, a very small fragment of the programming language Java (with essentially just classes) derived from the one given in [1]. For *NanoJava*, an operational semantics is given as well as a Hoare logic, which is proved both sound and (relatively) complete. The Hoare logic supports side-effecting expressions and implements a new approach for handling auxiliary variables. A more complex Hoare logic covering a much larger subset of Java is described in [3]. See also the homepage of project Bali at `https://isabelle.in.tum.de/Bali/` and the conference version of this document [2].

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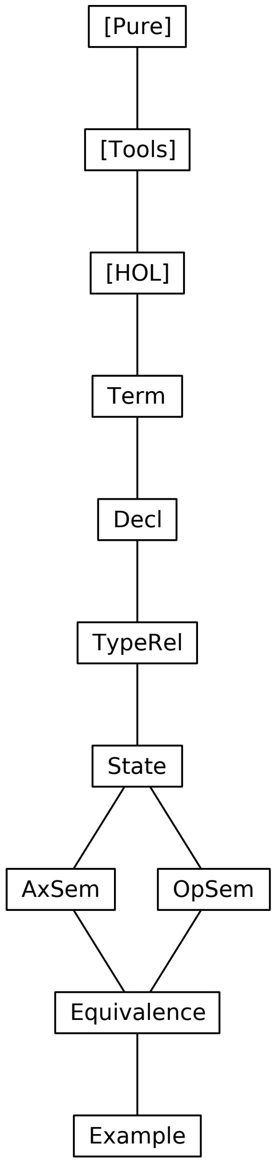
[Pure]
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Term
Decl
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1 Statements and expression emulations

theory *Term* **imports** *Main* **begin**

```
typedecl cname  — class name
typedecl mname  — method name
typedecl fname  — field name
typedecl vname  — variable name
```

```
axiomatization
  This — This pointer
  Par  — method parameter
  Res  :: vname — method result
 — Inequality axioms are not required for the meta theory.
```

```
datatype stmt
  = Skip                          — empty statement
  | Comp       stmt stmt   (‹_;; _›               [91,90   ] 90)
  | Cond expr  stmt stmt   (‹If ’(_’) _ Else _› [ 3,91,91] 91)
  | Loop vname stmt        (‹While ’(_’) _›      [ 3,91   ] 91)
  | LAss vname expr        (‹_ :== _›             [99,   95] 94) — local assignment
  | FAss expr  fname expr  (‹_.._:==_›            [95,99,95] 94) — field assignment
  | Meth "cname × mname"    — virtual method
  | Impl "cname × mname"    — method implementation
and expr
  = NewC cname       (‹new _›          [   99] 95) — object creation
  | Cast cname expr                               — type cast
  | LAcc vname                                    — local access
  | FAcc expr  fname (‹_.._›           [95,99] 95) — field access
  | Call cname expr mname expr
                     (‹{_}_.._’(_’)› [99,95,99,95] 95) — method call

end
```

2 Types, class Declarations, and whole programs

theory *Decl* **imports** *Term* **begin**

```
datatype ty
  = NT          — null type
  | Class cname — class type
```

Field declaration

```
type_synonym fdecl
        = "fname × ty"
```

```
record  methd
        = par :: ty
          res :: ty
          lcl ::"(vname × ty) list"
          bdy :: stmt
```

Method declaration

```
type_synonym mdecl
        = "mname × methd"
```

```
record  "class"
```

```
        = super   :: cname
          flds    ::"fdecl list"
          methods ::"mdecl list"
```

Class declaration

```
type_synonym cdecl
        = "cname × class"

type_synonym prog
        = "cdecl list"

translations
  (type) "fdecl" ↽ (type) "fname × ty"
  (type) "mdecl" ↽ (type) "mname × ty × ty × stmt"
  (type) "class" ↽ (type) "cname × fdecl list × mdecl list"
  (type) "cdecl" ↽ (type) "cname × class"
  (type) "prog " ↽ (type) "cdecl list"

axiomatization
  Prog    :: prog          — program as a global value
and
  Object  :: cname         — name of root class


definition "class" :: "cname ⇀ class" where
 "class      ≡ map_of Prog"

definition is_class   :: "cname => bool" where
 "is_class C ≡ class C ≠ None"

lemma finite_is_class: "finite {C. is_class C}"
⟨proof⟩

end
```

3 Type relations

```
theory TypeRel
imports Decl
begin
```

Direct subclass relation

```
definition subcls1 :: "(cname × cname) set"
where
  "subcls1 ≡ {(C,D). C≠Object ∧ (∃c. class C = Some c ∧ super c=D)}"

abbreviation
  subcls1_syntax :: "[cname, cname] => bool"  (‹_ ≺C1 _› [71,71] 70)
  where "C ≺C1 D == (C,D) ∈ subcls1"
abbreviation
  subcls_syntax  :: "[cname, cname] => bool" (‹_ ≼C _›  [71,71] 70)
  where "C ≼C D ≡ (C,D) ∈ subcls1*"
```

3.1 Declarations and properties not used in the meta theory

Widening, viz. method invocation conversion

```
inductive
```

```
  widen :: "ty => ty => bool"  (‹_ ⪯ _› [71,71] 70)
where
  refl [intro!, simp]: "T ⪯ T"
| subcls: "C⪯C D ⟹ Class C ⪯ Class D"
| null [intro!]: "NT ⪯ R"
```

lemma *subcls1D:*
`"C≺C1D ⟹ C ≠ Object ∧ (∃c. class C = Some c ∧ super c=D)"`
⟨*proof*⟩

lemma *subcls1I:* `"⟦class C = Some m; super m = D; C ≠ Object⟧ ⟹ C≺C1D"`
⟨*proof*⟩

lemma *subcls1_def2:*
```
  "subcls1 =
    (SIGMA C: {C. is_class C} . {D. C≠Object ∧ super (the (class C)) = D})"
```
⟨*proof*⟩

lemma *finite_subcls1:* `"finite subcls1"`
⟨*proof*⟩

definition *ws_prog* :: `"bool"` **where**
```
 "ws_prog ≡ ∀(C,c)∈set Prog. C≠Object ⟶
                              is_class (super c) ∧ (super c,C)∉subcls1⁺"
```

lemma *ws_progD:* `"⟦class C = Some c; C≠Object; ws_prog⟧ ⟹`
`  is_class (super c) ∧ (super c,C)∉subcls1⁺"`
⟨*proof*⟩

lemma *subcls1_irrefl_lemma1:* `"ws_prog ⟹ subcls1⁻¹ ∩ subcls1⁺ = {}"`
⟨*proof*⟩

lemma *irrefl_tranclI':* `"r⁻¹ ∩ r⁺ = {} ⟹ ∀x. (x, x) ∉ r⁺"`
⟨*proof*⟩

lemmas *subcls1_irrefl_lemma2 = subcls1_irrefl_lemma1 [THEN irrefl_tranclI']*

lemma *subcls1_irrefl:* `"⟦(x, y) ∈ subcls1; ws_prog⟧ ⟹ x ≠ y"`
⟨*proof*⟩

lemmas *subcls1_acyclic = subcls1_irrefl_lemma2 [THEN acyclicI]*

lemma *wf_subcls1:* `"ws_prog ⟹ wf (subcls1⁻¹)"`
⟨*proof*⟩

definition *class_rec* ::`"cname ⇒ (class ⇒ ('a × 'b) list) ⇒ ('a ⇀ 'b)"`
where
```
  "class_rec ≡ wfrec (subcls1⁻¹) (λrec C f.
     case class C of None ⇒ undefined
      | Some m ⇒ (if C = Object then Map.empty else rec (super m) f) ++ map_of (f m))"
```

lemma *class_rec:* `"⟦class C = Some m;  ws_prog⟧ ⟹`
```
 class_rec C f = (if C = Object then Map.empty else class_rec (super m) f) ++
                 map_of (f m)"
```
⟨*proof*⟩
definition `"method"` :: `"cname => (mname ⇀ methd)"` **where**
`  "method C ≡ class_rec C methods"`

lemma *method_rec:* "⟦*class C = Some m; ws_prog*⟧ ⟹
method C = (if C=Object then Map.empty else method (super m)) ++ map_of (methods m)"
⟨*proof*⟩
definition *field :: "cname => (fname ⇀ ty)"* **where**
"field C ≡ class_rec C flds"

lemma *flds_rec:* "⟦*class C = Some m; ws_prog*⟧ ⟹
field C = (if C=Object then Map.empty else field (super m)) ++ map_of (flds m)"
⟨*proof*⟩

end

4 Program State

theory *State* **imports** *TypeRel* **begin**

definition *body :: "cname × mname => stmt"* **where**
"body ≡ λ(C,m). bdy (the (method C m))"

Locations, i.e. abstract references to objects

typedecl *loc*

datatype *val*
= Null — null reference
| Addr loc — address, i.e. location of object

type_synonym *fields*
= "(fname ⇀ val)"

type_synonym
obj = "cname × fields"

translations
(type) "fields" ↽ (type) "fname => val option"
(type) "obj" ↽ (type) "cname × fields"

definition *init_vars :: "('a ⇀ 'b) => ('a ⇀ val)"* **where**
"init_vars m == map_option (λT. Null) o m"

private:

type_synonym *heap = "loc ⇀ obj"*
type_synonym *locals = "vname ⇀ val"*

private:

record *state*
= heap :: heap
locals :: locals

translations
(type) "heap" ↽ (type) "loc => obj option"
(type) "locals" ↽ (type) "vname => val option"
(type) "state" ↽ (type) "(|heap :: heap, locals :: locals|)"

definition *del_locs :: "state => state"* **where**
"del_locs s ≡ s (| locals := Map.empty |)"

definition *init_locs :: "cname => mname => state => state"* **where**

```
 "init_locs C m s ≡ s (| locals := locals s ++
                         init_vars (map_of (lcl (the (method C m)))) |)"
```

The first parameter of `set_locs` is of type `state` rather than `locals` in order to keep `locals` private.

```
definition set_locs :: "state => state => state" where
 "set_locs s s' ≡ s' (| locals := locals s |)"
```

```
definition get_local     :: "state => vname => val" (‹_<_>› [99,0] 99) where
 "get_local s x  ≡ the (locals s x)"
```

— local function:

```
definition get_obj       :: "state => loc => obj" where
 "get_obj s a ≡ the (heap s a)"
```

```
definition obj_class     :: "state => loc => cname" where
 "obj_class s a ≡ fst (get_obj s a)"
```

```
definition get_field     :: "state => loc => fname => val" where
 "get_field s a f ≡ the (snd (get_obj s a) f)"
```

— local function:

```
definition hupd       :: "loc => obj => state => state"   (‹hupd'(_↦_')› [10,10] 1000) where
 "hupd a obj s ≡ s (| heap   := ((heap   s)(a↦obj))|)"
```

```
definition lupd       :: "vname => val => state => state" (‹lupd'(_↦_')› [10,10] 1000) where
 "lupd x v s   ≡ s (| locals := ((locals s)(x↦v  ))|)"
```

```
definition new_obj :: "loc => cname => state => state" where
 "new_obj a C   ≡ hupd(a↦(C,init_vars (field C)))"
```

```
definition upd_obj    :: "loc => fname => val => state => state" where
 "upd_obj a f v s ≡ let (C,fs) = the (heap s a) in hupd(a↦(C,fs(f↦v))) s"
```

```
definition new_Addr      :: "state => val" where
 "new_Addr s == SOME v. (∃a. v = Addr a ∧ (heap s) a = None) | v = Null"
```

4.1 Properties not used in the meta theory

```
lemma locals_upd_id [simp]: "s⦇locals := locals s⦈ = s"
```
⟨*proof*⟩

```
lemma lupd_get_local_same [simp]: "lupd(x↦v) s<x> = v"
```
⟨*proof*⟩

```
lemma lupd_get_local_other [simp]: "x ≠ y ⟹ lupd(x↦v) s<y> = s<y>"
```
⟨*proof*⟩

```
lemma get_field_lupd [simp]:
  "get_field (lupd(x↦y) s) a f = get_field s a f"
```
⟨*proof*⟩

```
lemma get_field_set_locs [simp]:
  "get_field (set_locs l s) a f = get_field s a f"
```
⟨*proof*⟩

```
lemma get_field_del_locs [simp]:
  "get_field (del_locs s) a f = get_field s a f"
```
⟨*proof*⟩

lemma *new_obj_get_local [simp]: "new_obj a C s <x> = s<x>"*
⟨*proof*⟩

lemma *heap_lupd [simp]: "heap (lupd(x↦y) s) = heap s"*
⟨*proof*⟩

lemma *heap_hupd_same [simp]: "heap (hupd(a↦obj) s) a = Some obj"*
⟨*proof*⟩

lemma *heap_hupd_other [simp]: "aa ≠ a ⟹ heap (hupd(aa↦obj) s) a = heap s a"*
⟨*proof*⟩

lemma *hupd_hupd [simp]: "hupd(a↦obj) (hupd(a↦obj') s) = hupd(a↦obj) s"*
⟨*proof*⟩

lemma *heap_del_locs [simp]: "heap (del_locs s) = heap s"*
⟨*proof*⟩

lemma *heap_set_locs [simp]: "heap (set_locs l s) = heap s"*
⟨*proof*⟩

lemma *hupd_lupd [simp]:*
"hupd(a↦obj) (lupd(x↦y) s) = lupd(x↦y) (hupd(a↦obj) s)"
⟨*proof*⟩

lemma *hupd_del_locs [simp]:*
"hupd(a↦obj) (del_locs s) = del_locs (hupd(a↦obj) s)"
⟨*proof*⟩

lemma *new_obj_lupd [simp]:*
"new_obj a C (lupd(x↦y) s) = lupd(x↦y) (new_obj a C s)"
⟨*proof*⟩

lemma *new_obj_del_locs [simp]:*
"new_obj a C (del_locs s) = del_locs (new_obj a C s)"
⟨*proof*⟩

lemma *upd_obj_lupd [simp]:*
"upd_obj a f v (lupd(x↦y) s) = lupd(x↦y) (upd_obj a f v s)"
⟨*proof*⟩

lemma *upd_obj_del_locs [simp]:*
"upd_obj a f v (del_locs s) = del_locs (upd_obj a f v s)"
⟨*proof*⟩

lemma *get_field_hupd_same [simp]:*
"get_field (hupd(a↦(C, fs)) s) a = the ∘ fs"
⟨*proof*⟩

lemma *get_field_hupd_other [simp]:*
"aa ≠ a ⟹ get_field (hupd(aa↦obj) s) a = get_field s a"
⟨*proof*⟩

lemma *new_AddrD:*
"new_Addr s = v ⟹ (∃a. v = Addr a ∧ heap s a = None) | v = Null"
⟨*proof*⟩

end

5 Operational Evaluation Semantics

theory *OpSem* **imports** *State* **begin**

```
inductive
  exec :: "[state,stmt,    nat,state] => bool" (<_ -_-_→ _>  [98,90,   65,98] 89)
  and eval :: "[state,expr,val,nat,state] => bool" (<_ -_≻_-_→ _>[98,95,99,65,98] 89)
where
  Skip: "   s -Skip-n→ s"

| Comp: "[| s0 -c1-n→ s1; s1 -c2-n→ s2 |] ==>
          s0 -c1;; c2-n→ s2"

| Cond: "[| s0 -e≻v-n→ s1; s1 -(if v≠Null then c1 else c2)-n→ s2 |] ==>
          s0 -If(e) c1 Else c2-n→ s2"

| LoopF:"   s0<x> = Null ==>
          s0 -While(x) c-n→ s0"
| LoopT:"[| s0<x> ≠ Null; s0 -c-n→ s1; s1 -While(x) c-n→ s2 |] ==>
          s0 -While(x) c-n→ s2"

| LAcc: "   s -LAcc x≻s<x>-n→ s"

| LAss: "   s -e≻v-n→ s' ==>
          s -x:==e-n→ lupd(x↦v) s'"

| FAcc: "   s -e≻Addr a-n→ s' ==>
          s -e..f≻get_field s' a f-n→ s'"

| FAss: "[| s0 -e1≻Addr a-n→ s1;  s1 -e2≻v-n→ s2 |] ==>
          s0 -e1..f:==e2-n→ upd_obj a f v s2"

| NewC: "   new_Addr s = Addr a ==>
          s -new C≻Addr a-n→ new_obj a C s"

| Cast: "[| s -e≻v-n→ s';
          case v of Null => True | Addr a => obj_class s' a ≼C C |] ==>
          s -Cast C e≻v-n→ s'"

| Call: "[| s0 -e1≻a-n→ s1; s1 -e2≻p-n→ s2;
          lupd(This↦a)(lupd(Par↦p)(del_locs s2)) -Meth (C,m)-n→ s3
     |] ==> s0 -{C}e1..m(e2)≻s3<Res>-n→ set_locs s2 s3"

| Meth: "[| s<This> = Addr a; D = obj_class s a; D≼C C;
          init_locs D m s -Impl (D,m)-n→ s' |] ==>
          s -Meth (C,m)-n→ s'"

| Impl: "   s -body Cm-    n→ s' ==>
          s -Impl Cm-Suc n→ s'"


inductive_cases exec_elim_cases':
                               "s -Skip            -n→ t"
                               "s -c1;; c2         -n→ t"
                               "s -If(e) c1 Else c2-n→ t"
                               "s -While(x) c      -n→ t"
                               "s -x:==e           -n→ t"
                               "s -e1..f:==e2      -n→ t"
inductive_cases Meth_elim_cases:  "s -Meth Cm          -n→ t"
```

```
inductive_cases Impl_elim_cases:  "s -Impl Cm         -n→ t"
lemmas exec_elim_cases = exec_elim_cases' Meth_elim_cases Impl_elim_cases
inductive_cases eval_elim_cases:
                                   "s -new C         ≻v-n→ t"
                                   "s -Cast C e      ≻v-n→ t"
                                   "s -LAcc x        ≻v-n→ t"
                                   "s -e..f          ≻v-n→ t"
                                   "s -{C}e1..m(e2)  ≻v-n→ t"

lemma exec_eval_mono [rule_format]:
  "(s -c  -n→ t ⟶ (∀m. n ≤ m ⟶ s -c  -m→ t)) ∧
   (s -e≻v-n→ t ⟶ (∀m. n ≤ m ⟶ s -e≻v-m→ t))"
⟨proof⟩
lemmas exec_mono = exec_eval_mono [THEN conjunct1, rule_format]
lemmas eval_mono = exec_eval_mono [THEN conjunct2, rule_format]

lemma exec_exec_max: "⟦s1 -c1-    n1    → t1 ; s2 -c2-       n2→ t2⟧ ⟹
                       s1 -c1-max n1 n2→ t1 ∧ s2 -c2-max n1 n2→ t2"
⟨proof⟩

lemma eval_exec_max: "⟦s1 -c-    n1    → t1 ; s2 -e≻v-       n2→ t2⟧ ⟹
                       s1 -c-max n1 n2→ t1 ∧ s2 -e≻v-max n1 n2→ t2"
⟨proof⟩

lemma eval_eval_max: "⟦s1 -e1≻v1-    n1    → t1 ; s2 -e2≻v2-       n2→ t2⟧ ⟹
                       s1 -e1≻v1-max n1 n2→ t1 ∧ s2 -e2≻v2-max n1 n2→ t2"
⟨proof⟩

lemma eval_eval_exec_max:
 "⟦s1 -e1≻v1-n1→ t1; s2 -e2≻v2-n2→ t2; s3 -c-n3→ t3⟧ ⟹
   s1 -e1≻v1-max (max n1 n2) n3→ t1 ∧
   s2 -e2≻v2-max (max n1 n2) n3→ t2 ∧
   s3 -c    -max (max n1 n2) n3→ t3"
⟨proof⟩

lemma Impl_body_eq: "(λt. ∃n. Z -Impl M-n→ t) = (λt. ∃n. Z -body M-n→ t)"
⟨proof⟩

end
```

6 Axiomatic Semantics

```
theory AxSem imports State begin

type_synonym assn = "state => bool"
type_synonym vassn = "val => assn"
type_synonym triple = "assn × stmt ×  assn"
type_synonym etriple = "assn × expr × vassn"
translations
  (type) "assn" ↽ (type) "state => bool"
  (type) "vassn" ↽ (type) "val => assn"
  (type) "triple" ↽ (type) "assn × stmt × assn"
  (type) "etriple" ↽ (type) "assn × expr × vassn"
```

6.1 Hoare Logic Rules

```
inductive
 hoare :: "[triple set, triple set] => bool"  (<_ |⊢/ _> [61, 61] 60)
```

```
 and ehoare :: "[triple set, etriple] => bool"  (<_ |⊢e/ _> [61, 61] 60)
 and hoare1 :: "[triple set, assn,stmt,assn] => bool"
   (<_ ⊢/ ({(1_)}/ (_)/ {(1_)})> [61, 3, 90, 3] 60)
 and ehoare1 :: "[triple set, assn,expr,vassn]=> bool"
   (<_ ⊢e/ ({(1_)}/ (_)/ {(1_)})> [61, 3, 90, 3] 60)
where

  "A  ⊢ {P}c{Q} ≡ A |⊢ {(P,c,Q)}"
| "A  ⊢e {P}e{Q} ≡ A |⊢e (P,e,Q)"

| Skip:  "A ⊢ {P} Skip {P}"

| Comp: "[| A ⊢ {P} c1 {Q}; A ⊢ {Q} c2 {R} |] ==> A ⊢ {P} c1;;c2 {R}"

| Cond: "[| A ⊢e {P} e {Q};
            ∀v. A ⊢ {Q v} (if v ≠ Null then c1 else c2) {R} |] ==>
            A ⊢ {P} If(e) c1 Else c2 {R}"

| Loop: "A ⊢ {λs. P s ∧ s<x> ≠ Null} c {P} ==>
         A ⊢ {P} While(x) c {λs. P s ∧ s<x> = Null}"

| LAcc: "A ⊢e {λs. P (s<x>) s} LAcc x {P}"

| LAss: "A ⊢e {P} e {λv s.  Q (lupd(x↦v) s)} ==>
         A ⊢  {P} x:==e {Q}"

| FAcc: "A ⊢e {P} e {λv s. ∀a. v=Addr a --> Q (get_field s a f) s} ==>
         A ⊢e {P} e..f {Q}"

| FAss: "[| A ⊢e {P} e1 {λv s. ∀a. v=Addr a --> Q a s};
        ∀a. A ⊢e {Q a} e2 {λv s. R (upd_obj a f v s)} |] ==>
            A ⊢  {P} e1..f:==e2 {R}"

| NewC: "A ⊢e {λs. ∀a. new_Addr s = Addr a --> P (Addr a) (new_obj a C s)}
                new C {P}"

| Cast: "A ⊢e {P} e {λv s. (case v of Null => True
                                  | Addr a => obj_class s a ≼C C) --> Q v s} ==>
         A ⊢e {P} Cast C e {Q}"

| Call: "[| A ⊢e {P} e1 {Q}; ∀a. A ⊢e {Q a} e2 {R a};
    ∀a p ls. A ⊢ {λs'. ∃s. R a p s ∧ ls = s ∧
                    s' = lupd(This↦a)(lupd(Par↦p)(del_locs s))}
                  Meth (C,m) {λs. S (s<Res>) (set_locs ls s)} |] ==>
             A ⊢e {P} {C}e1..m(e2) {S}"

| Meth: "∀D. A ⊢ {λs'. ∃s a. s<This> = Addr a ∧ D = obj_class s a ∧ D ≼C C ∧
                        P s ∧ s' = init_locs D m s}
                  Impl (D,m) {Q} ==>
             A ⊢ {P} Meth (C,m) {Q}"
```

— $\bigcup Z$ instead of $\forall Z$ in the conclusion and
Z restricted to type state due to limitations of the inductive package

```
| Impl: "∀Z::state. A∪ (⋃Z. (λCm. (P Z Cm, Impl Cm, Q Z Cm))`Ms) |⊢
                        (λCm. (P Z Cm, body Cm, Q Z Cm))`Ms ==>
                   A |⊢ (λCm. (P Z Cm, Impl Cm, Q Z Cm))`Ms"
```

— structural rules

```
| Asm:  "   a ∈ A ==> A |⊢ {a}"

| ConjI: " ∀c ∈ C. A |⊢ {c} ==> A |⊢ C"

| ConjE: "[|A |⊢ C; c ∈ C |] ==> A |⊢ {c}"

  — Z restricted to type state due to limitations of the inductive package
| Conseq:"[| ∀Z::state. A ⊢ {P' Z} c {Q' Z};
            ∀s t. (∀Z. P' Z s --> Q' Z t) --> (P s --> Q t) |] ==>
           A ⊢ {P} c {Q }"

  — Z restricted to type state due to limitations of the inductive package
| eConseq:"[| ∀Z::state. A ⊢e {P' Z} e {Q' Z};
            ∀s v t. (∀Z. P' Z s --> Q' Z v t) --> (P s --> Q v t) |] ==>
           A ⊢e {P} e {Q }"
```

6.2 Fully polymorphic variants, required for Example only

```
axiomatization where
  Conseq:"[| ∀Z. A ⊢ {P' Z} c {Q' Z};
             ∀s t. (∀Z. P' Z s --> Q' Z t) --> (P s --> Q t) |] ==>
                 A ⊢ {P} c {Q }"

axiomatization where
  eConseq:"[| ∀Z. A ⊢e {P' Z} e {Q' Z};
             ∀s v t. (∀Z. P' Z s --> Q' Z v t) --> (P s --> Q v t) |] ==>
                 A ⊢e {P} e {Q }"

axiomatization where
  Impl: "∀Z. A∪ (⋃Z. (λCm. (P Z Cm, Impl Cm, Q Z Cm))`Ms) |⊢
                          (λCm. (P Z Cm, body Cm, Q Z Cm))`Ms ==>
                    A |⊢ (λCm. (P Z Cm, Impl Cm, Q Z Cm))`Ms"
```

6.3 Derived Rules

```
lemma Conseq1: "⟦A ⊢ {P'} c {Q}; ∀s. P s ⟶ P' s⟧ ⟹ A ⊢ {P} c {Q}"
⟨proof⟩

lemma Conseq2: "⟦A ⊢ {P} c {Q'}; ∀t. Q' t ⟶ Q t⟧ ⟹ A ⊢ {P} c {Q}"
⟨proof⟩

lemma eConseq1: "⟦A ⊢e {P'} e {Q}; ∀s. P s ⟶ P' s⟧ ⟹ A ⊢e {P} e {Q}"
⟨proof⟩

lemma eConseq2: "⟦A ⊢e {P} e {Q'}; ∀v t. Q' v t ⟶ Q v t⟧ ⟹ A ⊢e {P} e {Q}"
⟨proof⟩

lemma Weaken: "⟦A |⊢ C'; C ⊆ C'⟧ ⟹ A |⊢ C"
⟨proof⟩

lemma Thin_lemma:
  "(A' |⊢  C          ⟶ (∀A. A' ⊆ A ⟶ A |⊢  C       )) ∧
   (A'  ⊢e {P} e {Q} ⟶ (∀A. A' ⊆ A ⟶ A  ⊢e {P} e {Q}))"
⟨proof⟩

lemma cThin: "⟦A' |⊢ C; A' ⊆ A⟧ ⟹ A |⊢ C"
⟨proof⟩

lemma eThin: "⟦A' ⊢e {P} e {Q}; A' ⊆ A⟧ ⟹ A ⊢e {P} e {Q}"
```

⟨*proof*⟩

lemma *Union: "A |⊢ (⋃Z. C Z) = (∀Z. A |⊢ C Z)"*
⟨*proof*⟩

lemma *Impl1':*
```
  "⟦∀Z::state. A∪ (⋃Z. (λCm. (P Z Cm, Impl Cm, Q Z Cm))‘Ms) |⊢
                 (λCm. (P Z Cm, body Cm, Q Z Cm))‘Ms;
   Cm ∈ Ms⟧ ⟹
                A   ⊢  {P Z Cm} Impl Cm {Q Z Cm}"
```
⟨*proof*⟩

lemmas *Impl1 = AxSem.Impl [of _ _ _ "{Cm}", simplified]* **for** *Cm*

end

7 Equivalence of Operational and Axiomatic Semantics

theory *Equivalence* **imports** *OpSem AxSem* **begin**

7.1 Validity

definition *valid :: "[assn,stmt, assn] => bool" (‹⊨ {(1_)}/ (_)/ {(1_)}› [3,90,3] 60)* **where**
"⊨ {P} c {Q} ≡ ∀s   t. P s --> (∃n. s -c -n→ t) --> Q   t"

definition *evalid :: "[assn,expr,vassn] => bool" (‹⊨ₑ {(1_)}/ (_)/ {(1_)}› [3,90,3] 60)* **where**
"⊨ₑ {P} e {Q} ≡ ∀s v t. P s --> (∃n. s -e≻v-n→ t) --> Q v t"

definition *nvalid :: "[nat, triple   ] => bool" (‹⊨_: _› [61,61] 60)* **where**
"⊨n: t ≡ let (P,c,Q) = t in ∀s   t. s -c -n→ t --> P s --> Q   t"

definition *envalid :: "[nat,etriple   ] => bool" (‹⊨_:ₑ _› [61,61] 60)* **where**
"⊨n:ₑ t ≡ let (P,e,Q) = t in ∀s v t. s -e≻v-n→ t --> P s --> Q v t"

definition *nvalids :: "[nat,   triple set] => bool" (‹|⊨_: _› [61,61] 60)* **where**
"|⊨n: T ≡ ∀t∈T. ⊨n: t"

definition *cnvalids :: "[triple set,triple set] => bool" (‹_ |⊨/ _› [61,61] 60)* **where**
"A |⊨ C ≡ ∀n. |⊨n: A --> |⊨n: C"

definition *cenvalid :: "[triple set,etriple   ] => bool" (‹_ |⊨ₑ/ _›[61,61] 60)* **where**
"A |⊨ₑ t ≡ ∀n. |⊨n: A --> ⊨n:ₑ t"

lemma *nvalid_def2: "⊨n: (P,c,Q) ≡ ∀s t. s -c-n→ t ⟶ P s ⟶ Q t"*
⟨*proof*⟩

lemma *valid_def2: "⊨ {P} c {Q} = (∀n. ⊨n: (P,c,Q))"*
⟨*proof*⟩

lemma *envalid_def2: "⊨n:ₑ (P,e,Q) ≡ ∀s v t. s -e≻v-n→ t ⟶ P s ⟶ Q v t"*
⟨*proof*⟩

lemma *evalid_def2: "⊨ₑ {P} e {Q} = (∀n. ⊨n:ₑ (P,e,Q))"*
⟨*proof*⟩

lemma *cenvalid_def2:*
"A|⊨ₑ (P,e,Q) = (∀n. |⊨n: A ⟶ (∀s v t. s -e≻v-n→ t ⟶ P s ⟶ Q v t))"

⟨*proof*⟩

7.2 Soundness

declare *exec_elim_cases [elim!] eval_elim_cases [elim!]*

lemma *Impl_nvalid_0: "⊨0: (P,Impl M,Q)"*
⟨*proof*⟩

lemma *Impl_nvalid_Suc: "⊨n: (P,body M,Q) ⟹ ⊨Suc n: (P,Impl M,Q)"*
⟨*proof*⟩

lemma *nvalid_SucD: "⋀t. ⊨Suc n:t ⟹ ⊨n:t"*
⟨*proof*⟩

lemma *nvalids_SucD: "Ball A (nvalid (Suc n)) ⟹ Ball A (nvalid n)"*
⟨*proof*⟩

lemma *Loop_sound_lemma [rule_format (no_asm)]:*
"∀s t. s -c-n→ t ⟶ P s ∧ s<x> ≠ Null ⟶ P t ⟹
(s -c0-n0→ t ⟶ P s ⟶ c0 = While (x) c ⟶ n0 = n ⟶ P t ∧ t<x> = Null)"
⟨*proof*⟩

lemma *Impl_sound_lemma:*
"⟦∀z n. Ball (A ∪ B) (nvalid n) ⟶ Ball (f z ‘ Ms) (nvalid n);
Cm∈Ms; Ball A (nvalid na); Ball B (nvalid na)⟧ ⟹ nvalid na (f z Cm)"
⟨*proof*⟩

lemma *all_conjunct2: "∀l. P' l ∧ P l ⟹ ∀l. P l"*
⟨*proof*⟩

lemma *all3_conjunct2:*
"∀a p l. (P' a p l ∧ P a p l) ⟹ ∀a p l. P a p l"
⟨*proof*⟩

lemma *cnvalid1_eq:*
"A |⊨ {(P,c,Q)} ≡ ∀n. |⊨n: A ⟶ (∀s t. s -c-n→ t ⟶ P s ⟶ Q t)"
⟨*proof*⟩

lemma *hoare_sound_main:"⋀t. (A |⊢ C ⟶ A |⊨ C) ∧ (A |⊢$_e$ t ⟶ A |⊨$_e$ t)"*
⟨*proof*⟩

theorem *hoare_sound: "{} ⊢ {P} c {Q} ⟹ ⊨ {P} c {Q}"*
⟨*proof*⟩

theorem *ehoare_sound: "{} ⊢$_e$ {P} e {Q} ⟹ ⊨$_e$ {P} e {Q}"*
⟨*proof*⟩

7.3 (Relative) Completeness

definition *MGT :: "stmt => state => triple"* **where**
"MGT c Z ≡ (λs. Z = s, c, λ t. ∃n. Z -c- n→ t)"

definition *MGT$_e$ :: "expr => state => etriple"* **where**
"MGT$_e$ e Z ≡ (λs. Z = s, e, λv t. ∃n. Z -e≻v-n→ t)"

lemma *MGF_implies_complete:*
"∀Z. {} |⊢ { MGT c Z} ⟹ ⊨ {P} c {Q} ⟹ {} ⊢ {P} c {Q}"
⟨*proof*⟩

lemma *eMGF_implies_complete:*
"$\forall Z.$ {} |⊢$_e$ MGT$_e$ e Z ⟹ ⊨$_e$ {P} e {Q} ⟹ {} ⊢$_e$ {P} e {Q}"
⟨*proof*⟩

declare *exec_eval.intros[intro!]*

lemma *MGF_Loop:* "$\forall Z.$ A ⊢ {(=) Z} c {λt. ∃n. Z -c-n→ t} ⟹
A ⊢ {(=) Z} While (x) c {λt. ∃n. Z -While (x) c-n→ t}"
⟨*proof*⟩

lemma *MGF_lemma:* "∀M Z. A |⊢ {MGT (Impl M) Z} ⟹
(∀Z. A |⊢ {MGT c Z}) ∧ (∀Z. A |⊢$_e$ MGT$_e$ e Z)"
⟨*proof*⟩

lemma *MGF_Impl:* "{} |⊢ {MGT (Impl M) Z}"
⟨*proof*⟩

theorem *hoare_relative_complete:* "⊨ {P} c {Q} ⟹ {} ⊢ {P} c {Q}"
⟨*proof*⟩

theorem *ehoare_relative_complete:* "⊨$_e$ {P} e {Q} ⟹ {} ⊢$_e$ {P} e {Q}"
⟨*proof*⟩

lemma *cFalse:* "A ⊢ {λs. False} c {Q}"
⟨*proof*⟩

lemma *eFalse:* "A ⊢$_e$ {λs. False} e {Q}"
⟨*proof*⟩

end

8 Example

theory *Example*
imports *Equivalence*
begin

```
class Nat {

  Nat pred;

  Nat suc()
    { Nat n = new Nat(); n.pred = this; return n; }

  Nat eq(Nat n)
    { if (this.pred != null) if (n.pred != null) return this.pred.eq(n.pred);
                               else return n.pred; // false
      else if (n.pred != null) return this.pred; // false
           else return this.suc(); // true
    }

  Nat add(Nat n)
    { if (this.pred != null) return this.pred.add(n.suc()); else return n; }

  public static void main(String[] args) // test x+1=1+x
```

```
    {
        Nat one = new Nat().suc();
        Nat x   = new Nat().suc().suc().suc().suc();
        Nat ok = x.suc().eq(x.add(one));
        System.out.println(ok != null);
    }
}
```

```
axiomatization where
  This_neq_Par [simp]: "This ≠ Par" and
  Res_neq_This [simp]: "Res  ≠ This"
```

8.1 Program representation

```
axiomatization
  N    :: cname (‹Nat›)
  and pred :: fname
  and suc add :: mname
  and any  :: vname

abbreviation
  dummy :: expr (‹<>›)
  where "<> == LAcc any"

abbreviation
  one :: expr
  where "one == {Nat}new Nat..suc(<>)"
```

The following properties could be derived from a more complete program model, which we leave out for laziness.

```
axiomatization where Nat_no_subclasses [simp]: "D ≼C Nat = (D=Nat)"

axiomatization where method_Nat_add [simp]: "method Nat add = Some
  ⦇ par=Class Nat, res=Class Nat, lcl=[],
   bdy= If((LAcc This..pred))
          (Res :== {Nat}(LAcc This..pred)..add({Nat}LAcc Par..suc(<>)))
        Else Res :== LAcc Par ⦈"

axiomatization where method_Nat_suc [simp]: "method Nat suc = Some
  ⦇ par=NT, res=Class Nat, lcl=[],
   bdy= Res :== new Nat;; LAcc Res..pred :== LAcc This ⦈"

axiomatization where field_Nat [simp]: "field Nat = Map.empty(pred↦Class Nat)"

lemma init_locs_Nat_add [simp]: "init_locs Nat add s = s"
⟨proof⟩

lemma init_locs_Nat_suc [simp]: "init_locs Nat suc s = s"
⟨proof⟩

lemma upd_obj_new_obj_Nat [simp]:
  "upd_obj a pred v (new_obj a Nat s) = hupd(a↦(Nat, Map.empty(pred↦v))) s"
⟨proof⟩
```

8.2 "atleast" relation for interpretation of Nat "values"

```
primrec Nat_atleast :: "state ⇒ val ⇒ nat ⇒ bool" (‹_:_ ≥ _› [51, 51, 51] 50) where
  "s:x≥0     = (x≠Null)"
```

| "s:x≥Suc n = (∃a. x=Addr a ∧ heap s a ≠ None ∧ s:get_field s a pred≥n)"

lemma *Nat_atleast_lupd [rule_format, simp]:*
"∀s v::val. lupd(x↦y) s:v ≥ n = (s:v ≥ n)"
⟨*proof*⟩

lemma *Nat_atleast_set_locs [rule_format, simp]:*
"∀s v::val. set_locs l s:v ≥ n = (s:v ≥ n)"
⟨*proof*⟩

lemma *Nat_atleast_del_locs [rule_format, simp]:*
"∀s v::val. del_locs s:v ≥ n = (s:v ≥ n)"
⟨*proof*⟩

lemma *Nat_atleast_NullD [rule_format]: "s:Null ≥ n ⟶ False"*
⟨*proof*⟩

lemma *Nat_atleast_pred_NullD [rule_format]:*
"Null = get_field s a pred ⟹ s:Addr a ≥ n ⟶ n = 0"
⟨*proof*⟩

lemma *Nat_atleast_mono [rule_format]:*
"∀a. s:get_field s a pred ≥ n ⟶ heap s a ≠ None ⟶ s:Addr a ≥ n"
⟨*proof*⟩

lemma *Nat_atleast_newC [rule_format]:*
"heap s aa = None ⟹ ∀v::val. s:v ≥ n ⟶ hupd(aa↦obj) s:v ≥ n"
⟨*proof*⟩

8.3 Proof(s) using the Hoare logic

theorem *add_homomorph_lb:*
"{} ⊢ {λs. s:s<This> ≥ X ∧ s:s<Par> ≥ Y} Meth(Nat,add) {λs. s:s<Res> ≥ X+Y}"
⟨*proof*⟩

end

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