Homology

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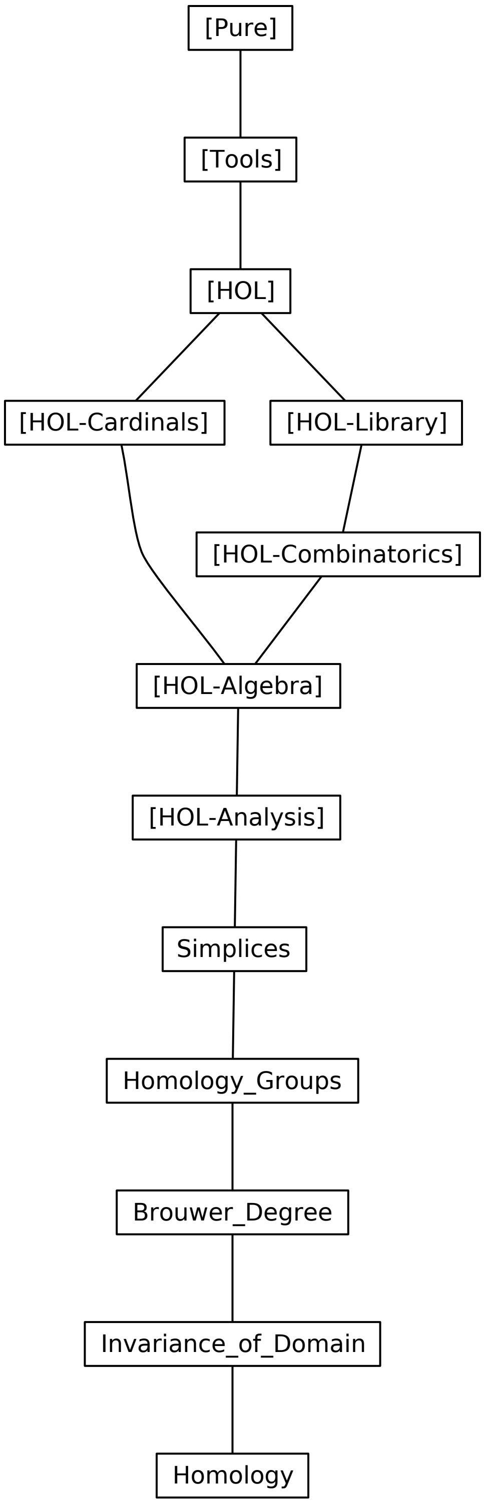
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0.1 Homology, I: Simplices

```
theory Simplices
  imports
    HOL−Analysis.Function_Metric
    HOL−Analysis.Abstract_Euclidean_Space
    HOL−Algebra.Free_Abelian_Groups
begin
```

0.1.1 Standard simplices, all of which are topological subspaces of R^n.

```
type_synonym 'a chain = ((nat ⇒ real) ⇒ 'a) ⇒₀ int

definition standard_simplex :: nat ⇒ (nat ⇒ real) set where
  standard_simplex p ≡
    {x. (∀ i. 0 ≤ x i ∧ x i ≤ 1) ∧ (∀ i>p. x i = 0) ∧ (∑ i≤p. x i) = 1}

lemma topspace_standard_simplex:
  topspace(subtopology (powertop_real UNIV) (standard_simplex p))
    = standard_simplex p
  by simp

lemma basis_in_standard_simplex [simp]:
   (λj. if j = i then 1 else 0) ∈ standard_simplex p ⟷ i ≤ p
  by (auto simp: standard_simplex_def)

lemma nonempty_standard_simplex: standard_simplex p ≠ {}
  using basis_in_standard_simplex by blast

lemma standard_simplex_0: standard_simplex 0 = {(λj. if j = 0 then 1 else 0)}
  by (auto simp: standard_simplex_def)

lemma standard_simplex_mono:
  assumes p ≤ q
  shows standard_simplex p ⊆ standard_simplex q
  using assms
proof (clarsimp simp: standard_simplex_def)
  fix x :: nat ⇒ real
  assume ∀ i. 0 ≤ x i ∧ x i ≤ 1 and ∀ i>p. x i = 0 and sum x {..p} = 1
  then show sum x {..q} = 1
    using sum.mono_neutral_left [of {..q} {..p} x] assms by auto
qed

lemma closedin_standard_simplex:
   closedin (powertop_real UNIV) (standard_simplex p)
    (is closedin ?X ?S)
proof −
  have eq: standard_simplex p =
```

```
              (⋂i. {x. x ∈ topspace ?X ∧ x i ∈ {0..1}}) ∩
              (⋂i ∈ {p<..}. {x ∈ topspace ?X. x i ∈ {0}}) ∩
              {x ∈ topspace ?X. (∑i≤p. x i) ∈ {1}}
    by (auto simp: standard_simplex_def topspace_product_topology)
  show ?thesis
    unfolding eq
    by (rule closedin_Int closedin_Inter continuous_map_sum
          continuous_map_product_projection closedin_continuous_map_preimage
| force | clarify)+
qed

lemma standard_simplex_01: standard_simplex p ⊆ UNIV →E {0..1}
  using standard_simplex_def by auto

lemma compactin_standard_simplex:
   compactin (powertop_real UNIV) (standard_simplex p)
proof (rule closed_compactin)
  show compactin (powertop_real UNIV) (UNIV →E {0..1})
    by (simp add: compactin_PiE)
  show standard_simplex p ⊆ UNIV →E {0..1}
    by (simp add: standard_simplex_01)
  show closedin (powertop_real UNIV) (standard_simplex p)
    by (simp add: closedin_standard_simplex)
qed

lemma convex_standard_simplex:
   ⟦x ∈ standard_simplex p; y ∈ standard_simplex p;
     0 ≤ u; u ≤ 1⟧
    ⟹ (λi. (1 − u) ∗ x i + u ∗ y i) ∈ standard_simplex p
 by (simp add: standard_simplex_def sum.distrib convex_bound_le flip: sum_distrib_left)

lemma path_connectedin_standard_simplex:
   path_connectedin (powertop_real UNIV) (standard_simplex p)
proof −
  define g where g ≡ λx y::nat⇒real. λu i. (1 − u) ∗ x i + u ∗ y i
  have continuous_map
              (subtopology euclideanreal {0..1}) (powertop_real UNIV)
              (g x y)
    if x ∈ standard_simplex p y ∈ standard_simplex p for x y
    unfolding g_def continuous_map_componentwise
    by (force intro: continuous_intros)
  moreover
  have g x y ∈ {0..1} → standard_simplex p g x y 0 = x g x y 1 = y
    if x ∈ standard_simplex p y ∈ standard_simplex p for x y
    using that by (auto simp: convex_standard_simplex g_def)
  ultimately
  show ?thesis
    unfolding path_connectedin_def path_connected_space_def pathin_def
   by (metis continuous_map_in_subtopology euclidean_product_topology top_greatest
```

```
topspace_euclidean topspace_euclidean_subtopology)
qed

lemma connectedin_standard_simplex:
  connectedin (powertop_real UNIV) (standard_simplex p)
 by (simp add: path_connectedin_imp_connectedin path_connectedin_standard_simplex)
```

0.1.2 Face map

```
definition simplical_face :: nat ⇒ (nat ⇒ 'a) ⇒ nat ⇒ 'a::comm_monoid_add
where
  simplical_face k x ≡ λi. if i < k then x i else if i = k then 0 else x(i −1)

lemma simplical_face_in_standard_simplex:
  assumes 1 ≤ p k ≤ p x ∈ standard_simplex (p − Suc 0)
  shows (simplical_face k x) ∈ standard_simplex p
proof −
  have x01: ⋀i. 0 ≤ x i ∧ x i ≤ 1 and sumx: sum x {..p − Suc 0} = 1
    using assms by (auto simp: standard_simplex_def simplical_face_def)
  have gg: ⋀g. sum g {..p} = sum g {..<k} + sum g {k..p}
    using ‹k ≤ p› sum.union_disjoint [of {..<k} {k..p}]
    by (force simp: ivl_disj_un ivl_disj_int)
  have eq: (∑i≤p. if i < k then x i else if i = k then 0 else x (i −1))
         = (∑i < k. x i) + (∑i ∈ {k..p}. if i = k then 0 else x (i −1))
    by (simp add: gg)
  consider k ≤ p − Suc 0 | k = p
    using ‹k ≤ p› by linarith
  then have (∑i≤p. if i < k then x i else if i = k then 0 else x (i −1)) = 1
  proof cases
    case 1
    have [simp]: Suc (p − Suc 0) = p
      using ‹1 ≤ p› by auto
   have (∑i = k..p. if i = k then 0 else x (i −1)) = (∑i = k+1..p. if i = k then
0 else x (i −1))
      by (rule sum.mono_neutral_right) auto
    also have ... = (∑i = k+1..p. x (i −1))
      by simp
    also have ... = (∑i = k..p−1. x i)
       using sum.atLeastAtMost_reindex [of Suc k p−1 λi. x (i − Suc 0)] 1 by
simp
   finally have eq2: (∑i = k..p. if i = k then 0 else x (i −1)) = (∑i = k..p−1.
x i) .
    with 1 show ?thesis
     by (metis (no_types, lifting) One_nat_def eq finite_atLeastAtMost finite_lessThan
ivl_disj_int(4) ivl_disj_un(10) sum.union_disjoint sumx)
  next
    case 2
    have [simp]: ({..p} ∩ {x. x < p}) = {..p − Suc 0}
      using assms by auto
```

have ($\sum i \leq p$. *if* $i < p$ *then* $x\ i$ *else if* $i = k$ *then* 0 *else* $x\ (i - 1)$) = ($\sum i \leq p$. *if* $i < p$ *then* $x\ i$ *else* 0)
by (*rule sum.cong*) (*auto simp*: *2*)
also have ... = *sum x* {..$p-1$}
by (*simp add*: *sum.If_cases*)
also have ... = *1*
by (*simp add*: *sumx*)
finally show *?thesis*
using *2* **by** *simp*
qed
then show *?thesis*
using *assms* **by** (*auto simp*: *standard_simplex_def simplical_face_def*)
qed

0.1.3 Singular simplices, forcing canonicity outside the intended domain

definition *singular_simplex* :: *nat* ⇒ *'a topology* ⇒ ((*nat* ⇒ *real*) ⇒ *'a*) ⇒ *bool*
where
singular_simplex p X f ≡
continuous_map(*subtopology* (*powertop_real UNIV*) (*standard_simplex p*)) *X f*
∧ *f* ∈ *extensional* (*standard_simplex p*)

abbreviation *singular_simplex_set* :: *nat* ⇒ *'a topology* ⇒ ((*nat* ⇒ *real*) ⇒ *'a*) *set* **where**
singular_simplex_set p X ≡ *Collect* (*singular_simplex p X*)

lemma *singular_simplex_empty*:
topspace X = {} ⟹ ¬ *singular_simplex p X f*
by (*simp add*: *singular_simplex_def continuous_map nonempty_standard_simplex*)

lemma *singular_simplex_mono*:
⟦*singular_simplex p* (*subtopology X T*) *f*; *T* ⊆ *S*⟧ ⟹ *singular_simplex p* (*subtopology X S*) *f*
by (*auto simp*: *singular_simplex_def continuous_map_in_subtopology*)

lemma *singular_simplex_subtopology*:
singular_simplex p (*subtopology X S*) *f* ⟷
singular_simplex p X f ∧ *f* ` (*standard_simplex p*) ⊆ *S*
by (*auto simp*: *singular_simplex_def continuous_map_in_subtopology*)

Singular face

definition *singular_face* :: *nat* ⇒ *nat* ⇒ ((*nat* ⇒ *real*) ⇒ *'a*) ⇒ (*nat* ⇒ *real*) ⇒ *'a*
where *singular_face p k f* ≡ *restrict* (*f* ∘ *simplical_face k*) (*standard_simplex* (*p* − *Suc 0*))

lemma *singular_simplex_singular_face*:

```
  assumes f: singular_simplex p X f and 1 ≤ p k ≤ p
  shows singular_simplex (p − Suc 0) X (singular_face p k f)
proof −
  let ?PT = (powertop_real UNIV)
  have 0: simplical_face k ∈ standard_simplex (p − Suc 0) → standard_simplex
p
    using assms simplical_face_in_standard_simplex by auto
  have 1: continuous_map (subtopology ?PT (standard_simplex (p − Suc 0)))
                         (subtopology ?PT (standard_simplex p))
                         (simplical_face k)
 proof (clarsimp simp add: continuous_map_in_subtopology simplical_face_in_standard_simplex
continuous_map_componentwise 0)
    fix i
     have continuous_map ?PT euclideanreal (λx. if i < k then x i else if i = k
then 0 else x (i −1))
       by (auto intro: continuous_map_product_projection)
     then show continuous_map (subtopology ?PT (standard_simplex (p − Suc
0))) euclideanreal
                              (λx. simplical_face k x i)
       by (simp add: simplical_face_def continuous_map_from_subtopology)
  qed
  have 2: continuous_map (subtopology ?PT (standard_simplex p)) X f
    using assms(1) singular_simplex_def by blast
  show ?thesis
   by (simp add: singular_simplex_def singular_face_def continuous_map_compose
[OF 1 2])
qed
```

0.1.4 Singular chains

```
definition singular_chain :: [nat, 'a topology, 'a chain] ⇒ bool
  where singular_chain p X c ≡ Poly_Mapping.keys c ⊆ singular_simplex_set p
X

abbreviation singular_chain_set :: [nat, 'a topology] ⇒ ('a chain) set
  where singular_chain_set p X ≡ Collect (singular_chain p X)

lemma singular_chain_empty:
   topspace X = {} ⟹ singular_chain p X c ⟷ c = 0
 by (auto simp: singular_chain_def singular_simplex_empty subset_eq poly_mapping_eqI)

lemma singular_chain_mono:
   ⟦singular_chain p (subtopology X T) c;  T ⊆ S⟧
        ⟹ singular_chain p (subtopology X S) c
  unfolding singular_chain_def using singular_simplex_mono by blast

lemma singular_chain_subtopology:
   singular_chain p (subtopology X S) c ⟷
        singular_chain p X c ∧ (∀f ∈ Poly_Mapping.keys c. f ` (standard_simplex
```

p) ⊆ *S*)
unfolding *singular_chain_def*
by (*fastforce simp add*: *singular_simplex_subtopology subset_eq*)

lemma *singular_chain_0* [*iff*]: *singular_chain p X 0*
by (*auto simp*: *singular_chain_def*)

lemma *singular_chain_of*:
singular_chain p X (*frag_of c*) ⟷ *singular_simplex p X c*
by (*auto simp*: *singular_chain_def*)

lemma *singular_chain_cmul*:
singular_chain p X c ⟹ *singular_chain p X* (*frag_cmul a c*)
by (*auto simp*: *singular_chain_def*)

lemma *singular_chain_minus*:
singular_chain p X (−*c*) ⟷ *singular_chain p X c*
by (*auto simp*: *singular_chain_def*)

lemma *singular_chain_add*:
⟦*singular_chain p X a*; *singular_chain p X b*⟧ ⟹ *singular_chain p X* (*a*+*b*)
unfolding *singular_chain_def*
using *keys_add* [*of a b*] **by** *blast*

lemma *singular_chain_diff*:
⟦*singular_chain p X a*; *singular_chain p X b*⟧ ⟹ *singular_chain p X* (*a*−*b*)
unfolding *singular_chain_def*
using *keys_diff* [*of a b*] **by** *blast*

lemma *singular_chain_sum*:
(⋀*i*. *i* ∈ *I* ⟹ *singular_chain p X* (*f i*)) ⟹ *singular_chain p X* (∑*i*∈*I*. *f i*)
unfolding *singular_chain_def*
using *keys_sum* [*of f I*] **by** *blast*

lemma *singular_chain_extend*:
(⋀*c*. *c* ∈ *Poly_Mapping.keys x* ⟹ *singular_chain p X* (*f c*))
⟹ *singular_chain p X* (*frag_extend f x*)
by (*simp add*: *frag_extend_def singular_chain_cmul singular_chain_sum*)

0.1.5 Boundary homomorphism for singular chains

definition *chain_boundary* :: *nat* ⇒ ('*a chain*) ⇒ '*a chain*
where *chain_boundary p c* ≡
(*if p* = *0 then 0 else*
frag_extend (λ*f*. (∑*k*≤*p*. *frag_cmul* ((−*1*) ^ *k*) (*frag_of*(*singular_face p k f*)))) *c*)

lemma *singular_chain_boundary*:
assumes *singular_chain p X c*

shows *singular_chain* $(p - Suc\ 0)$ *X* (*chain_boundary p c*)
unfolding *chain_boundary_def*
proof (*clarsimp intro*!: *singular_chain_extend singular_chain_sum singular_chain_cmul*)
show $\bigwedge d\ k.\ [\![0 < p;\ d \in Poly\_Mapping.keys\ c;\ k \le p]\!]$
$\Longrightarrow$ *singular_chain* $(p - Suc\ 0)$ *X* (*frag_of* (*singular_face p k d*))
using *assms* **by** (*auto simp*: *singular_chain_def intro*: *singular_simplex_singular_face*)
qed

lemma *singular_chain_boundary_alt*:
singular_chain (*Suc p*) *X c* $\Longrightarrow$ *singular_chain p X* (*chain_boundary* (*Suc p*) *c*)
using *singular_chain_boundary* **by** *force*

lemma *chain_boundary_0* [*simp*]: *chain_boundary p 0 = 0*
by (*simp add*: *chain_boundary_def*)

lemma *chain_boundary_cmul*:
chain_boundary p (*frag_cmul k c*) = *frag_cmul k* (*chain_boundary p c*)
by (*auto simp*: *chain_boundary_def frag_extend_cmul*)

lemma *chain_boundary_minus*:
chain_boundary p (− *c*) = − (*chain_boundary p c*)
by (*metis chain_boundary_cmul frag_cmul_minus_one*)

lemma *chain_boundary_add*:
chain_boundary p (*a+b*) = *chain_boundary p a* + *chain_boundary p b*
by (*simp add*: *chain_boundary_def frag_extend_add*)

lemma *chain_boundary_diff*:
chain_boundary p (*a−b*) = *chain_boundary p a* − *chain_boundary p b*
using *chain_boundary_add* [*of p a −b*]
by (*simp add*: *chain_boundary_minus*)

lemma *chain_boundary_sum*:
chain_boundary p (*sum g I*) = *sum* (*chain_boundary p* ∘ *g*) *I*
by (*induction I rule*: *infinite_finite_induct*) (*simp_all add*: *chain_boundary_add*)

lemma *chain_boundary_sum′*:
finite I $\Longrightarrow$ *chain_boundary p* (*sum′ g I*) = *sum′* (*chain_boundary p* ∘ *g*) *I*
by (*induction I rule*: *finite_induct*) (*simp_all add*: *chain_boundary_add*)

lemma *chain_boundary_of*:
chain_boundary p (*frag_of f*) =
(*if p = 0 then 0*
else ($\sum k \le p$. *frag_cmul* ((−*1*) ^ *k*) (*frag_of*(*singular_face p k f*))))
by (*simp add*: *chain_boundary_def*)

0.1.6 Factoring out chains in a subtopology for relative homology

definition *mod_subset*
  where *mod_subset p X* ≡ {(*a*,*b*). *singular_chain p X* (*a* − *b*)}

lemma *mod_subset_empty* [*simp*]:
  (*a*,*b*) ∈ (*mod_subset p* (*subtopology X* {})) ⟷ *a* = *b*
  by (*simp add*: *mod_subset_def singular_chain_empty*)

lemma *mod_subset_refl* [*simp*]: (*c*,*c*) ∈ *mod_subset p X*
  by (*auto simp*: *mod_subset_def*)

lemma *mod_subset_cmul*:
  assumes (*a*,*b*) ∈ *mod_subset p X*
  shows (*frag_cmul k a*, *frag_cmul k b*) ∈ *mod_subset p X*
  using *assms*
  by (*simp add*: *mod_subset_def*) (*metis* (*no_types*, *lifting*) *add_diff_cancel diff_add_cancel frag_cmul_distrib2 singular_chain_cmul*)

lemma *mod_subset_add*:
  ⟦(*c1*,*c2*) ∈ *mod_subset p X*; (*d1*,*d2*) ∈ *mod_subset p X*⟧ ⟹ (*c1*+*d1*, *c2*+*d2*) ∈ *mod_subset p X*
  by (*simp add*: *mod_subset_def add_diff_add singular_chain_add*)

0.1.7 Relative cycles $Z_pX(S)$ where X is a topology and S a subset

definition *singular_relcycle* :: *nat* ⇒ ′*a topology* ⇒ ′*a set* ⇒ (′*a chain*) ⇒ *bool*
  where *singular_relcycle* ≡
    λ*p X S c*. *singular_chain p X c* ∧ (*chain_boundary p c*, *0*) ∈ *mod_subset* (*p*−*1*) (*subtopology X S*)

abbreviation *singular_relcycle_set*
  where *singular_relcycle_set p X S* ≡ *Collect* (*singular_relcycle p X S*)

lemma *singular_relcycle_restrict* [*simp*]:
  singular_relcycle p X (*topspace X* ∩ *S*) = *singular_relcycle p X S*
proof −
  have *eq*: *subtopology X* (*topspace X* ∩ *S*) = *subtopology X S*
    by (*metis subtopology_subtopology subtopology_topspace*)
  show *?thesis*
    by (*force simp*: *singular_relcycle_def eq*)
qed

lemma *singular_relcycle*:
  singular_relcycle ≡
    λ*p X S c*. *singular_chain p X c* ∧ *singular_chain* (*p*−*1*) (*subtopology X S*) (*chain_boundary p c*)
  by (*simp add*: *singular_relcycle_def mod_subset_def*)

lemma *singular_relcycle_0* [*simp*]: *singular_relcycle p X S 0*
by (*auto simp*: *singular_relcycle_def*)

lemma *singular_relcycle_cmul*:
singular_relcycle p X S c ⟹ *singular_relcycle p X S* (*frag_cmul k c*)
by (*auto simp*: *singular_relcycle_def chain_boundary_cmul dest*: *singular_chain_cmul mod_subset_cmul*)

lemma *singular_relcycle_minus*:
singular_relcycle p X S (−*c*) ⟷ *singular_relcycle p X S c*
by (*simp add*: *chain_boundary_minus singular_chain_minus singular_relcycle*)

lemma *singular_relcycle_add*:
⟦*singular_relcycle p X S a*; *singular_relcycle p X S b*⟧
⟹ *singular_relcycle p X S* (*a*+*b*)
by (*simp add*: *singular_relcycle_def chain_boundary_add mod_subset_def singular_chain_add*)

lemma *singular_relcycle_sum*:
⟦⋀*i*. *i* ∈ *I* ⟹ *singular_relcycle p X S* (*f i*)⟧
⟹ *singular_relcycle p X S* (*sum f I*)
by (*induction I rule*: *infinite_finite_induct*) (*auto simp*: *singular_relcycle_add*)

lemma *singular_relcycle_diff*:
⟦*singular_relcycle p X S a*; *singular_relcycle p X S b*⟧
⟹ *singular_relcycle p X S* (*a*−*b*)
by (*metis singular_relcycle_add singular_relcycle_minus uminus_add_conv_diff*)

lemma *singular_cycle*:
singular_relcycle p X {} *c* ⟷ *singular_chain p X c* ∧ *chain_boundary p c* = *0*
using *mod_subset_empty* **by** (*auto simp*: *singular_relcycle_def*)

lemma *singular_cycle_mono*:
⟦*singular_relcycle p* (*subtopology X T*) {} *c*; *T* ⊆ *S*⟧
⟹ *singular_relcycle p* (*subtopology X S*) {} *c*
by (*auto simp*: *singular_cycle elim*: *singular_chain_mono*)

0.1.8 Relative boundaries $B_p X S$, where X is a topology and S a subset.

definition *singular_relboundary* :: *nat* ⇒ *'a topology* ⇒ *'a set* ⇒ (*'a chain*) ⇒ *bool*
where
singular_relboundary p X S ≡
λ*c*. ∃ *d*. *singular_chain* (*Suc p*) *X d* ∧ (*chain_boundary* (*Suc p*) *d*, *c*) ∈ (*mod_subset p* (*subtopology X S*))

abbreviation *singular_relboundary_set* :: *nat* ⇒ *'a topology* ⇒ *'a set* ⇒ (*'a chain*) *set*
where *singular_relboundary_set p X S* ≡ *Collect* (*singular_relboundary p X S*)

lemma *singular_relboundary_restrict* [*simp*]:
singular_relboundary p X (*topspace X* ∩ *S*) = *singular_relboundary p X S*
unfolding *singular_relboundary_def*
by (*metis* (*no_types*, *opaque_lifting*) *subtopology_subtopology subtopology_topspace*)

lemma *singular_relboundary_alt*:
singular_relboundary p X S c ⟷
(∃ *d e*. *singular_chain* (*Suc p*) *X d* ∧ *singular_chain p* (*subtopology X S*) *e* ∧
chain_boundary (*Suc p*) *d* = *c* + *e*)
unfolding *singular_relboundary_def mod_subset_def* **by** *fastforce*

lemma *singular_relboundary*:
singular_relboundary p X S c ⟷
(∃ *d e*. *singular_chain* (*Suc p*) *X d* ∧ *singular_chain p* (*subtopology X S*) *e* ∧
(*chain_boundary* (*Suc p*) *d*) + *e* = *c*)
using *singular_chain_minus*
by (*fastforce simp add*: *singular_relboundary_alt*)

lemma *singular_boundary*:
singular_relboundary p X {} *c* ⟷
(∃ *d*. *singular_chain* (*Suc p*) *X d* ∧ *chain_boundary* (*Suc p*) *d* = *c*)
by (*meson mod_subset_empty singular_relboundary_def*)

lemma *singular_boundary_imp_chain*:
singular_relboundary p X {} *c* ⟹ *singular_chain p X c*
by (*auto simp*: *singular_relboundary singular_chain_boundary_alt singular_chain_empty*)

lemma *singular_boundary_mono*:
⟦*T* ⊆ *S*; *singular_relboundary p* (*subtopology X T*) {} *c*⟧
⟹ *singular_relboundary p* (*subtopology X S*) {} *c*
by (*metis mod_subset_empty singular_chain_mono singular_relboundary_def*)

lemma *singular_relboundary_imp_chain*:
singular_relboundary p X S c ⟹ *singular_chain p X c*
unfolding *singular_relboundary singular_chain_subtopology*
by (*blast intro*: *singular_chain_add singular_chain_boundary_alt*)

lemma *singular_chain_imp_relboundary*:
singular_chain p (*subtopology X S*) *c* ⟹ *singular_relboundary p X S c*
unfolding *singular_relboundary_def*
using *mod_subset_def singular_chain_minus* **by** *fastforce*

lemma *singular_relboundary_0* [*simp*]: *singular_relboundary p X S 0*
unfolding *singular_relboundary_def*
by (*rule_tac x=0* **in** *exI*) *auto*

lemma *singular_relboundary_cmul*:
singular_relboundary p X S c $\Longrightarrow$ *singular_relboundary p X S* (*frag_cmul a c*)
unfolding *singular_relboundary_def*
by (*metis chain_boundary_cmul mod_subset_cmul singular_chain_cmul*)

lemma *singular_relboundary_minus*:
singular_relboundary p X S (−*c*) $\longleftrightarrow$ *singular_relboundary p X S c*
using *singular_relboundary_cmul*
by (*metis add.inverse_inverse frag_cmul_minus_one*)

lemma *singular_relboundary_add*:
⟦*singular_relboundary p X S a*; *singular_relboundary p X S b*⟧ $\Longrightarrow$ *singular_relboundary p X S* (*a*+*b*)
unfolding *singular_relboundary_def*
by (*metis chain_boundary_add mod_subset_add singular_chain_add*)

lemma *singular_relboundary_diff*:
⟦*singular_relboundary p X S a*; *singular_relboundary p X S b*⟧ $\Longrightarrow$ *singular_relboundary p X S* (*a*−*b*)
by (*metis uminus_add_conv_diff singular_relboundary_minus singular_relboundary_add*)

0.1.9 The (relative) homology relation

definition *homologous_rel* :: [*nat*,′*a topology*,′*a set*,′*a chain*,′*a chain*] ⇒ *bool*
where *homologous_rel p X S* ≡ λ*a b*. *singular_relboundary p X S* (*a*−*b*)

abbreviation *homologous_rel_set*
where *homologous_rel_set p X S a* ≡ *Collect* (*homologous_rel p X S a*)

lemma *homologous_rel_restrict* [*simp*]:
homologous_rel p X (*topspace X* ∩ *S*) = *homologous_rel p X S*
unfolding *homologous_rel_def* **by** (*metis singular_relboundary_restrict*)

lemma *homologous_rel_refl* [*simp*]: *homologous_rel p X S c c*
unfolding *homologous_rel_def* **by** *auto*

lemma *homologous_rel_sym*:
homologous_rel p X S a b = *homologous_rel p X S b a*
unfolding *homologous_rel_def*
using *singular_relboundary_minus* **by** *fastforce*

lemma *homologous_rel_trans*:
assumes *homologous_rel p X S b c homologous_rel p X S a b*
shows *homologous_rel p X S a c*
using *homologous_rel_def*
proof −
have *singular_relboundary p X S* (*b* − *c*)
using *assms* **unfolding** *homologous_rel_def* **by** *blast*

moreover have *singular_relboundary p X S* (*b* − *a*)
  using *assms* **by** (*meson homologous_rel_def homologous_rel_sym*)
ultimately have *singular_relboundary p X S* (*c* − *a*)
  using *singular_relboundary_diff* **by** *fastforce*
then show *?thesis*
  by (*meson homologous_rel_def homologous_rel_sym*)
qed

lemma *homologous_rel_eq*:
  homologous_rel p X S a = *homologous_rel p X S b* ⟷
   homologous_rel p X S a b
 using *homologous_rel_sym homologous_rel_trans* **by** *fastforce*

lemma *homologous_rel_set_eq*:
  homologous_rel_set p X S a = *homologous_rel_set p X S b* ⟷
   homologous_rel p X S a b
 by (*metis homologous_rel_eq mem_Collect_eq*)

lemma *homologous_rel_singular_chain*:
 homologous_rel p X S a b ⟹ (*singular_chain p X a* ⟷ *singular_chain p X b*)
 unfolding *homologous_rel_def*
 using *singular_chain_diff singular_chain_add*
 by (*fastforce dest*: *singular_relboundary_imp_chain*)

lemma *homologous_rel_add*:
  ⟦*homologous_rel p X S a a′*; *homologous_rel p X S b b′*⟧
    ⟹ *homologous_rel p X S* (*a*+*b*) (*a′*+*b′*)
 unfolding *homologous_rel_def*
 by (*simp add*: *add_diff_add singular_relboundary_add*)

lemma *homologous_rel_diff*:
 assumes *homologous_rel p X S a a′ homologous_rel p X S b b′*
 shows *homologous_rel p X S* (*a* − *b*) (*a′* − *b′*)
proof −
 have *singular_relboundary p X S* ((*a* − *a′*) − (*b* − *b′*))
   using *assms singular_relboundary_diff* **unfolding** *homologous_rel_def* **by** *blast*
 then show *?thesis*
  by (*simp add*: *homologous_rel_def algebra_simps*)
qed

lemma *homologous_rel_sum*:
 assumes *f*: *finite* {*i* ∈ *I*. *f i* ≠ *0*} **and** *g*: *finite* {*i* ∈ *I*. *g i* ≠ *0*}
  and *h*: ⋀*i*. *i* ∈ *I* ⟹ *homologous_rel p X S* (*f i*) (*g i*)
 shows *homologous_rel p X S* (*sum f I*) (*sum g I*)
proof (*cases finite I*)
 case *True*
 let *?L* = {*i* ∈ *I*. *f i* ≠ *0*} ∪ {*i* ∈ *I*. *g i* ≠ *0*}
 have *L*: *finite ?L ?L* ⊆ *I*

```
    using f g by blast+
  have sum f I = sum f ?L
    by (rule comm_monoid_add_class.sum.mono_neutral_right [OF True]) auto
  moreover have sum g I = sum g ?L
    by (rule comm_monoid_add_class.sum.mono_neutral_right [OF True]) auto
  moreover have *: homologous_rel p X S (f i) (g i) if i ∈ ?L for i
    using h that by auto
  have homologous_rel p X S (sum f ?L) (sum g ?L)
    using L
  proof induction
    case (insert j J)
    then show ?case
      by (simp add: h homologous_rel_add)
  qed auto
  ultimately show ?thesis
    by simp
qed auto
```

```
lemma chain_homotopic_imp_homologous_rel:
  assumes
   ⋀c. singular_chain p X c ⟹ singular_chain (Suc p) X' (h c)
  ⋀c. singular_chain (p -1) (subtopology X S) c ⟹ singular_chain p (subtopology
X' T) (h' c)
   ⋀c. singular_chain p X c
          ⟹ (chain_boundary (Suc p) (h c)) + (h'(chain_boundary p c)) = f c
- g c
    singular_relcycle p X S c
  shows homologous_rel p X' T (f c) (g c)
proof -
  have singular_chain p (subtopology X' T) (chain_boundary (Suc p) (h c) - (f
c - g c))
    using assms
   by (metis (no_types, lifting) add_diff_cancel_left' minus_diff_eq singular_chain_minus
singular_relcycle)
  then show ?thesis
  using assms
    by (metis homologous_rel_def singular_relboundary singular_relcycle)
qed
```

0.1.10 Show that all boundaries are cycles, the key "chain complex" property.

```
lemma chain_boundary_boundary:
  assumes singular_chain p X c
  shows chain_boundary (p - Suc 0) (chain_boundary p c) = 0
proof (cases p -1 = 0)
  case False
  then have 2 ≤ p
```

```
    by auto
  show ?thesis
    using assms
    unfolding singular_chain_def
  proof (induction rule: frag_induction)
    case (one g)
    then have ss: singular_simplex p X g
      by simp
    have eql: {..p} × {..p − Suc 0} ∩ {(x, y). y < x} = (λ(j,i). (Suc i, j)) ‘ {(i,j).
i ≤ j ∧ j ≤ p −1}
      using False
      by (auto simp: image_def) (metis One_nat_def diff_Suc_1 diff_le_mono
le_refl lessE less_imp_le_nat)
    have eqr: {..p} × {..p − Suc 0} − {(x, y). y < x} = {(i,j). i ≤ j ∧ j ≤ p −1}
      by auto
    have eqf: singular_face (p − Suc 0) i (singular_face p (Suc j) g) =
              singular_face (p − Suc 0) j (singular_face p i g) if i ≤ j j ≤ p − Suc
0 for i j
    proof (rule ext)
      fix t
      show singular_face (p − Suc 0) i (singular_face p (Suc j) g) t =
            singular_face (p − Suc 0) j (singular_face p i g) t
      proof (cases t ∈ standard_simplex (p −1 −1))
        case True
        have fi: simplical_face i t ∈ standard_simplex (p − Suc 0)
          using False True simplical_face_in_standard_simplex that by force
        have fj: simplical_face j t ∈ standard_simplex (p − Suc 0)
          by (metis False One_nat_def True simplical_face_in_standard_simplex
less_one not_less that(2))
          have eq: simplical_face (Suc j) (simplical_face i t) = simplical_face i
(simplical_face j t)
          using True that ss
          unfolding standard_simplex_def simplical_face_def by fastforce
        show ?thesis by (simp add: singular_face_def fi fj eq)
      qed (simp add: singular_face_def)
    qed
    show ?case
    proof (cases p = 1)
      case False
      have eq0: frag_cmul (−1) a = b ⟹ a + b = 0 for a b
        by (simp add: neg_eq_iff_add_eq_0)
      have ∗: (∑x≤p. ∑i≤p − Suc 0.
                frag_cmul ((−1) ^ (x + i)) (frag_of (singular_face (p − Suc 0) i
(singular_face p x g))))
              = 0
        apply (simp add: sum.cartesian_product sum.Int_Diff [of _ × _ _ {(x,y).
y < x}])
        apply (rule eq0)
      unfolding frag_cmul_sum prod.case_distrib [of frag_cmul (−1)] frag_cmul_cmul
```

```
eql eqr
      apply (force simp: inj_on_def sum.reindex add.commute eqf intro: sum.cong)
        done
      show ?thesis
            using False by (simp add: chain_boundary_of chain_boundary_sum
chain_boundary_cmul frag_cmul_sum * flip: power_add)
    qed (simp add: chain_boundary_def)
  next
    case (diff a b)
    then show ?case
      by (simp add: chain_boundary_diff)
  qed auto
qed (simp add: chain_boundary_def)


lemma chain_boundary_boundary_alt:
   singular_chain (Suc p) X c ⟹ chain_boundary p (chain_boundary (Suc p) c)
= 0
  using chain_boundary_boundary by force

lemma singular_relboundary_imp_relcycle:
  assumes singular_relboundary p X S c
  shows singular_relcycle p X S c
proof -
  obtain d e where d: singular_chain (Suc p) X d
             and e: singular_chain p (subtopology X S) e
             and c: c = chain_boundary (Suc p) d + e
    using assms by (auto simp: singular_relboundary singular_relcycle)
   have 1: singular_chain (p - Suc 0) (subtopology X S) (chain_boundary p
(chain_boundary (Suc p) d))
    using d chain_boundary_boundary_alt by fastforce
  have 2: singular_chain (p - Suc 0) (subtopology X S) (chain_boundary p e)
     using ‹singular_chain p (subtopology X S) e› singular_chain_boundary by
auto
  have singular_chain p X c
    using assms singular_relboundary_imp_chain by auto
  moreover have singular_chain (p - Suc 0) (subtopology X S) (chain_boundary
p c)
    by (simp add: c chain_boundary_add singular_chain_add 1 2)
  ultimately show ?thesis
    by (simp add: singular_relcycle)
qed

lemma homologous_rel_singular_relcycle_1:
  assumes homologous_rel p X S c1 c2 singular_relcycle p X S c1
  shows singular_relcycle p X S c2
  using assms
 by (metis diff_add_cancel homologous_rel_def homologous_rel_sym singular_relboundary_imp_relcycle
singular_relcycle_add)
```

lemma *homologous_rel_singular_relcycle*:
assumes *homologous_rel p X S c1 c2*
shows *singular_relcycle p X S c1 = singular_relcycle p X S c2*
using *assms homologous_rel_singular_relcycle_1*
using *homologous_rel_sym* **by** *blast*

0.1.11 Operations induced by a continuous map g between topological spaces

definition *simplex_map* :: *nat* ⇒ (*'b* ⇒ *'a*) ⇒ ((*nat* ⇒ *real*) ⇒ *'b*) ⇒ (*nat* ⇒ *real*) ⇒ *'a*
where *simplex_map p g c* ≡ *restrict* (*g* ∘ *c*) (*standard_simplex p*)

lemma *singular_simplex_simplex_map*:
⟦*singular_simplex p X f*; *continuous_map X X' g*⟧
⟹ *singular_simplex p X'* (*simplex_map p g f*)
unfolding *singular_simplex_def simplex_map_def*
by (*auto simp*: *continuous_map_compose*)

lemma *simplex_map_eq*:
⟦*singular_simplex p X c*;
⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧
⟹ *simplex_map p f c* = *simplex_map p g c*
by (*auto simp*: *singular_simplex_def simplex_map_def continuous_map_def Pi_iff*)

lemma *simplex_map_id_gen*:
⟦*singular_simplex p X c*;
⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *x*⟧
⟹ *simplex_map p f c* = *c*
unfolding *singular_simplex_def simplex_map_def continuous_map_def*
using *extensional_arb* **by** *fastforce*

lemma *simplex_map_id* [*simp*]:
simplex_map p id = (λ*c*. *restrict c* (*standard_simplex p*))
by (*auto simp*: *simplex_map_def*)

lemma *simplex_map_compose*:
simplex_map p (*h* ∘ *g*) = *simplex_map p h* ∘ *simplex_map p g*
unfolding *simplex_map_def* **by** *force*

lemma *singular_face_simplex_map*:
⟦*1* ≤ *p*; *k* ≤ *p*⟧
⟹ *singular_face p k* (*simplex_map p f c*) = *simplex_map* (*p* − *Suc 0*) *f* (*c* ∘ *simplical_face k*)
unfolding *simplex_map_def singular_face_def*
by (*force simp*: *simplical_face_in_standard_simplex*)

lemma *singular_face_restrict* [*simp*]:
  assumes $p > 0$ $i \leq p$
  shows *singular_face p i* (*restrict f* (*standard_simplex p*)) = *singular_face p i f*
  by (*metis assms One_nat_def Suc_leI simplex_map_id singular_face_def singular_face_simplex_map*)

definition *chain_map* :: *nat* ⇒ ('*b* ⇒ '*a*) ⇒ (((*nat* ⇒ *real*) ⇒ '*b*) $\Rightarrow_0$ *int*) ⇒ '*a chain*
  where *chain_map p g c* ≡ *frag_extend* (*frag_of* ∘ *simplex_map p g*) *c*

lemma *singular_chain_chain_map*:
  ⟦*singular_chain p X c*; *continuous_map X X' g*⟧ ⟹ *singular_chain p X'* (*chain_map p g c*)
  unfolding *chain_map_def*
  by (*force simp add*: *singular_chain_def subset_iff*
    intro!: *singular_chain_extend singular_simplex_simplex_map*)

lemma *chain_map_0* [*simp*]: *chain_map p g 0* = *0*
  by (*auto simp*: *chain_map_def*)

lemma *chain_map_of* [*simp*]: *chain_map p g* (*frag_of f*) = *frag_of* (*simplex_map p g f*)
  by (*simp add*: *chain_map_def*)

lemma *chain_map_cmul* [*simp*]:
  chain_map p g (*frag_cmul a c*) = *frag_cmul a* (*chain_map p g c*)
  by (*simp add*: *frag_extend_cmul chain_map_def*)

lemma *chain_map_minus*: *chain_map p g* (−*c*) = − (*chain_map p g c*)
  by (*simp add*: *frag_extend_minus chain_map_def*)

lemma *chain_map_add*:
  chain_map p g (*a*+*b*) = *chain_map p g a* + *chain_map p g b*
  by (*simp add*: *frag_extend_add chain_map_def*)

lemma *chain_map_diff*:
  chain_map p g (*a*−*b*) = *chain_map p g a* − *chain_map p g b*
  by (*simp add*: *frag_extend_diff chain_map_def*)

lemma *chain_map_sum*:
  finite I ⟹ *chain_map p g* (*sum f I*) = *sum* (*chain_map p g* ∘ *f*) *I*
  by (*simp add*: *frag_extend_sum chain_map_def*)

lemma *chain_map_eq*:
  ⟦*singular_chain p X c*; ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*⟧
  ⟹ *chain_map p f c* = *chain_map p g c*
  unfolding *singular_chain_def*
proof (*induction rule*: *frag_induction*)

```
  case (one x)
  then show ?case
   by (metis (no_types, lifting) chain_map_of mem_Collect_eq simplex_map_eq)
qed (auto simp: chain_map_diff)

lemma chain_map_id_gen:
   ⟦singular_chain p X c; ⋀x. x ∈ topspace X ⟹ f x = x⟧
    ⟹ chain_map p f c = c
  unfolding singular_chain_def
  by (erule frag_induction) (auto simp: chain_map_diff simplex_map_id_gen)

lemma chain_map_ident:
   singular_chain p X c ⟹ chain_map p id c = c
  by (simp add: chain_map_id_gen)

lemma chain_map_id:
   chain_map p id = frag_extend (frag_of ∘ (λf. restrict f (standard_simplex p)))
  by (auto simp: chain_map_def)

lemma chain_map_compose:
   chain_map p (h ∘ g) = chain_map p h ∘ chain_map p g
proof
  show chain_map p (h ∘ g) c = (chain_map p h ∘ chain_map p g) c for c
    using subset_UNIV
  proof (induction c rule: frag_induction)
    case (one x)
    then show ?case
      by simp (metis (mono_tags, lifting) comp_eq_dest_lhs restrict_apply simplex_map_def)
  next
    case (diff a b)
    then show ?case
      by (simp add: chain_map_diff)
  qed auto
qed

lemma singular_simplex_chain_map_id:
  assumes singular_simplex p X f
  shows chain_map p f (frag_of (restrict id (standard_simplex p))) = frag_of f
proof -
  have (restrict (f ∘ restrict id (standard_simplex p)) (standard_simplex p)) = f
   by (rule ext) (metis assms comp_apply extensional_arb id_apply restrict_apply singular_simplex_def)
  then show ?thesis
    by (simp add: simplex_map_def)
qed

lemma chain_boundary_chain_map:
  assumes singular_chain p X c
```

```
  shows chain_boundary p (chain_map p g c) = chain_map (p - Suc 0) g
(chain_boundary p c)
  using assms unfolding singular_chain_def
proof (induction c rule: frag_induction)
  case (one x)
  then have singular_face p i (simplex_map p g x) = simplex_map (p - Suc 0)
g (singular_face p i x)
    if 0 ≤ i i ≤ p p ≠ 0 for i
    using that
   by (fastforce simp add: singular_face_def simplex_map_def simplical_face_in_standard_simplex)
  then show ?case
    by (auto simp: chain_boundary_of chain_map_sum)
next
  case (diff a b)
  then show ?case
    by (simp add: chain_boundary_diff chain_map_diff)
qed auto

lemma singular_relcycle_chain_map:
  assumes singular_relcycle p X S c continuous_map X X' g g ∈ S → T
  shows singular_relcycle p X' T (chain_map p g c)
proof -
  have continuous_map (subtopology X S) (subtopology X' T) g
    using assms
   by (metis Pi_anti_mono continuous_map_from_subtopology continuous_map_in_subtopology

        openin_imp_subset openin_topspace subsetD)
  then show ?thesis
    using chain_boundary_chain_map [of p X c g]
    by (metis One_nat_def assms(1) assms(2) singular_chain_chain_map singu-
lar_relcycle)
qed

lemma singular_relboundary_chain_map:
  assumes singular_relboundary p X S c continuous_map X X' g g ∈ S → T
  shows singular_relboundary p X' T (chain_map p g c)
proof -
  obtain d e where d: singular_chain (Suc p) X d
    and e: singular_chain p (subtopology X S) e and c: c = chain_boundary (Suc
p) d + e
    using assms by (auto simp: singular_relboundary)
  have singular_chain (Suc p) X' (chain_map (Suc p) g d)
    using assms(2) d singular_chain_chain_map by blast
  moreover have singular_chain p (subtopology X' T) (chain_map p g e)
  proof -
    have ⋀Y. g ∈ topspace (subtopology Y S) → T
      using assms(3) by auto
    then show ?thesis
     by (metis assms(2) continuous_map_from_subtopology continuous_map_into_subtopology
```

e
singular_chain_chain_map)
qed
moreover have *chain_boundary* (*Suc p*) (*chain_map* (*Suc p*) *g d*) + *chain_map p g e* =
chain_map p g (*chain_boundary* (*Suc p*) *d* + *e*)
by (*metis One_nat_def chain_boundary_chain_map chain_map_add d diff_Suc_1*)
ultimately show *?thesis*
unfolding *singular_relboundary*
using *c* **by** *blast*
qed

0.1.12 Homology of one-point spaces degenerates except for $p = 0$.

lemma *singular_simplex_singleton*:
assumes *topspace X* = {*a*}
shows *singular_simplex p X f* ⟷ *f* = *restrict* (λ*x*. *a*) (*standard_simplex p*) (**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then show *?rhs*
proof −
have *continuous_map* (*subtopology* (*product_topology* (λ*n*. *euclideanreal*) *UNIV*) (*standard_simplex p*)) *X f*
using ‹*singular_simplex p X f*› *singular_simplex_def* **by** *blast*
then have ⋀*c*. *c* ∉ *standard_simplex p* ∨ *f c* = *a*
by (*simp add*: *assms continuous_map_def Pi_iff*)
then show *?thesis*
by (*metis* (*no_types*) *L extensional_restrict restrict_ext singular_simplex_def*)
qed
next
assume *?rhs*
with *assms* **show** *?lhs*
by (*auto simp*: *singular_simplex_def*)
qed

lemma *singular_chain_singleton*:
assumes *topspace X* = {*a*}
shows *singular_chain p X c* ⟷
(∃ *b*. *c* = *frag_cmul b* (*frag_of*(*restrict* (λ*x*. *a*) (*standard_simplex p*))))
(**is** *?lhs* = *?rhs*)
proof
let *?f* = *restrict* (λ*x*. *a*) (*standard_simplex p*)
assume *L*: *?lhs*
with *assms* **have** *Poly_Mapping.keys c* ⊆ {*?f*}
by (*auto simp*: *singular_chain_def singular_simplex_singleton*)
then consider *Poly_Mapping.keys c* = {} | *Poly_Mapping.keys c* = {*?f*}
by *blast*

```
  then show ?rhs
  proof cases
    case 1
    with L show ?thesis
      by (metis frag_cmul_zero keys_eq_empty)
  next
    case 2
   then have ∃ b. frag_extend frag_of c = frag_cmul b (frag_of (λx∈standard_simplex
p. a))
      by (force simp: frag_extend_def)
    then show ?thesis
      by (metis frag_expansion)
  qed
next
  assume ?rhs
  with assms show ?lhs
    by (auto simp: singular_chain_def singular_simplex_singleton)
qed

lemma chain_boundary_of_singleton:
  assumes tX: topspace X = {a} and sc: singular_chain p X c
  shows chain_boundary p c =
         (if p = 0 ∨ odd p then 0
         else frag_extend (λf. frag_of(restrict (λx. a) (standard_simplex (p −1))))
c)
    (is ?lhs = ?rhs)
proof (cases p = 0)
  case False
   have ?lhs = frag_extend (λf. if odd p then 0 else frag_of(restrict (λx. a)
(standard_simplex (p −1)))) c
  proof (simp only: chain_boundary_def False if_False, rule frag_extend_eq)
    fix f
    assume f ∈ Poly_Mapping.keys c
    with assms have singular_simplex p X f
      by (auto simp: singular_chain_def)
    then have ∗: ⋀k. k ≤ p ⟹ singular_face p k f = (λx∈standard_simplex (p
−1). a)
      using False singular_simplex_singular_face
      by (fastforce simp flip: singular_simplex_singleton [OF tX])
    define c where c ≡ frag_of (λx∈standard_simplex (p −1). a)
    have (∑ k≤p. frag_cmul ((−1) ^ k) (frag_of (singular_face p k f)))
       = (∑ k≤p. frag_cmul ((−1) ^ k) c)
      by (auto simp: c_def ∗ intro: sum.cong)
    also have ... = (if odd p then 0 else c)
      by (induction p) (auto simp: c_def restrict_def)
    finally show (∑ k≤p. frag_cmul ((−1) ^ k) (frag_of (singular_face p k f)))
              = (if odd p then 0 else frag_of (λx∈standard_simplex (p −1). a))
      unfolding c_def .
  qed
```

```
  also have ... = ?rhs
    by (auto simp: False frag_extend_eq_0)
  finally show ?thesis .
qed (simp add: chain_boundary_def)


lemma singular_cycle_singleton:
  assumes topspace X = {a}
  shows singular_relcycle p X {} c ⟷ singular_chain p X c ∧ (p = 0 ∨ odd p
∨ c = 0)
proof –
  have c = 0 if singular_chain p X c and chain_boundary p c = 0 and even p
and p ≠ 0
   using that assms singular_chain_singleton [of X a p c] chain_boundary_of_singleton
[OF assms]
    by (auto simp: frag_extend_cmul)
  moreover
  have chain_boundary p c = 0 if sc: singular_chain p X c and odd p
    by (simp add: chain_boundary_of_singleton [OF assms sc] that)
  moreover have chain_boundary 0 c = 0 if singular_chain 0 X c and p = 0
    by (simp add: chain_boundary_def)
  ultimately show ?thesis
  using assms by (auto simp: singular_cycle)
qed


lemma singular_boundary_singleton:
  assumes topspace X = {a}
  shows singular_relboundary p X {} c ⟷ singular_chain p X c ∧ (odd p ∨ c
= 0)
proof (cases singular_chain p X c)
  case True
  have ∃ d. singular_chain (Suc p) X d ∧ chain_boundary (Suc p) d = c
    if singular_chain p X c and odd p
  proof –
   obtain b where b: c = frag_cmul b (frag_of(restrict (λx. a) (standard_simplex
p)))
      by (metis True assms singular_chain_singleton)
    let ?d = frag_cmul b (frag_of (λx∈standard_simplex (Suc p). a))
    have scd: singular_chain (Suc p) X ?d
      by (metis assms singular_chain_singleton)
    moreover have chain_boundary (Suc p) ?d = c
        by (simp add: assms scd chain_boundary_of_singleton [of X a Suc p] b
frag_extend_cmul ‹odd p›)
    ultimately show ?thesis
      by metis
  qed
  with True assms show ?thesis
    by (auto simp: singular_boundary chain_boundary_of_singleton)
```

next
  case *False*
  with *assms singular_boundary_imp_chain* **show** *?thesis*
    by *metis*
qed

lemma *singular_boundary_eq_cycle_singleton*:
  assumes *topspace X* = {*a*} *1* ≤ *p*
  shows *singular_relboundary p X* {} *c* ⟷ *singular_relcycle p X* {} *c* (**is** *?lhs* = *?rhs*)
proof
  show *?lhs* ⟹ *?rhs*
    by (*simp add*: *singular_relboundary_imp_relcycle*)
  show *?rhs* ⟹ *?lhs*
    by (*metis assms not_one_le_zero singular_boundary_singleton singular_cycle_singleton*)
qed

lemma *singular_boundary_set_eq_cycle_singleton*:
  assumes *topspace X* = {*a*} *1* ≤ *p*
  shows *singular_relboundary_set p X* {} = *singular_relcycle_set p X* {}
  using *singular_boundary_eq_cycle_singleton* [*OF assms*]
  by *blast*

0.1.13 Simplicial chains

Simplicial chains, effectively those resulting from linear maps. We still allow the map to be singular, so the name is questionable. These are intended as building-blocks for singular subdivision, rather than as a axis for 1 simplicial homology.

definition *oriented_simplex*
  where *oriented_simplex p l* ≡ (λ*x*∈*standard_simplex p*. λ*i*. (∑*j*≤*p*. *l j i* ∗ *x j*))

definition *simplicial_simplex*
  where
  simplicial_simplex p S f ≡
      singular_simplex p (*subtopology* (*powertop_real UNIV*) *S*) *f* ∧
      (∃ *l*. *f* = *oriented_simplex p l*)

lemma *simplicial_simplex*:
  simplicial_simplex p S f ⟷ *f* ‘ (*standard_simplex p*) ⊆ *S* ∧ (∃ *l*. *f* = *oriented_simplex p l*)
  (**is** *?lhs* = *?rhs*)
proof
  assume *R*: *?rhs*
  have *continuous_map* (*subtopology* (*powertop_real UNIV*) (*standard_simplex p*))
      (*powertop_real UNIV*) (λ*x i*. ∑*j*≤*p*. *l j i* ∗ *x j*) **for** *l* :: *nat* ⇒ *'a* ⇒

real
unfolding *continuous_map_componentwise*
by (*force intro*: *continuous_intros continuous_map_from_subtopology continuous_map_product_projection*)
with *R* **show** *?lhs*
unfolding *simplicial_simplex_def singular_simplex_subtopology*
by (*auto simp add*: *singular_simplex_def oriented_simplex_def*)
qed (*simp add*: *simplicial_simplex_def singular_simplex_subtopology*)

lemma *simplicial_simplex_empty* [*simp*]: ¬ *simplicial_simplex p* {} *f*
by (*simp add*: *nonempty_standard_simplex simplicial_simplex*)

definition *simplicial_chain*
where *simplicial_chain p S c* ≡ *Poly_Mapping.keys c* ⊆ *Collect* (*simplicial_simplex p S*)

lemma *simplicial_chain_0* [*simp*]: *simplicial_chain p S 0*
by (*simp add*: *simplicial_chain_def*)

lemma *simplicial_chain_of* [*simp*]:
simplicial_chain p S (*frag_of c*) ⟷ *simplicial_simplex p S c*
by (*simp add*: *simplicial_chain_def*)

lemma *simplicial_chain_cmul*:
simplicial_chain p S c ⟹ *simplicial_chain p S* (*frag_cmul a c*)
by (*auto simp*: *simplicial_chain_def*)

lemma *simplicial_chain_diff*:
⟦*simplicial_chain p S c1*; *simplicial_chain p S c2*⟧ ⟹ *simplicial_chain p S* (*c1* − *c2*)
unfolding *simplicial_chain_def* **by** (*meson UnE keys_diff subset_iff*)

lemma *simplicial_chain_sum*:
(⋀*i*. *i* ∈ *I* ⟹ *simplicial_chain p S* (*f i*)) ⟹ *simplicial_chain p S* (*sum f I*)
unfolding *simplicial_chain_def*
using *order_trans* [*OF keys_sum* [*of f I*]]
by (*simp add*: *UN_least*)

lemma *simplicial_simplex_oriented_simplex*:
simplicial_simplex p S (*oriented_simplex p l*)
⟷ ((λ*x i*. ∑*j*≤*p*. *l j i* ∗ *x j*) ‘ *standard_simplex p* ⊆ *S*)
by (*auto simp*: *simplicial_simplex oriented_simplex_def*)

lemma *simplicial_imp_singular_simplex*:
simplicial_simplex p S f
⟹ *singular_simplex p* (*subtopology* (*powertop_real UNIV*) *S*) *f*
by (*simp add*: *simplicial_simplex_def*)

lemma *simplicial_imp_singular_chain*:

```
  simplicial_chain p S c
    ⟹ singular_chain p (subtopology (powertop_real UNIV) S) c
  unfolding simplicial_chain_def singular_chain_def
  by (auto intro: simplicial_imp_singular_simplex)

lemma oriented_simplex_eq:
  oriented_simplex p l = oriented_simplex p l' ⟷ (∀ i. i ≤ p ⟶ l i = l' i)
  (is ?lhs = ?rhs)
proof
  assume L: ?lhs
  show ?rhs
  proof clarify
    fix i
    assume i ≤ p
    let ?fi = (λj. if j = i then 1 else 0)
    have (∑j≤p. l j k ∗ ?fi j) = (∑j≤p. l' j k ∗ ?fi j) for k
      using L ‹i ≤ p›
      by (simp add: fun_eq_iff oriented_simplex_def split: if_split_asm)
    with ‹i ≤ p› show l i = l' i
      by (simp add: if_distrib ext cong: if_cong)
  qed
qed (auto simp: oriented_simplex_def)

lemma singular_face_oriented_simplex:
  assumes 1 ≤ p k ≤ p
  shows singular_face p k (oriented_simplex p l) =
        oriented_simplex (p −1) (λj. if j < k then l j else l (Suc j))
proof −
  have (∑j≤p. l j i ∗ simplical_face k x j)
     = (∑j≤p − Suc 0. (if j < k then l j else l (Suc j)) i ∗ x j)
    if x ∈ standard_simplex (p − Suc 0) for i x
  proof −
    show ?thesis
      unfolding simplical_face_def
      using sum.zero_middle [OF assms, where 'a=real, symmetric]
     by (simp add: if_distrib [of λx. _ ∗ x] if_distrib [of λf. f i ∗ _] atLeast0AtMost
cong: if_cong)
  qed
  then show ?thesis
    using simplical_face_in_standard_simplex assms
    by (auto simp: singular_face_def oriented_simplex_def restrict_def)
qed

lemma simplicial_simplex_singular_face:
  fixes f :: (nat ⇒ real) ⇒ nat ⇒ real
  assumes ss: simplicial_simplex p S f and p: 1 ≤ p k ≤ p
  shows simplicial_simplex (p − Suc 0) S (singular_face p k f)
proof −
  let ?X = subtopology (powertop_real UNIV) S
```

```
  obtain m where l: singular_simplex p ?X (oriented_simplex p m)
      and feq: f = oriented_simplex p m
    using assms by (force simp: simplicial_simplex_def)
  moreover
  have singular_face p k f = oriented_simplex (p − Suc 0) (λi. if i < k then m i
else m (Suc i))
    using feq p singular_face_oriented_simplex by auto
  ultimately
  show ?thesis
    using p simplicial_simplex_def singular_simplex_singular_face by blast
qed

lemma simplicial_chain_boundary:
   simplicial_chain p S c ⟹ simplicial_chain (p −1) S (chain_boundary p c)
  unfolding simplicial_chain_def
proof (induction rule: frag_induction)
  case (one f)
  then have simplicial_simplex p S f
    by simp
  have simplicial_chain (p − Suc 0) S (frag_of (singular_face p i f))
    if 0 < p i ≤ p for i
    using that one
    by (force simp: simplicial_simplex_def singular_simplex_singular_face singu-
lar_face_oriented_simplex)
  then have simplicial_chain (p − Suc 0) S (chain_boundary p (frag_of f))
    unfolding chain_boundary_def frag_extend_of
    by (auto intro!: simplicial_chain_cmul simplicial_chain_sum)
  then show ?case
    by (simp add: simplicial_chain_def [symmetric])
next
  case (diff a b)
  then show ?case
    by (metis chain_boundary_diff simplicial_chain_def simplicial_chain_diff)
qed auto
```

0.1.14 The cone construction on simplicial simplices.

```
consts simplex_cone :: [nat, nat ⇒ real, [nat ⇒ real, nat] ⇒ real, nat ⇒ real,
nat] ⇒ real
specification (simplex_cone)
  simplex_cone:
    ⋀p v l. simplex_cone p v (oriented_simplex p l) =
          oriented_simplex (Suc p) (λi. if i = 0 then v else l(i −1))
proof −
  have ∗: ⋀x. ∀xv. ∃y. (λl. oriented_simplex (Suc x)
                          (λi. if i = 0 then xv else l (i − 1))) =
                 y ∘ oriented_simplex x
    by (simp add: oriented_simplex_eq flip: choice_iff function_factors_left)
  then show ?thesis
```

```
    unfolding o_def by (metis(no_types))
qed

lemma simplicial_simplex_simplex_cone:
  assumes f: simplicial_simplex p S f
    and T: ⋀x u. ⟦0 ≤ u; u ≤ 1; x ∈ S⟧ ⟹ (λi. (1 − u) ∗ v i + u ∗ x i) ∈ T
  shows simplicial_simplex (Suc p) T (simplex_cone p v f)
proof −
  obtain l where l: ⋀x. x ∈ standard_simplex p ⟹ oriented_simplex p l x ∈ S
    and feq: f = oriented_simplex p l
    using f by (auto simp: simplicial_simplex)
  have oriented_simplex p l x ∈ S if x ∈ standard_simplex p for x
    using f that by (auto simp: simplicial_simplex feq)
  then have S: ⋀x. ⟦⋀i. 0 ≤ x i ∧ x i ≤ 1; ⋀i. i>p ⟹ x i = 0; sum x {..p} =
1⟧
                 ⟹ (λi. ∑j≤p. l j i ∗ x j) ∈ S
    by (simp add: oriented_simplex_def standard_simplex_def)
  have oriented_simplex (Suc p) (λi. if i = 0 then v else l (i −1)) x ∈ T
    if x ∈ standard_simplex (Suc p) for x
 proof (simp add: that oriented_simplex_def sum.atMost_Suc_shift del: sum.atMost_Suc)
    have x01: ⋀i. 0 ≤ x i ∧ x i ≤ 1 and x0: ⋀i. i > Suc p ⟹ x i = 0 and x1:
sum x {..Suc p} = 1
      using that by (auto simp: oriented_simplex_def standard_simplex_def)
    obtain a where a ∈ S
      using f by force
    show (λi. v i ∗ x 0 + (∑j≤p. l j i ∗ x (Suc j))) ∈ T
    proof (cases x 0 = 1)
      case True
      then have sum x {Suc 0..Suc p} = 0
        using x1 by (simp add: atMost_atLeast0 sum.atLeast_Suc_atMost)
      then have [simp]: x (Suc j) = 0 if j≤p for j
        unfolding sum.atLeast_Suc_atMost_Suc_shift
        using x01 that by (simp add: sum_nonneg_eq_0_iff)
      then show ?thesis
        using T [of 0 a] ‹a ∈ S› by (auto simp: True)
    next
      case False
       then have (λi. v i ∗ x 0 + (∑j≤p. l j i ∗ x (Suc j))) = (λi. (1 − (1 − x
0)) ∗ v i + (1 − x 0) ∗ (inverse (1 − x 0) ∗ (∑j≤p. l j i ∗ x (Suc j))))
        by (force simp: field_simps)
      also have ... ∈ T
      proof (rule T)
        have x 0 < 1
          by (simp add: False less_le x01)
        have xle: x (Suc i) ≤ (1 − x 0) for i
        proof (cases i ≤ p)
          case True
          have sum x {0, Suc i} ≤ sum x {..Suc p}
            by (rule sum_mono2) (auto simp: True x01)
```

```
          then show ?thesis
           using x1 x01 by (simp add: algebra_simps not_less)
        qed (simp add: x0 x01)
        have (λi. (∑j≤p. l j i ∗ (x (Suc j) ∗ inverse (1 − x 0)))) ∈ S
        proof (rule S)
          have x 0 + (∑j≤p. x (Suc j)) = sum x {..Suc p}
            by (metis sum.atMost_Suc_shift)
          with x1 have (∑j≤p. x (Suc j)) = 1 − x 0
            by simp
          with False show (∑j≤p. x (Suc j) ∗ inverse (1 − x 0)) = 1
              by (metis add_diff_cancel_left' diff_diff_eq2 diff_zero right_inverse
sum_distrib_right)
      qed (use x01 x0 xle ‹x 0 < 1› in ‹auto simp: field_split_simps›)
      then show (λi. inverse (1 − x 0) ∗ (∑j≤p. l j i ∗ x (Suc j))) ∈ S
        by (simp add: field_simps sum_divide_distrib)
    qed (use x01 in auto)
    finally show ?thesis .
  qed
qed
  then show ?thesis
    by (auto simp: simplicial_simplex feq  simplex_cone)
qed

definition simplicial_cone
  where simplicial_cone p v ≡ frag_extend (frag_of ∘ simplex_cone p v)

lemma simplicial_chain_simplicial_cone:
  assumes c: simplicial_chain p S c
    and T: ⋀x u. ⟦0 ≤ u; u ≤ 1; x ∈ S⟧ ⟹ (λi. (1 − u) ∗ v i + u ∗ x i) ∈ T
  shows simplicial_chain (Suc p) T (simplicial_cone p v c)
  using c unfolding simplicial_chain_def simplicial_cone_def
proof (induction rule: frag_induction)
  case (one x)
  then show ?case
    by (simp add: T simplicial_simplex_simplex_cone)
next
  case (diff a b)
  then show ?case
    by (metis frag_extend_diff simplicial_chain_def simplicial_chain_diff)
qed auto


lemma chain_boundary_simplicial_cone_of':
  assumes f = oriented_simplex p l
  shows chain_boundary (Suc p) (simplicial_cone p v (frag_of f)) =
       frag_of f
       − (if p = 0 then frag_of (λu∈standard_simplex p. v)
          else simplicial_cone (p −1) v (chain_boundary p (frag_of f)))
proof (simp, intro impI conjI)
```

```
  assume p = 0
  have eq: (oriented_simplex 0 (λj. if j = 0 then v else l j)) = (λu∈standard_simplex
0. v)
    by (force simp: oriented_simplex_def standard_simplex_def)
  show chain_boundary (Suc 0) (simplicial_cone 0 v (frag_of f))
        = frag_of f − frag_of (λu∈standard_simplex 0. v)
     by (simp add: assms simplicial_cone_def chain_boundary_of ‹p = 0› sim-
plex_cone singular_face_oriented_simplex eq cong: if_cong)
next
  assume 0 < p
  have 0: simplex_cone (p − Suc 0) v (singular_face p x (oriented_simplex p l))
         = oriented_simplex p
              (λj. if j < Suc x
                  then if j = 0 then v else l (j −1)
                  else if Suc j = 0 then v else l (Suc j −1)) if x ≤ p for x
    using ‹0 < p› that
      by (auto simp: Suc_leI singular_face_oriented_simplex simplex_cone ori-
ented_simplex_eq)
  have 1: frag_extend (frag_of ∘ simplex_cone (p − Suc 0) v)
                     (∑ k = 0..p. frag_cmul ((−1) ^ k) (frag_of (singular_face p k
(oriented_simplex p l))))
         = − (∑ k = Suc 0..Suc p. frag_cmul ((−1) ^ k)
              (frag_of (singular_face (Suc p) k (simplex_cone p v (oriented_simplex
p l)))))
    unfolding sum.atLeast_Suc_atMost_Suc_shift
   by (auto simp: 0 simplex_cone singular_face_oriented_simplex frag_extend_sum
frag_extend_cmul simp flip: sum_negf)
  moreover have 2: singular_face (Suc p) 0 (simplex_cone p v (oriented_simplex
p l))
                     = oriented_simplex p l
    by (simp add: simplex_cone singular_face_oriented_simplex)
  show chain_boundary (Suc p) (simplicial_cone p v (frag_of f))
        = frag_of f − simplicial_cone (p − Suc 0) v (chain_boundary p (frag_of
f))
    using ‹p > 0›
   apply (simp add: assms simplicial_cone_def chain_boundary_of atMost_atLeast0
del: sum.atMost_Suc)
    apply (subst sum.atLeast_Suc_atMost [of 0])
     apply (simp_all add: 1 2 del: sum.atMost_Suc)
    done
qed

lemma chain_boundary_simplicial_cone_of:
  assumes simplicial_simplex p S f
  shows chain_boundary (Suc p) (simplicial_cone p v (frag_of f)) =
         frag_of f
         − (if p = 0 then frag_of (λu∈standard_simplex p. v)
            else simplicial_cone (p −1) v (chain_boundary p (frag_of f)))
 using chain_boundary_simplicial_cone_of' assms unfolding simplicial_simplex_def
```

```
  by blast

lemma chain_boundary_simplicial_cone:
  simplicial_chain p S c
   ⟹ chain_boundary (Suc p) (simplicial_cone p v c) =
       c − (if p = 0 then frag_extend (λf. frag_of (λu∈standard_simplex p. v)) c
            else simplicial_cone (p −1) v (chain_boundary p c))
  unfolding simplicial_chain_def
proof (induction rule: frag_induction)
  case (one x)
  then show ?case
    by (auto simp: chain_boundary_simplicial_cone_of)
qed (auto simp: chain_boundary_diff simplicial_cone_def frag_extend_diff)

lemma simplex_map_oriented_simplex:
  assumes l: simplicial_simplex p (standard_simplex q) (oriented_simplex p l)
    and g: simplicial_simplex r S g and q ≤ r
  shows simplex_map p g (oriented_simplex p l) = oriented_simplex p (g ∘ l)
proof −
  obtain m where geq: g = oriented_simplex r m
    using g by (auto simp: simplicial_simplex_def)
  have g (λi. ∑j≤p. l j i ∗ x j) i = (∑j≤p. g (l j) i ∗ x j)
    if x ∈ standard_simplex p for x i
  proof −
    have ssr: (λi. ∑j≤p. l j i ∗ x j) ∈ standard_simplex r
      using l that standard_simplex_mono [OF ‹q ≤ r›]
      unfolding simplicial_simplex_oriented_simplex by auto
    have lss: l j ∈ standard_simplex r if j≤p for j
    proof −
     have q: (λx i. ∑j≤p. l j i ∗ x j) ‘ standard_simplex p ⊆ standard_simplex q
       using l by (simp add: simplicial_simplex_oriented_simplex)
      let ?x = (λi. if i = j then 1 else 0)
      have p: l j ∈ (λx i. ∑j≤p. l j i ∗ x j) ‘ standard_simplex p
      proof
        show l j = (λi. ∑j≤p. l j i ∗ ?x j)
          using ‹j≤p› by (force simp: if_distrib cong: if_cong)
        show ?x ∈ standard_simplex p
          by (simp add: that)
      qed
      show ?thesis
        using standard_simplex_mono [OF ‹q ≤ r›] q p
        by blast
    qed
    have g (λi. ∑j≤p. l j i ∗ x j) i = (∑j≤r. ∑n≤p. m j i ∗ (l n j ∗ x n))
      by (simp add: geq oriented_simplex_def sum_distrib_left ssr)
    also have ... = (∑j≤p. ∑n≤r. m n i ∗ (l j n ∗ x j))
      by (rule sum.swap)
    also have ... = (∑j≤p. g (l j) i ∗ x j)
      by (simp add: geq oriented_simplex_def sum_distrib_right mult.assoc lss)
```

```
    finally show ?thesis .
  qed
  then show ?thesis
    by (force simp: oriented_simplex_def simplex_map_def o_def)
qed


lemma chain_map_simplicial_cone:
  assumes g: simplicial_simplex r S g
      and c: simplicial_chain p (standard_simplex q) c
      and v: v ∈ standard_simplex q and q ≤ r
  shows chain_map (Suc p) g (simplicial_cone p v c) = simplicial_cone p (g v)
(chain_map p g c)
proof -
  have *: simplex_map (Suc p) g (simplex_cone p v f) = simplex_cone p (g v)
(simplex_map p g f)
    if f ∈ Poly_Mapping.keys c for f
  proof -
    have simplicial_simplex p (standard_simplex q) f
      using c that by (auto simp: simplicial_chain_def)
    then obtain m where feq: f = oriented_simplex p m
      by (auto simp: simplicial_simplex)
    have 0: simplicial_simplex p (standard_simplex q) (oriented_simplex p m)
      using ‹simplicial_simplex p (standard_simplex q) f› feq by blast
    then have 1: simplicial_simplex (Suc p) (standard_simplex q)
                     (oriented_simplex (Suc p) (λi. if i = 0 then v else m (i -1)))
      using convex_standard_simplex v
      by (simp flip: simplex_cone add: simplicial_simplex_simplex_cone)
    show ?thesis
      using simplex_map_oriented_simplex [OF 1 g ‹q ≤ r›]
            simplex_map_oriented_simplex [of p q m r S g, OF 0 g ‹q ≤ r›]
      by (simp add: feq oriented_simplex_eq simplex_cone)
  qed
  show ?thesis
    by (auto simp: chain_map_def simplicial_cone_def frag_extend_compose *
intro: frag_extend_eq)
qed
```

0.1.15 Barycentric subdivision of a linear ("simplicial") simplex's image

```
definition simplicial_vertex
  where simplicial_vertex i f = f(λj. if j = i then 1 else 0)

lemma simplicial_vertex_oriented_simplex:
   simplicial_vertex i (oriented_simplex p l) = (if i ≤ p then l i else undefined)
  by (simp add: simplicial_vertex_def oriented_simplex_def if_distrib cong: if_cong)
```

primrec *simplicial_subdivision*
where
simplicial_subdivision 0 = id
| *simplicial_subdivision (Suc p) =*
frag_extend
(λf. simplicial_cone p
(λi. (∑j≤Suc p. simplicial_vertex j f i) / (p + 2))
(simplicial_subdivision p (chain_boundary (Suc p) (frag_of f))))

lemma *simplicial_subdivision_0* [*simp*]:
simplicial_subdivision p 0 = 0
by (*induction p*) *auto*

lemma *simplicial_subdivision_diff*:
simplicial_subdivision p (c1−c2) = simplicial_subdivision p c1 − simplicial_subdivision p c2
by (*induction p*) (*auto simp*: *frag_extend_diff*)

lemma *simplicial_subdivision_of*:
simplicial_subdivision p (frag_of f) =
(if p = 0 then frag_of f
else simplicial_cone (p −1)
(λi. (∑j≤p. simplicial_vertex j f i) / (Suc p))
(simplicial_subdivision (p −1) (chain_boundary p (frag_of f))))
by (*induction p*) (*auto simp*: *add.commute*)

lemma *simplicial_chain_simplicial_subdivision*:
simplicial_chain p S c
⟹ simplicial_chain p S (simplicial_subdivision p c)
proof (*induction p arbitrary*: *S c*)
case (*Suc p*)
show *?case*
using *Suc.prems* [*unfolded simplicial_chain_def*]
proof (*induction c rule*: *frag_induction*)
case (*one f*)
then have *f*: *simplicial_simplex (Suc p) S f*
by *auto*
then have *simplicial_chain p (f ‘ standard_simplex (Suc p))*
(simplicial_subdivision p (chain_boundary (Suc p) (frag_of f)))
by (*metis Suc.IH diff_Suc_1 simplicial_chain_boundary simplicial_chain_of simplicial_simplex subsetI*)
moreover
obtain *l* **where** *l*: *⋀x. x ∈ standard_simplex (Suc p) ⟹ (λi. (∑j≤Suc p. l j i ∗ x j)) ∈ S*
and *feq*: *f = oriented_simplex (Suc p) l*
using *f* **by** (*fastforce simp*: *simplicial_simplex oriented_simplex_def simp del*: *sum.atMost_Suc*)

```
    have (λi. (1 − u) ∗ ((∑j≤Suc p. simplicial_vertex j f i) / (real p + 2)) + u
∗ y i) ∈ S
      if 0 ≤ u u ≤ 1 and y: y ∈ f ‘ standard_simplex (Suc p) for y u
    proof −
      obtain x where x: x ∈ standard_simplex (Suc p) and yeq: y = ori-
ented_simplex (Suc p) l x
        using y feq by blast
      have (λi. ∑j≤Suc p. l j i ∗ ((if j ≤ Suc p then (1 − u) ∗ inverse (p + 2)
+ u ∗ x j else 0))) ∈ S
      proof (rule l)
        have inverse (2 + real p) ≤ 1 (2 + real p) ∗ ((1 − u) ∗ inverse (2 + real
p)) + u = 1
          by (auto simp add: field_split_simps)
        then show (λj. if j ≤ Suc p then (1 − u) ∗ inverse (real (p + 2)) + u ∗ x
j else 0) ∈ standard_simplex (Suc p)
          using x ‹0 ≤ u› ‹u ≤ 1›
            by (simp add: sum.distrib standard_simplex_def linepath_le_1 flip:
sum_distrib_left del: sum.atMost_Suc)
      qed
      moreover have (λi. ∑j≤Suc p. l j i ∗ ((1 − u) ∗ inverse (2 + real p) + u
∗ x j))
                  = (λi. (1 − u) ∗ (∑j≤Suc p. l j i) / (real p + 2) + u ∗ (∑j≤Suc
p. l j i ∗ x j))
      proof
        fix i
        have (∑j≤Suc p. l j i ∗ ((1 − u) ∗ inverse (2 + real p) + u ∗ x j))
            = (∑j≤Suc p. (1 − u) ∗ l j i / (real p + 2) + u ∗ l j i ∗ x j) (is ?lhs
= _)
          by (simp add: field_simps cong: sum.cong)
        also have . . . = (1 − u) ∗ (∑j≤Suc p. l j i) / (real p + 2) + u ∗ (∑j≤Suc
p. l j i ∗ x j) (is _ = ?rhs)
          by (simp add: sum_distrib_left sum.distrib sum_divide_distrib mult.assoc
del: sum.atMost_Suc)
        finally show ?lhs = ?rhs .
      qed
      ultimately show ?thesis
        using feq x yeq
      by (simp add: simplicial_vertex_oriented_simplex) (simp add: oriented_simplex_def)
    qed
    ultimately show ?case
      by (simp add: simplicial_chain_simplicial_cone)
  next
    case (diff a b)
    then show ?case
      by (metis simplicial_chain_diff simplicial_subdivision_diff)
  qed auto
qed auto

lemma chain_boundary_simplicial_subdivision:
```

simplicial_chain p S c
⟹ *chain_boundary p* (*simplicial_subdivision p c*) = *simplicial_subdivision* (*p* −*1*) (*chain_boundary p c*)
proof (*induction p arbitrary*: *c*)
case (*Suc p*)
show *?case*
using *Suc.prems* [*unfolded simplicial_chain_def*]
proof (*induction c rule*: *frag_induction*)
case (*one f*)
then have *f*: *simplicial_simplex* (*Suc p*) *S f*
by *simp*
then have *simplicial_chain p S* (*simplicial_subdivision p* (*chain_boundary* (*Suc p*) (*frag_of f*)))
by (*metis diff_Suc_1 simplicial_chain_boundary simplicial_chain_of simplicial_chain_simplicial_subdivision*)
moreover have *simplicial_chain p S* (*chain_boundary* (*Suc p*) (*frag_of f*))
using *one simplicial_chain_boundary simplicial_chain_of* **by** *fastforce*
moreover have *simplicial_subdivision* (*p* − *Suc 0*) (*chain_boundary p* (*chain_boundary* (*Suc p*) (*frag_of f*))) = *0*
by (*metis f chain_boundary_boundary_alt simplicial_simplex_def simplicial_subdivision_0 singular_chain_of*)
ultimately show *?case*
using *chain_boundary_simplicial_cone Suc*
by (*auto simp*: *chain_boundary_of frag_extend_diff simplicial_cone_def*)
next
case (*diff a b*)
then show *?case*
by (*simp add*: *simplicial_subdivision_diff chain_boundary_diff frag_extend_diff*)
qed *auto*
qed *auto*

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lemma *simplicial_subdivision_shrinks*:
⟦*simplicial_chain p S c*;
⋀*f x y*. ⟦*f* ∈ *Poly_Mapping.keys c*; *x* ∈ *standard_simplex p*; *y* ∈ *standard_simplex p*⟧ ⟹ |*f x k* − *f y k*| ≤ *d*;
f ∈ *Poly_Mapping.keys*(*simplicial_subdivision p c*);
x ∈ *standard_simplex p*; *y* ∈ *standard_simplex p*⟧
⟹ |*f x k* − *f y k*| ≤ (*p* / (*Suc p*)) ∗ *d*
proof (*induction p arbitrary*: *d c f x y*)
case (*Suc p*)
define *Sigp* **where** *Sigp* ≡ λ*f*:: (*nat* ⇒ *real*) ⇒ *nat* ⇒ *real*. λ*i*. (∑*j*≤*Suc p*. *simplicial_vertex j f i*) / *real* (*p* + *2*)
define *CB* **where** *CB* ≡ λ*f*::(*nat* ⇒ *real*) ⇒ *nat* ⇒ *real*. *chain_boundary* (*Suc p*) (*frag_of f*)
have ∗: *Poly_Mapping.keys*
(*simplicial_cone p* (*Sigp f*)
(*simplicial_subdivision p* (*CB f*)))
⊆ {*f*. ∀ *x*∈*standard_simplex* (*Suc p*). ∀ *y*∈*standard_simplex* (*Suc p*).

```
                      |f x k − f y k| ≤ Suc p / (real p + 2) ∗ d} (is ?lhs ⊆ ?rhs)
    if f: f ∈ Poly_Mapping.keys c for f
  proof −
    have ssf: simplicial_simplex (Suc p) S f
      using Suc.prems(1) simplicial_chain_def that by auto
    have 2: ⋀x y. ⟦x ∈ standard_simplex (Suc p); y ∈ standard_simplex (Suc p)⟧
⟹ |f x k − f y k| ≤ d
     by (meson Suc.prems(2) f subsetD le_Suc_eq order_refl standard_simplex_mono)
    have sub: Poly_Mapping.keys ((frag_of ∘ simplex_cone p (Sigp f)) g) ⊆ ?rhs
      if g ∈ Poly_Mapping.keys (simplicial_subdivision p (CB f)) for g
    proof −
      have 1: simplicial_chain p S (CB f)
        unfolding CB_def
        using ssf simplicial_chain_boundary simplicial_chain_of by fastforce
      have simplicial_chain (Suc p) (f ` standard_simplex(Suc p)) (frag_of f)
        by (metis simplicial_chain_of simplicial_simplex ssf subset_refl)
      then have sc_sub: Poly_Mapping.keys (CB f)
                       ⊆ Collect (simplicial_simplex p (f ` standard_simplex (Suc p)))
           by (metis diff_Suc_1 simplicial_chain_boundary simplicial_chain_def
CB_def)
      have led: ⋀h x y. ⟦h ∈ Poly_Mapping.keys (CB f);
                          x ∈ standard_simplex p; y ∈ standard_simplex p⟧ ⟹ |h x k
− h y k| ≤ d
        using Suc.prems(2) f sc_sub
        by (simp add: simplicial_simplex subset_iff image_iff) metis
      have ⋀f' x y. ⟦f' ∈ Poly_Mapping.keys (simplicial_subdivision p (CB f));
                       x ∈ standard_simplex p; y ∈ standard_simplex p⟧
            ⟹ |f' x k − f' y k| ≤ (p / (Suc p)) ∗ d
        by (blast intro: led Suc.IH [of CB f, OF 1])
      then have g: ⋀x y. ⟦x ∈ standard_simplex p; y ∈ standard_simplex p⟧ ⟹
|g x k − g y k| ≤ (p / (Suc p)) ∗ d
        using that by blast
      have d ≥ 0
        using Suc.prems(2)[OF f] ‹x ∈ standard_simplex (Suc p)› by force
      have 3: simplex_cone p (Sigp f) g ∈ ?rhs
      proof −
        have simplicial_simplex p (f ` standard_simplex(Suc p)) g
           by (metis (mono_tags, opaque_lifting) sc_sub mem_Collect_eq simpli-
cial_chain_def simplicial_chain_simplicial_subdivision subsetD that)
         then obtain m where m: g ` standard_simplex p ⊆ f ` standard_simplex
(Suc p)
          and geq: g = oriented_simplex p m
          using ssf by (auto simp: simplicial_simplex)
        have m_in_gim: m i ∈ g ` standard_simplex p if i ≤ p for i
        proof
          show m i = g (λj. if j = i then 1 else 0)
            by (simp add: geq oriented_simplex_def that if_distrib cong: if_cong)
          show (λj. if j = i then 1 else 0) ∈ standard_simplex p
            by (simp add: oriented_simplex_def that)
```

qed
obtain *l* **where** *l*: *f* ' *standard_simplex* (*Suc p*) ⊆ *S*
and *feq*: *f* = *oriented_simplex* (*Suc p*) *l*
using *ssf* **by** (*auto simp*: *simplicial_simplex*)
show *?thesis*
proof (*clarsimp simp add*: *geq simp del*: *sum.atMost_Suc*)
fix *x y*
assume *x*: *x* ∈ *standard_simplex* (*Suc p*) **and** *y*: *y* ∈ *standard_simplex* (*Suc p*)
then have *x'*: (∀ *i*. *0* ≤ *x i* ∧ *x i* ≤ *1*) ∧ (∀ *i*>*Suc p*. *x i* = *0*) ∧ (∑ *i*≤*Suc p*. *x i*) = *1*
and *y'*: (∀ *i*. *0* ≤ *y i* ∧ *y i* ≤ *1*) ∧ (∀ *i*>*Suc p*. *y i* = *0*) ∧ (∑ *i*≤*Suc p*. *y i*) = *1*
by (*auto simp*: *standard_simplex_def*)
have |(∑ *j*≤*Suc p*. (*if j* = *0 then* λ*i*. (∑ *j*≤*Suc p*. *l j i*) / (*2* + *real p*) *else m* (*j* −*1*)) *k* ∗ *x j*) −
(∑ *j*≤*Suc p*. (*if j* = *0 then* λ*i*. (∑ *j*≤*Suc p*. *l j i*) / (*2* + *real p*) *else m* (*j* −*1*)) *k* ∗ *y j*)|
≤ (*1* + *real p*) ∗ *d* / (*2* + *real p*)
proof −
have *zero*: |*m* (*s* − *Suc 0*) *k* − (∑ *j*≤*Suc p*. *l j k*) / (*2* + *real p*)| ≤ (*1* + *real p*) ∗ *d* / (*2* + *real p*)
if *0* < *s* **and** *s* ≤ *Suc p* **for** *s*
proof −
have *m* (*s* − *Suc 0*) ∈ *f* ' *standard_simplex* (*Suc p*)
using *m m_in_gim that*(*2*) **by** *auto*
then obtain *z* **where** *eq*: *m* (*s* − *Suc 0*) = (λ*i*. ∑ *j*≤*Suc p*. *l j i* ∗ *z j*) **and** *z*: *z* ∈ *standard_simplex* (*Suc p*)
using *feq* **unfolding** *oriented_simplex_def* **by** *auto*
show *?thesis*
unfolding *eq*
proof (*rule convex_sum_bound_le*)
fix *i*
assume *i*: *i* ∈ {..*Suc p*}
then have [*simp*]: *card* ({..*Suc p*} − {*i*}) = *Suc p*
by (*simp add*: *card_Suc_Diff1*)
have (∑ *j*≤*Suc p*. |*l i k* / (*p* + *2*) − *l j k* / (*p* + *2*)|) = (∑ *j*≤*Suc p*. |*l i k* − *l j k*| / (*p* + *2*))
by (*rule sum.cong*) (*simp_all add*: *flip*: *diff_divide_distrib*)
also have ... = (∑ *j* ∈ {..*Suc p*} − {*i*}. |*l i k* − *l j k*| / (*p* + *2*))
by (*rule sum.mono_neutral_right*) *auto*
also have ... ≤ (*1* + *real p*) ∗ *d* / (*p* + *2*)
proof (*rule sum_bounded_above_divide*)
fix *i'* :: *nat*
assume *i'*: *i'* ∈ {..*Suc p*} − {*i*}
have *lf*: *l r* ∈ *f* ' *standard_simplex*(*Suc p*) **if** *r* ≤ *Suc p* **for** *r*
proof
show *l r* = *f* (λ*j*. *if j* = *r then 1 else 0*)
using *that* **by** (*simp add*: *feq oriented_simplex_def if_distrib*

```
cong: if_cong)
                    show (λj. if j = r then 1 else 0) ∈ standard_simplex (Suc p)
                      by (auto simp: oriented_simplex_def that)
                  qed
                  show |l i k − l i′ k| / real (p + 2) ≤ (1 + real p) ∗ d / real (p +
2) / real (card ({..Suc p} − {i}))
                    using i i′ lf [of i] lf [of i′] 2
                    by (auto simp: image_iff divide_simps)
                qed auto
                finally have (∑j≤Suc p. |l i k / (p + 2) − l j k / (p + 2)|) ≤ (1
+ real p) ∗ d / (p + 2) .
                then have |∑j≤Suc p. l i k / (p + 2) − l j k / (p + 2)| ≤ (1 +
real p) ∗ d / (p + 2)
                  by (rule order_trans [OF sum_abs])
                then show |l i k − (∑j≤Suc p. l j k) / (2 + real p)| ≤ (1 + real p)
∗ d / (2 + real p)
              by (simp add: sum_subtractf sum_divide_distrib del: sum.atMost_Suc)
              qed (use standard_simplex_def z in auto)
            qed
            have nonz: |m (s − Suc 0) k − m (r − Suc 0) k| ≤ (1 + real p) ∗ d /
(2 + real p) (is ?lhs ≤ ?rhs)
              if r < s and 0 < r and r ≤ Suc p and s ≤ Suc p for r s
            proof −
              have ?lhs ≤ (p / (Suc p)) ∗ d
                using m_in_gim [of r − Suc 0] m_in_gim [of s − Suc 0] that g by
fastforce
              also have . . . ≤ ?rhs
                by (simp add: field_simps ‹0 ≤ d›)
              finally show ?thesis .
            qed
            have jj: j ≤ Suc p ∧ j′ ≤ Suc p
                  ⟶ |(if j′ = 0 then λi. (∑j≤Suc p. l j i) / (2 + real p) else m (j′
−1)) k −
                    (if j = 0 then λi. (∑j≤Suc p. l j i) / (2 + real p) else m (j −1))
k|
                     ≤ (1 + real p) ∗ d / (2 + real p) for j j′
              using ‹0 ≤ d›
               by (rule_tac a=j and b = j′ in linorder_less_wlog; force simp: zero
nonz simp del: sum.atMost_Suc)
            show ?thesis
              apply (rule convex_sum_bound_le)
              using x′ apply blast
              using x′ apply blast
              apply (subst abs_minus_commute)
              apply (rule convex_sum_bound_le)
              using y′ apply blast
              using y′ apply blast
              using jj by blast
          qed
```

```
            then show |simplex_cone p (Sigp f) (oriented_simplex p m) x k −
simplex_cone p (Sigp f) (oriented_simplex p m) y k|
              ≤ (1 + real p) ∗ d / (real p + 2)
              apply (simp add: feq Sigp_def simplicial_vertex_oriented_simplex
simplex_cone del: sum.atMost_Suc)
          apply (simp add: oriented_simplex_def algebra_simps x y del: sum.atMost_Suc)
            done
        qed
      qed
      show ?thesis
        using Suc.IH [OF 1, where f=g] 2 3 by simp
    qed
    then show ?thesis
      unfolding simplicial_chain_def simplicial_cone_def
      by (simp add: order_trans [OF keys_frag_extend] sub UN_subset_iff)
  qed
  obtain ff where ff ∈ Poly_Mapping.keys c
          f ∈ Poly_Mapping.keys
                (simplicial_cone p
                  (λi. (∑j≤Suc p. simplicial_vertex j ff i) /
                        (real p + 2))
                  (simplicial_subdivision p (CB ff)))
    using Suc.prems(3) subsetD [OF keys_frag_extend]
    by (force simp: CB_def simp del: sum.atMost_Suc)
  then show ?case
    using Suc ∗ by (simp add: add.commute Sigp_def subset_iff)
qed (auto simp: standard_simplex_0)
```

0.1.16 Singular subdivision

```
definition singular_subdivision
  where singular_subdivision p ≡
        frag_extend
          (λf. chain_map p f
                (simplicial_subdivision p
                      (frag_of(restrict id (standard_simplex p)))))

lemma singular_subdivision_0 [simp]: singular_subdivision p 0 = 0
  by (simp add: singular_subdivision_def)

lemma singular_subdivision_add:
  singular_subdivision p (a + b) = singular_subdivision p a + singular_subdivision
p b
  by (simp add: singular_subdivision_def frag_extend_add)

lemma singular_subdivision_diff:
  singular_subdivision p (a − b) = singular_subdivision p a − singular_subdivision
p b
  by (simp add: singular_subdivision_def frag_extend_diff)
```

lemma *simplicial_simplex_id* [*simp*]:
simplicial_simplex p S (*restrict id* (*standard_simplex p*)) ⟷ *standard_simplex p* ⊆ *S*
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*simp add*: *simplicial_simplex*)
next
assume *R*: *?rhs*
then have *cm*: *continuous_map*
(*subtopology* (*powertop_real UNIV*) (*standard_simplex p*))
(*subtopology* (*powertop_real UNIV*) *S*) *id*
using *continuous_map_from_subtopology_mono continuous_map_id* **by** *blast*
moreover have ∃ *l*. *restrict id* (*standard_simplex p*) = *oriented_simplex p l*
proof
show *restrict id* (*standard_simplex p*) = *oriented_simplex p* (λ*i j*. *if i* = *j then 1 else 0*)
by (*force simp*: *oriented_simplex_def standard_simplex_def if_distrib* [*of* λ*u*. *u* ∗ _] *cong*: *if_cong*)
qed
ultimately show *?lhs*
by (*simp add*: *simplicial_simplex_def singular_simplex_def*)
qed

lemma *singular_chain_singular_subdivision*:
assumes *singular_chain p X c*
shows *singular_chain p X* (*singular_subdivision p c*)
unfolding *singular_subdivision_def*
proof (*rule singular_chain_extend*)
fix *ca*
assume *ca* ∈ *Poly_Mapping.keys c*
with *assms* **have** *singular_simplex p X ca*
by (*simp add*: *singular_chain_def subset_iff*)
then show *singular_chain p X* (*chain_map p ca* (*simplicial_subdivision p* (*frag_of* (*restrict id* (*standard_simplex p*)))))
unfolding *singular_simplex_def*
by (*metis order_refl simplicial_chain_of simplicial_chain_simplicial_subdivision simplicial_imp_singular_chain simplicial_simplex_id singular_chain_chain_map*)
qed

lemma *naturality_singular_subdivision*:
singular_chain p X c
⟹ *singular_subdivision p* (*chain_map p g c*) = *chain_map p g* (*singular_subdivision p c*)
unfolding *singular_chain_def*
proof (*induction rule*: *frag_induction*)
case (*one f*)

```
  then have singular_simplex p X f
    by auto
  have ⟦simplicial_chain p (standard_simplex p) d⟧
    ⟹ chain_map p (simplex_map p g f) d = chain_map p g (chain_map p f d)
for d
    unfolding simplicial_chain_def
  proof (induction rule: frag_induction)
    case (one x)
   then have simplex_map p (simplex_map p g f) x = simplex_map p g (simplex_map
p f x)
      by (force simp: simplex_map_def restrict_compose_left simplicial_simplex)
    then show ?case
      by auto
  qed (auto simp: chain_map_diff)
  then show ?case
   using simplicial_chain_simplicial_subdivision [of p standard_simplex p frag_of
(restrict id (standard_simplex p))]
    by (simp add: singular_subdivision_def)
next
  case (diff a b)
  then show ?case
    by (simp add: chain_map_diff singular_subdivision_diff)
qed auto

lemma simplicial_chain_chain_map:
 assumes f: simplicial_simplex q X f and c: simplicial_chain p (standard_simplex
q) c
  shows simplicial_chain p X (chain_map p f c)
  using c unfolding simplicial_chain_def
proof (induction c rule: frag_induction)
  case (one g)
  have ∃ n. simplex_map p (oriented_simplex q l)
              (oriented_simplex p m) = oriented_simplex p n
    if m: singular_simplex p
          (subtopology (powertop_real UNIV) (standard_simplex q)) (oriented_simplex
p m)
    for l m
  proof -
    have (λi. ∑j≤p. m j i * x j) ∈ standard_simplex q
      if x ∈ standard_simplex p for x
      using that m unfolding oriented_simplex_def singular_simplex_def
      by (auto simp: continuous_map_in_subtopology Pi_iff)
    then show ?thesis
      unfolding oriented_simplex_def simplex_map_def
      apply (rule_tac x=λj k. (∑i≤q. l i k * m j i) in exI)
        apply (force simp: sum_distrib_left sum_distrib_right mult.assoc intro:
sum.swap)
      done
  qed
```

```
  then show ?case
    using f one
    apply (simp add: simplicial_simplex_def)
    using singular_simplex_def singular_simplex_simplex_map by blast
next
  case (diff a b)
  then show ?case
    by (metis chain_map_diff simplicial_chain_def simplicial_chain_diff)
qed auto


lemma singular_subdivision_simplicial_simplex:
   simplicial_chain p S c
          ⟹ singular_subdivision p c = simplicial_subdivision p c
proof (induction p arbitrary: S c)
  case 0
  then show ?case
    unfolding simplicial_chain_def
  proof (induction rule: frag_induction)
    case (one x)
    then show ?case
      using singular_simplex_chain_map_id simplicial_imp_singular_simplex
      by (fastforce simp: singular_subdivision_def simplicial_subdivision_def)
  qed (auto simp: singular_subdivision_diff)
next
  case (Suc p)
  show ?case
    using Suc.prems unfolding simplicial_chain_def
  proof (induction rule: frag_induction)
    case (one f)
    then have ssf: simplicial_simplex (Suc p) S f
      by (auto simp: simplicial_simplex)
    then have 1: simplicial_chain p (standard_simplex (Suc p))
                   (simplicial_subdivision p
                     (chain_boundary (Suc p)
                       (frag_of (restrict id (standard_simplex (Suc p))))))
     by (metis diff_Suc_1 order_refl simplicial_chain_boundary simplicial_chain_of
simplicial_chain_simplicial_subdivision simplicial_simplex_id)
     have 2: (λi. (∑j≤Suc p. simplicial_vertex j (restrict id (standard_simplex
(Suc p))) i) / (real p + 2))
                  ∈ standard_simplex (Suc p)
     by (simp add: simplicial_vertex_def standard_simplex_def del: sum.atMost_Suc)
    have ss_Sp: (λi. (if i ≤ Suc p then 1 else 0) / (real p + 2)) ∈ standard_simplex
(Suc p)
      by (simp add: standard_simplex_def field_split_simps)
    obtain l where feq: f = oriented_simplex (Suc p) l
      using one unfolding simplicial_simplex by blast
   then have 3: f (λi. (∑j≤Suc p. simplicial_vertex j (restrict id (standard_simplex
(Suc p))) i) / (real p + 2))
```

```
              = (λi. (∑j≤Suc p. simplicial_vertex j f i) / (real p + 2))
      unfolding simplicial_vertex_def oriented_simplex_def
        by (simp add: ss_Sp if_distrib [of λx. _ * x] sum_divide_distrib del:
sum.atMost_Suc cong: if_cong)
    have scp: singular_chain (Suc p)
                (subtopology (powertop_real UNIV) (standard_simplex (Suc p)))
                (frag_of (restrict id (standard_simplex (Suc p))))
      by (simp add: simplicial_imp_singular_chain)
    have scps: simplicial_chain p (standard_simplex (Suc p))
                  (chain_boundary (Suc p) (frag_of (restrict id (standard_simplex
(Suc p)))))
      by (metis diff_Suc_1 order_refl simplicial_chain_boundary simplicial_chain_of
simplicial_simplex_id)
    have scpf: simplicial_chain p S
                (chain_map p f
                  (chain_boundary (Suc p) (frag_of (restrict id (standard_simplex
(Suc p))))))
      using scps simplicial_chain_chain_map ssf by blast
    have 4: chain_map p f
              (simplicial_subdivision p
                  (chain_boundary (Suc p) (frag_of (restrict id (standard_simplex
(Suc p))))))
           = simplicial_subdivision p (chain_boundary (Suc p) (frag_of f))
    proof -
      have singular_simplex (Suc p) (subtopology (powertop_real UNIV) S) f
        using simplicial_simplex_def ssf by blast
       then have chain_map (Suc p) f (frag_of (restrict id (standard_simplex
(Suc p)))) = frag_of f
        using singular_simplex_chain_map_id by blast
      then show ?thesis
         by (metis (no_types) Suc.IH chain_boundary_chain_map diff_Suc_Suc
diff_zero
         naturality_singular_subdivision scp scpf scps simplicial_imp_singular_chain)
    qed
    show ?case
      apply (simp add: singular_subdivision_def del: sum.atMost_Suc)
       apply (simp only: ssf 1 2 3 4 chain_map_simplicial_cone [of Suc p S _ p
Suc p])
      done
  qed (auto simp: frag_extend_diff singular_subdivision_diff)
qed


lemma naturality_simplicial_subdivision:
   ⟦simplicial_chain p (standard_simplex q) c; simplicial_simplex q S g⟧
   ⟹ simplicial_subdivision p (chain_map p g c) = chain_map p g (simplicial_subdivision
p c)
  by (metis naturality_singular_subdivision simplicial_chain_chain_map simpli-
cial_imp_singular_chain
```

singular_subdivision_simplicial_simplex)

lemma *chain_boundary_singular_subdivision*:
singular_chain p X c
⟹ *chain_boundary p* (*singular_subdivision p c*) =
singular_subdivision (*p* − *Suc 0*) (*chain_boundary p c*)
unfolding *singular_chain_def*
proof (*induction rule*: *frag_induction*)
case (*one f*)
then have *ssf*: *singular_simplex p X f*
by (*auto simp*: *singular_simplex_def*)
then have *scp*: *simplicial_chain p* (*standard_simplex p*) (*frag_of* (*restrict id* (*standard_simplex p*)))
by *simp*
have *scp1*: *simplicial_chain* (*p* − *Suc 0*) (*standard_simplex p*)
(*chain_boundary p* (*frag_of* (*restrict id* (*standard_simplex p*))))
using *simplicial_chain_boundary* **by** *force*
have *sgp1*: *singular_chain* (*p* − *Suc 0*)
(*subtopology* (*powertop_real UNIV*) (*standard_simplex p*))
(*chain_boundary p* (*frag_of* (*restrict id* (*standard_simplex p*))))
using *scp1 simplicial_imp_singular_chain* **by** *blast*
have *scpp*: *singular_chain p* (*subtopology* (*powertop_real UNIV*) (*standard_simplex p*))
(*frag_of* (*restrict id* (*standard_simplex p*)))
using *scp simplicial_imp_singular_chain* **by** *blast*
then show *?case*
unfolding *singular_subdivision_def*
using *chain_boundary_chain_map* [*of p subtopology* (*powertop_real UNIV*)
(*standard_simplex p*) _ *f*]
apply (*simp add*: *simplicial_chain_simplicial_subdivision*
simplicial_imp_singular_chain chain_boundary_simplicial_subdivision [*OF scp*]
flip: *singular_subdivision_simplicial_simplex* [*OF scp1*] *naturality_singular_subdivision* [*OF sgp1*])
by (*metis* (*full_types*) *singular_subdivision_def chain_boundary_chain_map* [*OF scpp*] *singular_simplex_chain_map_id* [*OF ssf*])
qed (*auto simp*: *singular_subdivision_def frag_extend_diff chain_boundary_diff*)

lemma *singular_subdivision_zero*:
singular_chain 0 X c ⟹ *singular_subdivision 0 c* = *c*
unfolding *singular_chain_def*
proof (*induction rule*: *frag_induction*)
case (*one f*)
then have *restrict* (*f* ∘ *restrict id* (*standard_simplex 0*)) (*standard_simplex 0*) = *f*
by (*simp add*: *extensional_restrict restrict_compose_right singular_simplex_def*)
then show *?case*
by (*auto simp*: *singular_subdivision_def simplex_map_def*)
qed (*auto simp*: *singular_subdivision_def frag_extend_diff*)

primrec *subd* **where**
subd 0 = (λ*x*. *0*)
| *subd* (*Suc p*) =
frag_extend
(λ*f*. *simplicial_cone* (*Suc p*) (λ*i*. (∑*j*≤*Suc p*. *simplicial_vertex j f i*) / *real* (*Suc p* + *1*))
(*simplicial_subdivision* (*Suc p*) (*frag_of f*) − *frag_of f* −
subd p (*chain_boundary* (*Suc p*) (*frag_of f*))))

lemma *subd_0* [*simp*]: *subd p 0* = *0*
by (*induction p*) *auto*

lemma *subd_diff* [*simp*]: *subd p* (*c1* − *c2*) = *subd p c1* − *subd p c2*
by (*induction p*) (*auto simp*: *frag_extend_diff*)

lemma *subd_uminus* [*simp*]: *subd p* (−*c*) = − *subd p c*
by (*metis diff_0 subd_0 subd_diff*)

lemma *subd_power_uminus*: *subd p* (*frag_cmul* ((−*1*) ^ *k*) *c*) = *frag_cmul* ((−*1*) ^ *k*) (*subd p c*)
proof (*induction k*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc k*)
then show *?case*
by (*metis frag_cmul_cmul frag_cmul_minus_one power_Suc subd_uminus*)
qed

lemma *subd_power_sum*: *subd p* (*sum f I*) = *sum* (*subd p* ∘ *f*) *I*
proof (*induction I rule*: *infinite_finite_induct*)
case (*insert i I*)
then show *?case*
by (*metis* (*no_types*, *lifting*) *comp_apply diff_minus_eq_add subd_diff subd_uminus sum.insert*)
qed *auto*

lemma *subd*: *simplicial_chain p* (*standard_simplex s*) *c*
⟹ (∀ *r g*. *simplicial_simplex s* (*standard_simplex r*) *g* ⟶ *chain_map* (*Suc p*) *g* (*subd p c*) = *subd p* (*chain_map p g c*))
∧ *simplicial_chain* (*Suc p*) (*standard_simplex s*) (*subd p c*)
∧ (*chain_boundary* (*Suc p*) (*subd p c*)) + (*subd* (*p* − *Suc 0*) (*chain_boundary p c*)) = (*simplicial_subdivision p c*) − *c*
proof (*induction p arbitrary*: *c*)
case (*Suc p*)
show *?case*
using *Suc.prems* [*unfolded simplicial_chain_def*]

```
  proof (induction rule: frag_induction)
    case (one f)
    then obtain l where l: (λx i. ∑j≤Suc p. l j i * x j) ` standard_simplex (Suc
p) ⊆ standard_simplex s
                  and feq: f = oriented_simplex (Suc p) l
     by (metis (mono_tags) mem_Collect_eq simplicial_simplex simplicial_simplex_oriented_simplex)
    have scf: simplicial_chain (Suc p) (standard_simplex s) (frag_of f)
      using one by simp
    have lss: l i ∈ standard_simplex s if i ≤ Suc p for i
    proof −
      have (λi'. ∑j≤Suc p. l j i' * (if j = i then 1 else 0)) ∈ standard_simplex s
        using subsetD [OF l] basis_in_standard_simplex that by blast
      moreover have (λi'. ∑j≤Suc p. l j i' * (if j = i then 1 else 0)) = l i
         using that by (simp add: if_distrib [of λx. _ * x] del: sum.atMost_Suc
cong: if_cong)
      ultimately show ?thesis
        by simp
    qed
    have *: (⋀i. i ≤ n ⟹ l i ∈ standard_simplex s)
     ⟹ (λi. (∑j≤n. l j i) / (Suc n)) ∈ standard_simplex s for n
    proof (induction n)
      case (Suc n)
      let ?x = λi. (1 − inverse (n + 2)) * ((∑j≤n. l j i) / (Suc n)) + inverse (n
+ 2) * l (Suc n) i
      have ?x ∈ standard_simplex s
      proof (rule convex_standard_simplex)
        show (λi. (∑j≤n. l j i) / real (Suc n)) ∈ standard_simplex s
          using Suc by simp
      qed (auto simp: lss Suc inverse_le_1_iff)
      moreover have ?x = (λi. (∑j≤Suc n. l j i) / real (Suc (Suc n)))
        by (force simp: divide_simps)
      ultimately show ?case
        by simp
    qed auto
     have **: (λi. (∑j≤Suc p. simplicial_vertex j f i) / (2 + real p)) ∈ stan-
dard_simplex s
      using * [of Suc p] lss by (simp add: simplicial_vertex_oriented_simplex feq)
    show ?case
    proof (intro conjI impI allI)
      fix r g
      assume g: simplicial_simplex s (standard_simplex r) g
      then obtain m where geq: g = oriented_simplex s m
        using simplicial_simplex by blast
     have 1: simplicial_chain (Suc p) (standard_simplex s) (simplicial_subdivision
(Suc p) (frag_of f))
      by (metis mem_Collect_eq one.hyps simplicial_chain_of simplicial_chain_simplicial_subdivision)
      have 2: (∑j≤Suc p. ∑i≤s. m i k * simplicial_vertex j f i)
            = (∑j≤Suc p. simplicial_vertex j
                          (simplex_map (Suc p) (oriented_simplex s m) f) k) for k
```

```
proof (rule sum.cong [OF refl])
  fix j
  assume j: j ∈ {..Suc p}
   have eq: simplex_map (Suc p) (oriented_simplex s m) (oriented_simplex
(Suc p) l)
          = oriented_simplex (Suc p) (oriented_simplex s m ∘ l)
  proof (rule simplex_map_oriented_simplex)
    show simplicial_simplex (Suc p) (standard_simplex s) (oriented_simplex
(Suc p) l)
      using one by (simp add: feq flip: oriented_simplex_def)
    show simplicial_simplex s (standard_simplex r) (oriented_simplex s m)
      using g by (simp add: geq)
  qed auto
  show (∑ i≤s. m i k * simplicial_vertex j f i)
     = simplicial_vertex j (simplex_map (Suc p) (oriented_simplex s m) f) k
    using one j
      apply (simp add: feq eq simplicial_vertex_oriented_simplex simpli-
cial_simplex_oriented_simplex image_subset_iff)
    apply (drule_tac x=(λi. if i = j then 1 else 0) in bspec)
     apply (auto simp: oriented_simplex_def lss)
    done
qed
have 4: chain_map (Suc p) g (subd p (chain_boundary (Suc p) (frag_of f)))
        = subd p (chain_boundary (Suc p) (frag_of (simplex_map (Suc p) g
f)))
  by (metis (no_types) One_nat_def scf Suc.IH chain_boundary_chain_map
chain_map_of diff_Suc_Suc diff_zero g simplicial_chain_boundary simplicial_imp_singular_chain)
show chain_map (Suc (Suc p)) g (subd (Suc p) (frag_of f)) = subd (Suc p)
(chain_map (Suc p) g (frag_of f))
  unfolding subd.simps frag_extend_of
  using g
  apply (subst chain_map_simplicial_cone [of s standard_simplex r _ Suc p
s], assumption)
   apply (metis 1 Suc.IH diff_Suc_1 scf simplicial_chain_boundary simpli-
cial_chain_diff)
  using ** apply auto[1]
   apply (rule order_refl)
  unfolding chain_map_of frag_extend_of
  apply (rule arg_cong2 [where f = simplicial_cone (Suc p)])
     apply (simp add: geq sum_distrib_left oriented_simplex_def ** del:
sum.atMost_Suc flip: sum_divide_distrib)
   using 2 apply (simp only: oriented_simplex_def sum.swap [where A =
{..s}])
   using naturality_simplicial_subdivision scf apply (fastforce simp add: 4
chain_map_diff)
  done
next
  have sc: simplicial_chain (Suc p) (standard_simplex s)
          (simplicial_cone p
```

```
                (λi. (∑j≤Suc p. simplicial_vertex j f i) / (Suc (Suc p)))
                (simplicial_subdivision p
                  (chain_boundary (Suc p) (frag_of f))))
          by (metis diff_Suc_1 nat.simps(3) simplicial_subdivision_of scf simpli-
cial_chain_simplicial_subdivision)
    have ff: simplicial_chain (Suc p) (standard_simplex s) (subd p (chain_boundary
(Suc p) (frag_of f)))
        by (metis (no_types) Suc.IH diff_Suc_1 scf simplicial_chain_boundary)
        show simplicial_chain (Suc (Suc p)) (standard_simplex s) (subd (Suc p)
(frag_of f))
        using one
        unfolding subd.simps frag_extend_of
      apply (rule_tac S=standard_simplex s in simplicial_chain_simplicial_cone)
      apply (meson ff scf simplicial_chain_diff simplicial_chain_simplicial_subdivision)
        using ** convex_standard_simplex by force
        have simplicial_chain p (standard_simplex s) (chain_boundary (Suc p)
(frag_of f))
        using scf simplicial_chain_boundary by fastforce
      then have chain_boundary (Suc p) (simplicial_subdivision (Suc p) (frag_of
f) − frag_of f
                                        − subd p (chain_boundary (Suc p) (frag_of f)))
= 0
        unfolding chain_boundary_diff
        using Suc.IH chain_boundary_boundary
      by (metis One_nat_def add_diff_cancel_left' chain_boundary_simplicial_subdivision
diff_Suc_1 scf
              simplicial_imp_singular_chain subd_0)
      moreover have simplicial_chain (Suc p) (standard_simplex s)
                      (simplicial_subdivision (Suc p) (frag_of f) − frag_of f −
                       subd p (chain_boundary (Suc p) (frag_of f)))
      by (meson ff scf simplicial_chain_diff simplicial_chain_simplicial_subdivision)
        ultimately show chain_boundary (Suc (Suc p)) (subd (Suc p) (frag_of f))
          + subd (Suc p − Suc 0) (chain_boundary (Suc p) (frag_of f))
          = simplicial_subdivision (Suc p) (frag_of f) − frag_of f
        unfolding subd.simps frag_extend_of
        apply (simp add: chain_boundary_simplicial_cone )
      apply (simp add: simplicial_cone_def del: sum.atMost_Suc simplicial_subdivision.simps)
        done
    qed
  next
    case (diff a b)
    then show ?case
      apply safe
        apply (metis chain_map_diff subd_diff)
       apply (metis simplicial_chain_diff subd_diff)
    by (smt (verit, ccfv_threshold) add_diff_add chain_boundary_diff diff_add_cancel
simplicial_subdivision_diff subd_diff)
  qed auto
qed simp
```

lemma *chain_homotopic_simplicial_subdivision1*:
⟦*simplicial_chain p* (*standard_simplex q*) *c*; *simplicial_simplex q* (*standard_simplex r*) *g*⟧
⟹ *chain_map* (*Suc p*) *g* (*subd p c*) = *subd p* (*chain_map p g c*)
by (*simp add*: *subd*)

lemma *chain_homotopic_simplicial_subdivision2*:
simplicial_chain p (*standard_simplex q*) *c*
⟹ *simplicial_chain* (*Suc p*) (*standard_simplex q*) (*subd p c*)
by (*simp add*: *subd*)

lemma *chain_homotopic_simplicial_subdivision3*:
simplicial_chain p (*standard_simplex q*) *c*
⟹ *chain_boundary* (*Suc p*) (*subd p c*) = (*simplicial_subdivision p c*) − *c* − *subd* (*p* − *Suc 0*) (*chain_boundary p c*)
by (*simp add*: *subd algebra_simps*)

lemma *chain_homotopic_simplicial_subdivision*:
∃ *h*. (∀ *p*. *h p 0* = *0*) ∧
(∀ *p c1 c2*. *h p* (*c1*−*c2*) = *h p c1* − *h p c2*) ∧
(∀ *p q r g c*.
simplicial_chain p (*standard_simplex q*) *c*
⟶ *simplicial_simplex q* (*standard_simplex r*) *g*
⟶ *chain_map* (*Suc p*) *g* (*h p c*) = *h p* (*chain_map p g c*)) ∧
(∀ *p q c*. *simplicial_chain p* (*standard_simplex q*) *c*
⟶ *simplicial_chain* (*Suc p*) (*standard_simplex q*) (*h p c*)) ∧
(∀ *p q c*. *simplicial_chain p* (*standard_simplex q*) *c*
⟶ *chain_boundary* (*Suc p*) (*h p c*) + *h* (*p* − *Suc 0*) (*chain_boundary p c*)
= (*simplicial_subdivision p c*) − *c*)
by (*rule_tac x*=*subd* **in** *exI*) (*fastforce simp*: *subd*)

lemma *chain_homotopic_singular_subdivision*:
obtains *h* **where**
⋀*p*. *h p 0* = *0*
⋀*p c1 c2*. *h p* (*c1*−*c2*) = *h p c1* − *h p c2*
⋀*p X c*. *singular_chain p X c* ⟹ *singular_chain* (*Suc p*) *X* (*h p c*)
⋀*p X c*. *singular_chain p X c*
⟹ *chain_boundary* (*Suc p*) (*h p c*) + *h* (*p* − *Suc 0*) (*chain_boundary p c*) = *singular_subdivision p c* − *c*
proof −
define *k* **where** *k* ≡ λ*p*. *frag_extend* (λ*f*:: (*nat* ⇒ *real*) ⇒ ′*a*. *chain_map* (*Suc p*) *f* (*subd p* (*frag_of*(*restrict id* (*standard_simplex p*)))))
show *?thesis*
proof
fix *p X* **and** *c* :: ′*a chain*
assume *c*: *singular_chain p X c*
have *singular_chain* (*Suc p*) *X* (*k p c*) ∧

```
          chain_boundary (Suc p) (k p c) + k (p − Suc 0) (chain_boundary p
c) = singular_subdivision p c − c
    using c [unfolded singular_chain_def]
  proof (induction rule: frag_induction)
    case (one f)
    let ?X = subtopology (powertop_real UNIV) (standard_simplex p)
    show ?case
    proof (simp add: k_def, intro conjI)
      show singular_chain (Suc p) X (chain_map (Suc p) f (subd p (frag_of
(restrict id (standard_simplex p)))))
      proof (rule singular_chain_chain_map)
      show singular_chain (Suc p) ?X (subd p (frag_of (restrict id (standard_simplex
p))))
        by (simp add: chain_homotopic_simplicial_subdivision2 simplicial_imp_singular_chain)
        show continuous_map ?X X f
          using one.hyps singular_simplex_def by auto
      qed
    next
    have scp: singular_chain (Suc p) ?X (subd p (frag_of (restrict id (standard_simplex
p))))
      by (simp add: chain_homotopic_simplicial_subdivision2 simplicial_imp_singular_chain)
      have feqf: frag_of (simplex_map p f (restrict id (standard_simplex p))) =
frag_of f
        using one.hyps singular_simplex_chain_map_id by auto
      have ∗: chain_map p f
              (subd (p − Suc 0)
               (∑k≤p. frag_cmul ((−1) ^ k) (frag_of (singular_face p k id))))
          = (∑x≤p. frag_cmul ((−1) ^ x)
                 (chain_map p (singular_face p x f)
                   (subd (p − Suc 0) (frag_of (restrict id (standard_simplex
(p − Suc 0)))))))
              (is ?lhs = ?rhs)
              if p > 0
      proof −
        have eqc: subd (p − Suc 0) (frag_of (singular_face p i id))
              = chain_map p (singular_face p i id)
                    (subd (p − Suc 0) (frag_of (restrict id (standard_simplex
(p − Suc 0)))))
          if i ≤ p for i
        proof −
          have 1: simplicial_chain (p − Suc 0) (standard_simplex (p − Suc 0))
                  (frag_of (restrict id (standard_simplex (p − Suc 0))))
            by simp
        have 2: simplicial_simplex (p − Suc 0) (standard_simplex p) (singular_face
p i id)
              by (metis One_nat_def Suc_leI ‹0 < p› simplicial_simplex_id
simplicial_simplex_singular_face singular_face_restrict subsetI that)
            have 3: simplex_map (p − Suc 0) (singular_face p i id) (restrict id
(standard_simplex (p − Suc 0)))
```

= *singular_face p i id*
by (*force simp*: *simplex_map_def singular_face_def*)
show *?thesis*
using *chain_homotopic_simplicial_subdivision1* [*OF 1 2*]
that ‹*p* > *0*› **by** (*simp add*: *3*)
qed
have *xx*: *simplicial_chain p* (*standard_simplex*(*p* − *Suc 0*))
(*subd* (*p* − *Suc 0*) (*frag_of* (*restrict id* (*standard_simplex* (*p* − *Suc 0*)))))
by (*metis Suc_pred chain_homotopic_simplicial_subdivision2 order_refl simplicial_chain_of simplicial_simplex_id that*)
have *yy*: ⋀*k*. *k* ≤ *p* ⟹
chain_map p f
(*chain_map p* (*singular_face p k id*) *h*) = *chain_map p* (*singular_face p k f*) *h*
if *simplicial_chain p* (*standard_simplex*(*p* − *Suc 0*)) *h* **for** *h*
using *that* **unfolding** *simplicial_chain_def*
proof (*induction h rule*: *frag_induction*)
case (*one x*)
then show *?case*
using *one*
apply (*simp add*: *chain_map_of singular_simplex_def simplicial_simplex_def*, *auto*)
apply (*rule arg_cong* [**where** *f*=*frag_of*])
by (*auto simp*: *image_subset_iff simplex_map_def simplicial_simplex singular_face_def*)

qed (*auto simp*: *chain_map_diff*)
have *?lhs*
= *chain_map p f*
(∑*k*≤*p*. *frag_cmul* ((−*1*) ^ *k*)
(*chain_map p* (*singular_face p k id*)
(*subd* (*p* − *Suc 0*) (*frag_of* (*restrict id* (*standard_simplex* (*p* − *Suc 0*)))))))
by (*simp add*: *subd_power_sum subd_power_uminus eqc*)
also have ... = *?rhs*
by (*simp add*: *chain_map_sum xx yy*)
finally show *?thesis* **.**
qed
have *chain_map p f*
(*simplicial_subdivision p* (*frag_of* (*restrict id* (*standard_simplex p*)))
− *subd* (*p* − *Suc 0*) (*chain_boundary p* (*frag_of* (*restrict id* (*standard_simplex p*)))))
= *singular_subdivision p* (*frag_of f*)
− *frag_extend*
(λ*f*. *chain_map* (*Suc* (*p* − *Suc 0*)) *f*
(*subd* (*p* − *Suc 0*) (*frag_of* (*restrict id* (*standard_simplex* (*p* − *Suc 0*))))))

```
                (chain_boundary p (frag_of f))
        apply (simp add: singular_subdivision_def chain_map_diff)
        apply (clarsimp simp add: chain_boundary_def)
        apply (simp add: frag_extend_sum frag_extend_cmul *)
        done
      then show chain_boundary (Suc p) (chain_map (Suc p) f (subd p (frag_of
(restrict id (standard_simplex p)))))
               + frag_extend
                (λf. chain_map (Suc (p - Suc 0)) f
                     (subd (p - Suc 0) (frag_of (restrict id (standard_simplex (p
- Suc 0))))))
                (chain_boundary p (frag_of f))
               = singular_subdivision p (frag_of f) - frag_of f
      by (simp add: chain_boundary_chain_map [OF scp] chain_homotopic_simplicial_subdivision3
[where q=p] chain_map_diff feqf)
     qed
   next
     case (diff a b)
     then show ?case
     apply (simp only: k_def singular_chain_diff chain_boundary_diff frag_extend_diff
singular_subdivision_diff)
       by (metis (no_types, lifting) add_diff_add diff_add_cancel)
   qed (auto simp: k_def)
   then show singular_chain (Suc p) X (k p c) chain_boundary (Suc p) (k p c)
+ k (p - Suc 0) (chain_boundary p c) = singular_subdivision p c - c
       by auto
  qed (auto simp: k_def frag_extend_diff)
qed


lemma homologous_rel_singular_subdivision:
  assumes singular_relcycle p X T c
  shows homologous_rel p X T (singular_subdivision p c) c
proof (cases p = 0)
  case True
  with assms show ?thesis
    by (auto simp: singular_relcycle_def singular_subdivision_zero)
next
  case False
  with assms show ?thesis
    unfolding homologous_rel_def singular_relboundary singular_relcycle
     by (metis One_nat_def Suc_diff_1 chain_homotopic_singular_subdivision
gr_zeroI)
qed
```

0.1.17 Excision argument that we keep doing singular subdivision

```
lemma singular_subdivision_power_0 [simp]: (singular_subdivision p ^^ n) 0 =
0
  by (induction n) auto

lemma singular_subdivision_power_diff:
  (singular_subdivision p ^^ n) (a − b) = (singular_subdivision p ^^ n) a −
(singular_subdivision p ^^ n) b
  by (induction n) (auto simp: singular_subdivision_diff)

lemma iterated_singular_subdivision:
  singular_chain p X c
     ⟹ (singular_subdivision p ^^ n) c =
        frag_extend
         (λf. chain_map p f
                ((simplicial_subdivision p ^^ n)
                  (frag_of(restrict id (standard_simplex p))))) c
proof (induction n arbitrary: c)
  case 0
  then show ?case
    unfolding singular_chain_def
  proof (induction c rule: frag_induction)
    case (one f)
    then have restrict f (standard_simplex p) = f
      by (simp add: extensional_restrict singular_simplex_def)
    then show ?case
      by (auto simp: simplex_map_def cong: restrict_cong)
  qed (auto simp: frag_extend_diff)
next
  case (Suc n)
  show ?case
    using Suc.prems unfolding singular_chain_def
  proof (induction c rule: frag_induction)
    case (one f)
    then have singular_simplex p X f
      by simp
    have scp: simplicial_chain p (standard_simplex p)
          ((simplicial_subdivision p ^^ n) (frag_of (restrict id (standard_simplex
p))))
    proof (induction n)
      case 0
      then show ?case
       by (metis funpow_0 order_refl simplicial_chain_of simplicial_simplex_id)
    next
      case (Suc n)
      then show ?case
        by (simp add: simplicial_chain_simplicial_subdivision)
    qed
```

```
    have scnp: simplicial_chain p (standard_simplex p)
                ((simplicial_subdivision p ^^ n) (frag_of (λx∈standard_simplex p.
x)))
    proof (induction n)
      case 0
      then show ?case
           by (metis eq_id_iff funpow_0 order_refl simplicial_chain_of simpli-
cial_simplex_id)
    next
      case (Suc n)
      then show ?case
        by (simp add: simplicial_chain_simplicial_subdivision)
    qed
    have sff: singular_chain p X (frag_of f)
      by (simp add: ‹singular_simplex p X f› singular_chain_of)
    then show ?case
     using Suc.IH [OF sff] naturality_singular_subdivision [OF simplicial_imp_singular_chain
[OF scp], of f] singular_subdivision_simplicial_simplex [OF scnp]
      by (simp add: singular_chain_of id_def del: restrict_apply)
  qed (auto simp: singular_subdivision_power_diff singular_subdivision_diff frag_extend_diff)
qed


lemma chain_homotopic_iterated_singular_subdivision:
  obtains h where
        ⋀p. h p 0 = (0 :: 'a chain)
        ⋀p c1 c2. h p (c1−c2) = h p c1 − h p c2
        ⋀p X c. singular_chain p X c ⟹ singular_chain (Suc p) X (h p c)
        ⋀p X c. singular_chain p X c
              ⟹ chain_boundary (Suc p) (h p c) + h (p − Suc 0) (chain_boundary
p c)
                  = (singular_subdivision p ^^ n) c − c
proof (induction n arbitrary: thesis)
  case 0
  show ?case
    by (rule 0 [of (λp x. 0)]) auto
next
  case (Suc n)
  then obtain k where k:
        ⋀p. k p 0 = (0 :: 'a chain)
        ⋀p c1 c2. k p (c1−c2) = k p c1 − k p c2
        ⋀p X c. singular_chain p X c ⟹ singular_chain (Suc p) X (k p c)
        ⋀p X c. singular_chain p X c
              ⟹ chain_boundary (Suc p) (k p c) + k (p − Suc 0) (chain_boundary
p c)
                  = (singular_subdivision p ^^ n) c − c
    by metis
  obtain h where h:
        ⋀p. h p 0 = (0 :: 'a chain)
```

```
        ⋀p c1 c2. h p (c1−c2) = h p c1 − h p c2
        ⋀p X c. singular_chain p X c ⟹ singular_chain (Suc p) X (h p c)
        ⋀p X c. singular_chain p X c
            ⟹ chain_boundary (Suc p) (h p c) + h (p − Suc 0) (chain_boundary
p c) = singular_subdivision p c − c
    by (blast intro: chain_homotopic_singular_subdivision)
  let ?h = (λp c. singular_subdivision (Suc p) (k p c) + h p c)
  show ?case
  proof (rule Suc.prems)
    fix p X and c :: 'a chain
    assume singular_chain p X c
    then show singular_chain (Suc p) X (?h p c)
      by (simp add: h k singular_chain_add singular_chain_singular_subdivision)
  next
    fix p :: nat and X :: 'a topology and c :: 'a chain
    assume sc: singular_chain p X c
     have f5: chain_boundary (Suc p) (singular_subdivision (Suc p) (k p c)) =
singular_subdivision p (chain_boundary (Suc p) (k p c))
      using chain_boundary_singular_subdivision k(3) sc by fastforce
   have [simp]: singular_subdivision (Suc (p − Suc 0)) (k (p − Suc 0) (chain_boundary
p c)) =
                singular_subdivision p (k (p − Suc 0) (chain_boundary p c))
    proof (cases p)
      case 0
      then show ?thesis
        by (simp add: k chain_boundary_def)
    qed auto
    show chain_boundary (Suc p) (?h p c) + ?h (p − Suc 0) (chain_boundary p
c) = (singular_subdivision p ^^ Suc n) c − c
      using chain_boundary_singular_subdivision [of Suc p X]
      apply (simp add: chain_boundary_add f5 h k algebra_simps)
       by (smt (verit, del_insts) add.commute add.left_commute diff_add_cancel
h(4) k(4) sc singular_subdivision_add)
  qed (auto simp: k h singular_subdivision_diff)
qed

lemma llemma:
  assumes p: standard_simplex p ⊆ ⋃𝒞
      and 𝒞: ⋀U. U ∈ 𝒞 ⟹ openin (powertop_real UNIV) U
  obtains d where 0 < d
                  ⋀K. ⟦K ⊆ standard_simplex p;
                        ⋀x y i. ⟦i ≤ p; x ∈ K; y ∈ K⟧ ⟹ |x i − y i| ≤ d⟧
                       ⟹ ∃U. U ∈ 𝒞 ∧ K ⊆ U
proof −
  have ∃e U. 0 < e ∧ U ∈ 𝒞 ∧ x ∈ U ∧
              (∀y. (∀i≤p. |y i − x i| ≤ 2 ∗ e) ∧ (∀i>p. y i = 0) ⟶ y ∈ U)
    if x: x ∈ standard_simplex p for x
  proof−
    obtain U where U: U ∈ 𝒞 x ∈ U
```

```
      using x p by blast
     then obtain V where finV: finite {i. V i ≠ UNIV} and openV: ⋀i. open
(V i)
                 and xV: x ∈ Pi_E UNIV V and UV: Pi_E UNIV V ⊆ U
      using 𝒞 unfolding openin_product_topology_alt by force
    have xVi: x i ∈ V i for i
      using PiE_mem [OF xV] by simp
    have ⋀i. ∃e>0. ∀x'. |x' − x i| < e ⟶ x' ∈ V i
      by (rule openV [unfolded open_real, rule_format, OF xVi])
    then obtain d where d: ⋀i. d i > 0 and dV: ⋀i x'. |x' − x i| < d i ⟹ x'
∈ V i
      by metis
    define e where e ≡ Inf (insert 1 (d ` {i. V i ≠ UNIV})) / 3
    have ed3: e ≤ d i / 3 if V i ≠ UNIV for i
      using that finV by (auto simp: e_def intro: cInf_le_finite)
    show ∃e U. 0 < e ∧ U ∈ 𝒞 ∧ x ∈ U ∧
            (∀y. (∀i≤p. |y i − x i| ≤ 2 ∗ e) ∧ (∀i>p. y i = 0) ⟶ y ∈ U)
    proof (intro exI conjI allI impI)
      show e > 0
        using d finV by (simp add: e_def finite_less_Inf_iff)
      fix y assume y: (∀i≤p. |y i − x i| ≤ 2 ∗ e) ∧ (∀i>p. y i = 0)
      have y ∈ Pi_E UNIV V
      proof
        show y i ∈ V i for i
        proof (cases p < i)
          case True
          then show ?thesis
          by (metis (mono_tags, lifting) y x mem_Collect_eq standard_simplex_def
xVi)
        next
          case False show ?thesis
          proof (cases V i = UNIV)
            case False show ?thesis
            proof (rule dV)
              have |y i − x i| ≤ 2 ∗ e
                using y ‹¬ p < i› by simp
              also have ... < d i
                using ed3 [OF False] ‹e > 0› by simp
              finally show |y i − x i| < d i .
            qed
          qed auto
        qed
      qed auto
      with UV show y ∈ U
        by blast
    qed (use U in auto)
  qed
  then obtain e U where
      eU: ⋀x. x ∈ standard_simplex p ⟹
```

```
            0 < e x ∧ U x ∈ 𝒞 ∧ x ∈ U x
    and UI: ⋀x y. ⟦x ∈ standard_simplex p; ⋀i. i ≤ p ⟹ |y i − x i| ≤ 2 ∗ e
x; ⋀i. i > p ⟹ y i = 0⟧
                ⟹ y ∈ U x
    by metis
  define F where F ≡ λx. Pi_E UNIV (λi. if i ≤ p then {x i − e x<..<x i + e
x} else UNIV)
  have ∀S ∈ F ‘ standard_simplex p. openin (powertop_real UNIV) S
    by (simp add: F_def openin_PiE_gen)
  moreover have pF: standard_simplex p ⊆ ⋃(F ‘ standard_simplex p)
    by (force simp: F_def PiE_iff eU)
  ultimately have ∃ℱ. finite ℱ ∧ ℱ ⊆ F ‘ standard_simplex p ∧ standard_simplex
p ⊆ ⋃ℱ
    using compactin_standard_simplex [of p]
    unfolding compactin_def by force
  then obtain S where finite S and ssp: S ⊆ standard_simplex p standard_simplex
p ⊆ ⋃(F ‘ S)
    unfolding ex_finite_subset_image by (auto simp: ex_finite_subset_image)
  then have S ≠ {}
    by (auto simp: nonempty_standard_simplex)
  show ?thesis
  proof
    show Inf (e ‘ S) > 0
      using ‹finite S› ‹S ≠ {}› ssp eU by (auto simp: finite_less_Inf_iff)
    fix k :: (nat ⇒ real) set
    assume k: k ⊆ standard_simplex p
       and kle: ⋀x y i. ⟦i ≤ p; x ∈ k; y ∈ k⟧ ⟹ |x i − y i| ≤ Inf (e ‘ S)
    show ∃U. U ∈ 𝒞 ∧ k ⊆ U
    proof (cases k = {})
      case True
      then show ?thesis
        using ‹S ≠ {}› eU equals0I ssp(1) subset_eq p by auto
    next
      case False
      with k ssp obtain x a where x ∈ k x ∈ standard_simplex p
                      and a: a ∈ S and Fa: x ∈ F a
        by blast
      then have le_ea: ⋀i. i ≤ p ⟹ abs (x i − a i) < e a
        by (simp add: F_def PiE_iff if_distrib abs_diff_less_iff cong: if_cong)
      show ?thesis
      proof (intro exI conjI)
        show U a ∈ 𝒞
          using a eU ssp(1) by auto
        show k ⊆ U a
        proof clarify
          fix y assume y ∈ k
          with k have y: y ∈ standard_simplex p
            by blast
          show y ∈ U a
```

```
          proof (rule UI)
            show a ∈ standard_simplex p
              using a ssp(1) by auto
            fix i :: nat
            assume i ≤ p
            then have |x i − y i| ≤ e a
              by (meson kle [OF ‹i ≤ p›] a ‹finite S› ‹x ∈ k› ‹y ∈ k› cInf_le_finite
finite_imageI imageI order_trans)
            then show |y i − a i| ≤ 2 ∗ e a
              using le_ea [OF ‹i ≤ p›] by linarith
          next
            fix i assume p < i
            then show y i = 0
              using standard_simplex_def y by auto
          qed
        qed
      qed
    qed
  qed
qed


proposition sufficient_iterated_singular_subdivision_exists:
  assumes 𝒞: ⋀U. U ∈ 𝒞 ⟹ openin X U
      and X: topspace X ⊆ ⋃𝒞
      and p: singular_chain p X c
  obtains n where ⋀m f. ⟦n ≤ m; f ∈ Poly_Mapping.keys ((singular_subdivision
p ^^ m) c)⟧
                      ⟹ ∃ V ∈ 𝒞. f ∈ (standard_simplex p) → V
proof (cases c = 0)
  case False
  then show ?thesis
  proof (cases topspace X = {})
    case True
    show ?thesis
      using p that by (force simp: singular_chain_empty True)
  next
    case False
    show ?thesis
    proof (cases 𝒞 = {})
      case True
      then show ?thesis
        using False X by blast
    next
      case False
      have ∃ e. 0 < e ∧
              (∀ K. K ⊆ standard_simplex p ⟶ (∀ x y i. x ∈ K ∧ y ∈ K ∧ i ≤ p
⟶ |x i − y i| ≤ e)
                      ⟶ (∃ V. V ∈ 𝒞 ∧ f ∈ K → V))
```

if *f*: *f* ∈ *Poly_Mapping.keys c* **for** *f*
proof −
have *ssf*: *singular_simplex p X f*
using *f p* **by** (*auto simp*: *singular_chain_def*)
then have *fp*: $\bigwedge x.\ x \in$ *standard_simplex p* $\Longrightarrow$ *f x* ∈ *topspace X*
by (*auto simp*: *singular_simplex_def image_subset_iff dest*: *continuous_map_image_subset_topspace*)
have ∃ *T*. *openin* (*powertop_real UNIV*) *T* ∧
standard_simplex p ∩ *f* −‘ *V* = *T* ∩ *standard_simplex p*
if *V*: *V* ∈ $\mathcal{C}$ **for** *V*
proof −
have *singular_simplex p X f*
using *p f* **unfolding** *singular_chain_def* **by** *blast*
then have *openin* (*subtopology* (*powertop_real UNIV*) (*standard_simplex p*))
{*x* ∈ *standard_simplex p*. *f x* ∈ *V*}
using $\mathcal{C}$ [*OF* ‹*V* ∈ $\mathcal{C}$›] **by** (*simp add*: *singular_simplex_def continuous_map_def*)
moreover have *standard_simplex p* ∩ *f* −‘ *V* = {*x* ∈ *standard_simplex p*. *f x* ∈ *V*}
by *blast*
ultimately show *?thesis*
by (*simp add*: *openin_subtopology*)
qed
then obtain *g* **where** *gope*: $\bigwedge V.\ V \in \mathcal{C} \Longrightarrow$ *openin* (*powertop_real UNIV*) (*g V*)
and *geq*: $\bigwedge V.\ V \in \mathcal{C} \Longrightarrow$ *standard_simplex p* ∩ *f* −‘ *V* = *g V* ∩ *standard_simplex p*
by *metis*
obtain *d* **where** *0* < *d*
and *d*: $\bigwedge K$. ⟦*K* ⊆ *standard_simplex p*; $\bigwedge$*x y i*. ⟦*i* ≤ *p*; *x* ∈ *K*; *y* ∈ *K*⟧ ⟹ |*x i* − *y i*| ≤ *d*⟧
⟹ ∃ *U*. *U* ∈ *g* ‘ $\mathcal{C}$ ∧ *K* ⊆ *U*
proof (*rule llemma* [*of p g* ‘ $\mathcal{C}$])
show *standard_simplex p* ⊆ ⋃(*g* ‘ $\mathcal{C}$)
using *geq X fp* **by** (*fastforce simp add*:)
show *openin* (*powertop_real UNIV*) *U* **if** *U* ∈ *g* ‘ $\mathcal{C}$ **for** *U* :: (*nat* ⇒ *real*) *set*
using *gope that* **by** *blast*
qed *auto*
show *?thesis*
proof (*rule exI*, *intro allI conjI impI*)
fix *K* :: (*nat* ⇒ *real*) *set*
assume *K*: *K* ⊆ *standard_simplex p*
and *Kd*: ∀ *x y i*. *x* ∈ *K* ∧ *y* ∈ *K* ∧ *i* ≤ *p* ⟶ |*x i* − *y i*| ≤ *d*
then have ∃ *U*. *U* ∈ *g* ‘ $\mathcal{C}$ ∧ *K* ⊆ *U*
using *d* [*OF K*] **by** *auto*
then show ∃ *V*. *V* ∈ $\mathcal{C}$ ∧ *f* ∈ *K* → *V*
using *K geq* **by** *fastforce*

```
    qed (rule ‹d > 0›)
  qed
  then obtain ψ where epos: ∀f ∈ Poly_Mapping.keys c. 0 < ψ f
           and e: ⋀f K. ⟦f ∈ Poly_Mapping.keys c; K ⊆ standard_simplex p;
                    ⋀x y i. x ∈ K ∧ y ∈ K ∧ i ≤ p ⟹ |x i − y i| ≤ ψ f⟧
                    ⟹ ∃V. V ∈ 𝒞 ∧ f ∈ K → V
    by metis
  obtain d where 0 < d
         and d: ⋀f K. ⟦f ∈ Poly_Mapping.keys c; K ⊆ standard_simplex p;
                  ⋀x y i. ⟦x ∈ K; y ∈ K; i ≤ p⟧ ⟹ |x i − y i| ≤ d⟧
                  ⟹ ∃V. V ∈ 𝒞 ∧ f ∈ K → V
  proof
    show Inf (ψ ‘ Poly_Mapping.keys c) > 0
      by (simp add: finite_less_Inf_iff ‹c ≠ 0› epos)
    fix f K
    assume fK: f ∈ Poly_Mapping.keys c K ⊆ standard_simplex p
       and le: ⋀x y i. ⟦x ∈ K; y ∈ K; i ≤ p⟧ ⟹ |x i − y i| ≤ Inf (ψ ‘
Poly_Mapping.keys c)
    then have lef: Inf (ψ ‘ Poly_Mapping.keys c) ≤ ψ f
      by (auto intro: cInf_le_finite)
    show ∃V. V ∈ 𝒞 ∧ f ∈ K → V
      using le lef by (blast intro: dual_order.trans e [OF fK])
  qed
 let ?d = λm. (simplicial_subdivision p ^^ m) (frag_of (restrict id (standard_simplex
p)))
  obtain n where n: (p / (Suc p)) ^ n < d
    using real_arch_pow_inv ‹0 < d› by fastforce
  show ?thesis
  proof
    fix m h
     assume n ≤ m and h ∈ Poly_Mapping.keys ((singular_subdivision p ^^
m) c)
     then obtain f where f ∈ Poly_Mapping.keys c h ∈ Poly_Mapping.keys
(chain_map p f (?d m))
       using subsetD [OF keys_frag_extend] iterated_singular_subdivision [OF
p, of m] by force
      then obtain g where g: g ∈ Poly_Mapping.keys (?d m) and heq: h =
restrict (f ∘ g) (standard_simplex p)
     using keys_frag_extend by (force simp: chain_map_def simplex_map_def)
    have xx: simplicial_chain p (standard_simplex p) (?d n) ∧
           (∀f ∈ Poly_Mapping.keys(?d n). ∀x ∈ standard_simplex p. ∀y ∈
standard_simplex p.
              |f x i − f y i| ≤ (p / (Suc p)) ^ n)
      for n i
    proof (induction n)
      case 0
     have simplicial_simplex p (standard_simplex p) (λa∈standard_simplex p.
a)
        by (metis eq_id_iff order_refl simplicial_simplex_id)
```

```
        moreover have (∀ x∈standard_simplex p. ∀ y∈standard_simplex p. |x i −
y i| ≤ 1)
          unfolding standard_simplex_def
          by (auto simp: abs_if dest!: spec [where x=i])
        ultimately show ?case
          unfolding power_0 funpow_0 by simp
      next
        case (Suc n)
        show ?case
          unfolding power_Suc funpow.simps o_def
        proof (intro conjI ballI)
          show simplicial_chain p (standard_simplex p) (simplicial_subdivision p
(?d n))
            by (simp add: Suc simplicial_chain_simplicial_subdivision)
          show |f x i − f y i| ≤ real p / real (Suc p) ∗ (real p / real (Suc p)) ^ n
            if f ∈ Poly_Mapping.keys (simplicial_subdivision p (?d n))
              and x ∈ standard_simplex p and y ∈ standard_simplex p for f x y
            using Suc that by (blast intro: simplicial_subdivision_shrinks)
        qed
      qed
      have g ‘ standard_simplex p ⊆ standard_simplex p
        using g xx [of m] unfolding simplicial_chain_def simplicial_simplex by
auto
      moreover
      have |g x i − g y i| ≤ d if i ≤ p x ∈ standard_simplex p y ∈ standard_simplex
p for x y i
      proof −
        have |g x i − g y i| ≤ (p / (Suc p)) ^ m
          using g xx [of m] that by blast
        also have . . . ≤ (p / (Suc p)) ^ n
          by (auto intro: power_decreasing [OF ‹n ≤ m›])
        finally show ?thesis using n by simp
      qed
      then have |x i − y i| ≤ d
        if x ∈ g ‘ (standard_simplex p) y ∈ g ‘ (standard_simplex p) i ≤ p for i
x y
        using that by blast
      ultimately show ∃ V∈C. h ∈ standard_simplex p → V
        using ‹f ∈ Poly_Mapping.keys c› d [of f g ‘ standard_simplex p]
        using heq image_subset_iff_funcset by fastforce
    qed
  qed
 qed
qed force


lemma small_homologous_rel_relcycle_exists:
 assumes C: ⋀U. U ∈ C =⇒ openin X U
   and X: topspace X ⊆ ⋃C
```

```
    and p: singular_relcycle p X S c
  obtains c′ where singular_relcycle p X S c′ homologous_rel p X S c c′
                ⋀f. f ∈ Poly_Mapping.keys c′ ⟹ ∃ V ∈ 𝒞. f ∈ (standard_simplex
p) → V
proof −
  have singular_chain p X c
      (chain_boundary p c, 0) ∈ (mod_subset (p − Suc 0) (subtopology X S))
    using p unfolding singular_relcycle_def by auto
  then obtain n where n: ⋀m f. ⟦n ≤ m; f ∈ Poly_Mapping.keys ((singular_subdivision
p ^^ m) c)⟧
                          ⟹ ∃ V ∈ 𝒞. f ∈ (standard_simplex p) → V
    by (blast intro: sufficient_iterated_singular_subdivision_exists [OF 𝒞 X])
  let ?c′ = (singular_subdivision p ^^ n) c
  show ?thesis
  proof
    show homologous_rel p X S c ?c′
    proof (induction n)
      case 0
      then show ?case by auto
    next
      case (Suc n)
      then show ?case
        by simp (metis homologous_rel_eq p  homologous_rel_singular_subdivision
homologous_rel_singular_relcycle)
    qed
    then show singular_relcycle p X S ?c′
      by (metis homologous_rel_singular_relcycle p)
  next
    fix f :: (nat ⇒ real) ⇒ 'a
    assume f ∈ Poly_Mapping.keys ?c′
    then show ∃ V∈𝒞. f ∈ (standard_simplex p) → V
      by (rule n [OF order_refl])
  qed
qed

lemma excised_chain_exists:
  fixes S :: 'a set
  assumes X closure_of U ⊆ X interior_of T T ⊆ S singular_chain p (subtopology
X S) c
  obtains n d e where singular_chain p (subtopology X (S − U)) d
                      singular_chain p (subtopology X T) e
                      (singular_subdivision p ^^ n) c = d + e
proof −
  have *: ∃ n d e. singular_chain p (subtopology X (S − U)) d ∧
                singular_chain p (subtopology X T) e ∧
                (singular_subdivision p ^^ n) c = d + e
    if c: singular_chain p (subtopology X S) c
      and X: X closure_of U ⊆ X interior_of T U ⊆ topspace X and S: T ⊆ S
S ⊆ topspace X
```

```
    for p X c S and T U :: 'a set
  proof -
   obtain n where n: ⋀m f. ⟦n ≤ m; f ∈ Poly_Mapping.keys ((singular_subdivision
p ^^ m) c)⟧
                        ⟹ ∃ V ∈ {S ∩ X interior_of T, S − X closure_of U}. f
∈ (standard_simplex p) → V
      apply (rule sufficient_iterated_singular_subdivision_exists
              [of {S ∩ X interior_of T, S − X closure_of U}])
      using X S c
     by (auto simp: topspace_subtopology openin_subtopology_Int2 openin_subtopology_diff_closed)
    let ?c' = λn. (singular_subdivision p ^^ n) c
    have singular_chain p (subtopology X S) (?c' m) for m
      by (induction m) (auto simp: singular_chain_singular_subdivision c)
    then have scp: singular_chain p (subtopology X S) (?c' n) .

    have SS: Poly_Mapping.keys (?c' n) ⊆ singular_simplex_set p (subtopology X
(S − U))
                        ∪ singular_simplex_set p (subtopology X T)
    proof (clarsimp)
      fix f
      assume f: f ∈ Poly_Mapping.keys ((singular_subdivision p ^^ n) c)
        and non: ¬ singular_simplex p (subtopology X T) f
      show singular_simplex p (subtopology X (S − U)) f
        using n [OF order_refl f] scp f non closure_of_subset [OF ‹U ⊆ topspace
X›] interior_of_subset [of X T]
        by (fastforce simp: image_subset_iff singular_simplex_subtopology singu-
lar_chain_def)
    qed
    show ?thesis
      unfolding singular_chain_def using frag_split [OF SS] by metis
  qed
  have (subtopology X (topspace X ∩ S)) = (subtopology X S)
    by (metis subtopology_subtopology subtopology_topspace)
  with assms have c: singular_chain p (subtopology X (topspace X ∩ S)) c
    by simp
  have Xsub: X closure_of (topspace X ∩ U) ⊆ X interior_of (topspace X ∩ T)
    using assms closure_of_restrict interior_of_restrict by fastforce
  obtain n d e where
    d: singular_chain p (subtopology X (topspace X ∩ S − topspace X ∩ U)) d
    and e: singular_chain p (subtopology X (topspace X ∩ T)) e
    and de: (singular_subdivision p ^^ n) c = d + e
    using *[OF c Xsub, simplified] assms by force
  show thesis
  proof
    show singular_chain p (subtopology X (S − U)) d
      by (metis d Diff_Int_distrib inf.cobounded2 singular_chain_mono)
    show singular_chain p (subtopology X T) e
      by (metis e inf.cobounded2 singular_chain_mono)
    show (singular_subdivision p ^^ n) c = d + e
```

```
    by (rule de)
  qed
qed


lemma excised_relcycle_exists:
  fixes S :: 'a set
  assumes X: X closure_of U ⊆ X interior_of T and T ⊆ S
      and c: singular_relcycle p (subtopology X S) T c
  obtains c' where singular_relcycle p (subtopology X (S − U)) (T − U) c'
                   homologous_rel p (subtopology X S) T c c'
proof −
  have [simp]: (S − U) ∩ (T − U) = T − U S ∩ T = T
    using ‹T ⊆ S› by auto
  have scc: singular_chain p (subtopology X S) c
    and scp1: singular_chain (p − Suc 0) (subtopology X T) (chain_boundary p
c)
   using c by (auto simp: singular_relcycle_def mod_subset_def subtopology_subtopology)
  obtain n d e where d: singular_chain p (subtopology X (S − U)) d
    and e: singular_chain p (subtopology X T) e
    and de: (singular_subdivision p ^^ n) c = d + e
    using excised_chain_exists [OF X ‹T ⊆ S› scc] .
  have scSUd: singular_chain (p − Suc 0) (subtopology X (S − U)) (chain_boundary
p d)
    by (simp add: singular_chain_boundary d)
  have sccn: singular_chain p (subtopology X S) ((singular_subdivision p ^^ n) c)
for n
    by (induction n) (auto simp: singular_chain_singular_subdivision scc)
  have singular_chain (p − Suc 0) (subtopology X T) (chain_boundary p ((singular_subdivision
p ^^ n) c))
  proof (induction n)
    case (Suc n)
    then show ?case
     by (simp add: singular_chain_singular_subdivision chain_boundary_singular_subdivision
[OF sccn])
  qed (auto simp: scp1)
  then have singular_chain (p − Suc 0) (subtopology X T) (chain_boundary p
((singular_subdivision p ^^ n) c − e))
   by (simp add: chain_boundary_diff singular_chain_diff singular_chain_boundary
e)
  with de have scTd: singular_chain (p − Suc 0) (subtopology X T) (chain_boundary
p d)
    by simp
  show thesis
  proof
    have singular_chain (p − Suc 0) X (chain_boundary p d)
      using scTd singular_chain_subtopology by blast
    with scSUd scTd have singular_chain (p − Suc 0) (subtopology X (T − U))
(chain_boundary p d)
```

by (*fastforce simp add*: *singular_chain_subtopology*)
then show *singular_relcycle p* (*subtopology X* (*S* − *U*)) (*T* − *U*) *d*
by (*auto simp*: *singular_relcycle_def mod_subset_def subtopology_subtopology d*)
have *homologous_rel p* (*subtopology X S*) *T* (*c*−*0*) ((*singular_subdivision p* ^^ *n*) *c* − *e*)
proof (*rule homologous_rel_diff*)
show *homologous_rel p* (*subtopology X S*) *T c* ((*singular_subdivision p* ^^ *n*) *c*)
proof (*induction n*)
case (*Suc n*)
then show *?case*
apply *simp*
by (*metis c homologous_rel_eq homologous_rel_singular_relcycle_1 homologous_rel_singular_subdivision*)
qed *auto*
show *homologous_rel p* (*subtopology X S*) *T 0 e*
unfolding *homologous_rel_def* **using** *e*
by (*intro singular_relboundary_diff singular_chain_imp_relboundary*; *simp add*: *subtopology_subtopology*)
qed
with *de* **show** *homologous_rel p* (*subtopology X S*) *T c d*
by *simp*
qed
qed

0.1.18 Homotopy invariance

theorem *homotopic_imp_homologous_rel_chain_maps*:
assumes *hom*: *homotopic_with* (λ*h*. *h* ∈ *T* → *V*) *S U f g* **and** *c*: *singular_relcycle p S T c*
shows *homologous_rel p U V* (*chain_map p f c*) (*chain_map p g c*)
proof −
note *sum.atMost_Suc* [*simp del*]
have *contf*: *continuous_map S U f* **and** *contg*: *continuous_map S U g*
using *homotopic_with_imp_continuous_maps* [*OF hom*] **by** *metis*+
obtain *h* **where** *conth*: *continuous_map* (*prod_topology* (*top_of_set* {*0*..*1*::*real*}) *S*) *U h*
and *h0*: ⋀*x*. *h*(*0*, *x*) = *f x*
and *h1*: ⋀*x*. *h*(*1*, *x*) = *g x*
and *hV*: ⋀*t x*. ⟦*0* ≤ *t*; *t* ≤ *1*; *x* ∈ *T*⟧ ⟹ *h*(*t*,*x*) ∈ *V*
using *hom* **by** (*fastforce simp*: *homotopic_with_def*)
define *vv* **where** *vv* ≡ λ*j i*. *if i* = *Suc j then 1 else* (*0*::*real*)
define *ww* **where** *ww* ≡ λ*j i*. *if i*=*0* ∨ *i* = *Suc j then 1 else* (*0*::*real*)
define *simp* **where** *simp* ≡ λ*q i*. *oriented_simplex* (*Suc q*) (λ*j*. *if j* ≤ *i then vv j else ww*(*j* −*1*))
define *pr* **where** *pr* ≡ λ*q c*. ∑ *i*≤*q*. *frag_cmul* ((−*1*) ^ *i*)
(*frag_of* (*simplex_map* (*Suc q*) (λ*z*. *h*(*z 0*, *c*(*z* ∘ *Suc*))) (*simp q i*)))

have *ss_ss*: *simplicial_simplex* (*Suc q*) ({*x*. *x 0* ∈ {*0..1*} ∧ (*x* ∘ *Suc*) ∈ *standard_simplex q*}) (*simp q i*)
if *i* ≤ *q* **for** *q i*
proof −
have (∑*j*≤*Suc q*. (*if j* ≤ *i then vv j 0 else ww* (*j* −*1*) *0*) ∗ *x j*) ∈ {*0..1*}
if *x* ∈ *standard_simplex* (*Suc q*) **for** *x*
proof −
have (∑*j*≤*Suc q*. *if j* ≤ *i then 0 else x j*) ≤ *sum x* {..*Suc q*}
using *that* **unfolding** *standard_simplex_def*
by (*force intro*!: *sum_mono*)
with ‹*i* ≤ *q*› *that* **show** *?thesis*
by (*simp add*: *vv_def ww_def standard_simplex_def if_distrib* [*of* λ*u*. *u* ∗ _] *sum_nonneg cong*: *if_cong*)
qed
moreover
have (λ*k*. ∑*j*≤*Suc q*. (*if j* ≤ *i then vv j k else ww* (*j* −*1*) *k*) ∗ *x j*) ∘ *Suc* ∈ *standard_simplex q*
if *x* ∈ *standard_simplex* (*Suc q*) **for** *x*
proof −
have *card*: ({..*q*} ∩ {*k*. *Suc k* = *j*}) = {*j*−*1*} **if** *0* < *j j* ≤ *Suc q* **for** *j*
using *that* **by** *auto*
have *eq*: (∑*j*≤*Suc q*. ∑*k*≤*q*. *if j* ≤ *i then if k* = *j then x j else 0 else if Suc k* = *j then x j else 0*)
= (∑*j*≤*Suc q*. *x j*)
by (*rule sum.cong* [*OF refl*]) (*use* ‹*i* ≤ *q*› **in** ‹*simp add*: *sum.If_cases card*›)
have (∑*j*≤*Suc q*. *if j* ≤ *i then if k* = *j then x j else 0 else if Suc k* = *j then x j else 0*)
≤ *sum x* {..*Suc q*} **for** *k*
using *that* **unfolding** *standard_simplex_def*
by (*force intro*!: *sum_mono*)
then show *?thesis*
using ‹*i* ≤ *q*› *that*
by (*simp add*: *vv_def ww_def standard_simplex_def if_distrib* [*of* λ*u*. *u* ∗ _] *sum_nonneg*
sum.swap [**where** *A* = *atMost q*] *eq cong*: *if_cong*)
qed
ultimately show *?thesis*
by (*simp add*: *that simplicial_simplex_oriented_simplex simp_def image_subset_iff if_distribR*)
qed
obtain *prism* **where** *prism*: ⋀*q*. *prism q 0* = *0*
⋀*q c*. *singular_chain q S c* ⟹ *singular_chain* (*Suc q*) *U* (*prism q c*)
⋀*q c*. *singular_chain q* (*subtopology S T*) *c*
⟹ *singular_chain* (*Suc q*) (*subtopology U V*) (*prism q c*)
⋀*q c*. *singular_chain q S c*
⟹ *chain_boundary* (*Suc q*) (*prism q c*) =
chain_map q g c − *chain_map q f c* − *prism* (*q* −*1*) (*chain_boundary q c*)
proof

show (*frag_extend* ∘ *pr*) *q 0* = *0* **for** *q*
by (*simp add*: *pr_def*)
next
show *singular_chain* (*Suc q*) *U* ((*frag_extend* ∘ *pr*) *q c*)
if *singular_chain q S c* **for** *q c*
using *that* [*unfolded singular_chain_def*]
proof (*induction c rule*: *frag_induction*)
case (*one m*)
show *?case*
proof (*simp add*: *pr_def*, *intro singular_chain_cmul singular_chain_sum*)
fix *i* :: *nat*
assume *i* ∈ {..*q*}
define *X* **where** *X* = *subtopology* (*powertop_real UNIV*) {*x*. *x 0* ∈ {*0*..*1*} ∧ (*x* ∘ *Suc*) ∈ *standard_simplex q*}
show *singular_chain* (*Suc q*) *U* (*frag_of* (*simplex_map* (*Suc q*) (λ*z*. *h* (*z 0*, *m* (*z* ∘ *Suc*))) (*simp q i*)))
unfolding *singular_chain_of*
proof (*rule singular_simplex_simplex_map*)
show *singular_simplex* (*Suc q*) *X* (*simp q i*)
unfolding *X_def* **using** ‹*i* ∈ {..*q*}› *simplicial_imp_singular_simplex ss_ss* **by** *blast*
have *0*: *continuous_map X* (*top_of_set* {*0*..*1*}) (λ*x*. *x 0*)
unfolding *continuous_map_in_subtopology topspace_subtopology X_def*
by (*auto intro*: *continuous_map_product_projection continuous_map_from_subtopology*)
have *1*: *continuous_map X S* (*m* ∘ (λ*x j*. *x* (*Suc j*)))
proof (*rule continuous_map_compose*)
have *continuous_map* (*powertop_real UNIV*) (*powertop_real UNIV*) (λ*x j*. *x* (*Suc j*))
by (*auto intro*: *continuous_map_product_projection*)
then show *continuous_map X* (*subtopology* (*powertop_real UNIV*) (*standard_simplex q*)) (λ*x j*. *x* (*Suc j*))
unfolding *X_def o_def*
by (*auto simp*: *continuous_map_in_subtopology intro*: *continuous_map_from_subtopology continuous_map_product_projection*)
qed (*use one* **in** ‹*simp add*: *singular_simplex_def*›)
show *continuous_map X U* (λ*z*. *h* (*z 0*, *m* (*z* ∘ *Suc*)))
apply (*rule continuous_map_compose* [*unfolded o_def*, *OF _ conth*])
using *0 1* **by** (*simp add*: *continuous_map_pairwise o_def*)
qed
qed
next
case (*diff a b*)
then show *?case*
by (*simp add*: *frag_extend_diff singular_chain_diff*)
qed *auto*
next
show *singular_chain* (*Suc q*) (*subtopology U V*) ((*frag_extend* ∘ *pr*) *q c*)
if *singular_chain q* (*subtopology S T*) *c* **for** *q c*

```
      using that [unfolded singular_chain_def]
    proof (induction c rule: frag_induction)
      case (one m)
      show ?case
      proof (simp add: pr_def, intro singular_chain_cmul singular_chain_sum)
        fix i :: nat
        assume i ∈ {..q}
        define X where X = subtopology (powertop_real UNIV) {x. x 0 ∈ {0..1}
∧ (x ∘ Suc) ∈ standard_simplex q}
        show singular_chain (Suc q) (subtopology U V)
                 (frag_of (simplex_map (Suc q) (λz. h (z 0, m (z ∘ Suc))) (simp q
i)))
          unfolding singular_chain_of
        proof (rule singular_simplex_simplex_map)
          show singular_simplex (Suc q) X (simp q i)
            unfolding X_def using ‹i ∈ {..q}› simplicial_imp_singular_simplex
ss_ss by blast
          have 0: continuous_map X (top_of_set {0..1}) (λx. x 0)
            unfolding continuous_map_in_subtopology topspace_subtopology X_def
          by (auto intro: continuous_map_product_projection continuous_map_from_subtopology)
          have 1: continuous_map X (subtopology S T) (m ∘ (λx j. x (Suc j)))
          proof (rule continuous_map_compose)
              have continuous_map (powertop_real UNIV) (powertop_real UNIV)
(λx j. x (Suc j))
              by (auto intro: continuous_map_product_projection)
              then show continuous_map X (subtopology (powertop_real UNIV)
(standard_simplex q)) (λx j. x (Suc j))
              unfolding X_def o_def
                 by (auto simp: continuous_map_in_subtopology intro: continu-
ous_map_from_subtopology continuous_map_product_projection)
            show continuous_map (subtopology (powertop_real UNIV) (standard_simplex
q)) (subtopology S T) m
              using one continuous_map_into_fulltopology by (auto simp: singu-
lar_simplex_def)
          qed
          have continuous_map X (subtopology U V) (h ∘ (λz. (z 0, m (z ∘ Suc))))
          proof (rule continuous_map_compose)
              show continuous_map X (prod_topology (top_of_set {0..1::real})
(subtopology S T)) (λz. (z 0, m (z ∘ Suc)))
              using 0 1 by (simp add: continuous_map_pairwise o_def)
              have continuous_map (subtopology (prod_topology euclideanreal S)
({0..1} × T)) U h
              by (metis conth continuous_map_from_subtopology subtopology_Times
subtopology_topspace)
          with hV show continuous_map (prod_topology (top_of_set {0..1::real})
(subtopology S T)) (subtopology U V) h
              by (force simp: topspace_subtopology continuous_map_in_subtopology
subtopology_restrict subtopology_Times)
          qed
```

then show *continuous_map X* (*subtopology U V*) (λz. *h* (*z 0*, *m* (*z* ∘ *Suc*)))
by (*simp add*: *o_def*)
qed
qed
next
case (*diff a b*)
then show *?case*
by (*metis comp_apply frag_extend_diff singular_chain_diff*)
qed *auto*
next
show *chain_boundary* (*Suc q*) ((*frag_extend* ∘ *pr*) *q c*) =
chain_map q g c − *chain_map q f c* − (*frag_extend* ∘ *pr*) (*q* −*1*) (*chain_boundary q c*)
if *singular_chain q S c* **for** *q c*
using *that* [*unfolded singular_chain_def*]
proof (*induction c rule*: *frag_induction*)
case (*one m*)
have *eq2*: *Sigma S T* = (λi. (*i*,*i*)) ‘ {*i* ∈ *S*. *i* ∈ *T i*} ∪ (*Sigma S* (λi. *T i* − {*i*})) **for** *S* :: *nat set* **and** *T*
by *force*
have *1*: (∑ (*i*,*j*)∈(λi. (*i*, *i*)) ‘ {*i*. *i* ≤ *q* ∧ *i* ≤ *Suc q*}.
frag_cmul (((−*1*) ^ *i*) ∗ (−*1*) ^ *j*)
(*frag_of*
(*singular_face* (*Suc q*) *j*
(*simplex_map* (*Suc q*) (λz. *h* (*z 0*, *m* (*z* ∘ *Suc*))) (*simp q i*)))))
+ (∑ (*i*,*j*)∈(λi. (*i*, *i*)) ‘ {*i*. *i* ≤ *q*}.
frag_cmul (− ((−*1*) ^ *i* ∗ (−*1*) ^ *j*))
(*frag_of*
(*singular_face* (*Suc q*) (*Suc j*)
(*simplex_map* (*Suc q*) (λz. *h* (*z 0*, *m* (*z* ∘ *Suc*))) (*simp q i*)))))
= *frag_of* (*simplex_map q g m*) − *frag_of* (*simplex_map q f m*)
proof −
have *restrict* ((λz. *h* (*z 0*, *m* (*z* ∘ *Suc*))) ∘ (*simp q 0* ∘ *simplical_face 0*)) (*standard_simplex q*)
= *restrict* (*g* ∘ *m*) (*standard_simplex q*)
proof (*rule restrict_ext*)
fix *x*
assume *x*: *x* ∈ *standard_simplex q*
have (∑ *j*≤*Suc q*. *if j* = *0 then 0 else x* (*j* − *Suc 0*)) = (∑ *j*≤*q*. *x j*)
by (*simp add*: *sum.atMost_Suc_shift*)
with *x* **have** *simp q 0* (*simplical_face 0 x*) *0* = *1*
apply (*simp add*: *oriented_simplex_def simp_def simplical_face_in_standard_simplex*)
apply (*simp add*: *simplical_face_def if_distrib ww_def standard_simplex_def cong*: *if_cong*)
done
moreover
have (λn. *if n* ≤ *q then x n else 0*) = *x*

using *standard_simplex_def x* **by** *auto*
then have ($\lambda n.$ *simp q 0* (*simplical_face 0 x*) (*Suc n*)) = *x*
unfolding *oriented_simplex_def simp_def ww_def* **using** *x*
apply (*simp add*: *simplical_face_in_standard_simplex*)
apply (*simp add*: *simplical_face_def if_distrib*)
apply (*simp add*: *if_distribR if_distrib cong*: *if_cong*)
done
ultimately show (($\lambda z.$ *h* (*z 0*, *m* (*z* ∘ *Suc*))) ∘ (*simp q 0* ∘ *simplical_face 0*)) *x* = (*g* ∘ *m*) *x*
by (*simp add*: *o_def h1*)
qed
then have *a*: *frag_of* (*singular_face* (*Suc q*) *0* (*simplex_map* (*Suc q*) ($\lambda z.$ *h* (*z 0*, *m* (*z* ∘ *Suc*))) (*simp q 0*)))
= *frag_of* (*simplex_map q g m*)
by (*simp add*: *singular_face_simplex_map*) (*simp add*: *simplex_map_def*)
have *restrict* (($\lambda z.$ *h* (*z 0*, *m* (*z* ∘ *Suc*))) ∘ (*simp q q* ∘ *simplical_face* (*Suc q*))) (*standard_simplex q*)
= *restrict* (*f* ∘ *m*) (*standard_simplex q*)
proof (*rule restrict_ext*)
fix *x*
assume *x*: *x* ∈ *standard_simplex q*
then have *simp q q* (*simplical_face* (*Suc q*) *x*) *0* = *0*
unfolding *oriented_simplex_def simp_def*
by (*simp add*: *simplical_face_in_standard_simplex sum.atMost_Suc*) (*simp add*: *simplical_face_def vv_def*)
moreover have ($\lambda n.$ *simp q q* (*simplical_face* (*Suc q*) *x*) (*Suc n*)) = *x*
unfolding *oriented_simplex_def simp_def vv_def* **using** *x*
apply (*simp add*: *simplical_face_in_standard_simplex*)
apply (*force simp*: *standard_simplex_def simplical_face_def if_distribR if_distrib* [*of* $\lambda x.$ *x* ∗ _] *sum.atMost_Suc cong*: *if_cong*)
done
ultimately show (($\lambda z.$ *h* (*z 0*, *m* (*z* ∘ *Suc*))) ∘ (*simp q q* ∘ *simplical_face* (*Suc q*))) *x* = (*f* ∘ *m*) *x*
by (*simp add*: *o_def h0*)
qed
then have *b*: *frag_of* (*singular_face* (*Suc q*) (*Suc q*)
(*simplex_map* (*Suc q*) ($\lambda z.$ *h* (*z 0*, *m* (*z* ∘ *Suc*))) (*simp q q*)))
= *frag_of* (*simplex_map q f m*)
by (*simp add*: *singular_face_simplex_map*) (*simp add*: *simplex_map_def*)
have *sfeq*: *simplex_map q* ($\lambda z.$ *h* (*z 0*, *m* (*z* ∘ *Suc*))) (*simp q* (*Suc i*) ∘ *simplical_face* (*Suc i*))
= *simplex_map q* ($\lambda z.$ *h* (*z 0*, *m* (*z* ∘ *Suc*))) (*simp q i* ∘ *simplical_face* (*Suc i*))
if *i* < *q* **for** *i*
unfolding *simplex_map_def*
proof (*rule restrict_ext*)
fix *x*
assume *x* ∈ *standard_simplex q*
then have (*simp q* (*Suc i*) ∘ *simplical_face* (*Suc i*)) *x* = (*simp q i* ∘

```
simplical_face (Suc i)) x
          unfolding oriented_simplex_def simp_def simplical_face_def
          by (force intro: sum.cong)
        then show ((λz. h (z 0, m (z ∘ Suc))) ∘ (simp q (Suc i) ∘ simplical_face
(Suc i))) x
              = ((λz. h (z 0, m (z ∘ Suc))) ∘ (simp q i ∘ simplical_face (Suc i))) x
          by simp
      qed
      have eqq: {i. i ≤ q ∧ i ≤ Suc q} = {..q}
        by force
      have qeq: {..q} = insert 0 ((λi. Suc i) ` {i. i < q}) {i. i ≤ q} = insert q
{i. i < q}
        using le_imp_less_Suc less_Suc_eq_0_disj by auto
      show ?thesis
        using a b
        apply (simp add: sum.reindex inj_on_def eqq)
       apply (simp add: qeq sum.insert_if sum.reindex sum_negf singular_face_simplex_map
sfeq)
        done
    qed
    have 2: (∑(i,j)∈(SIGMA i:{..q}. {0..min (Suc q) i} − {i}).
                  frag_cmul ((−1) ^ i ∗ (−1) ^ j)
                   (frag_of
                     (singular_face (Suc q) j
                     (simplex_map (Suc q) (λz. h (z 0, m (z ∘ Suc))) (simp q i)))))
            + (∑(i,j)∈(SIGMA i:{..q}. {i..q} − {i}).
               frag_cmul (− ((−1) ^ i ∗ (−1) ^ j))
                (frag_of
                  (singular_face (Suc q) (Suc j)
                    (simplex_map (Suc q) (λz. h (z 0, m (z ∘ Suc))) (simp q i)))))
            = − frag_extend (pr (q − Suc 0)) (chain_boundary q (frag_of m))
    proof (cases q=0)
      case True
      then show ?thesis
        by (simp add: chain_boundary_def flip: sum.Sigma)
    next
      case False
       have eq: {..q − Suc 0} × {..q} = Sigma {..q − Suc 0} (λi. {0..min q i})
∪ Sigma {..q} (λi. {i<..q})
        by force
      have I: (∑(i,j)∈(SIGMA i:{..q}. {0..min (Suc q) i} − {i}).
                 frag_cmul ((−1) ^ (i + j))
                   (frag_of
                     (singular_face (Suc q) j
                     (simplex_map (Suc q) (λz. h (z 0, m (z ∘ Suc))) (simp q i)))))
              = (∑(i,j)∈(SIGMA i:{..q − Suc 0}. {0..min q i}).
                 frag_cmul (− ((−1) ^ (j + i)))
                  (frag_of
                    (simplex_map q (λz. h (z 0, singular_face q j m (z ∘ Suc)))
```

```
                    (simp (q − Suc 0) i))))
      proof −
        have seq: simplex_map q (λz. h (z 0, singular_face q j m (z ∘ Suc)))
                    (simp (q − Suc 0) (i − Suc 0))
              = simplex_map q (λz. h (z 0, m (z ∘ Suc))) (simp q i ∘ simplical_face
j)
          if ij: i ≤ q j ≠ i j ≤ i for i j
          unfolding simplex_map_def
        proof (rule restrict_ext)
          fix x
          assume x: x ∈ standard_simplex q
          have i > 0
            using that by force
          then have iq: i − Suc 0 ≤ q − Suc 0
            using ‹i ≤ q› False by simp
          have q0_eq: {..Suc q} = insert 0 (Suc ‘ {..q})
            by (auto simp: image_def gr0_conv_Suc)
          have α: simp (q − Suc 0) (i − Suc 0) x 0 = simp q i (simplical_face j
x) 0
            using False x ij
            unfolding oriented_simplex_def simp_def vv_def ww_def
            apply (simp add: simplical_face_in_standard_simplex)
                apply (force simp: simplical_face_def q0_eq sum.reindex intro!:
sum.cong)
            done
         have β: simplical_face j (simp (q − Suc 0) (i − Suc 0) x ∘ Suc) = simp
q i (simplical_face j x) ∘ Suc
          proof
            fix k
            show simplical_face j (simp (q − Suc 0) (i − Suc 0) x ∘ Suc) k
                = (simp q i (simplical_face j x) ∘ Suc) k
              using False x ij
              unfolding oriented_simplex_def simp_def o_def vv_def ww_def
              apply (simp add: simplical_face_in_standard_simplex if_distribR)
              apply (simp add: simplical_face_def if_distrib [of λu. u ∗ _] cong:
if_cong)
              apply (intro impI conjI)
               apply (force simp: sum.atMost_Suc intro: sum.cong)
              apply (force simp: q0_eq sum.reindex intro!: sum.cong)
              done
          qed
          have simp (q − Suc 0) (i − Suc 0) x ∘ Suc ∈ standard_simplex (q −
Suc 0)
            using ss_ss [OF iq] ‹i ≤ q› False ‹i > 0›
            by (simp add: image_subset_iff simplicial_simplex x)
          then show ((λz. h (z 0, singular_face q j m (z ∘ Suc))) ∘ simp (q −
Suc 0) (i − Suc 0)) x
              = ((λz. h (z 0, m (z ∘ Suc))) ∘ (simp q i ∘ simplical_face j)) x
            by (simp add: singular_face_def α β)
```

qed
have [*simp*]: (−*1*::*int*) ^ (*i* + *j* − *Suc 0*) = − ((−*1*) ^ (*i* + *j*)) **if** *i* ≠ *j* **for** *i j*::*nat*
proof −
have *i* + *j* > *0*
using *that* **by** *blast*
then show *?thesis*
by (*metis* (*no_types*, *opaque_lifting*) *One_nat_def Suc_diff_1 add.inverse_inverse mult.left_neutral mult_minus_left power_Suc*)
qed
show *?thesis*
apply (*rule sum.eq_general_inverses* [**where** *h* = λ(*a*,*b*). (*a*−*1*,*b*) **and** *k* = λ(*a*,*b*). (*Suc a*,*b*)])
using *False* **apply** (*auto simp*: *singular_face_simplex_map seq add.commute*)
done
qed
have ∗: *singular_face* (*Suc q*) (*Suc j*) (*simplex_map* (*Suc q*) (λ*z*. *h* (*z 0*, *m* (*z* ∘ *Suc*))) (*simp q i*))
= *simplex_map q* (λ*z*. *h* (*z 0*, *singular_face q j m* (*z* ∘ *Suc*))) (*simp* (*q* − *Suc 0*) *i*)
if *ij*: *i* < *j j* ≤ *q* **for** *i j*
proof −
have *iq*: *i* ≤ *q* − *Suc 0*
using *that* **by** *auto*
have *sf_eqh*: *singular_face* (*Suc q*) (*Suc j*)
(λ*x*. *if x* ∈ *standard_simplex* (*Suc q*)
then ((λ*z*. *h* (*z 0*, *m* (*z* ∘ *Suc*))) ∘ *simp q i*) *x else undefined*) *x*
= *h* (*simp* (*q* − *Suc 0*) *i x 0*,
singular_face q j m (λ*xa*. *simp* (*q* − *Suc 0*) *i x* (*Suc xa*)))
if *x*: *x* ∈ *standard_simplex q* **for** *x*
proof −
let *?f* = λ*k*. ∑*j*≤*q*. *if j* ≤ *i then if k* = *j then x j else 0*
else if Suc k = *j then x j else 0*
have *fm*: *simplical_face* (*Suc j*) *x* ∈ *standard_simplex* (*Suc q*)
using *ss_ss* [*OF iq*] *that ij*
by (*simp add*: *simplical_face_in_standard_simplex*)
have *ss*: *?f* ∈ *standard_simplex* (*q* − *Suc 0*)
unfolding *standard_simplex_def*
proof (*intro CollectI conjI impI allI*)
fix *k*
show *0* ≤ *?f k*
using *that* **by** (*simp add*: *sum_nonneg standard_simplex_def*)
show *?f k* ≤ *1*
using *x sum_le_included* [*of* {..*q*} {..*q*} *x id*]
by (*simp add*: *standard_simplex_def*)
assume *k*: *q* − *Suc 0* < *k*
show *?f k* = *0*
by (*rule sum.neutral*) (*use that x iq k standard_simplex_def* **in** *auto*)

```
      next
        have (∑ k≤q − Suc 0. ?f k)
            = (∑ (k,j) ∈ ({..q − Suc 0} × {..q}) ∩ {(k,j). if j ≤ i then k = j
else Suc k = j}. x j)
          apply (simp add: sum.Sigma)
          by (rule sum.mono_neutral_cong) (auto simp: split: if_split_asm)
        also have ... = sum x {..q}
          apply (rule sum.eq_general_inverses
              [where h = λ(k,j). if j≤i ∧ k=j ∨ j>i ∧ Suc k = j then j else Suc
q
                  and k = λj. if j ≤ i then (j,j) else (j − Suc 0, j)])
          using ij by auto
        also have ... = 1
          using x by (simp add: standard_simplex_def)
        finally show (∑ k≤q − Suc 0. ?f k) = 1
          by (simp add: standard_simplex_def)
      qed
      let ?g = λk. if k ≤ i then 0
                   else if k < Suc j then x k
                        else if k = Suc j then 0 else x (k − Suc 0)
       have eq: {..Suc q} = {..j} ∪ {Suc j} ∪ Suc‘{j<..q} {..q} = {..j} ∪
{j<..q}
        using ij image_iff less_Suc_eq_0_disj less_Suc_eq_le
        by (force simp: image_iff)+
      then have (∑ k≤Suc q. ?g k) = (∑ k∈{..j} ∪ {Suc j} ∪ Suc‘{j<..q}.
?g k)
        by simp
      also have ... = (∑ k∈{..j} ∪ Suc‘{j<..q}. ?g k)
        by (rule sum.mono_neutral_right) auto
      also have ... = (∑ k∈{..j}. ?g k) + (∑ k∈Suc‘{j<..q}. ?g k)
        by (rule sum.union_disjoint) auto
      also have ... = (∑ k∈{..j}. ?g k) + (∑ k∈{j<..q}. ?g (Suc k))
        by (auto simp: sum.reindex)
      also have ... = (∑ k∈{..j}. if k ≤ i then 0 else x k)
                    + (∑ k∈{j<..q}. if k ≤ i then 0 else x k)
        by (intro sum.cong arg_cong2 [of concl: (+)]) (use ij in auto)
      also have ... = (∑ k≤q. if k ≤ i then 0 else x k)
        unfolding eq by (subst sum.union_disjoint) auto
      finally have (∑ k≤Suc q. ?g k) = (∑ k≤q. if k ≤ i then 0 else x k) .
      then have QQ: (∑ l≤Suc q. if l ≤ i then 0 else simplical_face (Suc j)
x l) = (∑ j≤q. if j ≤ i then 0 else x j)
        by (simp add: simplical_face_def cong: if_cong)
      have WW: (λk. ∑ l≤Suc q. if l ≤ i
                         then if k = l then simplical_face (Suc j) x l else 0
                         else if Suc k = l then simplical_face (Suc j) x l
                         else 0)
          = simplical_face j
             (λk. ∑ j≤q. if j ≤ i then if k = j then x j else 0
                        else if Suc k = j then x j else 0)
```

```
          proof -
            have *: (∑ l≤q. if l ≤ i then 0 else if Suc k = l then x (l - Suc 0)
else 0)
                 = (∑ l≤q. if l ≤ i then if k - Suc 0 = l then x l else 0 else if k =
l then x l else 0)
              (is ?lhs = ?rhs)
              if k ≠ q k > j for k
            proof (cases k ≤ q)
              case True
              have ?lhs = sum (λl. x (l - Suc 0)) {Suc k} ?rhs = sum x {k}
               by (rule sum.mono_neutral_cong_right; use True ij that in auto)+
              then show ?thesis
                by simp
            next
              case False
              have ?lhs = 0 ?rhs = 0
                by (rule sum.neutral; use False ij in auto)+
              then show ?thesis
                by simp
            qed
            have xq: x q = (∑ j≤q. if j ≤ i then if q - Suc 0 = j then x j else 0
                                  else if q = j then x j else 0) if q≠j
              using ij that
              by (force simp flip: ivl_disj_un(2) intro: sum.neutral)
            show ?thesis
              using ij unfolding simplical_face_def
              by (intro ext) (auto simp: * sum.atMost_Suc xq cong: if_cong)
          qed
          show ?thesis
            using False that iq
            unfolding oriented_simplex_def simp_def vv_def ww_def
           apply (simp add: if_distribR simplical_face_def if_distrib [of λu. u *
_] o_def cong: if_cong)
            apply (simp add: singular_face_def fm ss QQ WW)
            done
        qed
        show ?thesis
          unfolding simplex_map_def restrict_def
            apply (simp add: simplicial_simplex image_subset_iff o_def sf_eqh
fun_eq_iff)
          apply (simp add: singular_face_def)
          done
      qed
      have sgeq: (SIGMA i:{..q}. {i..q} - {i}) = (SIGMA i:{..q}. {i<..q})
        by force
      have II: (∑ (i,j)∈(SIGMA i:{..q}. {i..q} - {i}).
                  frag_cmul (- ((-1) ^ (i + j)))
                   (frag_of
                     (singular_face (Suc q) (Suc j)
```

```
                    (simplex_map (Suc q) (λz. h (z 0, m (z ∘ Suc))) (simp q
i))))) =
            (∑ (i,j)∈(SIGMA i:{..q}. {i<..q}).
              frag_cmul (− ((−1) ^ (j + i)))
               (frag_of
                 (simplex_map q (λz. h (z 0, singular_face q j m (z ∘ Suc)))
                   (simp (q − Suc 0) i))))
        by (force simp: ∗ sgeq add.commute intro: sum.cong)
      show ?thesis
        using False
      apply (simp add: chain_boundary_def frag_extend_sum frag_extend_cmul
frag_cmul_sum pr_def flip: sum_negf power_add)
        apply (subst sum.swap [where A = {..q}])
          apply (simp add: sum.cartesian_product eq sum.union_disjoint dis-
joint_iff_not_equal I II)
        done
    qed
    have ∗: ⟦a+b = w; c+d = −z⟧ =⇒ (a + c) + (b+d) = w−z for a b w c d z
:: 'c ⇒₀ int
      by (auto simp: algebra_simps)
    have eq: {..q} × {..Suc q} =
            Sigma {..q} (λi. {0..min (Suc q) i})
          ∪ Sigma {..q} (λi. {Suc i..Suc q})
      by force
    show ?case
      apply (subst pr_def)
      apply (simp add: chain_boundary_sum chain_boundary_cmul)
      apply (subst chain_boundary_def)
      apply simp
    apply (simp add: frag_cmul_sum sum.cartesian_product eq sum.union_disjoint
disjoint_iff_not_equal
          sum.atLeast_Suc_atMost_Suc_shift del: sum.cl_ivl_Suc min.absorb2
min.absorb4
        flip: comm_monoid_add_class.sum.Sigma)
      apply (simp add: sum.Sigma eq2 [of _ λi. {_ i.._ i}]
        del: min.absorb2 min.absorb4)
      apply (simp add: sum.union_disjoint disjoint_iff_not_equal ∗ [OF 1 2])
      done
  next
    case (diff a b)
    then show ?case
      by (simp add: chain_boundary_diff frag_extend_diff chain_map_diff)
  qed auto
qed
have ∗: singular_chain p (subtopology U V) (prism (p − Suc 0) (chain_boundary
p c))
  if singular_chain p S c singular_chain (p − Suc 0) (subtopology S T) (chain_boundary
p c)
proof (cases p)
```

case *0* **then show** *?thesis* **by** (*simp add*: *chain_boundary_def prism*)
next
case (*Suc p'*)
with *prism that* **show** *?thesis* **by** *auto*
qed
then show *?thesis*
using *c*
unfolding *singular_relcycle_def homologous_rel_def singular_relboundary_def mod_subset_def*
apply (*rule_tac x=− prism p c* **in** *exI*)
by (*simp add*: *chain_boundary_minus prism*(*2*) *prism*(*4*) *singular_chain_minus*)
qed

end

0.2 Homology, II: Homology Groups

theory *Homology_Groups*
imports *Simplices HOL−Algebra.Exact_Sequence*

begin

0.2.1 Homology Groups

Now actually connect to group theory and set up homology groups. Note that we define homomogy groups for all *integers p*, since this seems to avoid some special-case reasoning, though they are trivial for $p < 0$.

definition *chain_group* :: *nat* ⇒ *'a topology* ⇒ *'a chain monoid*
where *chain_group p X* ≡ *free_Abelian_group* (*singular_simplex_set p X*)

lemma *carrier_chain_group* [*simp*]: *carrier*(*chain_group p X*) = *singular_chain_set p X*
by (*auto simp*: *chain_group_def singular_chain_def free_Abelian_group_def*)

lemma *one_chain_group* [*simp*]: *one*(*chain_group p X*) = *0*
by (*auto simp*: *chain_group_def free_Abelian_group_def*)

lemma *mult_chain_group* [*simp*]: *monoid.mult*(*chain_group p X*) = (+)
by (*auto simp*: *chain_group_def free_Abelian_group_def*)

lemma *m_inv_chain_group* [*simp*]: *Poly_Mapping.keys a* ⊆ *singular_simplex_set p X* ⟹ $inv_{chain\_group\ p\ X}\ a = -a$
unfolding *chain_group_def* **by** *simp*

lemma *group_chain_group* [*simp*]: *Group.group* (*chain_group p X*)
by (*simp add*: *chain_group_def*)

lemma *abelian_chain_group*: *comm_group*(*chain_group p X*)

```
by (simp add: free_Abelian_group_def group.group_comm_groupI [OF group_chain_group])

lemma subgroup_singular_relcycle:
    subgroup (singular_relcycle_set p X S) (chain_group p X)
proof
  show x ⊗chain_group p X y ∈ singular_relcycle_set p X S
    if x ∈ singular_relcycle_set p X S and y ∈ singular_relcycle_set p X S for x
y
    using that by (simp add: singular_relcycle_add)
next
  show invchain_group p X x ∈ singular_relcycle_set p X S
    if x ∈ singular_relcycle_set p X S for x
    using that
    by clarsimp (metis m_inv_chain_group singular_chain_def singular_relcycle
singular_relcycle_minus)
qed (auto simp: singular_relcycle)


definition relcycle_group :: nat ⇒ 'a topology ⇒ 'a set ⇒ ('a chain) monoid
  where relcycle_group p X S ≡
        subgroup_generated (chain_group p X) (Collect(singular_relcycle p X S))

lemma carrier_relcycle_group [simp]:
  carrier (relcycle_group p X S) = singular_relcycle_set p X S
proof -
 have carrier (chain_group p X) ∩ singular_relcycle_set p X S = singular_relcycle_set
p X S
    using subgroup.subset subgroup_singular_relcycle by blast
  moreover have generate (chain_group p X) (singular_relcycle_set p X S) ⊆
singular_relcycle_set p X S
   by (simp add: group.generate_subgroup_incl group_chain_group subgroup_singular_relcycle)
  ultimately show ?thesis
    by (auto simp: relcycle_group_def subgroup_generated_def generate.incl)
qed

lemma one_relcycle_group [simp]: one(relcycle_group p X S) = 0
  by (simp add: relcycle_group_def)

lemma mult_relcycle_group [simp]: (⊗relcycle_group p X S) = (+)
  by (simp add: relcycle_group_def)

lemma abelian_relcycle_group [simp]:
   comm_group(relcycle_group p X S)
  unfolding relcycle_group_def
   by (intro group.abelian_subgroup_generated group_chain_group) (auto simp:
abelian_chain_group singular_relcycle)

lemma group_relcycle_group [simp]: group(relcycle_group p X S)
  by (simp add: comm_group.axioms(2))
```

lemma *relcycle_group_restrict* [*simp*]:
relcycle_group p X (*topspace X* ∩ *S*) = *relcycle_group p X S*
by (*metis relcycle_group_def singular_relcycle_restrict*)

definition *relative_homology_group* :: *int* ⇒ *'a topology* ⇒ *'a set* ⇒ (*'a chain*) *set monoid*
where
relative_homology_group p X S ≡
if p < *0 then singleton_group undefined else*
(*relcycle_group* (*nat p*) *X S*) *Mod* (*singular_relboundary_set* (*nat p*) *X S*)

abbreviation *homology_group*
where *homology_group p X* ≡ *relative_homology_group p X* {}

lemma *relative_homology_group_restrict* [*simp*]:
relative_homology_group p X (*topspace X* ∩ *S*) = *relative_homology_group p X S*
by (*simp add*: *relative_homology_group_def*)

lemma *nontrivial_relative_homology_group*:
fixes *p*::*nat*
shows *relative_homology_group p X S*
= *relcycle_group p X S Mod singular_relboundary_set p X S*
by (*simp add*: *relative_homology_group_def*)

lemma *singular_relboundary_ss*:
singular_relboundary p X S x ⟹ *Poly_Mapping.keys x* ⊆ *singular_simplex_set p X*
using *singular_chain_def singular_relboundary_imp_chain* **by** *blast*

lemma *trivial_relative_homology_group* [*simp*]:
p < *0* ⟹ *trivial_group*(*relative_homology_group p X S*)
by (*simp add*: *relative_homology_group_def*)

lemma *subgroup_singular_relboundary*:
subgroup (*singular_relboundary_set p X S*) (*chain_group p X*)
unfolding *chain_group_def*
proof *unfold_locales*
show *singular_relboundary_set p X S*
⊆ *carrier* (*free_Abelian_group* (*singular_simplex_set p X*))
using *singular_chain_def singular_relboundary_imp_chain* **by** *fastforce*
next
fix *x*
assume *x* ∈ *singular_relboundary_set p X S*
then show $inv_{free\_Abelian\_group\ (singular\_simplex\_set\ p\ X)}\ x$
∈ *singular_relboundary_set p X S*
by (*simp add*: *singular_relboundary_ss singular_relboundary_minus*)

qed (*auto simp*: *free_Abelian_group_def singular_relboundary_add*)

lemma *subgroup_singular_relboundary_relcycle*:
subgroup (*singular_relboundary_set p X S*) (*relcycle_group p X S*)
unfolding *relcycle_group_def*
by (*simp add*: *Collect_mono group.subgroup_of_subgroup_generated singular_relboundary_imp_relcycle subgroup_singular_relboundary*)

lemma *normal_subgroup_singular_relboundary_relcycle*:
(*singular_relboundary_set p X S*) $\lhd$ (*relcycle_group p X S*)
by (*simp add*: *comm_group.normal_iff_subgroup subgroup_singular_relboundary_relcycle*)

lemma *group_relative_homology_group* [*simp*]:
group (*relative_homology_group p X S*)
by (*simp add*: *relative_homology_group_def normal.factorgroup_is_group normal_subgroup_singular_relboundary_relcycle*)

lemma *right_coset_singular_relboundary*:
r_coset (*relcycle_group p X S*) (*singular_relboundary_set p X S*)
= (λa. {*b*. *homologous_rel p X S a b*})
using *singular_relboundary_minus*
by (*force simp*: *r_coset_def homologous_rel_def relcycle_group_def subgroup_generated_def*)

lemma *carrier_relative_homology_group*:
carrier(*relative_homology_group* (*int p*) *X S*)
= (*homologous_rel_set p X S*) ‘ *singular_relcycle_set p X S*
by (*auto simp*: *set_eq_iff image_iff relative_homology_group_def FactGroup_def RCOSETS_def right_coset_singular_relboundary*)

lemma *carrier_relative_homology_group_0*:
carrier(*relative_homology_group 0 X S*)
= (*homologous_rel_set 0 X S*) ‘ *singular_relcycle_set 0 X S*
using *carrier_relative_homology_group* [*of 0 X S*] **by** *simp*

lemma *one_relative_homology_group* [*simp*]:
one(*relative_homology_group* (*int p*) *X S*) = *singular_relboundary_set p X S*
by (*simp add*: *relative_homology_group_def FactGroup_def*)

lemma *mult_relative_homology_group*:
($\otimes_{relative\_homology\_group\ (int\ p)\ X\ S}$) = ($\lambda R\ S.$ ($\bigcup r{\in}R.\ \bigcup s{\in}S.\ \{r + s\}$))
unfolding *relcycle_group_def subgroup_generated_def chain_group_def free_Abelian_group_def set_mult_def relative_homology_group_def FactGroup_def*
by *force*

lemma *inv_relative_homology_group*:
assumes $R \in$ *carrier* (*relative_homology_group* (*int p*) *X S*)
shows *m_inv*(*relative_homology_group* (*int p*) *X S*) *R* = *uminus* ‘ *R*
proof (*rule group.inv_equality* [*OF group_relative_homology_group _ assms*])
obtain *c* **where** *c*: *R* = *homologous_rel_set p X S c singular_relcycle p X S c*

using *assms* **by** (*auto simp*: *carrier_relative_homology_group*)
have *singular_relboundary p X S* (*b* − *a*)
if *a* ∈ *R* **and** *b* ∈ *R* **for** *a b*
using *c that*
by *clarify* (*metis homologous_rel_def homologous_rel_eq*)
moreover
have *x* ∈ (⋃*x*∈*R*. ⋃*y*∈*R*. {*y* − *x*})
if *singular_relboundary p X S x* **for** *x*
using *c*
by *simp* (*metis diff_eq_eq homologous_rel_def homologous_rel_refl homologous_rel_sym that*)
ultimately
have (⋃*x*∈*R*. ⋃*xa*∈*R*. {*xa* − *x*}) = *singular_relboundary_set p X S*
by *auto*
then show *uminus* ‘ $R \otimes_{relative\_homology\_group\ (int\ p)\ X\ S} R = \mathbf{1}_{relative\_homology\_group\ (int\ p)\ X\ S}$
by (*auto simp*: *carrier_relative_homology_group mult_relative_homology_group*)
have *singular_relcycle p X S* (−*c*)
using *c* **by** (*simp add*: *singular_relcycle_minus*)
moreover have *homologous_rel p X S c x* ⟹ *homologous_rel p X S* (−*c*) (−*x*) **for** *x*
by (*metis homologous_rel_def homologous_rel_sym minus_diff_eq minus_diff_minus*)
moreover have *homologous_rel p X S* (−*c*) *x* ⟹ *x* ∈ *uminus* ‘ *homologous_rel_set p X S c* **for** *x*
by (*clarsimp simp*: *image_iff*) (*metis add.inverse_inverse diff_0 homologous_rel_diff homologous_rel_refl*)
ultimately show *uminus* ‘ *R* ∈ *carrier* (*relative_homology_group* (*int p*) *X S*)
using *c* **by** (*auto simp*: *carrier_relative_homology_group*)
qed

lemma *homologous_rel_eq_relboundary*:
homologous_rel p X S c = *singular_relboundary p X S*
⟷ *singular_relboundary p X S c* (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
unfolding *homologous_rel_def*
by (*metis diff_zero singular_relboundary_0*)
next
assume *R*: *?rhs*
show *?lhs*
unfolding *homologous_rel_def*
using *singular_relboundary_diff R* **by** *fastforce*
qed

lemma *homologous_rel_set_eq_relboundary*:
homologous_rel_set p X S c = *singular_relboundary_set p X S* ⟷ *singular_relboundary p X S c*
by (*auto simp flip*: *homologous_rel_eq_relboundary*)

Lift the boundary and induced maps to homology groups. We totalize both quite aggressively to the appropriate group identity in all "undefined" situations, which makes several of the properties cleaner and simpler.

```
lemma homomorphism_chain_boundary:
  chain_boundary p ∈ hom (relcycle_group p X S) (relcycle_group(p − Suc 0)
(subtopology X S) {})
   (is ?h ∈ hom ?G ?H)
proof (rule homI)
 show ⋀x. x ∈ carrier ?G ⟹ ?h x ∈ carrier ?H
  by (auto simp: singular_relcycle_def mod_subset_def chain_boundary_boundary)
qed (simp add: relcycle_group_def subgroup_generated_def chain_boundary_add)


lemma hom_boundary1:
   ∃ d. ∀ p X S.
        d p X S ∈ hom (relative_homology_group (int p) X S)
                    (homology_group (int (p − Suc 0)) (subtopology X S))
      ∧ (∀ c. singular_relcycle p X S c
           ⟶ d p X S (homologous_rel_set p X S c)
          = homologous_rel_set (p − Suc 0) (subtopology X S) {} (chain_boundary
p c))
   (is ∃ d. ∀ p X S. ?Φ (d p X S) p X S)
proof ((subst choice_iff [symmetric])+, clarify)
 fix p X and S :: 'a set
 define ϑ where ϑ ≡ r_coset (relcycle_group(p − Suc 0) (subtopology X S) {})
                (singular_relboundary_set (p − Suc 0) (subtopology X S) {}) ∘
chain_boundary p
 define H where H ≡ relative_homology_group (int (p − Suc 0)) (subtopology
X S) {}
 define J where J ≡ relcycle_group (p − Suc 0) (subtopology X S) {}

 have ϑ: ϑ ∈ hom (relcycle_group p X S) H
   unfolding ϑ_def
 proof (rule hom_compose)
   show chain_boundary p ∈ hom (relcycle_group p X S) J
    by (simp add: J_def homomorphism_chain_boundary)
   show (#>relcycle_group (p − Suc 0) (subtopology X S) {})
      (singular_relboundary_set (p − Suc 0) (subtopology X S) {}) ∈ hom J H
    by (simp add: H_def J_def nontrivial_relative_homology_group
       normal.r_coset_hom_Mod normal_subgroup_singular_relboundary_relcycle)
 qed
 have ∗: singular_relboundary (p − Suc 0) (subtopology X S) {} (chain_boundary
p c)
   if singular_relboundary p X S c for c
 proof (cases p=0)
   case True
   then show ?thesis
    by (metis chain_boundary_def singular_relboundary_0)
```

next
case *False*
with *that* **have** $\exists\, d.\ singular\_chain\ p\ (subtopology\ X\ S)\ d \wedge chain\_boundary\ p\ d = chain\_boundary\ p\ c$
by (*metis add.left_neutral chain_boundary_add chain_boundary_boundary_alt singular_relboundary*)
with *that False* **show** *?thesis*
by (*auto simp*: *singular_boundary*)
qed
have ϑ_eq: $\vartheta\ x = \vartheta\ y$
if x: $x \in singular\_relcycle\_set\ p\ X\ S$ **and** y: $y \in singular\_relcycle\_set\ p\ X\ S$
and eq: $singular\_relboundary\_set\ p\ X\ S\ \#{>}_{relcycle\_group\ p\ X\ S}\ x = singular\_relboundary\_set\ p\ X\ S\ \#{>}_{relcycle\_group\ p\ X\ S}\ y$ **for** $x\ y$
proof −
have *singular_relboundary p X S* $(x-y)$
by (*metis eq homologous_rel_def homologous_rel_eq mem_Collect_eq right_coset_singular_relboun*
with ∗ **have** (*singular_relboundary* $(p - Suc\ 0)$ (*subtopology X S*) {}) (*chain_boundary p* $(x-y)$)
by *blast*
then show *?thesis*
unfolding ϑ*_def comp_def*
by (*metis chain_boundary_diff homologous_rel_def homologous_rel_eq right_coset_singular_relboun*
qed
obtain *d*
where $d \in hom$ ((*relcycle_group p X S*) *Mod* (*singular_relboundary_set p X S*)) *H*
and d: $\bigwedge u.\ u \in singular\_relcycle\_set\ p\ X\ S \Longrightarrow d\ (homologous\_rel\_set\ p\ X\ S\ u) = \vartheta\ u$
by (*metis FactGroup_universal* [*OF* ϑ *normal_subgroup_singular_relboundary_relcycle* ϑ*_eq*] *right_coset_singular_relboundary carrier_relcycle_group*)
then have $d \in hom$ (*relative_homology_group p X S*) *H*
by (*simp add*: *nontrivial_relative_homology_group*)
then show $\exists\, d.\ ?\Phi\ d\ p\ X\ S$
by (*force simp*: *H_def right_coset_singular_relboundary d* ϑ*_def*)
qed

lemma *hom_boundary2*:
$\exists\, d.\ (\forall\, p\ X\ S.$
$(d\ p\ X\ S) \in hom\ (relative\_homology\_group\ p\ X\ S)$
$(homology\_group\ (p-1)\ (subtopology\ X\ S)))$
$\wedge\ (\forall\, p\ X\ S\ c.\ singular\_relcycle\ p\ X\ S\ c \wedge Suc\ 0 \le p$
$\longrightarrow d\ p\ X\ S\ (homologous\_rel\_set\ p\ X\ S\ c)$
$= homologous\_rel\_set\ (p - Suc\ 0)\ (subtopology\ X\ S)\ \{\}\ (chain\_boundary\ p\ c))$
(**is** $\exists\, d.\ ?\Phi\ d$)
proof −
have ∗: $\exists f.\ \Phi(\lambda p.\ if\ p \le 0\ then\ \lambda q\ r\ t.\ undefined\ else\ f(nat\ p)) \Longrightarrow \exists f.\ \Phi\ f$ **for** Φ
by *blast*

```
  show ?thesis
    apply (rule * [OF ex_forward [OF hom_boundary1]])
   apply (simp add: not_le relative_homology_group_def nat_diff_distrib' int_eq_iff
nat_diff_distrib flip: nat_1)
    by (simp add: hom_def singleton_group_def)
qed

lemma hom_boundary3:
  ∃ d. ((∀ p X S c. c ∉ carrier(relative_homology_group p X S)
              ⟶ d p X S c = one(homology_group (p−1) (subtopology X S))) ∧
       (∀ p X S.
          d p X S ∈ hom (relative_homology_group p X S)
                      (homology_group (p−1) (subtopology X S))) ∧
       (∀ p X S c.
           singular_relcycle p X S c ∧ 1 ≤ p
           ⟶ d p X S (homologous_rel_set p X S c)
          = homologous_rel_set (p − Suc 0) (subtopology X S) {} (chain_boundary
p c)) ∧
       (∀ p X S. d p X S = d p X (topspace X ∩ S))) ∧
       (∀ p X S c. d p X S c ∈ carrier(homology_group (p−1) (subtopology X S)))
∧
       (∀ p. p ≤ 0 ⟶ d p = (λq r t. undefined))
  (is ∃ x. ?P x ∧ ?Q x ∧ ?R x)
proof −
  have ⋀x. ?Q x ⟹ ?R x
    by (erule all_forward) (force simp: relative_homology_group_def)
  moreover have ∃ x. ?P x ∧ ?Q x
  proof −
    obtain d:: [int, 'a topology, 'a set, ('a chain) set] ⇒ ('a chain) set
      where 1: ⋀p X S. d p X S ∈ hom (relative_homology_group p X S)
                                (homology_group (p−1) (subtopology X S))
        and 2: ⋀n X S c. singular_relcycle n X S c ∧ Suc 0 ≤ n
                 ⟹ d n X S (homologous_rel_set n X S c)
                        = homologous_rel_set (n − Suc 0) (subtopology X S) {}
(chain_boundary n c)
      using hom_boundary2 by blast
    have 4: c ∈ carrier (relative_homology_group p X S) ⟹
          d p X (topspace X ∩ S) c ∈ carrier (relative_homology_group (p−1)
(subtopology X S) {})
      for p X S c
      using hom_carrier [OF 1 [of p X topspace X ∩ S]]
      by (simp add: image_subset_iff subtopology_restrict)
    show ?thesis
      apply (rule_tac x=λp X S c.
               if c ∈ carrier(relative_homology_group p X S)
               then d p X (topspace X ∩ S) c
               else one(homology_group (p−1) (subtopology X S)) in exI)
    apply (simp add: Int_left_absorb subtopology_restrict carrier_relative_homology_group
         group.is_monoid group.restrict_hom_iff 4 cong: if_cong)
```

by (*metis 1 2 homologous_rel_restrict relative_homology_group_restrict singular_relcycle_def subtopology_restrict*)
qed
ultimately show *?thesis*
by *auto*
qed

consts *hom_boundary* :: [*int*,′*a topology*,′*a set*,′*a chain set*] ⇒ ′*a chain set*
specification (*hom_boundary*)
hom_boundary:
((∀ *p X S c. c* ∉ *carrier*(*relative_homology_group p X S*)
⟶ *hom_boundary p X S c* = *one*(*homology_group* (*p*−*1*) (*subtopology X* (*S*::′*a set*)))) ∧
(∀ *p X S*.
hom_boundary p X S ∈ *hom* (*relative_homology_group p X S*)
(*homology_group* (*p*−*1*) (*subtopology X* (*S*::′*a set*)))) ∧
(∀ *p X S c*.
singular_relcycle p X S c ∧ *1* ≤ *p*
⟶ *hom_boundary p X S* (*homologous_rel_set p X S c*)
= *homologous_rel_set* (*p* − *Suc 0*) (*subtopology X* (*S*::′*a set*)) {} (*chain_boundary p c*)) ∧
(∀ *p X S*. *hom_boundary p X S* = *hom_boundary p X* (*topspace X* ∩ (*S*::′*a set*)))) ∧
(∀ *p X S c*. *hom_boundary p X S c* ∈ *carrier*(*homology_group* (*p*−*1*) (*subtopology X* (*S*::′*a set*)))) ∧
(∀ *p*. *p* ≤ *0* ⟶ *hom_boundary p* = (λ*q r*. λ*t*::′*a chain set*. *undefined*))
by (*fact hom_boundary3*)

lemma *hom_boundary_default*:
c ∉ *carrier*(*relative_homology_group p X S*)
⟹ *hom_boundary p X S c* = *one*(*homology_group* (*p*−*1*) (*subtopology X S*))
and *hom_boundary_hom*: *hom_boundary p X S* ∈ *hom* (*relative_homology_group p X S*) (*homology_group* (*p*−*1*) (*subtopology X S*))
and *hom_boundary_restrict* [*simp*]: *hom_boundary p X* (*topspace X* ∩ *S*) = *hom_boundary p X S*
and *hom_boundary_carrier*: *hom_boundary p X S c* ∈ *carrier*(*homology_group* (*p*−*1*) (*subtopology X S*))
and *hom_boundary_trivial*: *p* ≤ *0* ⟹ *hom_boundary p* = (λ*q r t*. *undefined*)
by (*metis hom_boundary*)+

lemma *hom_boundary_chain_boundary*:
⟦*singular_relcycle p X S c*; *1* ≤ *p*⟧
⟹ *hom_boundary* (*int p*) *X S* (*homologous_rel_set p X S c*) =
homologous_rel_set (*p* − *Suc 0*) (*subtopology X S*) {} (*chain_boundary p c*)
by (*metis hom_boundary*)+

lemma *hom_chain_map*:

⟦continuous_map X Y f; f ∈ S → T⟧
⟹ (chain_map p f) ∈ hom (relcycle_group p X S) (relcycle_group p Y T)
by (simp add: chain_map_add hom_def singular_relcycle_chain_map)

lemma hom_induced1:
∃ hom_relmap.
(∀ p X S Y T f.
continuous_map X Y f ∧ f ∈ (topspace X ∩ S) → T
⟶ (hom_relmap p X S Y T f) ∈ hom (relative_homology_group (int p) X S)
(relative_homology_group (int p) Y T)) ∧
(∀ p X S Y T f c.
continuous_map X Y f ∧ f ∈ (topspace X ∩ S) → T ∧
singular_relcycle p X S c
⟶ hom_relmap p X S Y T f (homologous_rel_set p X S c) =
homologous_rel_set p Y T (chain_map p f c))
proof −
have ∃ y. (y ∈ hom (relative_homology_group (int p) X S) (relative_homology_group (int p) Y T)) ∧
(∀ c. singular_relcycle p X S c ⟶
y (homologous_rel_set p X S c) = homologous_rel_set p Y T (chain_map p f c))
if contf: continuous_map X Y f **and** fim: f ∈ (topspace X ∩ S) → T
for p X S Y T **and** f :: 'a ⇒ 'b
proof −
let ?f = (#>$_{relcycle\_group\ p\ Y\ T}$) (singular_relboundary_set p Y T) ∘ chain_map p f
let ?F = λx. singular_relboundary_set p X S #>$_{relcycle\_group\ p\ X\ S}$ x
have chain_map p f ∈ hom (relcycle_group p X S) (relcycle_group p Y T)
by (metis contf fim hom_chain_map relcycle_group_restrict)
then have 1: ?f ∈ hom (relcycle_group p X S) (relative_homology_group (int p) Y T)
by (simp add: hom_compose normal.r_coset_hom_Mod normal_subgroup_singular_relboundary_relcycle relative_homology_group_def)
have 2: singular_relboundary_set p X S ⊲ relcycle_group p X S
using normal_subgroup_singular_relboundary_relcycle **by** blast
have 3: ?f x = ?f y
if singular_relcycle p X S x singular_relcycle p X S y ?F x = ?F y **for** x y
proof −
have homologous_rel p X S x y
by (metis (no_types) homologous_rel_set_eq right_coset_singular_relboundary that(3))
then have singular_relboundary p Y T (chain_map p f (x − y))
using singular_relboundary_chain_map [OF _ contf fim] **by** (simp add: homologous_rel_def)
then have singular_relboundary p Y T (chain_map p f x − chain_map p f y)
by (simp add: chain_map_diff)

```
      with that
      show ?thesis
      by (metis comp_apply homologous_rel_def homologous_rel_set_eq right_coset_singular_relbounda
    qed
   obtain g where g ∈ hom (relcycle_group p X S Mod singular_relboundary_set
p X S)
                         (relative_homology_group (int p) Y T)
                    ⋀x. x ∈ singular_relcycle_set p X S ⟹ g (?F x) = ?f x
      using FactGroup_universal [OF 1 2 3, unfolded carrier_relcycle_group] by
blast
    then show ?thesis
     by (force simp: right_coset_singular_relboundary nontrivial_relative_homology_group)
  qed
  then show ?thesis
    apply (simp flip: all_conj_distrib)
    apply ((subst choice_iff [symmetric])+)
    apply metis
    done
qed

lemma hom_induced2:
    ∃hom_relmap.
      (∀p X S Y T f.
          continuous_map X Y f ∧
          f ∈ (topspace X ∩ S) → T
          ⟶ (hom_relmap p X S Y T f) ∈ hom (relative_homology_group p X S)
                               (relative_homology_group p Y T)) ∧
      (∀p X S Y T f c.
          continuous_map X Y f ∧
          f ∈ (topspace X ∩ S) → T ∧
          singular_relcycle p X S c
          ⟶ hom_relmap p X S Y T f (homologous_rel_set p X S c) =
              homologous_rel_set p Y T (chain_map p f c)) ∧
      (∀p. p < 0 ⟶ hom_relmap p = (λX S Y T f c. undefined))
  (is ∃d. ?Φ d)
proof −
  have ∗: ∃f. Φ(λp. if p < 0 then λX S Y T f c. undefined else f(nat p)) ⟹ ∃f.
Φ f  for Φ
    by blast
  show ?thesis
    apply (rule ∗ [OF ex_forward [OF hom_induced1]])
   apply (simp add: not_le relative_homology_group_def nat_diff_distrib' int_eq_iff
nat_diff_distrib flip: nat_1)
    done
qed

lemma hom_induced3:
  ∃hom_relmap.
    ((∀p X S Y T f c.
```

```
      ~(continuous_map X Y f ∧ f ∈ (topspace X ∩ S) → T ∧
        c ∈ carrier(relative_homology_group p X S))
      ⟶ hom_relmap p X S Y T f c = one(relative_homology_group p Y T)) ∧
   (∀p X S Y T f.
        hom_relmap p X S Y T f ∈ hom (relative_homology_group p X S)
(relative_homology_group p Y T)) ∧
   (∀p X S Y T f c.
      continuous_map X Y f ∧ f ∈ (topspace X ∩ S) → T ∧ singular_relcycle p
X S c
      ⟶ hom_relmap p X S Y T f (homologous_rel_set p X S c) =
        homologous_rel_set p Y T (chain_map p f c)) ∧
   (∀p X S Y T.
      hom_relmap p X S Y T =
      hom_relmap p X (topspace X ∩ S) Y (topspace Y ∩ T))) ∧
   (∀p X S Y f T c.
      hom_relmap p X S Y T f c ∈ carrier(relative_homology_group p Y T)) ∧
   (∀p. p < 0 ⟶ hom_relmap p = (λX S Y T f c. undefined))
  (is ∃x. ?P x ∧ ?Q x ∧ ?R x)
proof -
  have ⋀x. ?Q x ⟹ ?R x
    by (erule all_forward) (fastforce simp: relative_homology_group_def)
  moreover have ∃x. ?P x ∧ ?Q x
  proof -
    obtain hom_relmap:: [int,'a topology,'a set,'b topology,'b set,'a ⇒ 'b,('a chain)
set] ⇒ ('b chain) set
      where 1: ⋀p X S Y T f. ⟦continuous_map X Y f; f ∈ (topspace X ∩ S) →
T⟧ ⟹
                hom_relmap p X S Y T f
              ∈ hom (relative_homology_group p X S) (relative_homology_group
p Y T)
        and 2: ⋀p X S Y T f c.
              ⟦continuous_map X Y f; f ∈ (topspace X ∩ S) → T; singular_relcycle
p X S c⟧
                ⟹
                hom_relmap (int p) X S Y T f (homologous_rel_set p X S c) =
                homologous_rel_set p Y T (chain_map p f c)
        and 3: (∀p. p < 0 ⟶ hom_relmap p = (λX S Y T f c. undefined))
      using hom_induced2 [where ?'a='a and ?'b='b]
      by (fastforce simp: Pi_iff)
     have 4: ⟦continuous_map X Y f; f ∈ (topspace X ∩ S) → T; c ∈ carrier
(relative_homology_group p X S)⟧ ⟹
        hom_relmap p X (topspace X ∩ S) Y (topspace Y ∩ T) f c
          ∈ carrier (relative_homology_group p Y T)
      for p X S Y f T c
      using hom_carrier [OF 1 [of X Y f topspace X ∩ S topspace Y ∩ T p]]
          continuous_map_image_subset_topspace by fastforce
    have inhom: (λc. if continuous_map X Y f ∧ f ∈ (topspace X ∩ S) → T ∧
                c ∈ carrier (relative_homology_group p X S)
          then hom_relmap p X (topspace X ∩ S) Y (topspace Y ∩ T) f c
```

```
            else 1relative_homology_group p Y T)
          ∈ hom (relative_homology_group p X S) (relative_homology_group p Y T)
(is ?h ∈ hom ?GX ?GY)
      for p X S Y T f
    proof (rule homI)
      show ⋀x. x ∈ carrier ?GX ⟹ ?h x ∈ carrier ?GY
        by (auto simp: 4 group.is_monoid)
      show ?h (x ⊗?GX y) = ?h x ⊗?GY ?h y if x ∈ carrier ?GX y ∈ carrier ?GX
for x y
      proof (cases p < 0)
        case True
        with that show ?thesis
          by (simp add: relative_homology_group_def singleton_group_def 3)
      next
        case False
        show ?thesis
        proof (cases continuous_map X Y f)
          case True
          then have f ∈ (topspace X ∩ S) → topspace Y
            using continuous_map_image_subset_topspace by blast
          then show ?thesis
            using True False that
            using 1 [of X Y f topspace X ∩ S topspace Y ∩ T p]
             by (simp add: 4 Pi_iff continuous_map_funspace hom_mult not_less
group.is_monoid monoid.m_closed Int_left_absorb)
        qed (simp add: group.is_monoid)
      qed
    qed
   have hrel: ⟦continuous_map X Y f; f ∈ (topspace X ∩ S) → T; singular_relcycle
p X S c⟧
            ⟹ hom_relmap (int p) X (topspace X ∩ S) Y (topspace Y ∩ T)
          f (homologous_rel_set p X S c) = homologous_rel_set p Y T (chain_map
p f c)
        for p X S Y T f c
      using 2 [of X Y f topspace X ∩ S topspace Y ∩ T p c]
            continuous_map_image_subset_topspace by fastforce
    show ?thesis
      apply (rule_tac x=λp X S Y T f c.
                if continuous_map X Y f ∧ f ∈ (topspace X ∩ S) → T ∧
                   c ∈ carrier(relative_homology_group p X S)
                then hom_relmap p X (topspace X ∩ S) Y (topspace Y ∩ T) f c
                else one(relative_homology_group p Y T) in exI)
     apply (simp add: Int_left_absorb subtopology_restrict carrier_relative_homology_group
           group.is_monoid group.restrict_hom_iff 4 inhom hrel split: if_splits)
      apply (intro ext strip)
      apply (auto simp: continuous_map_def)
      done
  qed
  ultimately show ?thesis
```

```
  by auto
qed


consts hom_induced:: [int,'a topology,'a set,'b topology,'b set,'a ⇒ 'b,('a chain)
set] ⇒ ('b chain) set
specification (hom_induced)
 hom_induced:
  ((∀ p X S Y T f c.
     ~(continuous_map X Y f ∧
      f ∈ (topspace X ∩ S) → T ∧
      c ∈ carrier(relative_homology_group p X S))
     ⟶ hom_induced p X (S::'a set) Y (T::'b set) f c =
        one(relative_homology_group p Y T)) ∧
  (∀ p X S Y T f.
    (hom_induced p X (S::'a set) Y (T::'b set) f) ∈ hom (relative_homology_group
p X S)
                          (relative_homology_group p Y T)) ∧
  (∀ p X S Y T f c.
     continuous_map X Y f ∧
     f ∈ (topspace X ∩ S) → T ∧
     singular_relcycle p X S c
     ⟶ hom_induced p X (S::'a set) Y (T::'b set) f (homologous_rel_set p X
S c) =
        homologous_rel_set p Y T (chain_map p f c)) ∧
  (∀ p X S Y T.
     hom_induced p X (S::'a set) Y (T::'b set) =
     hom_induced p X (topspace X ∩ S) Y (topspace Y ∩ T))) ∧
  (∀ p X S Y f T c.
     hom_induced p X (S::'a set) Y (T::'b set) f c ∈
     carrier(relative_homology_group p Y T)) ∧
  (∀ p. p < 0 ⟶ hom_induced p = (λX S Y T. λf::'a⇒'b. λc. undefined))
 by (fact hom_induced3)

lemma hom_induced_default:
  ~(continuous_map X Y f ∧ f ∈ (topspace X ∩ S) → T ∧ c ∈ carrier(relative_homology_group
p X S))
      ⟹ hom_induced p X S Y T f c = one(relative_homology_group p Y T)
 and hom_induced_hom:
  hom_induced p X S Y T f ∈ hom (relative_homology_group p X S) (relative_homology_group
p Y T)
 and hom_induced_restrict [simp]:
  hom_induced p X (topspace X ∩ S) Y (topspace Y ∩ T) = hom_induced p X
S Y T
 and hom_induced_carrier:
  hom_induced p X S Y T f c ∈ carrier(relative_homology_group p Y T)
 and hom_induced_trivial: p < 0 ⟹ hom_induced p = (λX S Y T f c. undefined)
 by (metis hom_induced)+
```

lemma *hom_induced_chain_map_gen*:
⟦*continuous_map X Y f*; *f* ∈ (*topspace X* ∩ *S*) → *T*; *singular_relcycle p X S c*⟧
⟹ *hom_induced p X S Y T f* (*homologous_rel_set p X S c*) = *homologous_rel_set p Y T* (*chain_map p f c*)
by (*metis hom_induced*)

lemma *hom_induced_chain_map*:
⟦*continuous_map X Y f*; *f* ∈ *S* → *T*; *singular_relcycle p X S c*⟧
⟹ *hom_induced p X S Y T f* (*homologous_rel_set p X S c*)
= *homologous_rel_set p Y T* (*chain_map p f c*)
by (*simp add*: *Pi_iff hom_induced_chain_map_gen*)

lemma *hom_induced_eq*:
assumes ⋀*x*. *x* ∈ *topspace X* ⟹ *f x* = *g x*
shows *hom_induced p X S Y T f* = *hom_induced p X S Y T g*
proof −
consider *p* < *0* | *n* **where** *p* = *int n*
by (*metis int_nat_eq not_less*)
then show *?thesis*
proof *cases*
case *1*
then show *?thesis*
by (*simp add*: *hom_induced_trivial*)
next
case *2*
have *hom_induced n X S Y T f C* = *hom_induced n X S Y T g C* **for** *C*
proof −
have *continuous_map X Y f* ∧ *f* ∈ (*topspace X* ∩ *S*) → *T* ∧ *C* ∈ *carrier* (*relative_homology_group n X S*)
⟷ *continuous_map X Y g* ∧ *g* ∈ (*topspace X* ∩ *S*) → *T* ∧ *C* ∈ *carrier* (*relative_homology_group n X S*)
(**is** *?P* = *?Q*)
using *assms Pi_iff continuous_map_eq* [*of X Y*]
by (*smt* (*verit*, *ccfv_SIG*) *Int_iff*)
then consider ¬ *?P* ∧ ¬ *?Q* | *?P* ∧ *?Q*
by *blast*
then show *?thesis*
proof *cases*
case *1*
then show *?thesis*
by (*simp add*: *hom_induced_default*)
next
case *2*
have *homologous_rel_set n Y T* (*chain_map n f c*) = *homologous_rel_set n Y T* (*chain_map n g c*)
if *continuous_map X Y f f* ∈ (*topspace X* ∩ *S*) → *T*
continuous_map X Y g g ∈ (*topspace X* ∩ *S*) → *T*
C = *homologous_rel_set n X S c singular_relcycle n X S c*

```
          for c
        proof -
          have chain_map n f c = chain_map n g c
            using assms chain_map_eq singular_relcycle that by metis
          then show ?thesis
            by simp
        qed
        with 2 show ?thesis
          by (force simp: relative_homology_group_def carrier_FactGroup
              right_coset_singular_relboundary hom_induced_chain_map_gen)
      qed
    qed
    with 2 show ?thesis
      by auto
  qed
qed
```

0.2.2 Towards the Eilenberg-Steenrod axioms

First prove we get functors into abelian groups with the boundary map being a natural transformation between them, and prove Eilenberg-Steenrod axioms (we also prove additivity a bit later on if one counts that).

```
lemma abelian_relative_homology_group [simp]:
   comm_group(relative_homology_group p X S)
  by (simp add: comm_group.abelian_FactGroup relative_homology_group_def
subgroup_singular_relboundary_relcycle)

lemma abelian_homology_group: comm_group(homology_group p X)
  by simp


lemma hom_induced_id_gen:
  assumes contf: continuous_map X X f and feq: ⋀x. x ∈ topspace X ⟹ f x =
x
    and c: c ∈ carrier (relative_homology_group p X S)
  shows hom_induced p X S X S f c = c
proof -
  consider p < 0 | n where p = int n
    by (metis int_nat_eq not_less)
  then show ?thesis
  proof cases
    case 1
    with c show ?thesis
      by (simp add: hom_induced_trivial relative_homology_group_def)
  next
    case 2
    have cm: chain_map n f d = d if singular_relcycle n X S d for d
      using that assms by (auto simp: chain_map_id_gen singular_relcycle)
    have f ` (topspace X ∩ S) ⊆ S
```

```
      using feq by auto
    with 2 c show ?thesis
      by (auto simp: nontrivial_relative_homology_group carrier_FactGroup
              cm right_coset_singular_relboundary hom_induced_chain_map_gen
assms)
  qed
qed


lemma hom_induced_id:
   c ∈ carrier (relative_homology_group p X S) ⟹ hom_induced p X S X S id c
= c
  by (rule hom_induced_id_gen) auto

lemma hom_induced_compose:
  assumes continuous_map X Y f f ∈ S → T continuous_map Y Z g g ∈ T → U
   shows hom_induced p X S Z U (g ∘ f) = hom_induced p Y T Z U g ∘
hom_induced p X S Y T f
proof −
  consider (neg) p < 0 | (int) n where p = int n
    by (metis int_nat_eq not_less)
  then show ?thesis
  proof cases
    case int
    have gf: continuous_map X Z (g ∘ f)
      using assms continuous_map_compose by fastforce
    have gfim: (g ∘ f) ∈ S → U
      unfolding o_def using assms by blast
   have sr: ⋀a. singular_relcycle n X S a ⟹ singular_relcycle n Y T (chain_map
n f a)
      by (simp add: assms singular_relcycle_chain_map)
    show ?thesis
    proof
      fix c
       show hom_induced p X S Z U (g ∘ f) c = (hom_induced p Y T Z U g ∘
hom_induced p X S Y T f) c
      proof (cases c ∈ carrier(relative_homology_group p X S))
        case True
        with gfim show ?thesis
          unfolding int
          by (auto simp: carrier_relative_homology_group gf gfim assms
              sr chain_map_compose hom_induced_chain_map)
      next
        case False
        then show ?thesis
          by (simp add: hom_induced_default hom_one [OF hom_induced_hom])
      qed
    qed
  qed (force simp: hom_induced_trivial)
```

```
qed

lemma hom_induced_compose':
  assumes continuous_map X Y f f ∈ S → T continuous_map Y Z g g ∈ T → U
  shows hom_induced p Y T Z U g (hom_induced p X S Y T f x) = hom_induced
p X S Z U (g ∘ f) x
  using hom_induced_compose [OF assms] by simp

lemma naturality_hom_induced:
  assumes continuous_map X Y f f ∈ S → T
  shows hom_boundary q Y T ∘ hom_induced q X S Y T f
      = hom_induced (q − 1) (subtopology X S) {} (subtopology Y T) {} f ∘
hom_boundary q X S
proof (cases q ≤ 0)
  case False
  then obtain p where p1: p ≥ Suc 0 and q: q = int p
    using zero_le_imp_eq_int by force
  show ?thesis
  proof
    fix c
    show (hom_boundary q Y T ∘ hom_induced q X S Y T f) c =
          (hom_induced (q − 1) (subtopology X S) {} (subtopology Y T) {} f ∘
hom_boundary q X S) c
    proof (cases c ∈ carrier(relative_homology_group p X S))
      case True
       then obtain a where ceq: c = homologous_rel_set p X S a and a: singu-
lar_relcycle p X S a
        by (force simp: carrier_relative_homology_group)
      then have sr: singular_relcycle p Y T (chain_map p f a)
        using assms singular_relcycle_chain_map by fastforce
    then have sb: singular_relcycle (p − Suc 0) (subtopology X S) {} (chain_boundary
p a)
          by (metis One_nat_def a chain_boundary_boundary singular_chain_0
singular_relcycle)
      have p1_eq: int p − 1 = int (p − Suc 0)
        using p1 by auto
      have cbm: (chain_boundary p (chain_map p f a))
            = (chain_map (p − Suc 0) f (chain_boundary p a))
        using a chain_boundary_chain_map singular_relcycle by metis
      have contf: continuous_map (subtopology X S) (subtopology Y T) f
        using assms
      by (auto simp: continuous_map_in_subtopology continuous_map_from_subtopology)
      show ?thesis
        unfolding q using assms p1 a
      by (simp add: cbm ceq contf hom_boundary_chain_boundary hom_induced_chain_map
p1_eq sb sr
                del: of_nat_diff)
    next
      case False
```

with *assms* **show** *?thesis*
unfolding *q o_def* **using** *assms*
apply (*simp add*: *hom_induced_default hom_boundary_default*)
by (*metis group_relative_homology_group hom_boundary hom_induced hom_one one_relative_homology_group*)
qed
qed
qed (*force simp*: *hom_induced_trivial hom_boundary_trivial*)

lemma *homology_exactness_axiom_1*:
exact_seq ([*homology_group* (*p*−*1*) (*subtopology X S*), *relative_homology_group p X S*, *homology_group p X*],
[*hom_boundary p X S*,*hom_induced p X* {} *X S id*])
proof −
consider (*neg*) *p* < *0* | (*int*) *n* **where** *p* = *int n*
by (*metis int_nat_eq not_less*)
then have (*hom_induced p X* {} *X S id*) ‘ *carrier* (*homology_group p X*)
= *kernel* (*relative_homology_group p X S*) (*homology_group* (*p*−*1*) (*subtopology X S*))
(*hom_boundary p X S*)
proof *cases*
case *neg*
then show *?thesis*
unfolding *kernel_def singleton_group_def relative_homology_group_def*
by (*auto simp*: *hom_induced_trivial hom_boundary_trivial*)
next
case *int*
have *hom_induced* (*int m*) *X* {} *X S id* ‘ *carrier* (*relative_homology_group* (*int m*) *X* {})
= *carrier* (*relative_homology_group* (*int m*) *X S*) ∩
{*c*. *hom_boundary* (*int m*) *X S c* = $\mathbf{1}_{relative\_homology\_group\ (int\ m\ -\ 1)\ (subtopology\ X\ S)\ \{\}}$}
for *m*
proof (*cases m*)
case *0*
have *hom_induced 0 X* {} *X S id* ‘ *carrier* (*relative_homology_group 0 X* {})
= *carrier* (*relative_homology_group 0 X S*) (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
using *hom_induced_hom* [*of 0 X* {} *X S id*]
by (*simp add*: *hom_induced_hom hom_carrier*)
show *?rhs* ⊆ *?lhs*
apply (*clarsimp simp add*: *image_iff carrier_relative_homology_group* [*of 0*, *simplified*] *singular_relcycle*)
apply (*force simp*: *chain_map_id_gen chain_boundary_def singular_relcycle*
hom_induced_chain_map [*of concl*: *0*, *simplified*])

```
        done
      qed
      with 0 show ?thesis
       by (simp add: hom_boundary_trivial relative_homology_group_def [of -1]
singleton_group_def)
    next
      case (Suc n)
      have (hom_induced (int (Suc n)) X {} X S id ∘
           homologous_rel_set (Suc n) X {}) ` singular_relcycle_set (Suc n) X {}
          = homologous_rel_set (Suc n) X S `
            (singular_relcycle_set (Suc n) X S ∩
             {c. hom_boundary (int (Suc n)) X S (homologous_rel_set (Suc n) X S
c)
                = singular_relboundary_set n (subtopology X S) {}})
        (is ?lhs = ?rhs)
      proof -
        have 1: (⋀x. x ∈ A ⟹ x ∈ B ⟷ x ∈ C) ⟹ f ` (A ∩ B) = f ` (A ∩ C)
for f A B C
          by blast
        have 2: ⟦⋀x. x ∈ A ⟹ ∃ y. y ∈ B ∧ f x = f y; ⋀x. x ∈ B ⟹ ∃ y. y ∈ A
∧ f x = f y⟧
    ⟹ f ` A = f ` B for f A B
          by blast
        have ?lhs = homologous_rel_set (Suc n) X S ` singular_relcycle_set (Suc
n) X {}
              using hom_induced_chain_map chain_map_ident [of _ X] singu-
lar_relcycle
          by (smt (verit, best) comp_apply continuous_map_id empty_iff funcsetI
              image_cong mem_Collect_eq)
        also have . . . = homologous_rel_set (Suc n) X S `
                         (singular_relcycle_set (Suc n) X S ∩
                     {c. singular_relboundary n (subtopology X S) {} (chain_boundary
(Suc n) c)})
        proof (rule 2)
          fix c
          assume c ∈ singular_relcycle_set (Suc n) X {}
          then show ∃ y. y ∈ singular_relcycle_set (Suc n) X S ∩
                     {c. singular_relboundary n (subtopology X S) {} (chain_boundary
(Suc n) c)} ∧
                    homologous_rel_set (Suc n) X S c = homologous_rel_set (Suc n)
X S y
            using singular_cycle singular_relcycle
            by (metis Int_Collect mem_Collect_eq singular_chain_0
                singular_relboundary_0)
        next
          fix c
          assume c: c ∈ singular_relcycle_set (Suc n) X S ∩
                    {c. singular_relboundary n (subtopology X S) {} (chain_boundary
(Suc n) c)}
```

```
        then obtain d where d: singular_chain (Suc n) (subtopology X S) d
          chain_boundary (Suc n) d = chain_boundary (Suc n) c
          by (auto simp: singular_boundary)
        with c have c − d ∈ singular_relcycle_set (Suc n) X {}
      by (auto simp: singular_cycle chain_boundary_diff singular_chain_subtopology
singular_relcycle singular_chain_diff)
      moreover have homologous_rel_set (Suc n) X S c = homologous_rel_set
(Suc n) X S (c − d)
        proof (simp add: homologous_rel_set_eq)
          show homologous_rel (Suc n) X S c (c − d)
        using d by (simp add: homologous_rel_def singular_chain_imp_relboundary)
        qed
        ultimately show ∃ y. y ∈ singular_relcycle_set (Suc n) X {} ∧
                homologous_rel_set (Suc n) X S c = homologous_rel_set (Suc n)
X S y
          by blast
      qed
      also have ... = ?rhs
     by (rule 1) (simp add: hom_boundary_chain_boundary homologous_rel_set_eq_relboundary
del: of_nat_Suc)
      finally show ?lhs = ?rhs .
    qed
    with Suc show ?thesis
      unfolding carrier_relative_homology_group image_comp id_def by auto
  qed
  then show ?thesis
    by (auto simp: kernel_def int)
 qed
 then show ?thesis
  using hom_boundary_hom hom_induced_hom
  by (force simp: group_hom_def group_hom_axioms_def)
qed


lemma homology_exactness_axiom_2:
  exact_seq ([homology_group (p−1) X, homology_group (p−1) (subtopology X
S), relative_homology_group p X S],
            [hom_induced (p−1) (subtopology X S) {} X {} id, hom_boundary p
X S])
proof −
 consider (neg) p ≤ 0 | (int) n where p = int (Suc n)
  by (metis linear not0_implies_Suc of_nat_0 zero_le_imp_eq_int)
 then have kernel (relative_homology_group (p−1) (subtopology X S) {})
               (relative_homology_group (p−1) X {})
               (hom_induced (p−1) (subtopology X S) {} X {} id)
         = hom_boundary p X S ‘ carrier (relative_homology_group p X S)
 proof cases
  case neg
  obtain x where x ∈ carrier (relative_homology_group p X S)
```

using *group_relative_homology_group group.is_monoid* **by** *blast*
with *neg* **show** *?thesis*
unfolding *kernel_def singleton_group_def relative_homology_group_def*
by (*force simp*: *hom_induced_trivial hom_boundary_trivial*)
next
case *int*
have *hom_boundary* (*int* (*Suc n*)) *X S* ‘ *carrier* (*relative_homology_group* (*int* (*Suc n*)) *X S*)
= *carrier* (*relative_homology_group n* (*subtopology X S*) {}) ∩
{*c*. *hom_induced n* (*subtopology X S*) {} *X* {} *id c* =
$\mathbf{1}_{relative\_homology\_group\ n\ X\ \{\}}$}
(**is** *?lhs* = *?rhs*)
proof −
have *1*: ($\bigwedge x.\ x \in A \Longrightarrow x \in B \longleftrightarrow x \in C) \Longrightarrow f$ ‘ (*A* ∩ *B*) = *f* ‘ (*A* ∩ *C*) **for** *f A B C*
by *blast*
have *2*: ($\bigwedge x.\ x \in A \Longrightarrow x \in B \longleftrightarrow x \in f$ −‘ *C*) ⟹ *f* ‘ (*A* ∩ *B*) = *f* ‘ *A* ∩ *C* **for** *f A B C*
by *blast*
have *?lhs* = *homologous_rel_set n* (*subtopology X S*) {}
‘ (*chain_boundary* (*Suc n*) ‘ *singular_relcycle_set* (*Suc n*) *X S*)
unfolding *carrier_relative_homology_group image_comp*
by (*rule image_cong* [*OF refl*]) (*simp add*: *o_def hom_boundary_chain_boundary del*: *of_nat_Suc*)
also have ... = *homologous_rel_set n* (*subtopology X S*) {} ‘
(*singular_relcycle_set n* (*subtopology X S*) {} ∩ *singular_relboundary_set n X* {})
by (*force simp*: *singular_relcycle singular_boundary chain_boundary_boundary_alt*)
also have ... = *?rhs*
unfolding *carrier_relative_homology_group vimage_def*
by (*intro 2*) (*auto simp*: *hom_induced_chain_map chain_map_ident homologous_rel_set_eq_relboundary singular_relcycle*)
finally show *?thesis* .
qed
then show *?thesis*
by (*auto simp*: *kernel_def int*)
qed
then show *?thesis*
using *hom_boundary_hom hom_induced_hom*
by (*force simp*: *group_hom_def group_hom_axioms_def*)
qed

lemma *homology_exactness_axiom_3*:
exact_seq ([*relative_homology_group p X S*, *homology_group p X*, *homology_group p* (*subtopology X S*)],
[*hom_induced p X* {} *X S id*, *hom_induced p* (*subtopology X S*) {} *X* {} *id*])
proof (*cases p* < *0*)

```
  case True
  then show ?thesis
    unfolding relative_homology_group_def
     by (simp add: group_hom.kernel_to_trivial_group group_hom_axioms_def
group_hom_def hom_induced_trivial)
next
  case False
  then obtain n where peq: p = int n
    by (metis int_ops(1) linorder_neqE_linordered_idom pos_int_cases)
  have hom_induced n (subtopology X S) {} X {} id `
        (homologous_rel_set n (subtopology X S) {} `
        singular_relcycle_set n (subtopology X S) {})
      = {c ∈ homologous_rel_set n X {} ` singular_relcycle_set n X {}.
         hom_induced n X {} X S id c = singular_relboundary_set n X S}
        (is ?lhs = ?rhs)
  proof −
    have 2: ⟦⋀x. x ∈ A ⟹ ∃ y. y ∈ B ∧ f x = f y; ⋀x. x ∈ B ⟹ ∃ y. y ∈ A ∧
f x = f y⟧
      ⟹ f ` A = f ` B for f A B
      by blast
    have ⋀f. singular_chain n (subtopology X S) f ∧
         singular_chain (n − Suc 0) trivial_topology (chain_boundary n f) ⟹
         hom_induced (int n) (subtopology X S) {} X {} id (homologous_rel_set n
(subtopology X S) {} f) =
         homologous_rel_set n X {} f
     by (auto simp: chain_map_ident hom_induced_chain_map singular_relcycle)
      then have ?lhs = homologous_rel_set n X {} ` (singular_relcycle_set n
(subtopology X S) {})
      by (simp add: singular_relcycle image_comp)
    also have ... = homologous_rel_set n X {} ` (singular_relcycle_set n X {}
∩ singular_relboundary_set n X S)
    proof (rule 2)
      fix c
      assume c ∈ singular_relcycle_set n (subtopology X S) {}
     then show ∃ y. y ∈ singular_relcycle_set n X {} ∩ singular_relboundary_set
n X S ∧
           homologous_rel_set n X {} c = homologous_rel_set n X {} y
        using singular_chain_imp_relboundary singular_relboundary_imp_chain
        by (fastforce simp: singular_cycle)
    next
      fix c
      assume c ∈ singular_relcycle_set n X {} ∩ singular_relboundary_set n X S
      then obtain d e where c: singular_relcycle n X {} c singular_relboundary
n X S c
        and d:  singular_chain n (subtopology X S) d
        and e: singular_chain (Suc n) X e chain_boundary (Suc n) e = c + d
        using singular_relboundary_alt by blast
      then have chain_boundary n (c + d) = 0
        using chain_boundary_boundary_alt by fastforce
```

```
      then have chain_boundary n c + chain_boundary n d = 0
        by (metis chain_boundary_add)
      with c have singular_relcycle n (subtopology X S) {} (− d)
       by (metis (no_types) d eq_add_iff singular_cycle singular_relcycle_minus)
      moreover have homologous_rel n X {} c (− d)
        using c
        by (metis diff_minus_eq_add e homologous_rel_def singular_boundary)
      ultimately
      show ∃ y. y ∈ singular_relcycle_set n (subtopology X S) {} ∧
           homologous_rel_set n X {} c = homologous_rel_set n X {} y
        by (force simp: homologous_rel_set_eq)
    qed
    also have ... = homologous_rel_set n X {} `
                (singular_relcycle_set n X {} ∩ homologous_rel_set n X {} −` {x.
hom_induced n X {} X S id x = singular_relboundary_set n X S})
     by (rule 2) (auto simp: hom_induced_chain_map homologous_rel_set_eq_relboundary
chain_map_ident [of _ X] singular_cycle cong: conj_cong)
    also have ... = ?rhs
      by blast
    finally show ?thesis .
  qed
  then have kernel (relative_homology_group p X {}) (relative_homology_group
p X S) (hom_induced p X {} X S id)
     = hom_induced p (subtopology X S) {} X {} id ` carrier (relative_homology_group
p (subtopology X S) {})
    by (simp add: kernel_def carrier_relative_homology_group peq)
  then show ?thesis
   by (simp add: not_less group_hom_def group_hom_axioms_def hom_induced_hom)
qed


lemma homology_dimension_axiom:
  assumes X: topspace X = {a} and p ≠ 0
  shows trivial_group(homology_group p X)
proof (cases p < 0)
  case True
  then show ?thesis
    by simp
next
  case False
  then obtain n where peq: p = int n n > 0
    by (metis assms(2) neq0_conv nonneg_int_cases not_less of_nat_0)
 have homologous_rel_set n X {} ` singular_relcycle_set n X {} = {singular_relcycle_set
n X {}}
        (is ?lhs = ?rhs)
  proof
    show ?lhs ⊆ ?rhs
      using peq assms
       by (auto simp: image_subset_iff homologous_rel_set_eq_relboundary simp
```

flip: *singular_boundary_set_eq_cycle_singleton*)
have *singular_relboundary n X* {} *0*
by *simp*
with *peq assms*
show *?rhs* ⊆ *?lhs*
by (*auto simp*: *image_iff simp flip*: *homologous_rel_eq_relboundary singular_boundary_set_eq_cycle_singleton*)
qed
with *peq assms* **show** *?thesis*
unfolding *trivial_group_def*
by (*simp add*: *carrier_relative_homology_group singular_boundary_set_eq_cycle_singleton* [*OF X*])
qed

proposition *homology_homotopy_axiom*:
assumes *homotopic_with* (λ*h*. *h* ∈ *S* → *T*) *X Y f g*
shows *hom_induced p X S Y T f* = *hom_induced p X S Y T g*
proof (*cases p* < *0*)
case *True*
then show *?thesis*
by (*simp add*: *hom_induced_trivial*)
next
case *False*
then obtain *n* **where** *peq*: *p* = *int n*
by (*metis int_nat_eq not_le*)
have *cont*: *continuous_map X Y f continuous_map X Y g*
using *assms homotopic_with_imp_continuous_maps* **by** *blast+*
have *im*: *f* ∈ (*topspace X* ∩ *S*) → *T g* ∈ (*topspace X* ∩ *S*) → *T*
using *homotopic_with_imp_property assms* **by** *blast+*
show *?thesis*
proof
fix *c* **show** *hom_induced p X S Y T f c* = *hom_induced p X S Y T g c*
proof (*cases c* ∈ *carrier*(*relative_homology_group p X S*))
case *True*
then obtain *a* **where** *a*: *c* = *homologous_rel_set n X S a singular_relcycle n X S a*
unfolding *carrier_relative_homology_group peq* **by** *auto*
with *assms homotopic_imp_homologous_rel_chain_maps* **show** *?thesis*
by (*force simp add*: *peq hom_induced_chain_map_gen cont im homologous_rel_set_eq*)
qed (*simp add*: *hom_induced_default*)
qed
qed

proposition *homology_excision_axiom*:
assumes *X closure_of U* ⊆ *X interior_of T T* ⊆ *S*
shows
hom_induced p (*subtopology X* (*S* − *U*)) (*T* − *U*) (*subtopology X S*) *T id*

```
  ∈ iso (relative_homology_group p (subtopology X (S − U)) (T − U))
        (relative_homology_group p (subtopology X S) T)
proof (cases p < 0)
  case True
  then show ?thesis
    unfolding iso_def bij_betw_def relative_homology_group_def by (simp add:
hom_induced_trivial)
next
  case False
  then obtain n where peq: p = int n
    by (metis int_nat_eq not_le)
  have cont: continuous_map (subtopology X (S − U)) (subtopology X S) id
      by (meson Diff_subset continuous_map_from_subtopology_mono continu-
ous_map_id)
  have TU: topspace X ∩ (S − U) ∩ (T − U) ⊆ T
    by auto
  show ?thesis
   proof (simp add: iso_def peq carrier_relative_homology_group bij_betw_def
hom_induced_hom, intro conjI)
    show inj_on (hom_induced n (subtopology X (S − U)) (T − U) (subtopology
X S) T id)
         (homologous_rel_set n (subtopology X (S − U)) (T − U) `
          singular_relcycle_set n (subtopology X (S − U)) (T − U))
      unfolding inj_on_def
    proof (clarsimp simp add: homologous_rel_set_eq)
      fix c d
      assume c: singular_relcycle n (subtopology X (S − U)) (T − U) c
        and d: singular_relcycle n (subtopology X (S − U)) (T − U) d
        and hh: hom_induced n (subtopology X (S − U)) (T − U) (subtopology X
S) T id
                  (homologous_rel_set n (subtopology X (S − U)) (T − U) c)
              = hom_induced n (subtopology X (S − U)) (T − U) (subtopology X
S) T id
                  (homologous_rel_set n (subtopology X (S − U)) (T − U) d)
      then obtain scc: singular_chain n (subtopology X (S − U)) c
           and  scd: singular_chain n (subtopology X (S − U)) d
        using singular_relcycle by metis
      have singular_relboundary n (subtopology X (S − U)) (T − U) c
        if srb: singular_relboundary n (subtopology X S) T c
          and src: singular_relcycle n (subtopology X (S − U)) (T − U) c for c
      proof −
        have [simp]: (S − U) ∩ (T − U) = T − U S ∩ T = T
          using ‹T ⊆ S› by blast+
        have c: singular_chain n (subtopology X (S − U)) c
            singular_chain (n − Suc 0) (subtopology X (T − U)) (chain_boundary
n c)
         using that by (auto simp: singular_relcycle_def mod_subset_def subtopol-
ogy_subtopology)
        obtain d e where d: singular_chain (Suc n) (subtopology X S) d
```

and *e*: *singular_chain n* (*subtopology X T*) *e*
and *dce*: *chain_boundary* (*Suc n*) *d* = *c* + *e*
using *srb* **by** (*auto simp*: *singular_relboundary_alt subtopology_subtopology*)
obtain *m f g* **where** *f*: *singular_chain* (*Suc n*) (*subtopology X* (*S* − *U*)) *f*
and *g*: *singular_chain* (*Suc n*) (*subtopology X T*) *g*
and *dfg*: (*singular_subdivision* (*Suc n*) ^^ *m*) *d* = *f* + *g*
using *excised_chain_exists* [*OF assms d*] **.**
obtain *h* **where**
h0: ⋀*p*. *h p 0* = (*0* :: ′*a chain*)
and *hdiff*: ⋀*p c1 c2*. *h p* (*c1*−*c2*) = *h p c1* − *h p c2*
and *hSuc*: ⋀*p X c*. *singular_chain p X c* ⟹ *singular_chain* (*Suc p*) *X* (*h p c*)
and *hchain*: ⋀*p X c*. *singular_chain p X c*
⟹ *chain_boundary* (*Suc p*) (*h p c*) + *h* (*p* − *Suc 0*) (*chain_boundary p c*)
= (*singular_subdivision p* ^^ *m*) *c* − *c*
using *chain_homotopic_iterated_singular_subdivision* **by** *blast*
have *hadd*: ⋀*p c1 c2*. *h p* (*c1* + *c2*) = *h p c1* + *h p c2*
by (*metis add_diff_cancel diff_add_cancel hdiff*)
define *c1* **where** *c1* ≡ *f* − *h n c*
define *c2* **where** *c2* ≡ *chain_boundary* (*Suc n*) (*h n e*) − (*chain_boundary* (*Suc n*) *g* − *e*)
show *?thesis*
unfolding *singular_relboundary_alt*
proof (*intro exI conjI*)
show *c1*: *singular_chain* (*Suc n*) (*subtopology X* (*S* − *U*)) *c1*
by (*simp add*: ‹*singular_chain n* (*subtopology X* (*S* − *U*)) *c*› *c1_def f hSuc singular_chain_diff*)
have *chain_boundary* (*Suc n*) (*chain_boundary* (*Suc* (*Suc n*)) (*h* (*Suc n*) *d*) + *h n* (*c*+*e*))
= *chain_boundary* (*Suc n*) (*f* + *g* − *d*)
using *hchain* [*OF d*] **by** (*simp add*: *dce dfg*)
then have *chain_boundary* (*Suc n*) (*h n* (*c* + *e*))
= *chain_boundary* (*Suc n*) *f* + *chain_boundary* (*Suc n*) *g* − (*c* + *e*)
using *chain_boundary_boundary_alt* [*of Suc n subtopology X S*]
by (*simp add*: *chain_boundary_add chain_boundary_diff d hSuc dce*)
then have *chain_boundary* (*Suc n*) (*h n c*) + *chain_boundary* (*Suc n*) (*h n e*)
= *chain_boundary* (*Suc n*) *f* + *chain_boundary* (*Suc n*) *g* − (*c* + *e*)
by (*simp add*: *chain_boundary_add hadd*)
then have ∗: *chain_boundary* (*Suc n*) (*f* − *h n c*) = *c* + (*chain_boundary* (*Suc n*) (*h n e*) − (*chain_boundary* (*Suc n*) *g* − *e*))
by (*simp add*: *algebra_simps chain_boundary_diff*)
then show *chain_boundary* (*Suc n*) *c1* = *c* + *c2*
unfolding *c1_def c2_def*
by (*simp add*: *algebra_simps chain_boundary_diff*)
obtain *singular_chain n* (*subtopology X* (*S* − *U*)) *c2 singular_chain n* (*subtopology X T*) *c2*
using *singular_chain_diff c c1* ∗

```
        unfolding c1_def c2_def
       by (metis add_diff_cancel_left' e g hSuc singular_chain_boundary_alt)
       then show singular_chain n (subtopology (subtopology X (S − U)) (T
− U)) c2
        by (fastforce simp add: singular_chain_subtopology)
     qed
    qed
    then have singular_relboundary n (subtopology X S) T (c − d) ⟹
             singular_relboundary n (subtopology X (S − U)) (T − U) (c − d)
     using c d singular_relcycle_diff by metis
    with hh show homologous_rel n (subtopology X (S − U)) (T − U) c d
     apply (simp add: hom_induced_chain_map cont c d chain_map_ident [OF
scc] chain_map_ident [OF scd])
     using homologous_rel_set_eq homologous_rel_def by metis
  qed
 next
  have h: homologous_rel_set n (subtopology X S) T a
       ∈ (λx. homologous_rel_set n (subtopology X S) T (chain_map n id x)) `
        singular_relcycle_set n (subtopology X (S − U)) (T − U)
    if a: singular_relcycle n (subtopology X S) T a for a
  proof −
    obtain c' where c': singular_relcycle n (subtopology X (S − U)) (T − U)
c'
                   homologous_rel n (subtopology X S) T a c'
     using a by (blast intro: excised_relcycle_exists [OF assms])
    then have scc': singular_chain n (subtopology X S) c'
     using homologous_rel_singular_chain that
     by (force simp: singular_relcycle)
    then show ?thesis
    using scc' chain_map_ident [of _ subtopology X S] c' homologous_rel_set_eq
     by fastforce
  qed
  have (λx. homologous_rel_set n (subtopology X S) T (chain_map n id x)) `
       singular_relcycle_set n (subtopology X (S − U)) (T − U) =
       homologous_rel_set n (subtopology X S) T `
       singular_relcycle_set n (subtopology X S) T
    by (force simp: cont h singular_relcycle_chain_map)
  then
  show hom_induced n (subtopology X (S − U)) (T − U) (subtopology X S) T
id `
       homologous_rel_set n (subtopology X (S − U)) (T − U) `
       singular_relcycle_set n (subtopology X (S − U)) (T − U)
       = homologous_rel_set n (subtopology X S) T ` singular_relcycle_set n
(subtopology X S) T
    by (simp add: image_comp o_def hom_induced_chain_map_gen cont TU
                 cong: image_cong_simp)
 qed
qed
```

0.2.3 Additivity axiom

Not in the original Eilenberg-Steenrod list but usually included nowadays, following Milnor's "On Axiomatic Homology Theory".

lemma *iso_chain_group_sum*:
  assumes *disj*: *pairwise disjnt* $\mathcal{U}$ **and** *UU*: $\bigcup\mathcal{U}$ = *topspace X*
    and *subs*: $\bigwedge$*C T*. ⟦*compactin X C*; *path_connectedin X C*; *T* ∈ $\mathcal{U}$; ~ *disjnt C T*⟧ ⟹ *C* ⊆ *T*
  shows (λ*f*. *sum′ f* $\mathcal{U}$) ∈ *iso* (*sum_group* $\mathcal{U}$ (λ*S*. *chain_group p* (*subtopology X S*))) (*chain_group p X*)
proof −
  have *pw*: *pairwise* (λ*i j*. *disjnt* (*singular_simplex_set p* (*subtopology X i*))
                    (*singular_simplex_set p* (*subtopology X j*))) $\mathcal{U}$
  proof
    fix *S T*
    assume *S* ∈ $\mathcal{U}$ *T* ∈ $\mathcal{U}$ *S* ≠ *T*
    then show *disjnt* (*singular_simplex_set p* (*subtopology X S*))
             (*singular_simplex_set p* (*subtopology X T*))
      using *nonempty_standard_simplex* [*of p*] *disj*
        by (*fastforce simp*: *pairwise_def disjnt_def singular_simplex_subtopology image_subset_iff*)
  qed
  have ∃ *S*∈$\mathcal{U}$. *singular_simplex p* (*subtopology X S*) *f*
    if *f*: *singular_simplex p X f* **for** *f*
  proof −
    obtain *x* **where** *x*: *x* ∈ *topspace X x* ∈ *f* ‘ *standard_simplex p*
     using *f nonempty_standard_simplex* [*of p*] *continuous_map_image_subset_topspace*
      unfolding *singular_simplex_def* **by** *fastforce*
    then obtain *S* **where** *S* ∈ $\mathcal{U}$ *x* ∈ *S*
      using *UU* **by** *auto*
    have *f* ‘ *standard_simplex p* ⊆ *S*
    proof (*rule subs*)
      have *cont*: *continuous_map* (*subtopology* (*powertop_real UNIV*)
                    (*standard_simplex p*)) *X f*
        using *f singular_simplex_def* **by** *auto*
      show *compactin X* (*f* ‘ *standard_simplex p*)
          by (*simp add*: *compactin_subtopology compactin_standard_simplex image_compactin* [*OF* _ *cont*])
      show *path_connectedin X* (*f* ‘ *standard_simplex p*)
      by (*simp add*: *path_connectedin_subtopology path_connectedin_standard_simplex path_connectedin_continuous_map_image* [*OF cont*])
      have *standard_simplex p* ≠ {}
        by (*simp add*: *nonempty_standard_simplex*)
      then
      show ¬ *disjnt* (*f* ‘ *standard_simplex p*) *S*
        using *x* ‹*x* ∈ *S*› **by** (*auto simp*: *disjnt_def*)
    qed (*auto simp*: ‹*S* ∈ $\mathcal{U}$›)
    then show *?thesis*
      by (*meson* ‹*S* ∈ $\mathcal{U}$› *singular_simplex_subtopology that*)

```
  qed
 then have (⋃ i∈𝒰. singular_simplex_set p (subtopology X i)) = singular_simplex_set
p X
    by (auto simp: singular_simplex_subtopology)
  then show ?thesis
   using iso_free_Abelian_group_sum [OF pw] by (simp add: chain_group_def)
qed

lemma relcycle_group_0_eq_chain_group: relcycle_group 0 X {} = chain_group
0 X
proof (rule monoid.equality)
  show carrier (relcycle_group 0 X {}) = carrier (chain_group 0 X)
   by (simp add: Collect_mono chain_boundary_def singular_cycle subset_antisym)
qed (simp_all add: relcycle_group_def chain_group_def)

proposition iso_cycle_group_sum:
  assumes disj: pairwise disjnt 𝒰 and UU: ⋃𝒰 = topspace X
    and subs: ⋀C T. ⟦compactin X C; path_connectedin X C; T ∈ 𝒰; ¬ disjnt C
T⟧ ⟹ C ⊆ T
  shows (λf. sum' f 𝒰) ∈ iso (sum_group 𝒰 (λT. relcycle_group p (subtopology
X T) {}))
                              (relcycle_group p X {})
proof (cases p = 0)
  case True
  then show ?thesis
    by (simp add: relcycle_group_0_eq_chain_group iso_chain_group_sum [OF
assms])
next
  case False
  let ?SG = (sum_group 𝒰 (λT. chain_group p (subtopology X T)))
  let ?PI = (Π_E T∈𝒰. singular_relcycle_set p (subtopology X T) {})
 have (λf. sum' f 𝒰) ∈ Group.iso (subgroup_generated ?SG (carrier ?SG ∩ ?PI))
                        (subgroup_generated (chain_group p X) (singular_relcycle_set
p X {}))
  proof (rule group_hom.iso_between_subgroups)
    have iso: (λf. sum' f 𝒰) ∈ Group.iso ?SG (chain_group p X)
      by (auto simp: assms iso_chain_group_sum)
    then show group_hom ?SG (chain_group p X) (λf. sum' f 𝒰)
     by (auto simp: iso_imp_homomorphism group_hom_def group_hom_axioms_def)
     have B: sum' f 𝒰 ∈ singular_relcycle_set p X {} ⟷ f ∈ (carrier ?SG ∩
?PI)
      if f ∈ (carrier ?SG) for f
    proof −
      have f: ⋀S. S ∈ 𝒰 ⟶ singular_chain p (subtopology X S) (f S)
              f ∈ extensional 𝒰 finite {i ∈ 𝒰. f i ≠ 0}
        using that by (auto simp: carrier_sum_group PiE_def Pi_def)
      then have rfin: finite {S ∈ 𝒰. restrict (chain_boundary p ∘ f) 𝒰 S ≠ 0}
        by (auto elim: rev_finite_subset)
      have chain_boundary p ((∑x | x ∈ 𝒰 ∧ f x ≠ 0. f x)) = 0
```

$\longleftrightarrow$ ($\forall$ $S \in \mathcal{U}$. *chain_boundary p* (*f S*) = *0*) (**is** *?cb* = *0* $\longleftrightarrow$ *?rhs*)
proof
assume *?cb* = *0*
moreover have *?cb* = *sum′* (λS. *chain_boundary p* (*f S*)) $\mathcal{U}$
unfolding *sum.G_def* **using** *rfin f*
by (*force simp*: *chain_boundary_sum intro*: *sum.mono_neutral_right cong*: *conj_cong*)
ultimately have *eq0*: *sum′* (λS. *chain_boundary p* (*f S*)) $\mathcal{U}$ = *0*
by *simp*
have (λf. *sum′ f* $\mathcal{U}$) $\in$ *hom* (*sum_group* $\mathcal{U}$ (λS. *chain_group* (*p* − *Suc 0*) (*subtopology X S*)))
(*chain_group* (*p* − *Suc 0*) *X*)
and *inj*: *inj_on* (λf. *sum′ f* $\mathcal{U}$) (*carrier* (*sum_group* $\mathcal{U}$ (λS. *chain_group* (*p* − *Suc 0*) (*subtopology X S*))))
using *iso_chain_group_sum* [*OF assms*, *of p*−*1*] **by** (*auto simp*: *iso_def bij_betw_def*)
then have *eq*: [[*f* $\in$ (Π_E *i*$\in\mathcal{U}$. *singular_chain_set* (*p* − *Suc 0*) (*subtopology X i*));
finite {$S \in \mathcal{U}$. *f S* $\neq$ *0*}; *sum′ f* $\mathcal{U}$ = *0*; $S \in \mathcal{U}$]] $\Longrightarrow$ *f S* = *0* **for** *f S*
apply (*simp add*: *group_hom_def group_hom_axioms_def group_hom.inj_on_one_iff* [*of _ chain_group* (*p*−*1*) *X*])
apply (*auto simp*: *carrier_sum_group fun_eq_iff that*)
done
show *?rhs*
proof *clarify*
fix *S* **assume** $S \in \mathcal{U}$
then show *chain_boundary p* (*f S*) = *0*
using *eq* [*of restrict* (*chain_boundary p* $\circ$ *f*) $\mathcal{U}$ *S*] *rfin f eq0*
by (*simp add*: *singular_chain_boundary cong*: *conj_cong*)
qed
next
assume *?rhs*
then show *?cb* = *0*
by (*force simp*: *chain_boundary_sum intro*: *sum.mono_neutral_right*)
qed
moreover
have ($\bigwedge S$. $S \in \mathcal{U}$ $\longrightarrow$ *singular_chain p* (*subtopology X S*) (*f S*))
$\Longrightarrow$ *singular_chain p X* ($\sum x$ | $x \in \mathcal{U} \wedge f\,x \neq 0$. *f x*)
by (*metis* (*no_types*, *lifting*) *mem_Collect_eq singular_chain_subtopology singular_chain_sum*)
ultimately show *?thesis*
using *f* **by** (*auto simp*: *carrier_sum_group sum.G_def singular_cycle PiE_iff*)
qed
have *singular_relcycle_set p X* {} $\subseteq$ *carrier* (*chain_group p X*)
using *subgroup.subset subgroup_singular_relcycle* **by** *blast*
then show (λf. *sum′ f* $\mathcal{U}$) ‘ (*carrier ?SG* $\cap$ *?PI*) = *singular_relcycle_set p X* {}
using *iso B* **unfolding** *Group.iso_def*

```
      by (smt (verit, del_insts) Int_iff bij_betw_def image_iff mem_Collect_eq
subset_antisym subset_iff)
  qed (auto simp: assms iso_chain_group_sum)
  then show ?thesis
   by (simp add: relcycle_group_def sum_group_subgroup_generated subgroup_singular_relcycle)
qed


proposition homology_additivity_axiom_gen:
  assumes disj: pairwise disjnt 𝒰 and UU: ⋃𝒰 = topspace X
    and subs: ⋀C T. ⟦compactin X C; path_connectedin X C; T ∈ 𝒰; ¬ disjnt C
T⟧ ⟹ C ⊆ T
  shows (λx. gfinprod (homology_group p X)
                      (λV. hom_induced p (subtopology X V) {} X {} id (x V)) 𝒰)
    ∈ iso (sum_group 𝒰 (λS. homology_group p (subtopology X S))) (homology_group
p X)
     (is  ?h ∈ iso ?SG ?HG)
proof (cases p < 0)
  case True
   then have [simp]: gfinprod (singleton_group undefined) (λv. undefined) 𝒰 =
undefined
   by (metis Pi_I carrier_singleton_group comm_group_def comm_monoid.gfinprod_closed
singletonD singleton_abelian_group)
  show ?thesis
    using True
    apply (simp add: iso_def relative_homology_group_def hom_induced_trivial
carrier_sum_group)
    apply (auto simp: singleton_group_def bij_betw_def inj_on_def fun_eq_iff)
    done
next
  case False
  then obtain n where peq: p = int n
    by (metis int_ops(1) linorder_neqE_linordered_idom pos_int_cases)
  interpret comm_group homology_group p X
    by (rule abelian_homology_group)
  show ?thesis
  proof (simp add: iso_def bij_betw_def, intro conjI)
    show ?h ∈ hom ?SG ?HG
      by (rule hom_group_sum) (simp_all add: hom_induced_hom)
    then interpret group_hom ?SG ?HG ?h
      by (simp add: group_hom_def group_hom_axioms_def)
    have carrSG: carrier ?SG
        = (λx. λS∈𝒰. homologous_rel_set n (subtopology X S) {} (x S))
          ‘ (carrier (sum_group 𝒰 (λS. relcycle_group n (subtopology X S) {}))) (is
?lhs = ?rhs)
    proof
      show ?lhs ⊆ ?rhs
      proof (clarsimp simp: carrier_sum_group carrier_relative_homology_group
peq)
```

fix z
assume z: z ∈ (Π_E S∈𝒰. homologous_rel_set n (subtopology X S) {} ‘ singular_relcycle_set n (subtopology X S) {})
and fin: finite {S ∈ 𝒰. z S ≠ singular_relboundary_set n (subtopology X S) {}}
then obtain c **where** c: ∀ S∈𝒰. singular_relcycle n (subtopology X S) {} (c S)
∧ z S = homologous_rel_set n (subtopology X S) {} (c S)
by (simp add: PiE_def Pi_def image_def) metis
let ?f = λS∈𝒰. if singular_relboundary n (subtopology X S) {} (c S) then 0 else c S
have z = (λS∈𝒰. homologous_rel_set n (subtopology X S) {} (?f S))
by (smt (verit) PiE_restrict c homologous_rel_eq_relboundary restrict_apply restrict_ext singular_relboundary_0 z)
moreover have ?f ∈ (Π_E i∈𝒰. singular_relcycle_set n (subtopology X i) {})
by (simp add: c fun_eq_iff PiE_arb [OF z])
moreover have finite {i ∈ 𝒰. ?f i ≠ 0}
using z c **by** (intro finite_subset [OF _ fin]) auto
ultimately
show z ∈ (λx. λS∈𝒰. homologous_rel_set n (subtopology X S) {} (x S)) ‘
{x ∈ Π_E i∈𝒰. singular_relcycle_set n (subtopology X i) {}. finite {i ∈ 𝒰. x i ≠ 0}}
by blast
qed
show ?rhs ⊆ ?lhs
by (force simp: peq carrier_sum_group carrier_relative_homology_group homologous_rel_set_eq_relboundary
elim: rev_finite_subset)
qed
have gf: gfinprod (homology_group p X)
(λV. hom_induced n (subtopology X V) {} X {} id
((λS∈𝒰. homologous_rel_set n (subtopology X S) {} (z S)) V)) 𝒰
= homologous_rel_set n X {} (sum' z 𝒰) (**is** ?lhs = ?rhs)
if z: z ∈ carrier (sum_group 𝒰 (λS. relcycle_group n (subtopology X S) {}))
for z
proof −
have hom_pi: (λS. homologous_rel_set n X {} (z S)) ∈ 𝒰 → carrier (homology_group p X)
using z
by (intro Pi_I) (force simp: peq carrier_sum_group carrier_relative_homology_group singular_chain_subtopology singular_cycle)
have fin: finite {S ∈ 𝒰. z S ≠ 0}
using that **by** (force simp: carrier_sum_group)
have ?lhs = gfinprod (homology_group p X) (λS. homologous_rel_set n X {} (z S)) 𝒰
proof (rule gfinprod_cong [OF refl Pi_I])
fix i

show *i* ∈ 𝒰 *=simp=> hom_induced* (*int n*) (*subtopology X i*) {} *X* {} *id*
((λ*S*∈𝒰. *homologous_rel_set n* (*subtopology X S*) {} (*z S*)) *i*)
= *homologous_rel_set n X* {} (*z i*)
using *that*
by (*auto simp*: *peq simp_implies_def carrier_sum_group PiE_def Pi_def chain_map_ident singular_cycle hom_induced_chain_map*)
qed (*simp add*: *hom_induced_carrier peq*)
also have ... = *gfinprod* (*homology_group p X*)
(λ*S*. *homologous_rel_set n X* {} (*z S*)) {*S* ∈ 𝒰. *z S* ≠ *0*}
proof −
have *homologous_rel_set n X* {} *0* = *singular_relboundary_set n X* {}
by (*metis homologous_rel_eq_relboundary singular_relboundary_0*)
with *hom_pi peq* **show** *?thesis*
by (*intro gfinprod_mono_neutral_cong_right*) *auto*
qed
also have ... = *?rhs*
proof −
have *gfinprod* (*homology_group p X*) (λ*S*. *homologous_rel_set n X* {} (*z S*)) ℱ
= *homologous_rel_set n X* {} (*sum z* ℱ)
if *finite* ℱ ℱ ⊆ {*S* ∈ 𝒰. *z S* ≠ *0*} **for** ℱ
using *that*
proof (*induction* ℱ)
case *empty*
have $\mathbf{1}_{homology\_group\ p\ X}$ = *homologous_rel_set n X* {} *0*
by (*metis homologous_rel_eq_relboundary one_relative_homology_group peq singular_relboundary_0*)
then show *?case*
by *simp*
next
case (*insert S* ℱ)
with *z* **have** *pi*: (λ*S*. *homologous_rel_set n X* {} (*z S*)) ∈ ℱ → *carrier* (*homology_group p X*)
homologous_rel_set n X {} (*z S*) ∈ *carrier* (*homology_group p X*)
by (*force simp*: *peq carrier_sum_group carrier_relative_homology_group singular_chain_subtopology singular_cycle*)+
have *hom*: *homologous_rel_set n X* {} (*z S*) ∈ *carrier* (*homology_group p X*)
using *insert z*
by (*force simp*: *peq carrier_sum_group carrier_relative_homology_group singular_chain_subtopology singular_cycle*)
show *?case*
using *insert z*
proof (*simp add*: *pi*)
have ⋀*x*. *homologous_rel n X* {} (*z S* + *sum z* ℱ) *x*
⟹ ∃ *u v*. *homologous_rel n X* {} (*z S*) *u* ∧ *homologous_rel n X* {} (*sum z* ℱ) *v* ∧ *x* = *u* + *v*
by (*metis* (*no_types, lifting*) *diff_add_cancel diff_diff_eq2 homologous_rel_def homologous_rel_refl*)

with *insert z*
show *homologous_rel_set n X* {} (*z S*) $\otimes_{homology\_group\ p\ X}$ *homologous_rel_set n X* {} (*sum z* $\mathcal{F}$)
= *homologous_rel_set n X* {} (*z S* + *sum z* $\mathcal{F}$)
using *insert z* **by** (*auto simp*: *peq homologous_rel_add mult_relative_homology_group*)
qed
qed
with *fin* **show** *?thesis*
by (*simp add*: *sum.G_def*)
qed
finally show *?thesis* **.**
qed
show *inj_on ?h* (*carrier ?SG*)
proof (*clarsimp simp add*: *inj_on_one_iff*)
fix *x*
assume *x*: *x* ∈ *carrier* (*sum_group* $\mathcal{U}$ (λ*S*. *homology_group p* (*subtopology X S*)))
and *1*: *gfinprod* (*homology_group p X*) (λ*V*. *hom_induced p* (*subtopology X V*) {} *X* {} *id* (*x V*)) $\mathcal{U}$
= $\mathbf{1}_{homology\_group\ p\ X}$
have *feq*: (λ*S*∈$\mathcal{U}$. *homologous_rel_set n* (*subtopology X S*) {} (*z S*))
= (λ*S*∈$\mathcal{U}$. $\mathbf{1}_{homology\_group\ p\ (subtopology\ X\ S)}$)
if *z*: *z* ∈ *carrier* (*sum_group* $\mathcal{U}$ (λ*S*. *relcycle_group n* (*subtopology X S*) {}))
and *eq*: *homologous_rel_set n X* {} (*sum′ z* $\mathcal{U}$) = $\mathbf{1}_{homology\_group\ p\ X}$
for *z*
proof −
have *z* ∈ (Π_E *S*∈$\mathcal{U}$. *singular_relcycle_set n* (*subtopology X S*) {}) *finite* {*S* ∈ $\mathcal{U}$. *z S* ≠ *0*}
using *z* **by** (*auto simp*: *carrier_sum_group*)
have *singular_relboundary n X* {} (*sum′ z* $\mathcal{U}$)
using *eq singular_chain_imp_relboundary* **by** (*auto simp*: *relative_homology_group_def peq*)
then obtain *d* **where** *scd*: *singular_chain* (*Suc n*) *X d* **and** *cbd*: *chain_boundary* (*Suc n*) *d* = *sum′ z* $\mathcal{U}$
by (*auto simp*: *singular_boundary*)
have ∗: ∃ *d*. *singular_chain* (*Suc n*) (*subtopology X S*) *d* ∧ *chain_boundary* (*Suc n*) *d* = *z S*
if *S* ∈ $\mathcal{U}$ **for** *S*
proof −
have *inj′*: *inj_on* (λ*f*. *sum′ f* $\mathcal{U}$) {*x* ∈ Π_E *S*∈$\mathcal{U}$. *singular_chain_set* (*Suc n*) (*subtopology X S*). *finite* {*S* ∈ $\mathcal{U}$. *x S* ≠ *0*}}
using *iso_chain_group_sum* [*OF assms*, *of Suc n*]
by (*simp add*: *iso_iff_mon_epi mon_def carrier_sum_group*)
obtain *w* **where** *w*: *w* ∈ (Π_E *S*∈$\mathcal{U}$. *singular_chain_set* (*Suc n*) (*subtopology X S*))
and *finw*: *finite* {*S* ∈ $\mathcal{U}$. *w S* ≠ *0*}
and *deq*: *d* = *sum′ w* $\mathcal{U}$
using *iso_chain_group_sum* [*OF assms*, *of Suc n*] *scd*
by (*auto simp*: *iso_iff_mon_epi epi_def carrier_sum_group set_eq_iff*)

with ‹*S* ∈ 𝒰› **have** *scwS*: *singular_chain* (*Suc n*) (*subtopology X S*) (*w S*)
by *blast*
have *inj_on* (λ*f*. *sum′ f* 𝒰) {*x* ∈ Π$_E$ *S*∈𝒰. *singular_chain_set n* (*subtopology X S*). *finite* {*S* ∈ 𝒰. *x S* ≠ *0*}}
using *iso_chain_group_sum* [*OF assms*, *of n*]
by (*simp add*: *iso_iff_mon_epi mon_def carrier_sum_group*)
then have (λ*S*∈𝒰. *chain_boundary* (*Suc n*) (*w S*)) = *z*
proof (*rule inj_onD*)
have *sum′* (λ*S*∈𝒰. *chain_boundary* (*Suc n*) (*w S*)) 𝒰 = *sum′* (*chain_boundary* (*Suc n*) ∘ *w*) {*S* ∈ 𝒰. *w S* ≠ *0*}
by (*auto simp*: *o_def intro*: *sum.mono_neutral_right′*)
also have ... = *chain_boundary* (*Suc n*) *d*
by (*auto simp*: *sum.G_def deq chain_boundary_sum finw intro*: *finite_subset* [*OF _ finw*] *sum.mono_neutral_left*)
finally show *sum′* (λ*S*∈𝒰. *chain_boundary* (*Suc n*) (*w S*)) 𝒰 = *sum′ z* 𝒰
by (*simp add*: *cbd*)
show (λ*S*∈𝒰. *chain_boundary* (*Suc n*) (*w S*)) ∈ {*x* ∈ Π$_E$ *S*∈𝒰. *singular_chain_set n* (*subtopology X S*). *finite* {*S* ∈ 𝒰. *x S* ≠ *0*}}
using *w* **by** (*auto simp*: *PiE_iff singular_chain_boundary_alt cong*: *rev_conj_cong intro*: *finite_subset* [*OF _ finw*])
show *z* ∈ {*x* ∈ Π$_E$ *S*∈𝒰. *singular_chain_set n* (*subtopology X S*). *finite* {*S* ∈ 𝒰. *x S* ≠ *0*}}
using *z* **by** (*simp_all add*: *carrier_sum_group PiE_iff singular_cycle*)
qed
with ‹*S* ∈ 𝒰› *scwS* **show** *?thesis*
by *force*
qed
show *?thesis*
using *that* ∗
by (*force intro*!: *restrict_ext simp add*: *singular_boundary relative_homology_group_def homologous_rel_set_eq_relboundary peq*)
qed
show *x* = (λ*S*∈𝒰. $\mathbf{1}_{homology\_group\ p\ (subtopology\ X\ S)}$)
using *x 1 carrSG gf*
by (*auto simp*: *peq feq*)
qed
show *?h* ‘ *carrier ?SG* = *carrier ?HG*
proof *safe*
fix *A*
assume *A* ∈ *carrier* (*homology_group p X*)
then obtain *y* **where** *y*: *singular_relcycle n X* {} *y* **and** *xeq*: *A* = *homologous_rel_set n X* {} *y*
by (*auto simp*: *peq carrier_relative_homology_group*)
then obtain *x* **where** *x* ∈ *carrier* (*sum_group* 𝒰 (λ*T*. *relcycle_group n* (*subtopology X T*) {}))
y = *sum′ x* 𝒰
using *iso_cycle_group_sum* [*OF assms*, *of n*] *that* **by** (*force simp*: *iso_iff_mon_epi epi_def*)

then show $A \in (\lambda x.\ gfinprod\ (homology\_group\ p\ X)\ (\lambda V.\ hom\_induced\ p\ (subtopology\ X\ V)\ \{\}\ X\ \{\}\ id\ (x\ V))\ \mathcal{U})$ ‘
$carrier\ (sum\_group\ \mathcal{U}\ (\lambda S.\ homology\_group\ p\ (subtopology\ X\ S)))$
apply (*simp add*: *carrSG image_comp o_def xeq*)
apply (*simp add*: *hom_induced_carrier peq flip*: *gf cong*: *gfinprod_cong*)
done
qed *auto*
qed
qed

corollary *homology_additivity_axiom*:
assumes *disj*: *pairwise disjnt* $\mathcal{U}$ **and** *UU*: $\bigcup\mathcal{U} = topspace\ X$
and *ope*: $\bigwedge v.\ v \in \mathcal{U} \Longrightarrow openin\ X\ v$
shows $(\lambda x.\ gfinprod\ (homology\_group\ p\ X)$
$(\lambda v.\ hom\_induced\ p\ (subtopology\ X\ v)\ \{\}\ X\ \{\}\ id\ (x\ v))\ \mathcal{U})$
$\in iso\ (sum\_group\ \mathcal{U}\ (\lambda S.\ homology\_group\ p\ (subtopology\ X\ S)))\ (homology\_group\ p\ X)$
proof (*rule homology_additivity_axiom_gen* [*OF disj UU*])
fix *C T*
assume
compactin X C **and**
path_connectedin X C **and**
$T \in \mathcal{U}$ **and**
$\neg$ *disjnt C T*
then have $*$: $\bigwedge B.\ [\![openin\ X\ T;\ T \cap B \cap C = \{\};\ C \subseteq T \cup B;\ openin\ X\ B]\!] \Longrightarrow B \cap C = \{\}$
by (*meson connectedin disjnt_def disjnt_sym path_connectedin_imp_connectedin*)
have $C \subseteq Union\ \mathcal{U}$
by (*simp add*: *UU* ‹*compactin X C*› *compactin_subset_topspace*)
moreover have $\bigcup\ (\mathcal{U} - \{T\}) \cap C = \{\}$
proof (*rule* $*$)
show $T \cap \bigcup\ (\mathcal{U} - \{T\}) \cap C = \{\}$
using ‹$T \in \mathcal{U}$› *disj disjointD* **by** *fastforce*
show $C \subseteq T \cup \bigcup\ (\mathcal{U} - \{T\})$
using ‹$C \subseteq \bigcup\ \mathcal{U}$› **by** *fastforce*
qed (*auto simp*: ‹$T \in \mathcal{U}$› *ope*)
ultimately show $C \subseteq T$
by *blast*
qed

0.2.4 Special properties of singular homology

In particular: the zeroth homology group is isomorphic to the free abelian group generated by the path components. So, the "coefficient group" is the integers.

lemma *iso_integer_zeroth_homology_group_aux*:
assumes *X*: *path_connected_space X* **and** *f*: *singular_simplex 0 X f* **and** f': *singular_simplex 0 X* f'

```
  shows homologous_rel 0 X {} (frag_of f) (frag_of f')
proof -
  let ?p = λj. if j = 0 then 1 else 0
  have f ?p ∈ topspace X f' ?p ∈ topspace X
  using assms by (auto simp: singular_simplex_def continuous_map_def)
  then obtain g where g: pathin X g
                and g0: g 0 = f ?p
                and g1: g 1 = f' ?p
    using assms by (force simp: path_connected_space_def)
  then have contg: continuous_map (subtopology euclideanreal {0..1}) X g
    by (simp add: pathin_def)
 have singular_chain (Suc 0) X (frag_of (restrict (g ∘ (λx. x 0)) (standard_simplex
1)))
  proof -
    have continuous_map (subtopology (powertop_real UNIV) (standard_simplex
(Suc 0)))
                        euclideanreal (λx. x 0)
      by (metis (mono_tags) UNIV_I continuous_map_from_subtopology contin-
uous_map_product_projection)
   then have continuous_map (subtopology (powertop_real UNIV) (standard_simplex
(Suc 0)))
                             (top_of_set {0..1}) (λx. x 0)
      unfolding continuous_map_in_subtopology g
      by (auto simp: continuous_map_in_subtopology standard_simplex_def g)
    moreover have continuous_map (top_of_set {0..1}) X g
      using contg by blast
    ultimately show ?thesis
      by (force simp: singular_chain_of chain_boundary_of singular_simplex_def
continuous_map_compose)
  qed
  moreover
 have chain_boundary (Suc 0) (frag_of (restrict (g ∘ (λx. x 0)) (standard_simplex
1))) =
      frag_of f - frag_of f'
  proof -
    have singular_face (Suc 0) 0 (g ∘ (λx. x 0)) = f
         singular_face (Suc 0) (Suc 0) (g ∘ (λx. x 0)) = f'
      using assms
     by (auto simp: singular_face_def singular_simplex_def extensional_def sim-
plical_face_def standard_simplex_0 g0 g1)
    then show ?thesis
      by (simp add: singular_chain_of chain_boundary_of)
  qed
  ultimately
  show ?thesis
    by (auto simp: homologous_rel_def singular_boundary)
qed
```

proposition *iso_integer_zeroth_homology_group*:
  assumes *X*: *path_connected_space X* **and** *f*: *singular_simplex 0 X f*
  shows *pow* (*homology_group 0 X*) (*homologous_rel_set 0 X* {} (*frag_of f*))
      ∈ *iso integer_group* (*homology_group 0 X*) (**is** *pow ?H ?q* ∈ *iso _ ?H*)
proof −
  have *srf*: *singular_relcycle 0 X* {} (*frag_of f*)
    by (*simp add*: *chain_boundary_def f singular_chain_of singular_cycle*)
  then have *qcarr*: *?q* ∈ *carrier ?H*
    by (*simp add*: *carrier_relative_homology_group_0*)
  have *1*: *homologous_rel_set 0 X* {} *a* ∈ *range* (λ*n*. *homologous_rel_set 0 X* {}
(*frag_cmul n* (*frag_of f*)))
    if *singular_relcycle 0 X* {} *a* **for** *a*
  proof −
    have *singular_chain 0 X d* ⟹
          homologous_rel_set 0 X {} *d* ∈ *range* (λ*n*. *homologous_rel_set 0 X* {}
(*frag_cmul n* (*frag_of f*))) **for** *d*
      unfolding *singular_chain_def*
    proof (*induction d rule*: *frag_induction*)
      case *zero*
      then show *?case*
        by (*metis frag_cmul_zero rangeI*)
    next
      case (*one x*)
      then have ∃*i*. *homologous_rel_set 0 X* {} (*frag_cmul i* (*frag_of f*))
              = *homologous_rel_set 0 X* {} (*frag_of x*)
        by (*metis* (*no_types*) *iso_integer_zeroth_homology_group_aux* [*OF X*] *f*
frag_cmul_one homologous_rel_eq mem_Collect_eq)
      with *one* **show** *?case*
        by *auto*
    next
      case (*diff a b*)
      then obtain *c d* **where**
        homologous_rel 0 X {} (*a* − *b*) (*frag_cmul c* (*frag_of f*) − *frag_cmul d*
(*frag_of f*))
       using *homologous_rel_diff* **by** (*fastforce simp add*: *homologous_rel_set_eq*)
      then show *?case*
        by (*rule_tac x*=*c*−*d* **in** *image_eqI*) (*auto simp*: *homologous_rel_set_eq*
frag_cmul_diff_distrib)
    qed
    with *that* **show** *?thesis*
      unfolding *singular_relcycle_def* **by** *blast*
  qed
  have *2*: *n* = *0*
   if *homologous_rel_set 0 X* {} (*frag_cmul n* (*frag_of f*)) = $\mathbf{1}_{relative\_homology\_group\ 0\ X\ \{\}}$
    for *n*
  proof −
    have *singular_chain* (*Suc 0*) *X d*
        ⟹ *frag_extend* (λ*x*. *frag_of f*) (*chain_boundary* (*Suc 0*) *d*) = *0* **for** *d*
      unfolding *singular_chain_def*

```
    proof (induction d rule: frag_induction)
      case (one x)
      then show ?case
        by (simp add: frag_extend_diff chain_boundary_of)
    next
      case (diff a b)
      then show ?case
        by (simp add: chain_boundary_diff frag_extend_diff)
    qed auto
    with that show ?thesis
        by (force simp: singular_boundary relative_homology_group_def homolo-
gous_rel_set_eq_relboundary frag_extend_cmul)
  qed
  interpret GH : group_hom integer_group ?H ([^]?H) ?q
   by (simp add: group_hom_def group_hom_axioms_def qcarr group.hom_integer_group_pow)
  have eq: pow ?H ?q = (λn. homologous_rel_set 0 X {} (frag_cmul n (frag_of
f)))
  proof
    fix n
    have frag_of f
          ∈ carrier (subgroup_generated
            (free_Abelian_group (singular_simplex_set 0 X)) (singular_relcycle_set
0 X {}))
        by (metis carrier_relcycle_group chain_group_def mem_Collect_eq relcy-
cle_group_def srf)
    then have ff: frag_of f [^]relcycle_group 0 X {} n = frag_cmul n (frag_of f)
     by (simp add: relcycle_group_def chain_group_def group.int_pow_subgroup_generated
f)
    show pow ?H ?q n = homologous_rel_set 0 X {} (frag_cmul n (frag_of f))
      apply (rule subst [OF right_coset_singular_relboundary])
     by (simp add: ff normal.FactGroup_int_pow normal_subgroup_singular_relboundary_relcycle
relative_homology_group_def srf)
  qed
  show ?thesis
    apply (subst GH.iso_iff)
    apply (simp add: eq)
    apply (auto simp: carrier_relative_homology_group_0 1 2)
    done
qed


corollary isomorphic_integer_zeroth_homology_group:
  assumes X: path_connected_space X topspace X ≠ {}
  shows homology_group 0 X ≅ integer_group
proof -
  obtain a where a: a ∈ topspace X
    using assms by blast
  have singular_simplex 0 X (restrict (λx. a) (standard_simplex 0))
    by (simp add: singular_simplex_def a)
```

then show *?thesis*
using *X group.iso_sym group_integer_group is_isoI iso_integer_zeroth_homology_group*
by *blast*
qed

corollary *homology_coefficients*:
topspace X = {*a*} ⟹ *homology_group 0 X* ≅ *integer_group*
using *isomorphic_integer_zeroth_homology_group path_connectedin_topspace*
by *fastforce*

proposition *zeroth_homology_group*:
homology_group 0 X ≅ *free_Abelian_group* (*path_components_of X*)
proof −
obtain *h* **where** *h*: *h* ∈ *iso* (*sum_group* (*path_components_of X*) (λ*S*. *homology_group 0* (*subtopology X S*)))
(*homology_group 0 X*)
proof (*rule that* [*OF homology_additivity_axiom_gen*])
show *disjoint* (*path_components_of X*)
by (*simp add*: *pairwise_disjoint_path_components_of*)
show ⋃(*path_components_of X*) = *topspace X*
by (*rule Union_path_components_of*)
next
fix *C T*
assume *path_connectedin X C T* ∈ *path_components_of X* ¬ *disjnt C T*
then show *C* ⊆ *T*
by (*metis path_components_of_maximal disjnt_sym*)+
qed
have *homology_group 0 X* ≅ *sum_group* (*path_components_of X*) (λ*S*. *homology_group 0* (*subtopology X S*))
by (*rule group.iso_sym*) (*use h is_iso_def* **in** *auto*)
also have ... ≅ *sum_group* (*path_components_of X*) (λ*i*. *integer_group*)
proof (*rule iso_sum_groupI*)
show *homology_group 0* (*subtopology X i*) ≅ *integer_group* **if** *i* ∈ *path_components_of X* **for** *i*
by (*metis that isomorphic_integer_zeroth_homology_group nonempty_path_components_of path_connectedin_def path_connectedin_path_components_of topspace_subtopology_subset*)
qed *auto*
also have ... ≅ *free_Abelian_group* (*path_components_of X*)
using *path_connectedin_path_components_of nonempty_path_components_of*
by (*simp add*: *isomorphic_sum_integer_group path_connectedin_def*)
finally show *?thesis* **.**
qed

lemma *isomorphic_homology_imp_path_components*:
assumes *homology_group 0 X* ≅ *homology_group 0 Y*
shows *path_components_of X* ≈ *path_components_of Y*
proof −

have *free_Abelian_group* (*path_components_of X*) ≅ *homology_group 0 X*
  by (*rule group.iso_sym*) (*auto simp*: *zeroth_homology_group*)
also have ... ≅ *homology_group 0 Y*
  by (*rule assms*)
also have ... ≅ *free_Abelian_group* (*path_components_of Y*)
  by (*rule zeroth_homology_group*)
finally have *free_Abelian_group* (*path_components_of X*) ≅ *free_Abelian_group* (*path_components_of Y*) **.**
then show *?thesis*
  by (*simp add*: *isomorphic_free_Abelian_groups*)
qed

lemma *isomorphic_homology_imp_path_connectedness*:
assumes *homology_group 0 X* ≅ *homology_group 0 Y*
shows *path_connected_space X* ⟷ *path_connected_space Y*
proof −
obtain *h* **where** *h*: *bij_betw h* (*path_components_of X*) (*path_components_of Y*)
  using *assms isomorphic_homology_imp_path_components eqpoll_def* **by** *blast*
have *1*: *path_components_of X* ⊆ {*a*} ⟹ *path_components_of Y* ⊆ {*h a*} **for** *a*
  using *h* **unfolding** *bij_betw_def* **by** *blast*
have *2*: *path_components_of Y* ⊆ {*a*}
    ⟹ *path_components_of X* ⊆ {*inv_into* (*path_components_of X*) *h a*} **for** *a*
  using *h* [*THEN bij_betw_inv_into*] **unfolding** *bij_betw_def* **by** *blast*
show *?thesis*
  unfolding *path_connected_space_iff_components_subset_singleton*
  by (*blast intro*: *dest*: *1 2*)
qed

0.2.5 More basic properties of homology groups, deduced from the E-S axioms

lemma *trivial_homology_group*:
p < *0* ⟹ *trivial_group*(*homology_group p X*)
by *simp*

lemma *hom_induced_empty_hom*:
(*hom_induced p X* {} *X'* {} *f*) ∈ *hom* (*homology_group p X*) (*homology_group p X'*)
by (*simp add*: *hom_induced_hom*)

lemma *hom_induced_compose_empty*:
⟦*continuous_map X Y f*; *continuous_map Y Z g*⟧
  ⟹ *hom_induced p X* {} *Z* {} (*g* ∘ *f*) = *hom_induced p Y* {} *Z* {} *g* ∘ *hom_induced p X* {} *Y* {} *f*
by (*simp add*: *hom_induced_compose*)

lemma *homology_homotopy_empty*:
homotopic_with ($\lambda h.$ *True*) X Y f g $\Longrightarrow$ *hom_induced* p X {} Y {} f = *hom_induced* p X {} Y {} g
by (*simp add*: *homology_homotopy_axiom*)

lemma *homotopy_equivalence_relative_homology_group_isomorphisms*:
assumes *contf*: *continuous_map* X Y f **and** *fim*: $f \in S \to T$
and *contg*: *continuous_map* Y X g **and** *gim*: $g \in T \to S$
and *gf*: *homotopic_with* ($\lambda h.$ $h \in S \to S$) X X ($g \circ f$) *id*
and *fg*: *homotopic_with* ($\lambda k.$ $k \in T \to T$) Y Y ($f \circ g$) *id*
shows *group_isomorphisms* (*relative_homology_group* p X S) (*relative_homology_group* p Y T)
(*hom_induced* p X S Y T f) (*hom_induced* p Y T X S g)
unfolding *group_isomorphisms_def*
proof (*intro conjI ballI*)
fix x
assume x: $x \in$ *carrier* (*relative_homology_group* p X S)
then show *hom_induced* p Y T X S g (*hom_induced* p X S Y T f x) = x
using *homology_homotopy_axiom* [*OF gf*, *of p*]
by (*simp add*: *contf contg fim gim hom_induced_compose′ hom_induced_id*)
next
fix y
assume $y \in$ *carrier* (*relative_homology_group* p Y T)
then show *hom_induced* p X S Y T f (*hom_induced* p Y T X S g y) = y
using *homology_homotopy_axiom* [*OF fg*, *of p*]
by (*simp add*: *contf contg fim gim hom_induced_compose′ hom_induced_id*)
qed (*auto simp*: *hom_induced_hom*)

lemma *homotopy_equivalence_relative_homology_group_isomorphism*:
assumes *continuous_map* X Y f **and** *fim*: $f \in S \to T$
and *continuous_map* Y X g **and** *gim*: $g \in T \to S$
and *homotopic_with* ($\lambda h.$ $h \in S \to S$) X X ($g \circ f$) *id*
and *homotopic_with* ($\lambda k.$ $k \in T \to T$) Y Y ($f \circ g$) *id*
shows (*hom_induced* p X S Y T f) $\in$ *iso* (*relative_homology_group* p X S) (*relative_homology_group* p Y T)
using *homotopy_equivalence_relative_homology_group_isomorphisms* [*OF assms*] *group_isomorphisms_imp_iso*
by *metis*

lemma *homotopy_equivalence_homology_group_isomorphism*:
assumes *continuous_map* X Y f
and *continuous_map* Y X g
and *homotopic_with* ($\lambda h.$ *True*) X X ($g \circ f$) *id*
and *homotopic_with* ($\lambda k.$ *True*) Y Y ($f \circ g$) *id*
shows (*hom_induced* p X {} Y {} f) $\in$ *iso* (*homology_group* p X) (*homology_group* p Y)
using *assms* **by** (*intro homotopy_equivalence_relative_homology_group_isomorphism*)

auto

lemma *homotopy_equivalent_space_imp_isomorphic_relative_homology_groups*:
  assumes *continuous_map X Y f* **and** *fim*: *f* ∈ *S* → *T*
    and *continuous_map Y X g* **and** *gim*: *g* ∈ *T* → *S*
    and *homotopic_with* (λ*h*. *h* ∈ *S* → *S*) *X X* (*g* ∘ *f*) *id*
    and *homotopic_with* (λ*k*. *k* ∈ *T* → *T*) *Y Y* (*f* ∘ *g*) *id*
   shows *relative_homology_group p X S* ≅ *relative_homology_group p Y T*
 using *homotopy_equivalence_relative_homology_group_isomorphism* [*OF assms*]
 unfolding *is_iso_def* **by** *blast*

lemma *homotopy_equivalent_space_imp_isomorphic_homology_groups*:
  X homotopy_equivalent_space Y ⟹ *homology_group p X* ≅ *homology_group p Y*
 unfolding *homotopy_equivalent_space_def*
 by (*auto intro*: *homotopy_equivalent_space_imp_isomorphic_relative_homology_groups*)

lemma *homeomorphic_space_imp_isomorphic_homology_groups*:
  X homeomorphic_space Y ⟹ *homology_group p X* ≅ *homology_group p Y*
 by (*simp add*: *homeomorphic_imp_homotopy_equivalent_space homotopy_equivalent_space_imp_isomorphic_ho*

lemma *trivial_relative_homology_group_gen*:
 assumes *continuous_map X* (*subtopology X S*) *f*
  homotopic_with (λ*h*. *True*) (*subtopology X S*) (*subtopology X S*) *f id*
  homotopic_with (λ*k*. *True*) *X X f id*
 shows *trivial_group*(*relative_homology_group p X S*)
proof (*rule exact_seq_imp_triviality*)
 show *exact_seq* ([*homology_group* (*p*−*1*) *X*,
   homology_group (*p*−*1*) (*subtopology X S*),
   relative_homology_group p X S, *homology_group p X*, *homology_group p* (*subtopology X S*)],
   [*hom_induced* (*p*−*1*) (*subtopology X S*) {} *X* {} *id*,
   hom_boundary p X S,
   hom_induced p X {} *X S id*,
   hom_induced p (*subtopology X S*) {} *X* {} *id*])
  using *homology_exactness_axiom_1 homology_exactness_axiom_2 homology_exactness_axiom_3*
  by (*metis exact_seq_cons_iff*)
next
 show *hom_induced p* (*subtopology X S*) {} *X* {} *id*
   ∈ *iso* (*homology_group p* (*subtopology X S*)) (*homology_group p X*)
  hom_induced (*p*−*1*) (*subtopology X S*) {} *X* {} *id*
   ∈ *iso* (*homology_group* (*p*−*1*) (*subtopology X S*)) (*homology_group* (*p*−*1*) *X*)
  using *assms*
  by (*auto intro*: *homotopy_equivalence_relative_homology_group_isomorphism*)
qed

lemma *trivial_relative_homology_group_topspace*:
trivial_group(*relative_homology_group p X* (*topspace X*))
by (*rule trivial_relative_homology_group_gen* [**where** *f*=*id*]) *auto*

lemma *trivial_relative_homology_group_empty*:
topspace X = {} ⟹ *trivial_group*(*relative_homology_group p X S*)
by (*metis Int_absorb2 empty_subsetI relative_homology_group_restrict trivial_relative_homology_group_topspace*)

lemma *trivial_homology_group_empty*:
topspace X = {} ⟹ *trivial_group*(*homology_group p X*)
by (*simp add*: *trivial_relative_homology_group_empty*)

lemma *homeomorphic_maps_relative_homology_group_isomorphisms*:
assumes *homeomorphic_maps X Y f g* **and** *im*: *f* ∈ *S* → *T g* ∈ *T* → *S*
shows *group_isomorphisms* (*relative_homology_group p X S*) (*relative_homology_group p Y T*)
(*hom_induced p X S Y T f*) (*hom_induced p Y T X S g*)
proof −
have *fg*: *continuous_map X Y f continuous_map Y X g*
(∀ *x*∈*topspace X*. *g* (*f x*) = *x*) (∀ *y*∈*topspace Y*. *f* (*g y*) = *y*)
using *assms* **by** (*simp_all add*: *homeomorphic_maps_def*)
have *group_isomorphisms*
(*relative_homology_group p X* (*topspace X* ∩ *S*))
(*relative_homology_group p Y* (*topspace Y* ∩ *T*))
(*hom_induced p X* (*topspace X* ∩ *S*) *Y* (*topspace Y* ∩ *T*) *f*)
(*hom_induced p Y* (*topspace Y* ∩ *T*) *X* (*topspace X* ∩ *S*) *g*)
proof (*rule homotopy_equivalence_relative_homology_group_isomorphisms*)
show *homotopic_with* (λ*h*. *h* ∈ (*topspace X* ∩ *S*) → *topspace X* ∩ *S*) *X X* (*g* ∘ *f*) *id*
using *fg im* **by** (*auto intro*: *homotopic_with_equal continuous_map_compose*)
next
show *homotopic_with* (λ*k*. *k* ∈ (*topspace Y* ∩ *T*) → *topspace Y* ∩ *T*) *Y Y* (*f* ∘ *g*) *id*
using *fg im* **by** (*auto intro*: *homotopic_with_equal continuous_map_compose*)
qed (*use im fg* **in** ‹*auto simp*: *continuous_map_def*›)
then show *?thesis*
by *simp*
qed

lemma *homeomorphic_map_relative_homology_iso*:
assumes *f*: *homeomorphic_map X Y f* **and** *S*: *S* ⊆ *topspace X f* ‘ *S* = *T*
shows (*hom_induced p X S Y T f*) ∈ *iso* (*relative_homology_group p X S*) (*relative_homology_group p Y T*)
proof −
obtain *g* **where** *g*: *homeomorphic_maps X Y f g*
using *homeomorphic_map_maps f* **by** *metis*
then have *group_isomorphisms* (*relative_homology_group p X S*) (*relative_homology_group p Y T*)

```
                              (hom_induced p X S Y T f) (hom_induced p Y T X S g)
    using S g by (auto simp: homeomorphic_maps_def intro!: homeomorphic_maps_relative_homology_group_isom
  then show ?thesis
    by (rule group_isomorphisms_imp_iso)
qed

lemma inj_on_hom_induced_section_map:
  assumes section_map X Y f
  shows inj_on (hom_induced p X {} Y {} f) (carrier (homology_group p X))
proof –
  obtain g where cont: continuous_map X Y f continuous_map Y X g
           and gf: ⋀x. x ∈ topspace X ⟹ g (f x) = x
    using assms by (auto simp: section_map_def retraction_maps_def)
  show ?thesis
  proof (rule inj_on_inverseI)
    fix x
    assume x: x ∈ carrier (homology_group p X)
    have continuous_map X X (λx. g (f x))
        by (metis (no_types, lifting) continuous_map_eq continuous_map_id gf
id_apply)
    with x show hom_induced p Y {} X {} g (hom_induced p X {} Y {} f x) =
x
      using hom_induced_compose_empty [OF cont, symmetric]
     by (metis comp_apply cont continuous_map_compose gf hom_induced_id_gen)
  qed
qed

corollary mon_hom_induced_section_map:
  assumes section_map X Y f
 shows (hom_induced p X {} Y {} f) ∈ mon (homology_group p X) (homology_group
p Y)
  by (simp add: hom_induced_empty_hom inj_on_hom_induced_section_map
[OF assms] mon_def)

lemma surj_hom_induced_retraction_map:
  assumes retraction_map X Y f
  shows carrier (homology_group p Y) = (hom_induced p X {} Y {} f) ` carrier
(homology_group p X)
         (is ?lhs = ?rhs)
proof –
  obtain g where cont: continuous_map Y X g  continuous_map X Y f
    and fg: ⋀x. x ∈ topspace Y ⟹ f (g x) = x
    using assms by (auto simp: retraction_map_def retraction_maps_def)
  have x = hom_induced p X {} Y {} f (hom_induced p Y {} X {} g x)
    if x: x ∈ carrier (homology_group p Y) for x
  proof –
    have continuous_map Y Y (λx. f (g x))
        by (metis (no_types, lifting) continuous_map_eq continuous_map_id fg
id_apply)
```

```
    with x show ?thesis
      using hom_induced_compose_empty [OF cont, symmetric]
     by (metis comp_def cont continuous_map_compose fg hom_induced_id_gen)
  qed
  moreover
  have (hom_induced p Y {} X {} g x) ∈ carrier (homology_group p X)
    if x ∈ carrier (homology_group p Y) for x
    by (metis hom_induced)
  ultimately have ?lhs ⊆ ?rhs
    by auto
  moreover have ?rhs ⊆ ?lhs
    using hom_induced_hom [of p X {} Y {} f]
    by (simp add: hom_def flip: image_subset_iff_funcset)
  ultimately show ?thesis
    by auto
qed


corollary epi_hom_induced_retraction_map:
  assumes retraction_map X Y f
 shows (hom_induced p X {} Y {} f) ∈ epi (homology_group p X) (homology_group
p Y)
 using assms epi_iff_subset hom_induced_empty_hom surj_hom_induced_retraction_map
by fastforce


lemma homeomorphic_map_homology_iso:
  assumes homeomorphic_map X Y f
 shows (hom_induced p X {} Y {} f) ∈ iso (homology_group p X) (homology_group
p Y)
  using assms by (simp add: homeomorphic_map_relative_homology_iso)


lemma inj_on_hom_induced_inclusion:
  assumes S = {} ∨ S retract_of_space X
 shows inj_on (hom_induced p (subtopology X S) {} X {} id) (carrier (homology_group
p (subtopology X S)))
  using assms
proof
  assume S = {}
  then have trivial_group(homology_group p (subtopology X S))
    by (auto simp: topspace_subtopology intro: trivial_homology_group_empty)
  then show ?thesis
    by (auto simp: inj_on_def trivial_group_def)
next
  assume S retract_of_space X
  then show ?thesis
   by (simp add: retract_of_space_section_map inj_on_hom_induced_section_map)
qed
```

lemma *trivial_homomorphism_hom_boundary_inclusion*:
assumes $S = \{\} \vee S$ *retract_of_space X*
shows *trivial_homomorphism*
(*relative_homology_group p X S*) (*homology_group* (*p*−*1*) (*subtopology X S*))
(*hom_boundary p X S*)
using *exact_seq_mon_eq_triviality inj_on_hom_induced_inclusion* [*OF assms*]
by (*metis exact_seq_cons_iff homology_exactness_axiom_1 homology_exactness_axiom_2*)

lemma *epi_hom_induced_relativization*:
assumes $S = \{\} \vee S$ *retract_of_space X*
shows (*hom_induced p X* {} *X S id*) ' *carrier* (*homology_group p X*) = *carrier* (*relative_homology_group p X S*)
using *exact_seq_epi_eq_triviality trivial_homomorphism_hom_boundary_inclusion*
by (*metis assms exact_seq_cons_iff homology_exactness_axiom_1 homology_exactness_axiom_2*)

lemmas *short_exact_sequence_hom_induced_inclusion* = *homology_exactness_axiom_3*

lemma *group_isomorphisms_homology_group_prod_retract*:
assumes $S = \{\} \vee S$ *retract_of_space X*
obtains $\mathcal{H}$ $\mathcal{K}$ **where**
subgroup $\mathcal{H}$ (*homology_group p X*)
subgroup $\mathcal{K}$ (*homology_group p X*)
(λ(*x*, *y*). *x* $\otimes_{homology\_group\ p\ X}$ *y*)
∈ *iso* (*DirProd* (*subgroup_generated* (*homology_group p X*) $\mathcal{H}$) (*subgroup_generated* (*homology_group p X*) $\mathcal{K}$))
(*homology_group p X*)
(*hom_induced p* (*subtopology X S*) {} *X* {} *id*)
∈ *iso* (*homology_group p* (*subtopology X S*)) (*subgroup_generated* (*homology_group p X*) $\mathcal{H}$)
(*hom_induced p X* {} *X S id*)
∈ *iso* (*subgroup_generated* (*homology_group p X*) $\mathcal{K}$) (*relative_homology_group p X S*)
using *assms*
proof
assume *S* = {}
show *thesis*
proof (*rule splitting_lemma_left* [*OF homology_exactness_axiom_3* [*of p*]])
let *?f* = λ*x*. *one*(*homology_group p* (*subtopology X* {}))
show *?f* ∈ *hom* (*homology_group p X*) (*homology_group p* (*subtopology X* {}))
by (*simp add*: *trivial_hom*)
have *tg*: *trivial_group* (*homology_group p* (*subtopology X* {}))
by (*auto simp*: *topspace_subtopology trivial_homology_group_empty*)
then have [*simp*]: *carrier* (*homology_group p* (*subtopology X* {})) = {*one* (*homology_group p* (*subtopology X* {}))}
by (*auto simp*: *trivial_group_def*)
then show *?f* (*hom_induced p* (*subtopology X* {}) {} *X* {} *id x*) = *x*

if x ∈ carrier (homology_group p (subtopology X {})) **for** x
using that **by** auto
show inj_on (hom_induced p (subtopology X {}) {} X {} id)
(carrier (homology_group p (subtopology X {})))
by (meson inj_on_hom_induced_inclusion)
show hom_induced p X {} X {} id ‘ carrier (homology_group p X) = carrier (homology_group p X)
by (metis epi_hom_induced_relativization)
next
fix $\mathcal{H}$ $\mathcal{K}$
assume ∗: $\mathcal{H}$ ◁ homology_group p X $\mathcal{K}$ ◁ homology_group p X
$\mathcal{H} \cap \mathcal{K} \subseteq \{\mathbf{1}_{homology\_group\ p\ X}\}$
hom_induced p (subtopology X {}) {} X {} id
∈ Group.iso (homology_group p (subtopology X {})) (subgroup_generated (homology_group p X) $\mathcal{H}$)
hom_induced p X {} X {} id
∈ Group.iso (subgroup_generated (homology_group p X) $\mathcal{K}$) (relative_homology_group p X {})
$\mathcal{H}$ <#>$_{homology\_group\ p\ X}$ $\mathcal{K}$ = carrier (homology_group p X)
show thesis
proof (rule that)
show (λ(x, y). x $\otimes_{homology\_group\ p\ X}$ y)
∈ iso (subgroup_generated (homology_group p X) $\mathcal{H}$ ×× subgroup_generated (homology_group p X) $\mathcal{K}$)
(homology_group p X)
using ∗ **by** (simp add: group_disjoint_sum.iso_group_mul normal_def group_disjoint_sum_def)
qed (use ‹S = {}› ∗ **in** ‹auto simp: normal_def›)
qed
next
assume S retract_of_space X
then obtain r **where** S ⊆ topspace X **and** r: continuous_map X (subtopology X S) r
and req: ∀ x ∈ S. r x = x
by (auto simp: retract_of_space_def)
show thesis
proof (rule splitting_lemma_left [OF homology_exactness_axiom_3 [of p]])
let ?f = hom_induced p X {} (subtopology X S) {} r
show ?f ∈ hom (homology_group p X) (homology_group p (subtopology X S))
by (simp add: hom_induced_empty_hom)
show eqx: ?f (hom_induced p (subtopology X S) {} X {} id x) = x
if x ∈ carrier (homology_group p (subtopology X S)) **for** x
proof −
have hom_induced p (subtopology X S) {} (subtopology X S) {} r x = x
by (metis ‹S ⊆ topspace X› continuous_map_from_subtopology hom_induced_id_gen inf.absorb_iff2 r req that topspace_subtopology)
then show ?thesis
by (simp add: r hom_induced_compose [unfolded o_def fun_eq_iff, rule_format, symmetric])

```
    qed
    then show inj_on (hom_induced p (subtopology X S) {} X {} id)
              (carrier (homology_group p (subtopology X S)))
      unfolding inj_on_def by metis
    show hom_induced p X {} X S id ` carrier (homology_group p X) = carrier
(relative_homology_group p X S)
      by (simp add: ‹S retract_of_space X› epi_hom_induced_relativization)
  next
    fix ℋ 𝒦
    assume *: ℋ ⊲ homology_group p X 𝒦 ⊲ homology_group p X
      ℋ ∩ 𝒦 ⊆ {𝟏_homology_group p X}
      ℋ <#>_homology_group p X 𝒦 = carrier (homology_group p X)
      hom_induced p (subtopology X S) {} X {} id
       ∈ Group.iso (homology_group p (subtopology X S)) (subgroup_generated
(homology_group p X) ℋ)
      hom_induced p X {} X S id
   ∈ Group.iso (subgroup_generated (homology_group p X) 𝒦) (relative_homology_group
p X S)
    show thesis
    proof (rule that)
      show (λ(x, y). x ⊗_homology_group p X y)
        ∈ iso (subgroup_generated (homology_group p X) ℋ ×× subgroup_generated
(homology_group p X) 𝒦)
                (homology_group p X)
        using *
      by (simp add: group_disjoint_sum.iso_group_mul normal_def group_disjoint_sum_def)
    qed (use * in ‹auto simp: normal_def›)
  qed
qed


lemma isomorphic_group_homology_group_prod_retract:
  assumes S = {} ∨ S retract_of_space X
  shows homology_group p X ≅ homology_group p (subtopology X S) ×× rela-
tive_homology_group p X S
        (is ?lhs ≅ ?rhs)
proof −
  obtain ℋ 𝒦 where
    subgroup ℋ (homology_group p X)
    subgroup 𝒦 (homology_group p X)
   and 1: (λ(x, y). x ⊗_homology_group p X y)
   ∈ iso (DirProd (subgroup_generated (homology_group p X) ℋ) (subgroup_generated
(homology_group p X) 𝒦))
          (homology_group p X)
    (hom_induced p (subtopology X S) {} X {} id)
   ∈ iso (homology_group p (subtopology X S)) (subgroup_generated (homology_group
p X) ℋ)
    (hom_induced p X {} X S id)
```

∈ *iso* (*subgroup_generated* (*homology_group p X*) 𝒦) (*relative_homology_group p X S*)
using *group_isomorphisms_homology_group_prod_retract* [*OF assms*] **by** *blast*
have *?lhs* ≅ *subgroup_generated* (*homology_group p X*) ℋ ×× *subgroup_generated* (*homology_group p X*) 𝒦
by (*meson DirProd_group 1 abelian_homology_group comm_group_def group.abelian_subgroup_gene group.iso_sym is_isoI*)
also have ... ≅ *?rhs*
by (*meson 1*(*2*) *1*(*3*) *abelian_homology_group comm_group_def group.DirProd_iso_trans group.abelian_subgroup_generated group.iso_sym is_isoI*)
finally show *?thesis* **.**
qed

lemma *homology_additivity_explicit*:
assumes *openin X S openin X T disjnt S T* **and** *SUT*: *S* ∪ *T* = *topspace X*
shows (λ(*a*,*b*).(*hom_induced p* (*subtopology X S*) {} *X* {} *id a*)
⊗*homology_group p X*
(*hom_induced p* (*subtopology X T*) {} *X* {} *id b*))
∈ *iso* (*DirProd* (*homology_group p* (*subtopology X S*)) (*homology_group p* (*subtopology X T*)))
(*homology_group p X*)
proof −
have *closedin X S closedin X T*
using *assms Un_commute disjnt_sym*
by (*metis Diff_cancel Diff_triv Un_Diff disjnt_def openin_closedin_eq sup_bot.right_neutral*)+
with ‹*openin X S*› ‹*openin X T*› **have** *SS*: *X closure_of S* ⊆ *X interior_of S*
and *TT*: *X closure_of T* ⊆ *X interior_of T*
by (*simp_all add*: *closure_of_closedin interior_of_openin*)
have [*simp*]: *S* ∪ *T* − *T* = *S S* ∪ *T* − *S* = *T*
using ‹*disjnt S T*›
by (*auto simp*: *Diff_triv Un_Diff disjnt_def*)
let *?f* = *hom_induced p X* {} *X T id*
let *?g* = *hom_induced p X* {} *X S id*
let *?h* = *hom_induced p* (*subtopology X S*) {} *X T id*
let *?i* = *hom_induced p* (*subtopology X S*) {} *X* {} *id*
let *?j* = *hom_induced p* (*subtopology X T*) {} *X* {} *id*
let *?k* = *hom_induced p* (*subtopology X T*) {} *X S id*
let *?A* = *homology_group p* (*subtopology X S*)
let *?B* = *homology_group p* (*subtopology X T*)
let *?C* = *relative_homology_group p X T*
let *?D* = *relative_homology_group p X S*
let *?G* = *homology_group p X*
have *h*: *?h* ∈ *iso ?A ?C* **and** *k*: *?k* ∈ *iso ?B ?D*
using *homology_excision_axiom* [*OF TT*, *of S* ∪ *T p*]
using *homology_excision_axiom* [*OF SS*, *of S* ∪ *T p*]
by *auto* (*simp_all add*: *SUT*)
have *1*: ⋀*x*. (*hom_induced p X* {} *X T id* ∘ *hom_induced p* (*subtopology X S*) {} *X* {} *id*) *x*

= *hom_induced p* (*subtopology X S*) {} *X T id x*
by (*simp flip*: *hom_induced_compose*)
have *2*: $\bigwedge x$. (*hom_induced p X* {} *X S id* ∘ *hom_induced p* (*subtopology X T*) {} *X* {} *id*) *x*
= *hom_induced p* (*subtopology X T*) {} *X S id x*
by (*simp flip*: *hom_induced_compose*)
show *?thesis*
using *exact_sequence_sum_lemma*
[*OF abelian_homology_group h k homology_exactness_axiom_3 homology_exactness_axiom_3*] *1 2*
by *auto*
qed

0.2.6 Generalize exact homology sequence to triples

definition *hom_relboundary* :: [*int*,'*a topology*,'*a set*,'*a set*,'*a chain set*] ⇒ '*a chain set*
where
hom_relboundary p X S T =
hom_induced (*p*−*1*) (*subtopology X S*) {} (*subtopology X S*) *T id* ∘
hom_boundary p X S

lemma *group_homomorphism_hom_relboundary*:
hom_relboundary p X S T
∈ *hom* (*relative_homology_group p X S*) (*relative_homology_group* (*p*−*1*) (*subtopology X S*) *T*)
unfolding *hom_relboundary_def*
proof (*rule hom_compose*)
show *hom_boundary p X S* ∈ *hom* (*relative_homology_group p X S*) (*homology_group*(*p*−*1*) (*subtopology X S*))
by (*simp add*: *hom_boundary_hom*)
show *hom_induced* (*p*−*1*) (*subtopology X S*) {} (*subtopology X S*) *T id*
∈ *hom* (*homology_group*(*p*−*1*) (*subtopology X S*)) (*relative_homology_group* (*p*−*1*) (*subtopology X S*) *T*)
by (*simp add*: *hom_induced_hom*)
qed

lemma *hom_relboundary*:
hom_relboundary p X S T c ∈ *carrier* (*relative_homology_group* (*p*−*1*) (*subtopology X S*) *T*)
by (*simp add*: *hom_relboundary_def hom_induced_carrier*)

lemma *hom_relboundary_empty*: *hom_relboundary p X S* {} = *hom_boundary p X S*
by (*simp add*: *ext hom_boundary_carrier hom_induced_id hom_relboundary_def*)

lemma *naturality_hom_induced_relboundary*:
assumes *continuous_map X Y f f* ∈ *S* → *U f* ∈ *T* → *V*
shows *hom_relboundary p Y U V* ∘

```
        hom_induced p X S Y (U) f =
        hom_induced (p−1) (subtopology X S) T (subtopology Y U) V f ∘
        hom_relboundary p X S T
proof −
  have [simp]: continuous_map (subtopology X S) (subtopology Y U) f
   using assms continuous_map_from_subtopology continuous_map_in_subtopology
topspace_subtopology
    by (fastforce simp: Pi_iff)
  have hom_induced (p−1) (subtopology Y U) {} (subtopology Y U) V id ∘
       hom_induced (p−1) (subtopology X S) {} (subtopology Y U) {} f
     = hom_induced (p−1) (subtopology X S) T (subtopology Y U) V f ∘
       hom_induced (p−1) (subtopology X S) {} (subtopology X S) T id
    using assms by (simp flip: hom_induced_compose)
  with assms show ?thesis
    unfolding hom_relboundary_def
    by (metis (no_types, lifting) ext fun.map_comp naturality_hom_induced)
qed

proposition homology_exactness_triple_1:
  assumes T ⊆ S
  shows exact_seq ([relative_homology_group(p−1) (subtopology X S) T,
                    relative_homology_group p X S,
                    relative_homology_group p X T],
                   [hom_relboundary p X S T, hom_induced p X T X S id])
    (is exact_seq ([?G1,?G2,?G3], [?h1,?h2]))
proof −
  have iTS: id ∈ T → S and [simp]: S ∩ T = T
    using assms by auto
  have ?h2 B ∈ kernel ?G2 ?G1 ?h1 for B
  proof −
     have hom_boundary p X T B ∈ carrier (relative_homology_group (p−1)
(subtopology X T) {})
      by (metis (no_types) hom_boundary)
    then have ∗: hom_induced (p−1) (subtopology X S) {} (subtopology X S) T
id
        (hom_induced (p−1) (subtopology X T) {} (subtopology X S) {} id
        (hom_boundary p X T B))
      = 1_?G1
      using homology_exactness_axiom_3 [of p−1 subtopology X S T]
      by (auto simp: subtopology_subtopology kernel_def)
    show ?thesis
      using naturality_hom_induced [OF continuous_map_id iTS]
     by (smt (verit, best) ∗ comp_apply hom_induced_carrier hom_relboundary_def
kernel_def mem_Collect_eq)
  qed
  moreover have B ∈ ?h2 ‘ carrier ?G3 if B ∈ kernel ?G2 ?G1 ?h1 for B
  proof −
    have Bcarr: B ∈ carrier ?G2
      and Beq: ?h1 B = 1_?G1
```

using *that* **by** (*auto simp*: *kernel_def*)
have ∃ *A'* ∈ *carrier* (*homology_group* (*p*−*1*) (*subtopology X T*)). *hom_induced* (*p*−*1*) (*subtopology X T*) {} (*subtopology X S*) {} *id A'* = *A*
if *A* ∈ *carrier* (*homology_group* (*p*−*1*) (*subtopology X S*))
hom_induced (*p*−*1*) (*subtopology X S*) {} (*subtopology X S*) *T id A* = $\mathbf{1}_{?G1}$ **for** *A*
using *homology_exactness_axiom_3* [*of p*−*1 subtopology X S T*] *that*
by (*simp add*: *kernel_def subtopology_subtopology image_iff set_eq_iff*) *meson*
then obtain *C* **where** *Ccarr*: *C* ∈ *carrier* (*homology_group* (*p*−*1*) (*subtopology X T*))
and *Ceq*: *hom_induced* (*p*−*1*) (*subtopology X T*) {} (*subtopology X S*) {} *id C* = *hom_boundary p X S B*
using *Beq* **by** (*simp add*: *hom_relboundary_def*) (*metis hom_boundary_carrier*)
let *?hi_XT* = *hom_induced* (*p*−*1*) (*subtopology X T*) {} *X* {} *id*
have *?hi_XT*
= *hom_induced* (*p*−*1*) (*subtopology X S*) {} *X* {} *id*
∘ (*hom_induced* (*p*−*1*) (*subtopology X T*) {} (*subtopology X S*) {} *id*)
by (*metis assms comp_id continuous_map_id_subt hom_induced_compose_empty inf.absorb_iff2 subtopology_subtopology*)
then have *?hi_XT C*
= *hom_induced* (*p*−*1*) (*subtopology X S*) {} *X* {} *id* (*hom_boundary p X S B*)
by (*simp add*: *Ceq*)
also have *eq*: ... = $\mathbf{1}_{homology\_group\ (p-1)\ X}$
using *homology_exactness_axiom_2* [*of p X S*] *Bcarr* **by** (*auto simp*: *kernel_def*)
finally have *?hi_XT C* = $\mathbf{1}_{homology\_group\ (p-1)\ X}$.
then obtain *D* **where** *Dcarr*: *D* ∈ *carrier ?G3* **and** *Deq*: *hom_boundary p X T D* = *C*
using *homology_exactness_axiom_2* [*of p X T*] *Ccarr*
by (*auto simp*: *kernel_def image_iff set_eq_iff*) *meson*
interpret *hb*: *group_hom ?G2 homology_group* (*p*−*1*) (*subtopology X S*)
hom_boundary p X S
using *hom_boundary_hom group_hom_axioms_def group_hom_def* **by** *fastforce*
let *?A* = *B* $\otimes_{?G2}$ $inv_{?G2}$ *?h2 D*
have ∃ *A'* ∈ *carrier* (*homology_group p X*). *hom_induced p X* {} *X S id A'* = *A*
if *A* ∈ *carrier ?G2*
hom_boundary p X S A = *one* (*homology_group* (*p*−*1*) (*subtopology X S*)) **for** *A*
using *that homology_exactness_axiom_1* [*of p X S*]
by (*simp add*: *kernel_def subtopology_subtopology image_iff set_eq_iff*) *meson*
moreover
have *?A* ∈ *carrier ?G2*
by (*simp add*: *Bcarr abelian_relative_homology_group comm_groupE*(*1*) *hom_induced_carrier*)

```
    moreover have hom_boundary p X S (?h2 D) = hom_boundary p X S B
      by (metis (mono_tags, lifting) Ceq Deq comp_eq_dest continuous_map_id
iTS naturality_hom_induced)
   then have hom_boundary p X S ?A = one (homology_group (p−1) (subtopology
X S))
      by (simp add: hom_induced_carrier Bcarr)
    ultimately obtain W where Wcarr: W ∈ carrier (homology_group p X)
      and Weq: hom_induced p X {} X S id W = ?A
      by blast
    let ?W = D ⊗?G3 hom_induced p X {} X T id W
    show ?thesis
    proof
      interpret comm_group ?G2
        by (rule abelian_relative_homology_group)
        have hom_induced p X T X S id (hom_induced p X {} X T id W) =
hom_induced p X {} X S id W
        using assms iTS by (simp add: hom_induced_compose')
      then have B = (?h2 ∘ hom_induced p X {} X T id) W ⊗?G2 ?h2 D
        by (simp add: Bcarr Weq hb.G.m_assoc hom_induced_carrier)
      then show B = ?h2 ?W
      by (metis hom_mult [OF hom_induced_hom] Dcarr comp_apply hom_induced_carrier
m_comm)
      show ?W ∈ carrier ?G3
        by (simp add: Dcarr comm_groupE(1) hom_induced_carrier)
    qed
  qed
  ultimately show ?thesis
     by (auto simp: group_hom_def group_hom_axioms_def hom_induced_hom
group_homomorphism_hom_relboundary)
qed


proposition homology_exactness_triple_2:
  assumes T ⊆ S
  shows exact_seq ([relative_homology_group(p−1) X T,
                    relative_homology_group(p−1) (subtopology X S) T,
                    relative_homology_group p X S],
                 [hom_induced (p−1) (subtopology X S) T X T id, hom_relboundary
p X S T])
    (is exact_seq ([?G1,?G2,?G3], [?h1,?h2]))
proof −
  let ?H2 = homology_group (p−1) (subtopology X S)
  have iTS: id ∈ T → S and [simp]: S ∩ T = T
    using assms by auto
  have ?h2 C ∈ kernel ?G2 ?G1 ?h1 for C
  proof −
    have ?h1 (?h2 C)
       = (hom_induced (p−1) X {} X T id ∘ hom_induced (p−1) (subtopology X
S) {} X {} id ∘ hom_boundary p X S) C
```

unfolding *hom_relboundary_def*
by (*metis Pi_empty comp_eq_dest_lhs continuous_map_id continuous_map_id_subt funcsetI hom_induced_compose' id_apply*)
also have ... = $\mathbf{1}_{?G1}$
proof −
have ∗: *hom_boundary p X S C* ∈ *carrier ?H2*
by (*simp add*: *hom_boundary_carrier*)
moreover have *hom_boundary p X S C* ∈ *hom_boundary p X S* ‘ *carrier ?G3*
using *homology_exactness_axiom_2* [*of p X S*] ∗
apply (*simp add*: *kernel_def set_eq_iff*)
by (*metis group_relative_homology_group hom_boundary_default hom_one image_eqI*)
ultimately
have *1*: *hom_induced* (*p*−*1*) (*subtopology X S*) {} *X* {} *id* (*hom_boundary p X S C*)
= $\mathbf{1}_{homology\_group\ (p-1)\ X}$
using *homology_exactness_axiom_2* [*of p X S*] **by** (*simp add*: *kernel_def*) *blast*
show *?thesis*
by (*simp add*: *1 hom_one* [*OF hom_induced_hom*])
qed
finally have *?h1* (*?h2 C*) = $\mathbf{1}_{?G1}$.
then show *?thesis*
by (*simp add*: *kernel_def hom_relboundary_def hom_induced_carrier*)
qed
moreover have *x* ∈ *?h2* ‘ *carrier ?G3* **if** *x* ∈ *kernel ?G2 ?G1 ?h1* **for** *x*
proof −
let *?homX* = *hom_induced* (*p*−*1*) (*subtopology X S*) {} *X* {} *id*
let *?homXS* = *hom_induced* (*p*−*1*) (*subtopology X S*) {} (*subtopology X S*) *T id*
have *x* ∈ *carrier* (*relative_homology_group* (*p*−*1*) (*subtopology X S*) *T*)
using *that* **by** (*simp add*: *kernel_def*)
moreover
have *hom_boundary* (*p*−*1*) *X T* ∘ *hom_induced* (*p*−*1*) (*subtopology X S*) *T X T id* = *hom_boundary* (*p*−*1*) (*subtopology X S*) *T*
by (*metis funcsetI* ‹*S* ∩ *T* = *T*› *continuous_map_id_subt hom_relboundary_def hom_relboundary_empty id_apply naturality_hom_induced subtopology_subtopology*)
then have *hom_boundary* (*p*−*1*) (*subtopology X S*) *T x* = $\mathbf{1}_{homology\_group\ (p\ -\ 2)\ (subtopology\ (subtopology\ X\ S)}$
using *naturality_hom_induced* [*of subtopology X S X id T T p*−*1*] *that hom_one* [*OF hom_boundary_hom group_relative_homology_group group_relative_homology_group, of p*−*1 X T*]
by (*smt* (*verit*) *assms comp_apply inf.absorb_iff2 kernel_def mem_Collect_eq subtopology_subtopology*)
ultimately
obtain *y* **where** *ycarr*: *y* ∈ *carrier ?H2*
and *yeq*: *?homXS y* = *x*
using *homology_exactness_axiom_1* [*of p*−*1 subtopology X S T*]

by (*simp add*: *kernel_def image_def set_eq_iff*) *meson*
have *?homX y* ∈ *carrier* (*homology_group* (*p*−*1*) *X*)
by (*simp add*: *hom_induced_carrier*)
moreover
have (*hom_induced* (*p*−*1*) *X* {} *X T id* ∘ *?homX*) *y* = $\mathbf{1}_{relative\_homology\_group\ (p-1)\ X\ T}$
using *that*
apply (*simp add*: *kernel_def flip*: *hom_induced_compose*)
using *hom_induced_compose* [*of subtopology X S subtopology X S id* {} *T X id T p*−*1*] *yeq*
by *auto*
then have *hom_induced* (*p*−*1*) *X* {} *X T id* (*?homX y*) = $\mathbf{1}_{relative\_homology\_group\ (p-1)\ X\ T}$
by *simp*
ultimately obtain *z* **where** *zcarr*: *z* ∈ *carrier* (*homology_group* (*p*−*1*) (*subtopology X T*))
and *zeq*: *hom_induced* (*p*−*1*) (*subtopology X T*) {} *X* {} *id z* = *?homX y*
using *homology_exactness_axiom_3* [*of p*−*1 X T*]
by (*auto simp*: *kernel_def dest*!: *equalityD1* [*of Collect* _])
have ∗: ⋀*t*. ⟦*t* ∈ *carrier ?H2*;
hom_induced (*p*−*1*) (*subtopology X S*) {} *X* {} *id t* = $\mathbf{1}_{homology\_group\ (p-1)\ X}$⟧
⟹ *t* ∈ *hom_boundary p X S* ‘ *carrier ?G3*
using *homology_exactness_axiom_2* [*of p X S*]
by (*auto simp*: *kernel_def dest*!: *equalityD1* [*of Collect* _])
interpret *comm_group ?H2*
by (*rule abelian_relative_homology_group*)
interpret *gh*: *group_hom ?H2 homology_group* (*p*−*1*) *X hom_induced* (*p*−*1*) (*subtopology X S*) {} *X* {} *id*
by (*meson group_hom_axioms_def group_hom_def group_relative_homology_group hom_induced*)
let *?yz* = *y* $\otimes_{?H2}$ $inv_{?H2}$ *hom_induced* (*p*−*1*) (*subtopology X T*) {} (*subtopology X S*) {} *id z*
have *yzcarr*: *?yz* ∈ *carrier ?H2*
by (*simp add*: *hom_induced_carrier ycarr*)
have *hom_induced* (*p*−*1*) (*subtopology X S*) {} *X* {} *id y* =
hom_induced (*p*−*1*) (*subtopology X S*) {} *X* {} *id*
(*hom_induced* (*p*−*1*) (*subtopology X T*) {} (*subtopology X S*) {} *id z*)
by (*metis assms continuous_map_id_subt hom_induced_compose_empty inf.absorb_iff2 o_apply o_id subtopology_subtopology zeq*)
then have *yzeq*: *hom_induced* (*p*−*1*) (*subtopology X S*) {} *X* {} *id ?yz* = $\mathbf{1}_{homology\_group\ (p-1)\ X}$
by (*simp add*: *hom_induced_carrier ycarr gh.inv_solve_right*′)
obtain *w* **where** *wcarr*: *w* ∈ *carrier ?G3* **and** *weq*: *hom_boundary p X S w* = *?yz*
using ∗ [*OF yzcarr yzeq*] **by** *blast*
interpret *gh2*: *group_hom ?H2 ?G2 ?homXS*
by (*simp add*: *group_hom_axioms_def group_hom_def hom_induced_hom*)
have *?homXS* (*hom_induced* (*p*−*1*) (*subtopology X T*) {} (*subtopology X S*) {} *id z*)
= $\mathbf{1}_{relative\_homology\_group\ (p-1)\ (subtopology\ X\ S)\ T}$

```
      using homology_exactness_axiom_3 [of p−1 subtopology X S T] zcarr
      by (auto simp: kernel_def subtopology_subtopology)
    then show ?thesis
      apply (rule_tac x=w in image_eqI)
       apply (simp_all add: hom_relboundary_def weq wcarr)
     by (metis gh2.hom_inv gh2.hom_mult gh2.inv_one gh2.r_one group.inv_closed
group_l_invI hom_induced_carrier l_inv_ex ycarr yeq)
  qed
  ultimately show ?thesis
   by (auto simp: group_hom_axioms_def group_hom_def group_homomorphism_hom_relboundary
hom_induced_hom)
qed

proposition homology_exactness_triple_3:
  assumes T ⊆ S
  shows exact_seq ([relative_homology_group p X S,
                    relative_homology_group p X T,
                    relative_homology_group p (subtopology X S) T],
                   [hom_induced p X T X S id, hom_induced p (subtopology X S) T
X T id])
    (is exact_seq ([?G1,?G2,?G3], [?h1,?h2]))
proof −
  have iTS: id ∈ T → S and [simp]: S ∩ T = T
    using assms by auto
  have 1: ?h2 x ∈ kernel ?G2 ?G1 ?h1 for x
  proof −
    have ?h1 (?h2 x)
        = (hom_induced p (subtopology X S) S X S id ∘
          hom_induced p (subtopology X S) T (subtopology X S) S id) x
      by (simp add: hom_induced_compose' iTS)
    also have ... = 𝟭relative_homology_group p X S
    proof −
      have trivial_group (relative_homology_group p (subtopology X S) S)
        using trivial_relative_homology_group_topspace [of p subtopology X S]
       by (metis inf_right_idem relative_homology_group_restrict topspace_subtopology)
      then have 1: hom_induced p (subtopology X S) T (subtopology X S) S id x
         = 𝟭relative_homology_group p (subtopology X S) S
        using hom_induced_carrier by (fastforce simp add: trivial_group_def)
      show ?thesis
        by (simp add: 1 hom_one [OF hom_induced_hom])
    qed
    finally have ?h1 (?h2 x) = 𝟭relative_homology_group p X S .
    then show ?thesis
      by (simp add: hom_induced_carrier kernel_def)
  qed
  moreover have x ∈ ?h2 ` carrier ?G3 if x: x ∈ kernel ?G2 ?G1 ?h1 for x
  proof −
    have xcarr: x ∈ carrier ?G2
      using that by (auto simp: kernel_def)
```

interpret *G2*: *comm_group ?G2*
by (*rule abelian_relative_homology_group*)
let *?b* = *hom_boundary p X T x*
have *bcarr*: *?b* ∈ *carrier*(*homology_group*(*p*−*1*) (*subtopology X T*))
by (*simp add*: *hom_boundary_carrier*)
have *hom_boundary p X S* (*hom_induced p X T X S id x*)
= *hom_induced* (*p*−*1*) (*subtopology X T*) {} (*subtopology X S*) {} *id*
(*hom_boundary p X T x*)
using *naturality_hom_induced* [*of X X id T S p*] *iTS*
by (*simp add*: *assms o_def*) *meson*
with *bcarr* **have** *hom_boundary p X T x* ∈ *hom_boundary p* (*subtopology X S*) *T* ‘ *carrier ?G3*
using *homology_exactness_axiom_2* [*of p subtopology X S T*] *x*
apply (*simp add*: *kernel_def set_eq_iff subtopology_subtopology*)
by (*metis group_relative_homology_group hom_boundary_hom hom_one set_eq_iff*)
then obtain *u* **where** *ucarr*: *u* ∈ *carrier ?G3*
and *ueq*: *hom_boundary p X T x* = *hom_boundary p* (*subtopology X S*) *T u*
by (*auto simp*: *kernel_def set_eq_iff subtopology_subtopology hom_boundary_carrier*)
define *y* **where** $y = x \otimes_{?G2} inv_{?G2}$ *?h2 u*
have *ycarr*: *y* ∈ *carrier ?G2*
using *x* **by** (*simp add*: *y_def kernel_def hom_induced_carrier*)
interpret *hb*: *group_hom ?G2 homology_group* (*p*−*1*) (*subtopology X T*) *hom_boundary p X T*
by (*simp add*: *group_hom_axioms_def group_hom_def hom_boundary_hom*)
have *yyy*: *hom_boundary p X T y* = $\mathbf{1}_{homology\_group\ (p-1)\ (subtopology\ X\ T)}$
apply (*simp add*: *y_def bcarr xcarr hom_induced_carrier hom_boundary_carrier hb.inv_solve_right′*)
using *naturality_hom_induced* [*of concl*: *p X T subtopology X S T id*]
by (*metis* ‹*S* ∩ *T* = *T*› *comp_eq_dest_lhs continuous_map_id_subt*
hom_relboundary_def hom_relboundary_empty id_apply image_id
image_subset_iff_funcset subsetI subtopology_subtopology ueq)
then have *y* ∈ *hom_induced p X* {} *X T id* ‘ *carrier* (*homology_group p X*)
using *homology_exactness_axiom_1* [*of p X T*] *x ycarr* **by** (*auto simp*: *kernel_def*)
then obtain *z* **where** *zcarr*: *z* ∈ *carrier* (*homology_group p X*)
and *zeq*: *hom_induced p X* {} *X T id z* = *y*
by *auto*
interpret *gh1*: *group_hom ?G2 ?G1 ?h1*
by (*meson group_hom_axioms_def group_hom_def group_relative_homology_group hom_induced*)

have *hom_induced p X* {} *X S id z* = (*hom_induced p X T X S id* ∘ *hom_induced p X* {} *X T id*) *z*
using *iTS* **by** (*simp add*: *assms flip*: *hom_induced_compose*)
also have … = $\mathbf{1}_{relative\_homology\_group\ p\ X\ S}$
using *x 1* **by** (*simp add*: *kernel_def zeq y_def*)
finally have *hom_induced p X* {} *X S id z* = $\mathbf{1}_{relative\_homology\_group\ p\ X\ S}$ **.**

```
    then have z ∈ hom_induced p (subtopology X S) {} X {} id `
                  carrier (homology_group p (subtopology X S))
      using homology_exactness_axiom_3 [of p X S] zcarr by (auto simp: kernel_def)
    then obtain w where wcarr: w ∈ carrier (homology_group p (subtopology X S))
              and weq: hom_induced p (subtopology X S) {} X {} id w = z
      by blast
    let ?u = hom_induced p (subtopology X S) {} (subtopology X S) T id w ⊗?G3 u
    show ?thesis
    proof
      have *: x = z ⊗?G2 u
        if z = x ⊗?G2 inv?G2 u z ∈ carrier ?G2 u ∈ carrier ?G2 for z u
        using that by (simp add: group.inv_solve_right xcarr)
      have eq: ?h2 ∘ hom_induced p (subtopology X S) {} (subtopology X S) T id
           = hom_induced p X {} X T id ∘ hom_induced p (subtopology X S) {} X {} id
        by (simp flip: hom_induced_compose)
      show x = hom_induced p (subtopology X S) T X T id ?u
        using hom_mult [OF hom_induced_hom] hom_induced_carrier *
        by (smt (verit, best) comp_eq_dest eq ucarr weq y_def zeq)
      show ?u ∈ carrier (relative_homology_group p (subtopology X S) T)
      by (simp add: abelian_relative_homology_group comm_groupE(1) hom_induced_carrier ucarr)
    qed
  qed
  ultimately show ?thesis
    by (auto simp: group_hom_axioms_def group_hom_def hom_induced_hom)
qed

end
```

0.3 Homology, III: Brouwer Degree

```
theory Brouwer_Degree
  imports Homology_Groups HOL−Algebra.Multiplicative_Group

begin
```

0.3.1 Reduced Homology

```
definition reduced_homology_group :: int ⇒ 'a topology ⇒ 'a chain set monoid
  where reduced_homology_group p X ≡
          subgroup_generated (homology_group p X)
             (kernel (homology_group p X) (homology_group p (discrete_topology {()}))
                   (hom_induced p X {} (discrete_topology {()}) {} (λx. ())))
```

lemma *one_reduced_homology_group*: $\mathbf{1}_{reduced\_homology\_group\ p\ X} = \mathbf{1}_{homology\_group\ p\ X}$
  by (*simp add*: *reduced_homology_group_def*)

lemma *group_reduced_homology_group* [*simp*]: *group* (*reduced_homology_group p X*)
  by (*simp add*: *reduced_homology_group_def group.group_subgroup_generated*)

lemma *carrier_reduced_homology_group*:
  carrier (*reduced_homology_group p X*) =
  kernel (*homology_group p X*) (*homology_group p* (*discrete_topology* {()}))
    (*hom_induced p X* {} (*discrete_topology* {()}) {} (λ*x*. ()))
  (**is** _ = *kernel ?G ?H ?h*)
proof −
  interpret *subgroup kernel ?G ?H ?h ?G*
  by (*simp add*: *hom_induced_empty_hom group_hom_axioms_def group_hom_def group_hom.subgroup_kernel*)
  show *?thesis*
    unfolding *reduced_homology_group_def*
    using *carrier_subgroup_generated_subgroup* **by** *blast*
qed

lemma *carrier_reduced_homology_group_subset*:
  carrier (*reduced_homology_group p X*) ⊆ *carrier* (*homology_group p X*)
  by (*simp add*: *group.carrier_subgroup_generated_subset reduced_homology_group_def*)

lemma *un_reduced_homology_group*:
  assumes $p \neq 0$
  shows *reduced_homology_group p X* = *homology_group p X*
proof −
  have (*kernel* (*homology_group p X*) (*homology_group p* (*discrete_topology* {()}))
      (*hom_induced p X* {} (*discrete_topology* {()}) {} (λ*x*. ())))
    = *carrier* (*homology_group p X*)
  proof (*rule group_hom.kernel_to_trivial_group*)
    show *group_hom* (*homology_group p X*) (*homology_group p* (*discrete_topology* {()}))
      (*hom_induced p X* {} (*discrete_topology* {()}) {} (λ*x*. ()))
    by (*auto simp*: *hom_induced_empty_hom group_hom_def group_hom_axioms_def*)
    show *trivial_group* (*homology_group p* (*discrete_topology* {()}))
      by (*simp add*: *homology_dimension_axiom* [*OF* _ *assms*])
  qed
  then show *?thesis*
  by (*simp add*: *reduced_homology_group_def group.subgroup_generated_group_carrier*)
qed

lemma *trivial_reduced_homology_group*:
  $p < 0 \Longrightarrow$ *trivial_group*(*reduced_homology_group p X*)
  by (*simp add*: *trivial_homology_group un_reduced_homology_group*)

lemma *hom_induced_reduced_hom*:

```
  (hom_induced p X {} Y {} f) ∈ hom (reduced_homology_group p X) (reduced_homology_group
p Y)
proof (cases continuous_map X Y f)
  case True
  have eq: continuous_map X Y f
        ⟹ hom_induced p X {} (discrete_topology {()}) {} (λx. ())
        = (hom_induced p Y {} (discrete_topology {()}) {} (λx. ()) ∘ hom_induced
p X {} Y {} f)
    by (simp flip: hom_induced_compose_empty)
  interpret subgroup kernel (homology_group p X)
                      (homology_group p (discrete_topology {()}))
                        (hom_induced p X {} (discrete_topology {()}) {} (λx. ()))
                    homology_group p X
   by (meson group_hom.subgroup_kernel group_hom_axioms_def group_hom_def
group_relative_homology_group hom_induced)
  have sb: hom_induced p X {} Y {} f ` carrier (homology_group p X) ⊆ carrier
(homology_group p Y)
    using hom_induced_carrier by blast
    show ?thesis
    using True
    unfolding reduced_homology_group_def
   apply (simp add: hom_into_subgroup_eq group_hom.subgroup_kernel hom_induced_empty_hom
group.hom_from_subgroup_generated group_hom_def group_hom_axioms_def)
    unfolding kernel_def using eq sb by auto
next
  case False
  then have hom_induced p X {} Y {} f = (λc. one(reduced_homology_group p
Y))
    by (force simp: hom_induced_default reduced_homology_group_def)
  then show ?thesis
    by (simp add: trivial_hom)
qed


lemma hom_induced_reduced:
   c ∈ carrier(reduced_homology_group p X)
       ⟹ hom_induced p X {} Y {} f c ∈ carrier(reduced_homology_group p Y)
  by (meson hom_in_carrier hom_induced_reduced_hom)

lemma hom_boundary_reduced_hom:
   hom_boundary p X S
 ∈ hom (relative_homology_group p X S) (reduced_homology_group (p−1) (subtopology
X S))
proof −
  have *: continuous_map X (discrete_topology {()}) (λx. ()) (λx. ()) ∈ S → {()}
    by auto
  interpret group_hom relative_homology_group p (discrete_topology {()}) {()}
                     homology_group (p−1) (discrete_topology {()})
                     hom_boundary p (discrete_topology {()}) {()}
```

apply (*clarsimp simp*: *group_hom_def group_hom_axioms_def*)
by (*metis UNIV_unit hom_boundary_hom subtopology_UNIV*)
have *hom_boundary p X S* ‘
carrier (*relative_homology_group p X S*)
⊆ *kernel* (*homology_group* (*p* − *1*) (*subtopology X S*))
(*homology_group* (*p* − *1*) (*discrete_topology* {()}))
(*hom_induced* (*p* − *1*) (*subtopology X S*) {}
(*discrete_topology* {()}) {} (λ*x*. ()))
proof (*clarsimp simp add*: *kernel_def hom_boundary_carrier*)
fix *c*
assume *c*: *c* ∈ *carrier* (*relative_homology_group p X S*)
have *triv*: *trivial_group* (*relative_homology_group p* (*discrete_topology* {()}) {()})
by (*metis topspace_discrete_topology trivial_relative_homology_group_topspace*)
have *hom_boundary p* (*discrete_topology* {()}) {()}
(*hom_induced p X S* (*discrete_topology* {()}) {()} (λ*x*. ()) *c*)
= $\mathbf{1}_{homology\_group\ (p\ -\ 1)\ (discrete\_topology\ \{()\})}$
by (*metis hom_induced_carrier local.hom_one singletonD triv trivial_group_def*)
then show *hom_induced* (*p* − *1*) (*subtopology X S*) {} (*discrete_topology* {()}) {} (λ*x*. ()) (*hom_boundary p X S c*) =
$\mathbf{1}_{homology\_group\ (p\ -\ 1)\ (discrete\_topology\ \{()\})}$
using *naturality_hom_induced* [*OF* ∗, *of p*, *symmetric*] **by** (*simp add*: *o_def fun_eq_iff*)
qed
then show *?thesis*
by (*simp add*: *reduced_homology_group_def hom_boundary_hom hom_into_subgroup*)
qed

lemma *homotopy_equivalence_reduced_homology_group_isomorphisms*:
assumes *contf*: *continuous_map X Y f* **and** *contg*: *continuous_map Y X g*
and *gf*: *homotopic_with* (λ*h*. *True*) *X X* (*g* ∘ *f*) *id*
and *fg*: *homotopic_with* (λ*k*. *True*) *Y Y* (*f* ∘ *g*) *id*
shows *group_isomorphisms* (*reduced_homology_group p X*) (*reduced_homology_group p Y*)
(*hom_induced p X* {} *Y* {} *f*) (*hom_induced p Y* {} *X* {} *g*)
proof (*simp add*: *hom_induced_reduced_hom group_isomorphisms_def*, *intro conjI ballI*)
fix *a*
assume *a* ∈ *carrier* (*reduced_homology_group p X*)
then have (*hom_induced p Y* {} *X* {} *g* ∘ *hom_induced p X* {} *Y* {} *f*) *a* = *a*
apply (*simp add*: *contf contg flip*: *hom_induced_compose*)
using *carrier_reduced_homology_group_subset gf hom_induced_id homology_homotopy_empty* **by** *fastforce*
then show *hom_induced p Y* {} *X* {} *g* (*hom_induced p X* {} *Y* {} *f a*) = *a*
by *simp*
next
fix *b*

assume b ∈ carrier (reduced_homology_group p Y)
then have (hom_induced p X {} Y {} f ∘ hom_induced p Y {} X {} g) b = b
apply (simp add: contf contg flip: hom_induced_compose)
using carrier_reduced_homology_group_subset fg hom_induced_id homology_homotopy_empty **by** fastforce
then show hom_induced p X {} Y {} f (hom_induced p Y {} X {} g b) = b
by (simp add: carrier_reduced_homology_group)
qed

lemma homotopy_equivalence_reduced_homology_group_isomorphism:
assumes continuous_map X Y f continuous_map Y X g
and homotopic_with (λh. True) X X (g ∘ f) id homotopic_with (λk. True) Y Y (f ∘ g) id
shows (hom_induced p X {} Y {} f)
∈ iso (reduced_homology_group p X) (reduced_homology_group p Y)
proof (rule group_isomorphisms_imp_iso)
show group_isomorphisms (reduced_homology_group p X) (reduced_homology_group p Y)
(hom_induced p X {} Y {} f) (hom_induced p Y {} X {} g)
by (simp add: assms homotopy_equivalence_reduced_homology_group_isomorphisms)
qed

lemma homotopy_equivalent_space_imp_isomorphic_reduced_homology_groups:
X homotopy_equivalent_space Y
⟹ reduced_homology_group p X ≅ reduced_homology_group p Y
unfolding homotopy_equivalent_space_def
using homotopy_equivalence_reduced_homology_group_isomorphism is_isoI **by** blast

lemma homeomorphic_space_imp_isomorphic_reduced_homology_groups:
X homeomorphic_space Y ⟹ reduced_homology_group p X ≅ reduced_homology_group p Y
by (simp add: homeomorphic_imp_homotopy_equivalent_space homotopy_equivalent_space_imp_isomorphic_re

lemma trivial_reduced_homology_group_empty:
topspace X = {} ⟹ trivial_group(reduced_homology_group p X)
by (metis carrier_reduced_homology_group_subset group.trivial_group_alt group_reduced_homology_group trivial_group_def trivial_homology_group_empty)

lemma homology_dimension_reduced:
assumes topspace X = {a}
shows trivial_group (reduced_homology_group p X)
proof –
have iso: (hom_induced p X {} (discrete_topology {()}) {} (λx. ()))
∈ iso (homology_group p X) (homology_group p (discrete_topology {()}))
apply (rule homeomorphic_map_homology_iso)
apply (force simp: homeomorphic_map_maps homeomorphic_maps_def assms)
done
show ?thesis

unfolding *reduced_homology_group_def*
by (*rule group.trivial_group_subgroup_generated*) (*use iso* **in** ‹*auto simp*: *iso_kernel_image*›)
qed

lemma *trivial_reduced_homology_group_contractible_space*:
contractible_space X $\Longrightarrow$ *trivial_group* (*reduced_homology_group p X*)
apply (*simp add*: *contractible_eq_homotopy_equivalent_singleton_subtopology*)
apply (*auto simp*: *trivial_reduced_homology_group_empty*)
using *isomorphic_group_triviality*
by (*metis* (*full_types*) *group_reduced_homology_group homology_dimension_reduced homotopy_equivalent_space_imp_isomorphic_reduced_homology_groups path_connectedin_def path_connectedin_singleton topspace_subtopology_subset*)

lemma *image_reduced_homology_group*:
assumes *topspace* $X \cap S \neq \{\}$
shows *hom_induced p X* {} *X S id* ' *carrier* (*reduced_homology_group p X*)
= *hom_induced p X* {} *X S id* ' *carrier* (*homology_group p X*)
(**is** *?h* ' *carrier ?G* = *?h* ' *carrier ?H*)
proof −
obtain *a* **where** *a*: $a \in$ *topspace X* **and** $a \in S$
using *assms* **by** *blast*
have [*simp*]: $A \cap \{x \in A.\ P\ x\} = \{x \in A.\ P\ x\}$ **for** *A P*
by *blast*
interpret *comm_group homology_group p X*
by (*rule abelian_relative_homology_group*)
have $*$: $\exists x'.$ *?h y* = *?h* x' $\wedge$
$x' \in$ *carrier ?H* $\wedge$
hom_induced p X {} (*discrete_topology* {()}) {} ($\lambda x.$ ()) x'
= $\mathbf{1}_{homology\_group\ p\ (discrete\_topology\ \{()\})}$
if $y \in$ *carrier ?H* **for** *y*
proof −
let *?f* = *hom_induced p* (*discrete_topology* {()}) {} *X* {} ($\lambda x.$ *a*)
let *?g* = *hom_induced p X* {} (*discrete_topology* {()}) {} ($\lambda x.$ ())
have *bcarr*: *?f* (*?g y*) $\in$ *carrier ?H*
by (*simp add*: *hom_induced_carrier*)
interpret *gh1*:
group_hom relative_homology_group p X S relative_homology_group p (*discrete_topology* {()}) {()}
hom_induced p X S (*discrete_topology* {()}) {()} ($\lambda x.$ ())
by (*meson group_hom_axioms_def group_hom_def hom_induced_hom group_relative_homology_group*)
interpret *gh2*:
group_hom relative_homology_group p (*discrete_topology* {()}) {()} *relative_homology_group p X S*
hom_induced p (*discrete_topology* {()}) {()} *X S* ($\lambda x.$ *a*)
by (*meson group_hom_axioms_def group_hom_def hom_induced_hom*

group_relative_homology_group)
interpret *gh3*:
group_hom homology_group p X relative_homology_group p X S ?h
by (*meson group_hom_axioms_def group_hom_def hom_induced_hom group_relative_homology_group*)
interpret *gh4*:
group_hom homology_group p X homology_group p (*discrete_topology* {()})
?g
by (*meson group_hom_axioms_def group_hom_def hom_induced_hom group_relative_homology_group*)
interpret *gh5*:
group_hom homology_group p (*discrete_topology* {()}) *homology_group p X*
?f
by (*meson group_hom_axioms_def group_hom_def hom_induced_hom group_relative_homology_group*)
interpret *gh6*:
group_hom homology_group p (*discrete_topology* {()}) *relative_homology_group p* (*discrete_topology* {()}) {()}
hom_induced p (*discrete_topology* {()}) {} (*discrete_topology* {()}) {()} *id*
by (*meson group_hom_axioms_def group_hom_def hom_induced_hom group_relative_homology_group*)
show *?thesis*
proof (*intro exI conjI*)
have (*?h* ∘ *?f* ∘ *?g*) *y*
= (*hom_induced p* (*discrete_topology* {()}) {()} *X S* (λ*x*. *a*) ∘
hom_induced p (*discrete_topology* {()}) {} (*discrete_topology* {()}) {()} *id* ∘ *?g*) *y*
by (*simp add*: *a* ‹*a* ∈ *S*› *flip*: *hom_induced_compose*)
also have ... = $\mathbf{1}_{relative\_homology\_group\ p\ X\ S}$
using *trivial_relative_homology_group_topspace* [*of p discrete_topology* {()}]
apply *simp*
by (*metis* (*full_types*) *empty_iff gh1.H.one_closed gh1.H.trivial_group gh2.hom_one hom_induced_carrier insert_iff*)
finally have *?h* (*?f* (*?g y*)) = $\mathbf{1}_{relative\_homology\_group\ p\ X\ S}$
by *simp*
then show *?h y* = *?h* (*y* $\otimes_{?H}$ *inv*$_{?H}$ *?f* (*?g y*))
by (*simp add*: *that hom_induced_carrier*)
show (*y* $\otimes_{?H}$ *inv*$_{?H}$ *?f* (*?g y*)) ∈ *carrier* (*homology_group p X*)
by (*simp add*: *hom_induced_carrier that*)
have ∗: (*?g* ∘ *hom_induced p X* {} *X* {} (λ*x*. *a*)) *y* = *hom_induced p X* {} (*discrete_topology* {()}) {} (λ*a*. ()) *y*
by (*simp add*: *a* ‹*a* ∈ *S*› *flip*: *hom_induced_compose*)
have *?g* (*y* $\otimes_{?H}$ *inv*$_{?H}$ (*?f* ∘ *?g*) *y*)
= $\mathbf{1}_{homology\_group\ p\ (discrete\_topology\ \{()\})}$
by (*simp add*: *a* ‹*a* ∈ *S*› *that hom_induced_carrier flip*: *hom_induced_compose* ∗ [*unfolded o_def*])
then show *?g* (*y* $\otimes_{?H}$ *inv*$_{?H}$ *?f* (*?g y*))

```
        = 1homology_group p (discrete_topology {()})
      by simp
    qed
  qed
  show ?thesis
     apply (auto simp: reduced_homology_group_def carrier_subgroup_generated
kernel_def image_iff)
    apply (metis (no_types, lifting) generate_in_carrier mem_Collect_eq subsetI)
    apply (force simp: dest: ∗ intro: generate.incl)
    done
qed


lemma homology_exactness_reduced_1:
  assumes topspace X ∩ S ≠ {}
  shows  exact_seq([reduced_homology_group(p − 1) (subtopology X S),
                    relative_homology_group p X S,
                    reduced_homology_group p X],
                   [hom_boundary p X S, hom_induced p X {} X S id])
    (is exact_seq ([?G1,?G2,?G3], [?h1,?h2]))
proof −
  have ∗: ?h2 ‘ carrier (homology_group p X)
        = kernel ?G2 (homology_group (p − 1) (subtopology X S)) ?h1
    using homology_exactness_axiom_1 [of p X S] by simp
  have gh: group_hom ?G3 ?G2 ?h2
   by (simp add: reduced_homology_group_def group_hom_def group_hom_axioms_def
     group.group_subgroup_generated group.hom_from_subgroup_generated hom_induced_hom)
  show ?thesis
   apply (simp add: hom_boundary_reduced_hom gh ∗ image_reduced_homology_group
[OF assms])
    apply (simp add: kernel_def one_reduced_homology_group)
    done
qed


lemma homology_exactness_reduced_2:
    exact_seq([reduced_homology_group(p − 1) X,
                 reduced_homology_group(p − 1) (subtopology X S),
                 relative_homology_group p X S],
               [hom_induced (p − 1) (subtopology X S) {} X {} id, hom_boundary
p X S])
    (is exact_seq ([?G1,?G2,?G3], [?h1,?h2]))
  using homology_exactness_axiom_2 [of p X S]
 apply (simp add: group_hom_axioms_def group_hom_def hom_boundary_reduced_hom
hom_induced_reduced_hom)
 apply (simp add: reduced_homology_group_def group_hom.subgroup_kernel group_hom_axioms_def
group_hom_def hom_induced_hom)
  using hom_boundary_reduced_hom [of p X S]
  apply (auto simp: image_def set_eq_iff)
```

```
  by (metis carrier_reduced_homology_group hom_in_carrier set_eq_iff)


lemma homology_exactness_reduced_3:
   exact_seq([relative_homology_group p X S,
              reduced_homology_group p X,
              reduced_homology_group p (subtopology X S)],
             [hom_induced p X {} X S id, hom_induced p (subtopology X S) {} X
{} id])
    (is exact_seq ([?G1,?G2,?G3], [?h1,?h2]))
proof −
  have kernel ?G2 ?G1 ?h1 =
      ?h2 ‘ carrier ?G3
  proof −
    obtain U where U:
      (hom_induced p (subtopology X S) {} X {} id) ‘ carrier ?G3 ⊆ U
      (hom_induced p (subtopology X S) {} X {} id) ‘ carrier ?G3
      ⊆ (hom_induced p (subtopology X S) {} X {} id) ‘ carrier (homology_group
p (subtopology X S))
      U ∩ kernel (homology_group p X) ?G1 (hom_induced p X {} X S id)
     = kernel ?G2 ?G1 (hom_induced p X {} X S id)
     U ∩ (hom_induced p (subtopology X S) {} X {} id) ‘ carrier (homology_group
p (subtopology X S))
    ⊆ (hom_induced p (subtopology X S) {} X {} id) ‘ carrier ?G3
    proof
      show ?h2 ‘ carrier ?G3 ⊆ carrier ?G2
        by (simp add: hom_induced_reduced image_subset_iff)
      show ?h2 ‘ carrier ?G3 ⊆ ?h2 ‘ carrier (homology_group p (subtopology X
S))
        by (meson carrier_reduced_homology_group_subset image_mono)
    have subgroup (kernel (homology_group p X) (homology_group p (discrete_topology
{()}))
                          (hom_induced p X {} (discrete_topology {()}) {} (λx. ())))
                    (homology_group p X)
      by (simp add: group.normal_invE(1) group_hom.normal_kernel group_hom_axioms_def
group_hom_def hom_induced_empty_hom)
      then show carrier ?G2 ∩ kernel (homology_group p X) ?G1 ?h1 = kernel
?G2 ?G1 ?h1
        unfolding carrier_reduced_homology_group
        by (auto simp: reduced_homology_group_def)
    show carrier ?G2 ∩ ?h2 ‘ carrier (homology_group p (subtopology X S))
      ⊆ ?h2 ‘ carrier ?G3
    by (force simp: carrier_reduced_homology_group kernel_def hom_induced_compose')
  qed
  with homology_exactness_axiom_3 [of p X S] show ?thesis
    by (fastforce simp add:)
qed
  then show ?thesis
   apply (simp add: group_hom_axioms_def group_hom_def hom_boundary_reduced_hom
```

hom_induced_reduced_hom)
apply (*simp add*: *group.hom_from_subgroup_generated hom_induced_hom reduced_homology_group_def*)
done
qed

0.3.2 More homology properties of deformations, retracts, contractible spaces

lemma *iso_relative_homology_of_contractible*:
⟦*contractible_space X*; *topspace X* ∩ *S* ≠ {}⟧
⟹ *hom_boundary p X S*
∈ *iso* (*relative_homology_group p X S*) (*reduced_homology_group*(*p* − *1*) (*subtopology X S*))
using *very_short_exact_sequence*
[*of reduced_homology_group* (*p* − *1*) *X*
reduced_homology_group (*p* − *1*) (*subtopology X S*)
relative_homology_group p X S
reduced_homology_group p X
hom_induced (*p* − *1*) (*subtopology X S*) {} *X* {} *id*
hom_boundary p X S
hom_induced p X {} *X S id*]
by (*meson exact_seq_cons_iff homology_exactness_reduced_1 homology_exactness_reduced_2 trivial_reduced_homology_group_contractible_space*)

lemma *isomorphic_group_relative_homology_of_contractible*:
⟦*contractible_space X*; *topspace X* ∩ *S* ≠ {}⟧
⟹ *relative_homology_group p X S* ≅
reduced_homology_group(*p* − *1*) (*subtopology X S*)
by (*meson iso_relative_homology_of_contractible is_isoI*)

lemma *isomorphic_group_reduced_homology_of_contractible*:
⟦*contractible_space X*; *topspace X* ∩ *S* ≠ {}⟧
⟹ *reduced_homology_group p* (*subtopology X S*) ≅ *relative_homology_group*(*p* + *1*) *X S*
by (*metis add.commute add_diff_cancel_left′ group.iso_sym group_relative_homology_group isomorphic_group_relative_homology_of_contractible*)

lemma *iso_reduced_homology_by_contractible*:
⟦*contractible_space*(*subtopology X S*); *topspace X* ∩ *S* ≠ {}⟧
⟹ (*hom_induced p X* {} *X S id*) ∈ *iso* (*reduced_homology_group p X*) (*relative_homology_group p X S*)
using *very_short_exact_sequence*
[*of reduced_homology_group* (*p* − *1*) (*subtopology X S*)
relative_homology_group p X S
reduced_homology_group p X
reduced_homology_group p (*subtopology X S*)
hom_boundary p X S
hom_induced p X {} *X S id*

```
      hom_induced p (subtopology X S) {} X {} id]
  by (meson exact_seq_cons_iff homology_exactness_reduced_1 homology_exactness_reduced_3
trivial_reduced_homology_group_contractible_space)

lemma isomorphic_reduced_homology_by_contractible:
   ⟦contractible_space(subtopology X S); topspace X ∩ S ≠ {}⟧
     ⟹ reduced_homology_group p X ≅ relative_homology_group p X S
  using is_isoI iso_reduced_homology_by_contractible by blast

lemma isomorphic_relative_homology_by_contractible:
   ⟦contractible_space(subtopology X S); topspace X ∩ S ≠ {}⟧
     ⟹ relative_homology_group p X S ≅ reduced_homology_group p X
  using group.iso_sym group_reduced_homology_group isomorphic_reduced_homology_by_contractible
by blast

lemma isomorphic_reduced_homology_by_singleton:
    a ∈ topspace X ⟹ reduced_homology_group p X ≅ relative_homology_group
p X ({a})
  by (simp add: contractible_space_subtopology_singleton isomorphic_reduced_homology_by_contractible)

lemma isomorphic_relative_homology_by_singleton:
   a ∈ topspace X ⟹ relative_homology_group p X ({a}) ≅ reduced_homology_group
p X
  by (simp add: group.iso_sym isomorphic_reduced_homology_by_singleton)

lemma reduced_homology_group_pair:
  assumes t1_space X and a: a ∈ topspace X and b: b ∈ topspace X and a ≠ b
  shows reduced_homology_group p (subtopology X {a,b}) ≅ homology_group p
(subtopology X {a})
        (is ?lhs ≅ ?rhs)
proof −
  have ?lhs ≅ relative_homology_group p (subtopology X {a,b}) {b}
   by (simp add: b isomorphic_reduced_homology_by_singleton topspace_subtopology)
  also have . . . ≅ ?rhs
  proof −
    have sub: subtopology X {a, b} closure_of {b} ⊆ subtopology X {a, b} interior_of {b}
       by (simp add: assms t1_space_subtopology closure_of_singleton subtopology_eq_discrete_topology_finite discrete_topology_closure_of)
    show ?thesis
      using homology_excision_axiom [OF sub, of {a,b} p]
      by (simp add: assms(4) group.iso_sym is_isoI subtopology_subtopology)
  qed
  finally show ?thesis .
qed


lemma deformation_retraction_relative_homology_group_isomorphisms:
   ⟦retraction_maps X Y r s; r ∈ U → V; s ∈ V → U; homotopic_with (λh. h ‘
```

U ⊆ U) X X (s ∘ r) id⟧
⟹ group_isomorphisms (relative_homology_group p X U) (relative_homology_group p Y V)
(hom_induced p X U Y V r) (hom_induced p Y V X U s)
apply (simp add: retraction_maps_def)
apply (rule homotopy_equivalence_relative_homology_group_isomorphisms)
apply (auto simp: image_subset_iff_funcset Pi_iff continuous_map_compose homotopic_with_equal)
done

lemma deformation_retract_relative_homology_group_isomorphisms:
⟦retraction_maps X Y r id; V ⊆ U; r ∈ U → V; homotopic_with (λh. h ‘ U ⊆ U) X X r id⟧
⟹ group_isomorphisms (relative_homology_group p X U) (relative_homology_group p Y V)
(hom_induced p X U Y V r) (hom_induced p Y V X U id)
by (simp add: deformation_retraction_relative_homology_group_isomorphisms in_mono)

lemma deformation_retract_relative_homology_group_isomorphism:
⟦retraction_maps X Y r id; V ⊆ U; r ∈ U → V; homotopic_with (λh. h ‘ U ⊆ U) X X r id⟧
⟹ (hom_induced p X U Y V r) ∈ iso (relative_homology_group p X U) (relative_homology_group p Y V)
by (metis deformation_retract_relative_homology_group_isomorphisms group_isomorphisms_imp_is

lemma deformation_retract_relative_homology_group_isomorphism_id:
⟦retraction_maps X Y r id; V ⊆ U; r ∈ U → V; homotopic_with (λh. h ‘ U ⊆ U) X X r id⟧
⟹ (hom_induced p Y V X U id) ∈ iso (relative_homology_group p Y V) (relative_homology_group p X U)
by (metis deformation_retract_relative_homology_group_isomorphisms group_isomorphisms_imp_is
group_isomorphisms_sym)

lemma deformation_retraction_imp_isomorphic_relative_homology_groups:
⟦retraction_maps X Y r s; r ∈ U → V; s ‘ V ⊆ U; homotopic_with (λh. h ‘ U ⊆ U) X X (s ∘ r) id⟧
⟹ relative_homology_group p X U ≅ relative_homology_group p Y V
by (blast intro: is_isoI group_isomorphisms_imp_iso deformation_retraction_relative_homology_grou

lemma deformation_retraction_imp_isomorphic_homology_groups:
⟦retraction_maps X Y r s; homotopic_with (λh. True) X X (s ∘ r) id⟧
⟹ homology_group p X ≅ homology_group p Y
by (simp add: deformation_retraction_imp_homotopy_equivalent_space homotopy_equivalent_space_imp_isomorphic_homology_groups)

lemma deformation_retract_imp_isomorphic_relative_homology_groups:
⟦retraction_maps X X′ r id; V ⊆ U; r ∈ U → V; homotopic_with (λh. h ‘ U

⊆ *U*) *X X r id*⟧
⟹ *relative_homology_group p X U* ≅ *relative_homology_group p X′ V*
by (*simp add*: *deformation_retraction_imp_isomorphic_relative_homology_groups*)

lemma *deformation_retract_imp_isomorphic_homology_groups*:
⟦*retraction_maps X X′ r id*; *homotopic_with* (λ*h*. *True*) *X X r id*⟧
⟹ *homology_group p X* ≅ *homology_group p X′*
by (*simp add*: *deformation_retraction_imp_isomorphic_homology_groups*)

lemma *epi_hom_induced_inclusion*:
assumes *homotopic_with* (λ*x*. *True*) *X X id f* **and** *f* ∈ *topspace X* → *S*
shows (*hom_induced p* (*subtopology X S*) {} *X* {} *id*)
∈ *epi* (*homology_group p* (*subtopology X S*)) (*homology_group p X*)
proof (*rule epi_right_invertible*)
show *hom_induced p* (*subtopology X S*) {} *X* {} *id*
∈ *hom* (*homology_group p* (*subtopology X S*)) (*homology_group p X*)
by (*simp add*: *hom_induced_empty_hom*)
show *hom_induced p X* {} (*subtopology X S*) {} *f*
∈ *carrier* (*homology_group p X*) → *carrier* (*homology_group p* (*subtopology X S*))
by (*simp add*: *hom_induced_carrier*)
fix *x*
assume *x*: *x* ∈ *carrier* (*homology_group p X*)
show *hom_induced p* (*subtopology X S*) {} *X* {} *id* (*hom_induced p X* {} (*subtopology X S*) {} *f x*) = *x*
proof (*subst hom_induced_compose′*)
show *continuous_map X* (*subtopology X S*) *f*
by (*meson assms continuous_map_into_subtopology homotopic_with_imp_continuous_maps*)
show *hom_induced p X* {} *X* {} (*id* ∘ *f*) *x* = *x*
by (*metis assms*(*1*) *hom_induced_id homology_homotopy_empty id_comp x*)
qed (*use assms* **in** *auto*)
qed

lemma *trivial_homomorphism_hom_induced_relativization*:
assumes *homotopic_with* (λ*x*. *True*) *X X id f* **and** *f* ∈ *topspace X* → *S*
shows *trivial_homomorphism* (*homology_group p X*) (*relative_homology_group p X S*)
(*hom_induced p X* {} *X S id*)
proof –
have (*hom_induced p* (*subtopology X S*) {} *X* {} *id*)
∈ *epi* (*homology_group p* (*subtopology X S*)) (*homology_group p X*)
by (*metis assms epi_hom_induced_inclusion*)
then show *?thesis*
using *homology_exactness_axiom_3* [*of p X S*] *homology_exactness_axiom_1* [*of p X S*]

by (*simp add*: *epi_def group.trivial_homomorphism_image group_hom.trivial_hom_iff*)
qed

lemma *mon_hom_boundary_inclusion*:
assumes *homotopic_with* (λ*x*. *True*) *X X id f* **and** *f* ∈ *topspace X* → *S*
shows (*hom_boundary p X S*) ∈ *mon*
(*relative_homology_group p X S*) (*homology_group* (*p* − *1*) (*subtopology X S*))
proof −
have (*hom_induced p* (*subtopology X S*) {} *X* {} *id*)
∈ *epi* (*homology_group p* (*subtopology X S*)) (*homology_group p X*)
by (*metis assms epi_hom_induced_inclusion*)
then show *?thesis*
using *homology_exactness_axiom_3* [*of p X S*] *homology_exactness_axiom_1* [*of p X S*]
apply (*simp add*: *mon_def epi_def hom_boundary_hom*)
by (*metis* (*no_types*, *opaque_lifting*) *group_hom.trivial_hom_iff group_hom.trivial_ker_imp_inj group_hom_axioms_def group_hom_def group_relative_homology_group hom_boundary_hom*)
qed

lemma *short_exact_sequence_hom_induced_relativization*:
assumes *homotopic_with* (λ*x*. *True*) *X X id f* **and** *f* ∈ *topspace X* → *S*
shows *short_exact_sequence* (*homology_group* (*p*−*1*) *X*) (*homology_group* (*p*−*1*) (*subtopology X S*)) (*relative_homology_group p X S*)
(*hom_induced* (*p*−*1*) (*subtopology X S*) {} *X* {} *id*) (*hom_boundary p X S*)
unfolding *short_exact_sequence_iff*
by (*intro conjI homology_exactness_axiom_2 epi_hom_induced_inclusion* [*OF assms*] *mon_hom_boundary_inclusion* [*OF assms*])

lemma *group_isomorphisms_homology_group_prod_deformation*:
fixes *p*::*int*
assumes *homotopic_with* (λ*x*. *True*) *X X id f* **and** *f* ∈ *topspace X* → *S*
obtains *H K* **where**
subgroup H (*homology_group p* (*subtopology X S*))
subgroup K (*homology_group p* (*subtopology X S*))
(λ(*x*, *y*). $x \otimes_{homology\_group\ p\ (subtopology\ X\ S)} y$)
∈ *Group.iso* (*subgroup_generated* (*homology_group p* (*subtopology X S*)) *H* ××
subgroup_generated (*homology_group p* (*subtopology X S*)) *K*)
(*homology_group p* (*subtopology X S*))
hom_boundary (*p* + *1*) *X S*
∈ *Group.iso* (*relative_homology_group* (*p* + *1*) *X S*)
(*subgroup_generated* (*homology_group p* (*subtopology X S*)) *H*)
hom_induced p (*subtopology X S*) {} *X* {} *id*
∈ *Group.iso*
(*subgroup_generated* (*homology_group p* (*subtopology X S*)) *K*)

```
      (homology_group p X)
proof −
  let ?rhs = relative_homology_group (p + 1) X S
  let ?pXS = homology_group p (subtopology X S)
  let ?pX = homology_group p X
  let ?hb = hom_boundary (p + 1) X S
  let ?hi = hom_induced p (subtopology X S) {} X {} id
  have x: short_exact_sequence (?pX) ?pXS ?rhs ?hi ?hb
    using short_exact_sequence_hom_induced_relativization [OF assms, of p +
1] by simp
  have contf: continuous_map X (subtopology X S) f
   by (metis assms continuous_map_into_subtopology homotopic_with_imp_continuous_maps)
  obtain H K where HK: H ◁ ?pXS subgroup K ?pXS H ∩ K ⊆ {one ?pXS}
set_mult ?pXS H K = carrier ?pXS
   and iso: ?hb ∈ iso ?rhs (subgroup_generated ?pXS H) ?hi ∈ iso (subgroup_generated
?pXS K) ?pX
   proof (rule splitting_lemma_right [OF x, where g' = hom_induced p X {}
(subtopology X S) {} f])
    show hom_induced p X {} (subtopology X S) {} f ∈ hom (homology_group p
X) (homology_group p (subtopology X S))
      using hom_induced_empty_hom by blast
  next
    fix z
    assume z ∈ carrier (homology_group p X)
    then show hom_induced p (subtopology X S) {} X {} id (hom_induced p X
{} (subtopology X S) {} f z) = z
      using assms(1) contf hom_induced_id homology_homotopy_empty
      by (fastforce simp add: hom_induced_compose')
  qed blast
  show ?thesis
  proof
    show subgroup H ?pXS
      using HK(1) normal_imp_subgroup by blast
    then show (λ(x, y). x ⊗?pXS y)
       ∈ Group.iso (subgroup_generated (?pXS) H ×× subgroup_generated (?pXS)
K) (?pXS)
     by (meson HK abelian_relative_homology_group group_disjoint_sum.iso_group_mul
group_disjoint_sum_def group_relative_homology_group)
    show subgroup K ?pXS
      by (rule HK)
     show hom_boundary (p + 1) X S ∈ Group.iso ?rhs (subgroup_generated
(?pXS) H)
      using iso int_ops(4) by presburger
   show hom_induced p (subtopology X S) {} X {} id ∈ Group.iso (subgroup_generated
(?pXS) K) (?pX)
      by (simp add: iso(2))
  qed
qed
```

lemma *iso_homology_group_prod_deformation*:
assumes *homotopic_with* (*λx. True*) *X X id f* **and** *f* ∈ *topspace X* → *S*
shows *homology_group p* (*subtopology X S*)
≅ *DirProd* (*homology_group p X*) (*relative_homology_group*(*p* + *1*) *X S*)
(**is** *?G* ≅ *DirProd ?H ?R*)
proof −
obtain *H K* **where** *HK*:
(λ(*x*, *y*). *x* ⊗$_{?G}$ *y*)
∈ *Group.iso* (*subgroup_generated* (*?G*) *H* ×× *subgroup_generated* (*?G*) *K*) (*?G*)
hom_boundary (*p* + *1*) *X S* ∈ *Group.iso* (*?R*) (*subgroup_generated* (*?G*) *H*)
hom_induced p (*subtopology X S*) {} *X* {} *id* ∈ *Group.iso* (*subgroup_generated* (*?G*) *K*) (*?H*)
by (*blast intro*: *group_isomorphisms_homology_group_prod_deformation* [*OF assms*])
have *?G* ≅ *DirProd* (*subgroup_generated* (*?G*) *H*) (*subgroup_generated* (*?G*) *K*)
by (*meson DirProd_group HK*(*1*) *group.group_subgroup_generated group.iso_sym group_relative_homology_group is_isoI*)
also have ... ≅ *DirProd ?R ?H*
by (*meson HK group.DirProd_iso_trans group.group_subgroup_generated group.iso_sym group_relative_homology_group is_isoI*)
also have ... ≅ *DirProd ?H ?R*
by (*simp add*: *DirProd_commute_iso*)
finally show *?thesis* **.**
qed

lemma *iso_homology_contractible_space_subtopology1*:
assumes *contractible_space X S* ⊆ *topspace X S* ≠ {}
shows *homology_group 0* (*subtopology X S*) ≅ *DirProd integer_group* (*relative_homology_group*(*1*) *X S*)
proof −
obtain *f* **where** *homotopic_with* (*λx. True*) *X X id f* **and** *f* ∈ *topspace X* → *S*
using *assms contractible_space_alt* **by** *fastforce*
then have *homology_group 0* (*subtopology X S*) ≅ *homology_group 0 X* ×× *relative_homology_group 1 X S*
using *iso_homology_group_prod_deformation* [*of X _ S 0*] **by** *auto*
also have ... ≅ *integer_group* ×× *relative_homology_group 1 X S*
using *assms contractible_imp_path_connected_space group.DirProd_iso_trans group_relative_homology_group iso_refl isomorphic_integer_zeroth_homology_group* **by** *blast*
finally show *?thesis* **.**
qed

lemma *iso_homology_contractible_space_subtopology2*:
⟦*contractible_space X*; *S* ⊆ *topspace X*; *p* ≠ *0*; *S* ≠ {}⟧
⟹ *homology_group p* (*subtopology X S*) ≅ *relative_homology_group* (*p* + *1*)

X S
by (*metis* (*no_types*, *opaque_lifting*) *add.commute isomorphic_group_reduced_homology_of_contractible topspace_subtopology topspace_subtopology_subset un_reduced_homology_group*)

lemma *trivial_relative_homology_group_contractible_spaces*:
⟦*contractible_space X*; *contractible_space*(*subtopology X S*); *topspace X* ∩ *S* ≠ {}⟧
⟹ *trivial_group*(*relative_homology_group p X S*)
using *group_reduced_homology_group group_relative_homology_group isomorphic_group_triviality isomorphic_relative_homology_by_contractible trivial_reduced_homology_group_contractible*
by *blast*

lemma *trivial_relative_homology_group_alt*:
assumes *contf*: *continuous_map X* (*subtopology X S*) *f* **and** *hom*: *homotopic_with* (λ*k*. *k* ‘ *S* ⊆ *S*) *X X f id*
shows *trivial_group* (*relative_homology_group p X S*)
proof (*rule trivial_relative_homology_group_gen* [*OF contf*])
show *homotopic_with* (λ*h*. *True*) (*subtopology X S*) (*subtopology X S*) *f id*
using *hom* **unfolding** *homotopic_with_def*
apply (*rule ex_forward*)
apply (*auto simp*: *prod_topology_subtopology continuous_map_in_subtopology continuous_map_from_subtopology image_subset_iff topspace_subtopology*)
done
show *homotopic_with* (λ*k*. *True*) *X X f id*
using *assms* **by** (*force simp*: *homotopic_with_def*)
qed

lemma *iso_hom_induced_relativization_contractible*:
assumes *contractible_space*(*subtopology X S*) *contractible_space*(*subtopology X T*) *T* ⊆ *S topspace X* ∩ *T* ≠ {}
shows (*hom_induced p X T X S id*) ∈ *iso* (*relative_homology_group p X T*) (*relative_homology_group p X S*)
proof (*rule very_short_exact_sequence*)
show *exact_seq*
([*relative_homology_group*(*p* − *1*) (*subtopology X S*) *T*, *relative_homology_group p X S*, *relative_homology_group p X T*, *relative_homology_group p* (*subtopology X S*) *T*],
[*hom_relboundary p X S T*, *hom_induced p X T X S id*, *hom_induced p* (*subtopology X S*) *T X T id*])
using *homology_exactness_triple_1* [*OF* ‹*T* ⊆ *S*›] *homology_exactness_triple_3* [*OF* ‹*T* ⊆ *S*›]
by *fastforce*
show *trivial_group* (*relative_homology_group p* (*subtopology X S*) *T*) *trivial_group* (*relative_homology_group*(*p* − *1*) (*subtopology X S*) *T*)
using *assms*
by (*force simp*: *inf.absorb_iff2 subtopology_subtopology topspace_subtopology intro*!: *trivial_relative_homology_group_contractible_spaces*)+
qed

corollary *isomorphic_relative_homology_groups_relativization_contractible*:
assumes *contractible_space*(*subtopology X S*) *contractible_space*(*subtopology X T*) *T* ⊆ *S topspace X* ∩ *T* ≠ {}
shows *relative_homology_group p X T* ≅ *relative_homology_group p X S*
by (*rule is_isoI*) (*rule iso_hom_induced_relativization_contractible* [*OF assms*])

lemma *iso_hom_induced_inclusion_contractible*:
assumes *contractible_space X contractible_space*(*subtopology X S*) *T* ⊆ *S topspace X* ∩ *S* ≠ {}
shows (*hom_induced p* (*subtopology X S*) *T X T id*)
∈ *iso* (*relative_homology_group p* (*subtopology X S*) *T*) (*relative_homology_group p X T*)
proof (*rule very_short_exact_sequence*)
show *exact_seq*
([*relative_homology_group p X S*, *relative_homology_group p X T*,
relative_homology_group p (*subtopology X S*) *T*, *relative_homology_group* (*p*+*1*) *X S*],
[*hom_induced p X T X S id*, *hom_induced p* (*subtopology X S*) *T X T id*, *hom_relboundary* (*p*+*1*) *X S T*])
using *homology_exactness_triple_2* [*OF* ‹*T* ⊆ *S*›] *homology_exactness_triple_3* [*OF* ‹*T* ⊆ *S*›]
by (*metis add_diff_cancel_left′ diff_add_cancel exact_seq_cons_iff*)
show *trivial_group* (*relative_homology_group* (*p*+*1*) *X S*) *trivial_group* (*relative_homology_group p X S*)
using *assms*
by (*auto simp*: *subtopology_subtopology topspace_subtopology intro*!: *trivial_relative_homology_group_*
qed

corollary *isomorphic_relative_homology_groups_inclusion_contractible*:
assumes *contractible_space X contractible_space*(*subtopology X S*) *T* ⊆ *S topspace X* ∩ *S* ≠ {}
shows *relative_homology_group p* (*subtopology X S*) *T* ≅ *relative_homology_group p X T*
by (*rule is_isoI*) (*rule iso_hom_induced_inclusion_contractible* [*OF assms*])

lemma *iso_hom_relboundary_contractible*:
assumes *contractible_space X contractible_space*(*subtopology X T*) *T* ⊆ *S topspace X* ∩ *T* ≠ {}
shows *hom_relboundary p X S T*
∈ *iso* (*relative_homology_group p X S*) (*relative_homology_group* (*p* − *1*) (*subtopology X S*) *T*)
proof (*rule very_short_exact_sequence*)
show *exact_seq*
([*relative_homology_group* (*p* − *1*) *X T*, *relative_homology_group* (*p* − *1*) (*subtopology X S*) *T*, *relative_homology_group p X S*, *relative_homology_group p X T*],
[*hom_induced* (*p* − *1*) (*subtopology X S*) *T X T id*, *hom_relboundary p X S T*, *hom_induced p X T X S id*])

```
  using homology_exactness_triple_1 [OF ‹T ⊆ S›] homology_exactness_triple_2
[OF ‹T ⊆ S›] by simp
 show trivial_group (relative_homology_group p X T) trivial_group (relative_homology_group
(p − 1) X T)
   using assms
  by (auto simp: subtopology_subtopology topspace_subtopology intro!: trivial_relative_homology_group_contractib
qed

corollary isomorphic_relative_homology_groups_relboundary_contractible:
 assumes contractible_space X contractible_space(subtopology X T) T ⊆ S topspace
X ∩ T ≠ {}
  shows relative_homology_group p X S ≅ relative_homology_group (p − 1)
(subtopology X S) T
  by (rule is_isoI) (rule iso_hom_relboundary_contractible [OF assms])

lemma isomorphic_relative_contractible_space_imp_homology_groups:
 assumes contractible_space X contractible_space Y S ⊆ topspace X T ⊆ topspace
Y
    and ST: S = {} ⟷ T = {}
    and iso: ⋀p. relative_homology_group p X S ≅ relative_homology_group p Y
T
 shows homology_group p (subtopology X S) ≅ homology_group p (subtopology Y
T)
proof (cases T = {})
  case True
  have homology_group p (subtopology X {}) ≅ homology_group p (subtopology Y
{})
   by (simp add: homeomorphic_empty_space_eq homeomorphic_space_imp_isomorphic_homology_groups)
  then show ?thesis
    using ST True by blast
next
  case False
  show ?thesis
  proof (cases p = 0)
    case True
   have homology_group p (subtopology X S) ≅ integer_group ×× relative_homology_group
1 X S
      using assms True ‹T ≠ {}›
      by (simp add: iso_homology_contractible_space_subtopology1)
    also have ... ≅ integer_group ×× relative_homology_group 1 Y T
      by (simp add: assms group.DirProd_iso_trans iso_refl)
    also have ... ≅ homology_group p (subtopology Y T)
     by (simp add: True ‹T ≠ {}› assms group.iso_sym iso_homology_contractible_space_subtopology1)
    finally show ?thesis .
  next
    case False
    have homology_group p (subtopology X S) ≅ relative_homology_group (p+1)
X S
      using assms False ‹T ≠ {}›
```

```
    by (simp add: iso_homology_contractible_space_subtopology2)
  also have ... ≅ relative_homology_group (p+1) Y T
    by (simp add: assms)
  also have ... ≅ homology_group p (subtopology Y T)
   by (simp add: False ‹T ≠ {}› assms group.iso_sym iso_homology_contractible_space_subtopology2)
  finally show ?thesis .
 qed
qed
```

0.3.3 Homology groups of spheres

```
lemma iso_reduced_homology_group_lower_hemisphere:
 assumes k ≤ n
 shows hom_induced p (nsphere n) {} (nsphere n) {x. x k ≤ 0} id
     ∈ iso (reduced_homology_group p (nsphere n)) (relative_homology_group p
(nsphere n) {x. x k ≤ 0})
proof (rule iso_reduced_homology_by_contractible)
 show contractible_space (subtopology (nsphere n) {x. x k ≤ 0})
  by (simp add: assms contractible_space_lower_hemisphere)
 have (λi. if i = k then −1 else 0) ∈ topspace (nsphere n) ∩ {x. x k ≤ 0}
  using assms by (simp add: nsphere if_distrib [of λx. x ^ 2] cong: if_cong)
 then show topspace (nsphere n) ∩ {x. x k ≤ 0} ≠ {}
  by blast
qed


lemma topspace_nsphere_1:
 assumes x ∈ topspace (nsphere n) shows (x k)² ≤ 1
proof (cases k ≤ n)
 case True
 have (∑ i ∈ {..n} − {k}. (x i)²) = (∑ i≤n. (x i)²) − (x k)²
  using ‹k ≤ n› by (simp add: sum_diff)
 then show ?thesis
  using assms
  apply (simp add: nsphere)
  by (metis diff_ge_0_iff_ge sum_nonneg zero_le_power2)
next
 case False
 then show ?thesis
  using assms by (simp add: nsphere)
qed

lemma topspace_nsphere_1_eq_0:
 fixes x :: nat ⇒ real
 assumes x: x ∈ topspace (nsphere n) and xk: (x k)² = 1 and i ≠ k
 shows x i = 0
proof (cases i ≤ n)
 case True
 have k ≤ n
```

```
    using x
    by (simp add: nsphere) (metis not_less xk zero_neq_one zero_power2)
  have (∑ i ∈ {..n} − {k}. (x i)^2) = (∑ i≤n. (x i)^2) − (x k)^2
    using ‹k ≤ n› by (simp add: sum_diff)
  also have ... = 0
    using assms by (simp add: nsphere)
  finally have ∀ i∈{..n} − {k}. (x i)^2 = 0
    by (simp add: sum_nonneg_eq_0_iff)
  then show ?thesis
    using True ‹i ≠ k› by auto
next
  case False
  with x show ?thesis
    by (simp add: nsphere)
qed


proposition iso_relative_homology_group_upper_hemisphere:
   (hom_induced p (subtopology (nsphere n) {x. x k ≥ 0}) {x. x k = 0} (nsphere
n) {x. x k ≤ 0} id)
  ∈ iso (relative_homology_group p (subtopology (nsphere n) {x. x k ≥ 0}) {x. x
k = 0})
        (relative_homology_group p (nsphere n) {x. x k ≤ 0}) (is ?h ∈ iso ?G ?H)
proof −
  have topspace (nsphere n) ∩ {x. x k < − 1 / 2} ⊆ {x ∈ topspace (nsphere n).
x k ∈ {y. y ≤ − 1 / 2}}
    by force
  moreover have closedin (nsphere n) {x ∈ topspace (nsphere n). x k ∈ {y. y ≤
− 1 / 2}}
   apply (rule closedin_continuous_map_preimage [OF continuous_map_nsphere_projection])
    using closed_Collect_le [of id λx::real. −1/2] apply simp
    done
   ultimately have nsphere n closure_of {x. x k < −1/2} ⊆ {x ∈ topspace
(nsphere n). x k ∈ {y. y ≤ −1/2}}
   by (metis (no_types, lifting) closure_of_eq closure_of_mono closure_of_restrict)
  also have ... ⊆ {x ∈ topspace (nsphere n). x k ∈ {y. y < 0}}
    by force
  also have ... ⊆ nsphere n interior_of {x. x k ≤ 0}
  proof (rule interior_of_maximal)
    show {x ∈ topspace (nsphere n). x k ∈ {y. y < 0}} ⊆ {x. x k ≤ 0}
      by force
    show openin (nsphere n) {x ∈ topspace (nsphere n). x k ∈ {y. y < 0}}
     apply (rule openin_continuous_map_preimage [OF continuous_map_nsphere_projection])
      using open_Collect_less [of id λx::real. 0] apply simp
      done
  qed
  finally have nn: nsphere n closure_of {x. x k < −1/2} ⊆ nsphere n interior_of
{x. x k ≤ 0} .
  have [simp]: {x::nat⇒real. x k ≤ 0} − {x. x k < − (1/2)} = {x. −1/2 ≤ x k
```

∧ *x k* ≤ *0*}
UNIV − {*x*::*nat*⇒*real*. *x k* < *a*} = {*x*. *a* ≤ *x k*} **for** *a*
by *auto*
let *?T01* = *top_of_set* {*0*..*1*::*real*}
let *?X12* = *subtopology* (*nsphere n*) {*x*. −*1*/*2* ≤ *x k*}
have *1*: *hom_induced p ?X12* {*x*. −*1*/*2* ≤ *x k* ∧ *x k* ≤ *0*} (*nsphere n*) {*x*. *x k* ≤ *0*} *id*
∈ *iso* (*relative_homology_group p ?X12* {*x*. −*1*/*2* ≤ *x k* ∧ *x k* ≤ *0*})
?H
using *homology_excision_axiom* [*OF nn subset_UNIV*, *of p*] **by** *simp*
define *h* **where** *h* ≡ λ(*T*,*x*). *let y* = *max* (*x k*) (−*T*) *in*
(λ*i*. *if i* = *k then y else sqrt*(*1* − *y* ^ *2*) / *sqrt*(*1* − *x k* ^ *2*) ∗ *x i*)
have *h*: *h*(*T*,*x*) = *x* **if** *0* ≤ *T T* ≤ *1* ($\sum$ *i*≤*n*. (*x i*)^2) = *1* **and** *0*: ∀ *i*>*n*. *x i* = *0* −*T* ≤ *x k* **for** *T x*
using *that* **by** (*force simp*: *nsphere h_def Let_def max_def intro*!: *topspace_nsphere_1_eq_0*)
have *continuous_map* (*prod_topology ?T01 ?X12*) *euclideanreal* (λ*x*. *h x i*) **for** *i*
proof −
show *?thesis*
proof (*rule continuous_map_eq*)
show *continuous_map* (*prod_topology ?T01 ?X12*)
euclideanreal (λ(*T*, *x*). *if 0* ≤ *x k then x i else h* (*T*, *x*) *i*)
unfolding *case_prod_unfold*
proof (*rule continuous_map_cases_le*)
show *continuous_map* (*prod_topology ?T01 ?X12*) *euclideanreal* (λ*x*. *snd x k*)
apply (*subst continuous_map_of_snd* [*unfolded o_def*])
by (*simp add*: *continuous_map_from_subtopology continuous_map_nsphere_projection*)
next
show *continuous_map* (*subtopology* (*prod_topology ?T01 ?X12*) {*p* ∈ *topspace* (*prod_topology ?T01 ?X12*). *0* ≤ *snd p k*})
euclideanreal (λ*x*. *snd x i*)
apply (*rule continuous_map_from_subtopology*)
apply (*subst continuous_map_of_snd* [*unfolded o_def*])
by (*simp add*: *continuous_map_from_subtopology continuous_map_nsphere_projection*)
next
note *fst* = *continuous_map_into_fulltopology* [*OF continuous_map_subtopology_fst*]
have *snd*: *continuous_map* (*subtopology* (*prod_topology ?T01* (*subtopology* (*nsphere n*) *T*)) *S*) *euclideanreal* (λ*x*. *snd x k*) **for** *k S T*
apply (*simp add*: *nsphere*)
apply (*rule continuous_map_from_subtopology*)
apply (*subst continuous_map_of_snd* [*unfolded o_def*])
using *continuous_map_from_subtopology continuous_map_nsphere_projection nsphere* **by** *fastforce*
show *continuous_map* (*subtopology* (*prod_topology ?T01 ?X12*) {*p* ∈ *topspace* (*prod_topology ?T01 ?X12*). *snd p k* ≤ *0*})
euclideanreal (λ*x*. *h* (*fst x*, *snd x*) *i*)
apply (*simp add*: *h_def case_prod_unfold Let_def*)

```
        apply (intro conjI impI fst snd continuous_intros)
        apply (auto simp: nsphere power2_eq_1_iff)
        done
      qed (auto simp: nsphere h)
    qed (auto simp: nsphere h)
  qed
  moreover
  have h ` ({0..1} × (topspace (nsphere n) ∩ {x. − (1/2) ≤ x k}))
     ⊆ {x. (∑ i≤n. (x i)^2) = 1 ∧ (∀ i>n. x i = 0)}
  proof −
    have (∑ i≤n. (h (T,x) i)^2) = 1
      if x: x ∈ topspace (nsphere n) and xk: − (1/2) ≤ x k and T: 0 ≤ T T ≤ 1
for T x
    proof (cases −T ≤ x k )
      case True
      then show ?thesis
        using that by (auto simp: nsphere h)
    next
      case False
      with x ‹0 ≤ T› have k ≤ n
        apply (simp add: nsphere)
        by (metis neg_le_0_iff_le not_le)
      have 1 − (x k)^2 ≥ 0
        using topspace_nsphere_1 x by auto
      with False T ‹k ≤ n›
      have (∑ i≤n. (h (T,x) i)^2) = T^2 + (1 − T^2) ∗ (∑ i∈{..n} − {k}. (x i)^2 /
(1 − (x k)^2))
        unfolding h_def Let_def max_def
          by (simp add: not_le square_le_1 power_mult_distrib power_divide
if_distrib [of λx. x ^ 2]
             sum.delta_remove sum_distrib_left)
      also have ... = 1
        using x False xk ‹0 ≤ T›
          by (simp add: nsphere sum_diff not_le ‹k ≤ n› power2_eq_1_iff flip:
sum_divide_distrib)
      finally show ?thesis .
    qed
    moreover
    have h (T,x) i = 0
      if x ∈ topspace (nsphere n) − (1/2) ≤ x k and n < i 0 ≤ T T ≤ 1
      for T x i
    proof (cases −T ≤ x k )
      case False
      then show ?thesis
        using that by (auto simp: nsphere h_def Let_def not_le max_def)
    qed (use that in ‹auto simp: nsphere h›)
    ultimately show ?thesis
      by auto
  qed
```

ultimately
have *cmh*: *continuous_map* (*prod_topology ?T01 ?X12*) (*nsphere n*) *h*
proof (*subst* (*2*) *nsphere*)
qed (*fastforce simp add*: *continuous_map_in_subtopology continuous_map_componentwise_UNIV*)
have *hom_induced p* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*})
(*topspace* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) ∩ {*x*. *x k* = *0*}) *?X12*
(*topspace ?X12* ∩ {*x*. − *1/2* ≤ *x k* ∧ *x k* ≤ *0*}) *id*
∈ *iso* (*relative_homology_group p* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*})
(*topspace* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) ∩ {*x*. *x k* = *0*}))
(*relative_homology_group p ?X12* (*topspace ?X12* ∩ {*x*. − *1/2* ≤ *x k* ∧ *x k* ≤ *0*}))
proof (*rule deformation_retract_relative_homology_group_isomorphism_id*)
show *retraction_maps ?X12* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) (*h* ∘ (λ*x*. (*0*,*x*))) *id*
unfolding *retraction_maps_def*
proof (*intro conjI ballI*)
show *continuous_map ?X12* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) (*h* ∘ *Pair 0*)
apply (*simp add*: *continuous_map_in_subtopology*)
apply (*intro conjI continuous_map_compose* [*OF _ cmh*] *continuous_intros*)
apply (*auto simp*: *h_def Let_def*)
done
show *continuous_map* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) *?X12 id*
by (*simp add*: *continuous_map_in_subtopology*)
qed (*simp add*: *nsphere h*)
next
have *h0*: ⋀*xa*. ⟦*xa* ∈ *topspace* (*nsphere n*); − (*1/2*) ≤ *xa k*; *xa k* ≤ *0*⟧ ⟹ *h* (*0*, *xa*) *k* = *0*
by (*simp add*: *h_def Let_def*)
show (*h* ∘ (λ*x*. (*0*,*x*))) ∈ (*topspace ?X12* ∩ {*x*. − *1* / *2* ≤ *x k* ∧ *x k* ≤ *0*})
→ *topspace* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) ∩ {*x*. *x k* = *0*}
apply (*auto simp*: *h0*)
apply (*rule subsetD* [*OF continuous_map_image_subset_topspace* [*OF cmh*]])
apply (*force simp*: *nsphere*)
done
have *hin*: ⋀*t x*. ⟦*x* ∈ *topspace* (*nsphere n*); − (*1/2*) ≤ *x k*; *0* ≤ *t*; *t* ≤ *1*⟧ ⟹ *h* (*t*,*x*) ∈ *topspace* (*nsphere n*)
apply (*rule subsetD* [*OF continuous_map_image_subset_topspace* [*OF cmh*]])
apply (*force simp*: *nsphere*)
done
have *h1*: ⋀*x*. ⟦*x* ∈ *topspace* (*nsphere n*); − (*1/2*) ≤ *x k*⟧ ⟹ *h* (*1*, *x*) = *x*
by (*simp add*: *h nsphere*)
have *continuous_map* (*prod_topology ?T01 ?X12*) (*nsphere n*) *h*
using *cmh* **by** *force*
then show *homotopic_with*
(λ*h*. *h* ‘ (*topspace ?X12* ∩ {*x*. − *1* / *2* ≤ *x k* ∧ *x k* ≤ *0*}) ⊆ *topspace ?X12* ∩ {*x*. − *1* / *2* ≤ *x k* ∧ *x k* ≤ *0*})
?X12 ?X12 (*h* ∘ (λ*x*. (*0*,*x*))) *id*

```
    apply (subst homotopic_with, force)
    apply (rule_tac x=h in exI)
    apply (auto simp: hin h1 continuous_map_in_subtopology)
      apply (auto simp: h_def Let_def max_def)
    done
  qed auto
  then have 2: hom_induced p (subtopology (nsphere n) {x. 0 ≤ x k}) {x. x k =
0}
             ?X12 {x. - 1/2 ≤ x k ∧ x k ≤ 0} id
           ∈ Group.iso
               (relative_homology_group p (subtopology (nsphere n) {x. 0 ≤ x k})
{x. x k = 0})
              (relative_homology_group p ?X12 {x. - 1/2 ≤ x k ∧ x k ≤ 0})
   by (metis hom_induced_restrict relative_homology_group_restrict topspace_subtopology)
  show ?thesis
    using iso_set_trans [OF 2 1]
   by (simp add: subset_iff continuous_map_in_subtopology flip: hom_induced_compose)
qed


corollary iso_upper_hemisphere_reduced_homology_group:
   (hom_boundary (1 + p) (subtopology (nsphere (Suc n)) {x. x(Suc n) ≥ 0}) {x.
x(Suc n) = 0})
  ∈ iso (relative_homology_group (1 + p) (subtopology (nsphere (Suc n)) {x. x(Suc
n) ≥ 0}) {x. x(Suc n) = 0})
        (reduced_homology_group p (nsphere n))
proof -
  have {x. 0 ≤ x (Suc n)} ∩ {x. x (Suc n) = 0} = {x. x (Suc n) = (0::real)}
    by auto
  then have n: nsphere n = subtopology (subtopology (nsphere (Suc n)) {x. x(Suc
n) ≥ 0}) {x. x(Suc n) = 0}
    by (simp add: subtopology_nsphere_equator subtopology_subtopology)
  have ne: (λi. if i = n then 1 else 0) ∈ topspace (subtopology (nsphere (Suc n))
{x. 0 ≤ x (Suc n)}) ∩ {x. x (Suc n) = 0}
    by (simp add: nsphere if_distrib [of λx. x ^ 2] cong: if_cong)
  show ?thesis
    unfolding n
    using iso_relative_homology_of_contractible [where p = 1 + p, simplified]
    by (metis contractible_space_upper_hemisphere dual_order.refl empty_iff ne)
qed

corollary iso_reduced_homology_group_upper_hemisphere:
  assumes k ≤ n
  shows hom_induced p (nsphere n) {} (nsphere n) {x. x k ≥ 0} id
      ∈ iso (reduced_homology_group p (nsphere n)) (relative_homology_group p
(nsphere n) {x. x k ≥ 0})
proof (rule iso_reduced_homology_by_contractible [OF contractible_space_upper_hemisphere
[OF assms]])
  have (λi. if i = k then 1 else 0) ∈ topspace (nsphere n) ∩ {x. 0 ≤ x k}
```

using *assms* **by** (*simp add*: *nsphere if_distrib* [*of* λ*x*. *x* ^ *2*] *cong*: *if_cong*)
then show *topspace* (*nsphere n*) ∩ {*x*. *0* ≤ *x k*} ≠ {}
by *blast*
qed

lemma *iso_relative_homology_group_lower_hemisphere*:
hom_induced p (*subtopology* (*nsphere n*) {*x*. *x k* ≤ *0*}) {*x*. *x k* = *0*} (*nsphere n*) {*x*. *x k* ≥ *0*} *id*
∈ *iso* (*relative_homology_group p* (*subtopology* (*nsphere n*) {*x*. *x k* ≤ *0*}) {*x*. *x k* = *0*})
(*relative_homology_group p* (*nsphere n*) {*x*. *x k* ≥ *0*}) (**is** *?k* ∈ *iso ?G ?H*)
proof −
define *r* **where** *r* ≡ λ*x i*. *if i* = *k then* −*x i else* (*x i*::*real*)
then have [*simp*]: *r* ∘ *r* = *id*
by *force*
have *cmr*: *continuous_map* (*subtopology* (*nsphere n*) *S*) (*nsphere n*) *r* **for** *S*
using *continuous_map_nsphere_reflection* [*of n k*]
by (*simp add*: *continuous_map_from_subtopology r_def*)
let *?f* = *hom_induced p* (*subtopology* (*nsphere n*) {*x*. *x k* ≤ *0*}) {*x*. *x k* = *0*}
(*subtopology* (*nsphere n*) {*x*. *x k* ≥ *0*}) {*x*. *x k* = *0*} *r*
let *?g* = *hom_induced p* (*subtopology* (*nsphere n*) {*x*. *x k* ≥ *0*}) {*x*. *x k* = *0*} (*nsphere n*) {*x*. *x k* ≤ *0*} *id*
let *?h* = *hom_induced p* (*nsphere n*) {*x*. *x k* ≤ *0*} (*nsphere n*) {*x*. *x k* ≥ *0*} *r*
obtain *f h* **where**
f: *f* ∈ *iso ?G* (*relative_homology_group p* (*subtopology* (*nsphere n*) {*x*. *x k* ≥ *0*}) {*x*. *x k* = *0*})
and *h*: *h* ∈ *iso* (*relative_homology_group p* (*nsphere n*) {*x*. *x k* ≤ *0*}) *?H*
and *eq*: *h* ∘ *?g* ∘ *f* = *?k*
proof
have *hmr*: *homeomorphic_map* (*nsphere n*) (*nsphere n*) *r*
unfolding *homeomorphic_map_maps*
by (*metis* ‹*r* ∘ *r* = *id*› *cmr homeomorphic_maps_involution pointfree_idE subtopology_topspace*)
then have *hmrs*: *homeomorphic_map* (*subtopology* (*nsphere n*) {*x*. *x k* ≤ *0*}) (*subtopology* (*nsphere n*) {*x*. *x k* ≥ *0*}) *r*
by (*simp add*: *homeomorphic_map_subtopologies_alt r_def*)
have *rimeq*: *r* ‘ (*topspace* (*subtopology* (*nsphere n*) {*x*. *x k* ≤ *0*}) ∩ {*x*. *x k* = *0*})
= *topspace* (*subtopology* (*nsphere n*) {*x*. *0* ≤ *x k*}) ∩ {*x*. *x k* = *0*}
using *continuous_map_eq_topcontinuous_at continuous_map_nsphere_reflection topcontinuous_at_atin*
by (*fastforce simp*: *r_def Pi_iff*)
show *?f* ∈ *iso ?G* (*relative_homology_group p* (*subtopology* (*nsphere n*) {*x*. *x k* ≥ *0*}) {*x*. *x k* = *0*})
using *homeomorphic_map_relative_homology_iso* [*OF hmrs Int_lower1 rimeq*]
by (*metis hom_induced_restrict relative_homology_group_restrict*)
have *rimeq*: *r* ‘ (*topspace* (*nsphere n*) ∩ {*x*. *x k* ≤ *0*}) = *topspace* (*nsphere n*)

∩ {x. 0 ≤ x k}
    by (*metis hmrs homeomorphic_imp_surjective_map topspace_subtopology*)
  show *?h* ∈ *Group.iso* (*relative_homology_group p* (*nsphere n*) {x. x k ≤ 0})
?H
  using *homeomorphic_map_relative_homology_iso* [*OF hmr Int_lower1 rimeq*]
by *simp*
  have [*simp*]: ⋀x. x k = 0 ⟹ r x k = 0
    by (*auto simp*: *r_def*)
  have *?h* ∘ *?g* ∘ *?f*
      = *hom_induced p* (*subtopology* (*nsphere n*) {x. 0 ≤ x k}) {x. x k = 0}
(*nsphere n*) {x. 0 ≤ x k} r ∘
        hom_induced p (*subtopology* (*nsphere n*) {x. x k ≤ 0}) {x. x k = 0}
(*subtopology* (*nsphere n*) {x. 0 ≤ x k}) {x. x k = 0} r
    apply (*subst hom_induced_compose* [*symmetric*])
    using *continuous_map_nsphere_reflection* **apply** (*force simp*: *r_def*)+
    done
  also have ... = *?k*
    apply (*subst hom_induced_compose* [*symmetric*])
      apply (*simp_all add*: *image_subset_iff cmr*)
    using *hmrs homeomorphic_imp_continuous_map* **apply** *blast*
    done
  finally show *?h* ∘ *?g* ∘ *?f* = *?k* **.**
 qed
 with *iso_relative_homology_group_upper_hemisphere* [*of p n k*]
 have h ∘ *hom_induced p* (*subtopology* (*nsphere n*) {f. 0 ≤ f k}) {f. f k = 0}
(*nsphere n*) {f. f k ≤ 0} *id* ∘ f
 ∈ *Group.iso ?G* (*relative_homology_group p* (*nsphere n*) {f. 0 ≤ f k})
  using f h *iso_set_trans* **by** *blast*
 then show *?thesis*
  by (*simp add*: *eq*)
qed

lemma *iso_lower_hemisphere_reduced_homology_group*:
 hom_boundary (1 + p) (*subtopology* (*nsphere* (*Suc n*)) {x. x(*Suc n*) ≤ 0}) {x.
x(*Suc n*) = 0}
 ∈ *iso* (*relative_homology_group* (1 + p) (*subtopology* (*nsphere* (*Suc n*)) {x. x(*Suc*
n) ≤ 0})
            {x. x(*Suc n*) = 0})
    (*reduced_homology_group p* (*nsphere n*))
proof −
 have {x. (∑ i≤n. (x i)²) = 1 ∧ (∀ i>n. x i = 0)} =
    ({x. (∑ i≤n. (x i)²) + (x (*Suc n*))² = 1 ∧ (∀ i>*Suc n*. x i = 0)} ∩ {x. x
(*Suc n*) ≤ 0} ∩
    {x. x (*Suc n*) = (0::*real*)})
  by (*force simp*: *dest*: *Suc_lessI*)
 then have n: *nsphere n* = *subtopology* (*subtopology* (*nsphere* (*Suc n*)) {x. x(*Suc*
n) ≤ 0}) {x. x(*Suc n*) = 0}
  by (*simp add*: *nsphere subtopology_subtopology*)

have *ne*: (λ*i*. *if i* = *n then 1 else 0*) ∈ *topspace* (*subtopology* (*nsphere* (*Suc n*)) {*x*. *x* (*Suc n*) ≤ *0*}) ∩ {*x*. *x* (*Suc n*) = *0*}
by (*simp add*: *nsphere if_distrib* [*of* λ*x*. *x* ^ *2*] *cong*: *if_cong*)
show *?thesis*
unfolding *n*
apply (*rule iso_relative_homology_of_contractible* [**where** *p* = *1* + *p*, *simplified*])
using *contractible_space_lower_hemisphere ne* **apply** *blast*+
done
qed

lemma *isomorphism_sym*:
⟦*f* ∈ *iso G1 G2*; ⋀*x*. *x* ∈ *carrier G1* ⟹ *r*′(*f x*) = *f*(*r x*);
⋀*x*. *x* ∈ *carrier G1* ⟹ *r x* ∈ *carrier G1*; *group G1*; *group G2*⟧
⟹ ∃*f* ∈ *iso G2 G1*. ∀*x* ∈ *carrier G2*. *r*(*f x*) = *f*(*r*′ *x*)
apply (*clarsimp simp add*: *group.iso_iff_group_isomorphisms Bex_def*)
by (*metis* (*full_types*) *group_isomorphisms_def group_isomorphisms_sym hom_in_carrier*)

lemma *isomorphism_trans*:
⟦∃*f* ∈ *iso G1 G2*. ∀*x* ∈ *carrier G1*. *r2*(*f x*) = *f*(*r1 x*); ∃*f* ∈ *iso G2 G3*. ∀*x* ∈ *carrier G2*. *r3*(*f x*) = *f*(*r2 x*)⟧
⟹ ∃*f* ∈ *iso G1 G3*. ∀*x* ∈ *carrier G1*. *r3*(*f x*) = *f*(*r1 x*)
apply *clarify*
by (*smt* (*verit*, *ccfv_threshold*) *Group.iso_iff hom_in_carrier iso_set_trans o_apply*)

lemma *reduced_homology_group_nsphere_step*:
∃*f* ∈ *iso*(*reduced_homology_group p* (*nsphere n*))
(*reduced_homology_group* (*1* + *p*) (*nsphere* (*Suc n*))).
∀*c* ∈ *carrier*(*reduced_homology_group p* (*nsphere n*)).
hom_induced (*1* + *p*) (*nsphere*(*Suc n*)) {} (*nsphere*(*Suc n*)) {}
(λ*x i*. *if i* = *0 then* −*x i else x i*) (*f c*)
= *f* (*hom_induced p* (*nsphere n*) {} (*nsphere n*) {} (λ*x i*. *if i* = *0 then* −*x i else x i*) *c*)
proof −
define *r* **where** *r* ≡ λ*x*::*nat*⇒*real*. λ*i*. *if i* = *0 then* −*x i else x i*
have *cmr*: *continuous_map* (*nsphere n*) (*nsphere n*) *r* **for** *n*
unfolding *r_def* **by** (*rule continuous_map_nsphere_reflection*)
have *rsub*: *r* ∈ {*x*. *0* ≤ *x* (*Suc n*)} → {*x*. *0* ≤ *x* (*Suc n*)}
r ∈ {*x*. *x* (*Suc n*) ≤ *0*} → {*x*. *x* (*Suc n*) ≤ *0*}
r ∈ {*x*. *x* (*Suc n*) = *0*} → {*x*. *x* (*Suc n*) = *0*}
by (*force simp*: *r_def*)+
let *?sub* = *subtopology* (*nsphere* (*Suc n*)) {*x*. *x* (*Suc n*) ≥ *0*}
let *?G2* = *relative_homology_group* (*1* + *p*) *?sub* {*x*. *x* (*Suc n*) = *0*}
let *?r2* = *hom_induced* (*1* + *p*) *?sub* {*x*. *x* (*Suc n*) = *0*} *?sub* {*x*. *x* (*Suc n*) = *0*} *r*
let *?j* = λ*p n*. *hom_induced p* (*nsphere n*) {} (*nsphere n*) {} *r*
show *?thesis*
unfolding *r_def* [*symmetric*]

```
  proof (rule isomorphism_trans)
    let ?f = hom_boundary (1 + p) ?sub {x. x (Suc n) = 0}
    show ∃f∈Group.iso (reduced_homology_group p (nsphere n)) ?G2.
           ∀c∈carrier (reduced_homology_group p (nsphere n)). ?r2 (f c) = f (?j p
n c)
    proof (rule isomorphism_sym)
      show ?f ∈ Group.iso ?G2 (reduced_homology_group p (nsphere n))
        using iso_upper_hemisphere_reduced_homology_group
        by (metis add.commute)
    next
      fix c
      assume c ∈ carrier ?G2
      have cmrs: continuous_map ?sub ?sub r
      by (metis (no_types, lifting) IntE Pi_iff cmr continuous_map_from_subtopology
           continuous_map_into_subtopology rsub(1) topspace_subtopology)
      have hom_induced p (nsphere n) {} (nsphere n) {} r ∘ hom_boundary (1 +
p) ?sub {x. x (Suc n) = 0}
           = hom_boundary (1 + p) ?sub {x. x (Suc n) = 0} ∘
            hom_induced (1 + p) ?sub {x. x (Suc n) = 0} ?sub {x. x (Suc n) = 0}
r
           using naturality_hom_induced [OF cmrs rsub(3), symmetric, of 1+p,
simplified]
         by (simp add: Pi_iff subtopology_subtopology subtopology_nsphere_equator
flip: Collect_conj_eq cong: rev_conj_cong)
      then show ?j p n (?f c) = ?f (hom_induced (1 + p) ?sub {x. x (Suc n) =
0} ?sub {x. x (Suc n) = 0} r c)
        by (metis comp_def)
    next
      fix c
      assume c ∈ carrier ?G2
      show hom_induced (1 + p) ?sub {x. x (Suc n) = 0} ?sub {x. x (Suc n) =
0} r c ∈ carrier ?G2
        using hom_induced_carrier by blast
    qed auto
  next
    let ?H2 = relative_homology_group (1 + p) (nsphere (Suc n)) {x. x (Suc n)
≤ 0}
    let ?s2 = hom_induced (1 + p) (nsphere (Suc n)) {x. x (Suc n) ≤ 0} (nsphere
(Suc n)) {x. x (Suc n) ≤ 0} r
     show ∃f∈Group.iso ?G2 (reduced_homology_group (1 + p) (nsphere (Suc
n))). ∀c∈carrier ?G2. ?j (1 + p) (Suc n) (f c)
            = f (?r2 c)
    proof (rule isomorphism_trans)
      show ∃f∈Group.iso ?G2 ?H2.
                ∀c∈carrier ?G2.
                  ?s2 (f c) = f (hom_induced (1 + p) ?sub {x. x (Suc n) = 0} ?sub
{x. x (Suc n) = 0} r c)
      proof (intro ballI bexI)
        fix c
```

assume *c* ∈ *carrier* (*relative_homology_group* (*1* + *p*) *?sub* {*x*. *x* (*Suc n*) = *0*})
show *?s2* (*hom_induced* (*1* + *p*) *?sub* {*x*. *x* (*Suc n*) = *0*} (*nsphere* (*Suc n*)) {*x*. *x* (*Suc n*) ≤ *0*} *id c*)
= *hom_induced* (*1* + *p*) *?sub* {*x*. *x* (*Suc n*) = *0*} (*nsphere* (*Suc n*)) {*x*. *x* (*Suc n*) ≤ *0*} *id* (*?r2 c*)
apply (*simp add*: *rsub hom_induced_compose′ Collect_mono_iff cmr*)
apply (*subst hom_induced_compose′*)
apply (*simp_all add*: *continuous_map_in_subtopology continuous_map_from_subtopology* [*OF cmr*] *rsub*)
apply (*auto simp*: *r_def*)
done
qed (*simp add*: *iso_relative_homology_group_upper_hemisphere*)
next
let *?h* = *hom_induced* (*1* + *p*) (*nsphere*(*Suc n*)) {} (*nsphere* (*Suc n*)) {*x*. *x*(*Suc n*) ≤ *0*} *id*
show ∃*f*∈*Group.iso ?H2* (*reduced_homology_group* (*1* + *p*) (*nsphere* (*Suc n*))).
∀ *c*∈*carrier ?H2*. *?j* (*1* + *p*) (*Suc n*) (*f c*) = *f* (*?s2 c*)
proof (*rule isomorphism_sym*)
show *?h* ∈ *Group.iso* (*reduced_homology_group* (*1* + *p*) (*nsphere* (*Suc n*)))
(*relative_homology_group* (*1* + *p*) (*nsphere* (*Suc n*)) {*x*. *x* (*Suc n*) ≤ *0*})
using *iso_reduced_homology_group_lower_hemisphere* **by** *blast*
next
fix *c*
assume *c* ∈ *carrier* (*reduced_homology_group* (*1* + *p*) (*nsphere* (*Suc n*)))
show *?s2* (*?h c*) = *?h* (*?j* (*1* + *p*) (*Suc n*) *c*)
by (*simp add*: *hom_induced_compose′ cmr rsub*)
next
fix *c*
assume *c* ∈ *carrier* (*reduced_homology_group* (*1* + *p*) (*nsphere* (*Suc n*)))
then show *hom_induced* (*1* + *p*) (*nsphere* (*Suc n*)) {} (*nsphere* (*Suc n*)) {} *r c*
∈ *carrier* (*reduced_homology_group* (*1* + *p*) (*nsphere* (*Suc n*)))
by (*simp add*: *hom_induced_reduced*)
qed *auto*
qed
qed
qed

lemma *reduced_homology_group_nsphere_aux*:
if p = *int n then reduced_homology_group n* (*nsphere n*) ≅ *integer_group*
else trivial_group(*reduced_homology_group p* (*nsphere n*))
proof (*induction n arbitrary*: *p*)
case *0*
let *?a* = *λi*::*nat*. *if i* = *0 then 1 else* (*0*::*real*)
let *?b* = *λi*::*nat*. *if i* = *0 then* −*1 else* (*0*::*real*)

have *st*: *subtopology* (*powertop_real UNIV*) {*?a*, *?b*} = *nsphere 0*
proof −
have {*?a*, *?b*} = {*x*. (*x 0*)^2 = *1* ∧ (∀ *i*>*0*. *x i* = *0*)}
using *power2_eq_iff* **by** *fastforce*
then show *?thesis*
by (*simp add*: *nsphere*)
qed
have *t1_space* (*powertop_real UNIV*)
using *t1_space_euclidean t1_space_product_topology* **by** *blast*
then have ∗: *reduced_homology_group p* (*subtopology* (*powertop_real UNIV*) {*?a*, *?b*}) ≅
homology_group p (*subtopology* (*powertop_real UNIV*) {*?a*})
by (*intro reduced_homology_group_pair*) (*auto simp*: *fun_eq_iff*)
have *reduced_homology_group 0* (*nsphere 0*) ≅ *integer_group* **if** *p*=*0*
proof −
have *reduced_homology_group 0* (*nsphere 0*) ≅ *homology_group 0* (*top_of_set* {*?a*}) **if** *p*=*0*
by (*metis* ∗ *euclidean_product_topology st that*)
also have ... ≅ *integer_group*
by (*simp add*: *homology_coefficients*)
finally show *?thesis*
using *that* **by** *blast*
qed
moreover have *trivial_group* (*reduced_homology_group p* (*nsphere 0*)) **if** *p*≠*0*
using ∗ *that homology_dimension_axiom* [*of subtopology* (*powertop_real UNIV*) {*?a*} *?a p*]
using *isomorphic_group_triviality st* **by** *force*
ultimately show *?case*
by *auto*
next
case (*Suc n*)
have *eq*: *reduced_homology_group* (*int n*) (*nsphere n*) ≅ *integer_group* **if** *p*−*1* = *n*
by (*simp add*: *Suc.IH*)
have *neq*: *trivial_group* (*reduced_homology_group* (*p*−*1*) (*nsphere n*)) **if** *p*−*1* ≠ *n*
by (*simp add*: *Suc.IH that*)
have *iso*: *reduced_homology_group p* (*nsphere* (*Suc n*)) ≅ *reduced_homology_group* (*p*−*1*) (*nsphere n*)
using *reduced_homology_group_nsphere_step* [*of p*−*1 n*] *group.iso_sym* [*OF _ is_isoI*] *group_reduced_homology_group*
by *fastforce*
then show *?case*
using *eq iso_trans iso isomorphic_group_triviality neq*
by (*metis* (*no_types*, *opaque_lifting*) *add.commute add_left_cancel diff_add_cancel group_reduced_homology_group of_nat_Suc*)
qed

lemma reduced_homology_group_nsphere:
reduced_homology_group n (nsphere n) ≅ integer_group
p ≠ n ⟹ trivial_group(reduced_homology_group p (nsphere n))
using reduced_homology_group_nsphere_aux **by** auto

lemma cyclic_reduced_homology_group_nsphere:
cyclic_group(reduced_homology_group p (nsphere n))
by (metis reduced_homology_group_nsphere trivial_imp_cyclic_group cyclic_integer_group group_integer_group group_reduced_homology_group isomorphic_group_cyclicity)

lemma trivial_reduced_homology_group_nsphere:
trivial_group(reduced_homology_group p (nsphere n)) ⟷ (p ≠ n)
using group_integer_group isomorphic_group_triviality nontrivial_integer_group reduced_homology_group_nsphere(1) reduced_homology_group_nsphere(2) trivial_group_def **by** blast

lemma non_contractible_space_nsphere: ¬ (contractible_space(nsphere n))
proof (clarsimp simp add: contractible_eq_homotopy_equivalent_singleton_subtopology)
fix a :: nat ⇒ real
assume a: a ∈ topspace (nsphere n)
and he: nsphere n homotopy_equivalent_space subtopology (nsphere n) {a}
have trivial_group (reduced_homology_group (int n) (subtopology (nsphere n) {a}))
by (simp add: a homology_dimension_reduced [**where** a=a])
then show False
using isomorphic_group_triviality [OF homotopy_equivalent_space_imp_isomorphic_reduced_homo [OF he, of n]]
by (simp add: trivial_reduced_homology_group_nsphere)
qed

0.3.4 Brouwer degree of a Map

definition Brouwer_degree2 :: nat ⇒ ((nat ⇒ real) ⇒ nat ⇒ real) ⇒ int
where
Brouwer_degree2 p f ≡
@d::int. ∀ x ∈ carrier(reduced_homology_group p (nsphere p)).
hom_induced p (nsphere p) {} (nsphere p) {} f x = pow (reduced_homology_group p (nsphere p)) x d

lemma Brouwer_degree2_eq:
(⋀x. x ∈ topspace(nsphere p) ⟹ f x = g x) ⟹ Brouwer_degree2 p f = Brouwer_degree2 p g
unfolding Brouwer_degree2_def Ball_def
apply (intro Eps_cong all_cong)
by (metis (mono_tags, lifting) hom_induced_eq)

lemma Brouwer_degree2:
assumes x ∈ carrier(reduced_homology_group p (nsphere p))
shows hom_induced p (nsphere p) {} (nsphere p) {} f x

```
    = pow (reduced_homology_group p (nsphere p)) x (Brouwer_degree2 p f)
    (is ?h x = pow ?G x _)
proof (cases continuous_map(nsphere p) (nsphere p) f)
  case True
  interpret group ?G
    by simp
  interpret group_hom ?G ?G ?h
     using hom_induced_reduced_hom group_hom_axioms_def group_hom_def
is_group by blast
  obtain a where a: a ∈ carrier ?G
    and aeq: subgroup_generated ?G {a} = ?G
   using cyclic_reduced_homology_group_nsphere [of p p] by (auto simp: cyclic_group_def)
   then have carra: carrier (subgroup_generated ?G {a}) = range (λn::int. pow
?G a n)
    using carrier_subgroup_generated_by_singleton by blast
  moreover have ?h a ∈ carrier (subgroup_generated ?G {a})
    by (simp add: a aeq hom_induced_reduced)
  ultimately obtain d::int where d: ?h a = pow ?G a d
    by auto
  have *: hom_induced (int p) (nsphere p) {} (nsphere p) {} f x = x [^]?G d
    if x: x ∈ carrier ?G for x
  proof –
    obtain n::int where xeq: x = pow ?G a n
      using carra x aeq by auto
    show ?thesis
      by (simp add: xeq a d hom_int_pow int_pow_pow mult.commute)
  qed
  show ?thesis
    unfolding Brouwer_degree2_def
    apply (rule someI2 [where a=d])
    using assms * apply blast+
    done
next
  case False
  show ?thesis
    unfolding Brouwer_degree2_def
     by (rule someI2 [where a=0]) (simp_all add: hom_induced_default False
one_reduced_homology_group assms)
qed



lemma Brouwer_degree2_iff:
  assumes f: continuous_map (nsphere p) (nsphere p) f
    and x: x ∈ carrier(reduced_homology_group p (nsphere p))
  shows (hom_induced (int p) (nsphere p) {} (nsphere p) {} f x =
          x [^]reduced_homology_group (int p) (nsphere p) d)
     ⟷ (x = 1reduced_homology_group (int p) (nsphere p) ∨ Brouwer_degree2 p f
= d)
```

(**is** (*?h x* = *x* [^]$_{?G}$ *d*) ⟷ _)
proof –
interpret *group ?G*
by *simp*
obtain *a* **where** *a*: *a* ∈ *carrier ?G*
and *aeq*: *subgroup_generated ?G* {*a*} = *?G*
using *cyclic_reduced_homology_group_nsphere* [*of p p*] **by** (*auto simp*: *cyclic_group_def*)
then obtain *i*::*int* **where** *i*: *x* = (*a* [^]$_{?G}$ *i*)
using *carrier_subgroup_generated_by_singleton x* **by** *fastforce*
then have *a* [^]$_{?G}$ *i* ∈ *carrier ?G*
using *x* **by** *blast*
have [*simp*]: *ord a* = *0*
by (*simp add*: *a aeq iso_finite* [*OF reduced_homology_group_nsphere*(*1*)] *flip*: *infinite_cyclic_subgroup_order*)
show *?thesis*
by (*auto simp*: *Brouwer_degree2 int_pow_eq_id x i a int_pow_pow int_pow_eq*)
qed

lemma *Brouwer_degree2_unique*:
assumes *f*: *continuous_map* (*nsphere p*) (*nsphere p*) *f*
and *hi*: ⋀*x*. *x* ∈ *carrier*(*reduced_homology_group p* (*nsphere p*))
⟹ *hom_induced p* (*nsphere p*) {} (*nsphere p*) {} *f x* = *pow* (*reduced_homology_group p* (*nsphere p*)) *x d*
(**is** ⋀*x*. *x* ∈ *carrier ?G* ⟹ *?h x* = _)
shows *Brouwer_degree2 p f* = *d*
proof –
obtain *a* **where** *a*: *a* ∈ *carrier ?G*
and *aeq*: *subgroup_generated ?G* {*a*} = *?G*
using *cyclic_reduced_homology_group_nsphere* [*of p p*] **by** (*auto simp*: *cyclic_group_def*)
show *?thesis*
using *hi* [*OF a*] **unfolding** *Brouwer_degree2 a*
by (*metis Brouwer_degree2_iff a aeq f group.trivial_group_subgroup_generated group_reduced_homology_group subsetI trivial_reduced_homology_group_nsphere*)
qed

lemma *Brouwer_degree2_unique_generator*:
assumes *f*: *continuous_map* (*nsphere p*) (*nsphere p*) *f*
and *eq*: *subgroup_generated* (*reduced_homology_group p* (*nsphere p*)) {*a*}
= *reduced_homology_group p* (*nsphere p*)
and *hi*: *hom_induced p* (*nsphere p*) {} (*nsphere p*) {} *f a* = *pow* (*reduced_homology_group p* (*nsphere p*)) *a d*
(**is** *?h a* = *pow ?G a* _)
shows *Brouwer_degree2 p f* = *d*
proof (*cases a* ∈ *carrier ?G*)
case *True*
then show *?thesis*
by (*metis Brouwer_degree2_iff hi eq f group.trivial_group_subgroup_generated group_reduced_homology_group*

subset_singleton_iff trivial_reduced_homology_group_nsphere)
next
case *False*
then show *?thesis*
using *trivial_reduced_homology_group_nsphere* [*of p p*]
by (*metis group.trivial_group_subgroup_generated_eq disjoint_insert*(*1*) *eq group_reduced_homology_group inf_bot_right subset_singleton_iff*)
qed

lemma *Brouwer_degree2_homotopic*:
assumes *homotopic_with* (λ*x*. *True*) (*nsphere p*) (*nsphere p*) *f g*
shows *Brouwer_degree2 p f* = *Brouwer_degree2 p g*
proof −
have *continuous_map* (*nsphere p*) (*nsphere p*) *f*
using *homotopic_with_imp_continuous_maps* [*OF assms*] **by** *auto*
show *?thesis*
using *Brouwer_degree2_def assms homology_homotopy_empty* **by** *fastforce*
qed

lemma *Brouwer_degree2_id* [*simp*]: *Brouwer_degree2 p id* = *1*
proof (*rule Brouwer_degree2_unique*)
fix *x*
assume *x*: *x* ∈ *carrier* (*reduced_homology_group* (*int p*) (*nsphere p*))
then have *x* ∈ *carrier* (*homology_group* (*int p*) (*nsphere p*))
using *carrier_reduced_homology_group_subset* **by** *blast*
then show *hom_induced* (*int p*) (*nsphere p*) {} (*nsphere p*) {} *id x* =
x [^]$_{reduced\_homology\_group\ (int\ p)\ (nsphere\ p)}$ (*1::int*)
by (*simp add*: *hom_induced_id group.int_pow_1 x*)
qed *auto*

lemma *Brouwer_degree2_compose*:
assumes *f*: *continuous_map* (*nsphere p*) (*nsphere p*) *f* **and** *g*: *continuous_map* (*nsphere p*) (*nsphere p*) *g*
shows *Brouwer_degree2 p* (*g* ∘ *f*) = *Brouwer_degree2 p g* ∗ *Brouwer_degree2 p f*
proof (*rule Brouwer_degree2_unique*)
show *continuous_map* (*nsphere p*) (*nsphere p*) (*g* ∘ *f*)
by (*meson continuous_map_compose f g*)
next
fix *x*
assume *x*: *x* ∈ *carrier* (*reduced_homology_group* (*int p*) (*nsphere p*))
have *hom_induced* (*int p*) (*nsphere p*) {} (*nsphere p*) {} (*g* ∘ *f*) =
hom_induced (*int p*) (*nsphere p*) {} (*nsphere p*) {} *g* ∘
hom_induced (*int p*) (*nsphere p*) {} (*nsphere p*) {} *f*
by (*blast intro*: *hom_induced_compose* [*OF f _ g*])
with *x* **show** *hom_induced* (*int p*) (*nsphere p*) {} (*nsphere p*) {} (*g* ∘ *f*) *x* =
x [^]$_{reduced\_homology\_group\ (int\ p)\ (nsphere\ p)}$ (*Brouwer_degree2 p g* ∗ *Brouwer_degree2 p f*)
by (*simp add*: *mult.commute hom_induced_reduced flip*: *Brouwer_degree2*

group.int_pow_pow)
qed

lemma *Brouwer_degree2_homotopy_equivalence*:
assumes *f*: *continuous_map* (*nsphere p*) (*nsphere p*) *f* **and** *g*: *continuous_map* (*nsphere p*) (*nsphere p*) *g*
and *hom*: *homotopic_with* (λ*x*. *True*) (*nsphere p*) (*nsphere p*) (*f* ∘ *g*) *id*
obtains |*Brouwer_degree2 p f*| = *1* |*Brouwer_degree2 p g*| = *1* *Brouwer_degree2 p g* = *Brouwer_degree2 p f*
using *Brouwer_degree2_homotopic* [*OF hom*] *Brouwer_degree2_compose f g zmult_eq_1_iff* **by** *auto*

lemma *Brouwer_degree2_homeomorphic_maps*:
assumes *homeomorphic_maps* (*nsphere p*) (*nsphere p*) *f g*
obtains |*Brouwer_degree2 p f*| = *1* |*Brouwer_degree2 p g*| = *1* *Brouwer_degree2 p g* = *Brouwer_degree2 p f*
using *assms*
by (*auto simp*: *homeomorphic_maps_def homotopic_with_equal continuous_map_compose intro*: *Brouwer_degree2_homotopy_equivalence*)

lemma *Brouwer_degree2_retraction_map*:
assumes *retraction_map* (*nsphere p*) (*nsphere p*) *f*
shows |*Brouwer_degree2 p f*| = *1*
proof −
obtain *g* **where** *g*: *retraction_maps* (*nsphere p*) (*nsphere p*) *f g*
using *assms* **by** (*auto simp*: *retraction_map_def*)
show *?thesis*
proof (*rule Brouwer_degree2_homotopy_equivalence*)
show *homotopic_with* (λ*x*. *True*) (*nsphere p*) (*nsphere p*) (*f* ∘ *g*) *id*
using *g* **apply** (*auto simp*: *retraction_maps_def*)
by (*simp add*: *homotopic_with_equal continuous_map_compose*)
show *continuous_map* (*nsphere p*) (*nsphere p*) *f* *continuous_map* (*nsphere p*) (*nsphere p*) *g*
using *g retraction_maps_def* **by** *blast*+
qed
qed

lemma *Brouwer_degree2_section_map*:
assumes *section_map* (*nsphere p*) (*nsphere p*) *f*
shows |*Brouwer_degree2 p f*| = *1*
proof −
obtain *g* **where** *g*: *retraction_maps* (*nsphere p*) (*nsphere p*) *g f*
using *assms* **by** (*auto simp*: *section_map_def*)
show *?thesis*
proof (*rule Brouwer_degree2_homotopy_equivalence*)
show *homotopic_with* (λ*x*. *True*) (*nsphere p*) (*nsphere p*) (*g* ∘ *f*) *id*
using *g* **apply** (*auto simp*: *retraction_maps_def*)
by (*simp add*: *homotopic_with_equal continuous_map_compose*)

show *continuous_map (nsphere p) (nsphere p) g continuous_map (nsphere p) (nsphere p) f*
using *g retraction_maps_def* **by** *blast+*
qed
qed

lemma *Brouwer_degree2_homeomorphic_map*:
homeomorphic_map (nsphere p) (nsphere p) f $\Longrightarrow$ *|Brouwer_degree2 p f| = 1*
using *Brouwer_degree2_retraction_map section_and_retraction_eq_homeomorphic_map*
by *blast*

lemma *Brouwer_degree2_nullhomotopic*:
assumes *homotopic_with* (λx. *True*) *(nsphere p) (nsphere p) f* (λx. *a*)
shows *Brouwer_degree2 p f = 0*
proof –
have *contf*: *continuous_map (nsphere p) (nsphere p) f*
and *contc*: *continuous_map (nsphere p) (nsphere p)* (λx. *a*)
using *homotopic_with_imp_continuous_maps [OF assms]* **by** *metis+*
have *Brouwer_degree2 p f = Brouwer_degree2 p* (λx. *a*)
using *Brouwer_degree2_homotopic [OF assms]* **.**
moreover
let *?G = reduced_homology_group (int p) (nsphere p)*
interpret *group ?G*
by *simp*
have *Brouwer_degree2 p* (λx. *a*) *= 0*
proof *(rule Brouwer_degree2_unique [OF contc])*
fix *c*
assume *c*: *c* $\in$ *carrier ?G*
have *continuous_map (nsphere p) (subtopology (nsphere p) {a})* (λf. *a*)
using *contc continuous_map_in_subtopology* **by** *blast*
then have *he*: *hom_induced p (nsphere p) {} (nsphere p) {}* (λx. *a*)
= hom_induced p (subtopology (nsphere p) {a}) {} (nsphere p) {} id $\circ$
hom_induced p (nsphere p) {} (subtopology (nsphere p) {a}) {} (λx. *a*)
by *(metis continuous_map_id_subt fun.map_id hom_induced_compose_empty)*
have *1*: *hom_induced p (nsphere p) {} (subtopology (nsphere p) {a}) {}* (λx. *a*) *c =*
$\mathbf{1}_{reduced\_homology\_group\ (int\ p)\ (subtopology\ (nsphere\ p)\ \{a\})}$
using *c trivial_reduced_homology_group_contractible_space [of subtopology (nsphere p) {a} p]*
by *(simp add: hom_induced_reduced contractible_space_subtopology_singleton trivial_group_subset group.trivial_group_subset subset_iff)*
show *hom_induced (int p) (nsphere p) {} (nsphere p) {}* (λx. *a*) *c =*
c $[\urcorner]_{?G}$ *(0::int)*
apply *(simp add: he 1)*
using *hom_induced_reduced_hom group_hom.hom_one group_hom_axioms_def group_hom_def group_reduced_homology_group* **by** *blast*

qed
ultimately show *?thesis*
by *metis*
qed

lemma *Brouwer_degree2_const*: *Brouwer_degree2 p* (*λx. a*) = *0*
proof (*cases continuous_map*(*nsphere p*) (*nsphere p*) (*λx. a*))
case *True*
then show *?thesis*
by (*auto intro*: *Brouwer_degree2_nullhomotopic* [**where** *a*=*a*])
next
case *False*
let *?G* = *reduced_homology_group* (*int p*) (*nsphere p*)
let *?H* = *homology_group* (*int p*) (*nsphere p*)
interpret *group ?G*
by *simp*
have *eq1*: $\mathbf{1}_{?H} = \mathbf{1}_{?G}$
by (*simp add*: *one_reduced_homology_group*)
have ∗: ∀*x*∈*carrier ?G*. *hom_induced* (*int p*) (*nsphere p*) {} (*nsphere p*) {} (*λx. a*) *x* = $\mathbf{1}_{?H}$
by (*metis False hom_induced_default one_relative_homology_group*)
obtain *c* **where** *c*: *c* ∈ *carrier ?G* **and** *ceq*: *subgroup_generated ?G* {*c*} = *?G*
using *cyclic_reduced_homology_group_nsphere* [*of p p*] **by** (*force simp*: *cyclic_group_def*)
have [*simp*]: *ord c* = *0*
by (*simp add*: *c ceq iso_finite* [*OF reduced_homology_group_nsphere*(*1*)] *flip*: *infinite_cyclic_subgroup_order*)
show *?thesis*
unfolding *Brouwer_degree2_def*
proof (*rule some_equality*)
fix *d* :: *int*
assume ∀*x*∈*carrier ?G*. *hom_induced* (*int p*) (*nsphere p*) {} (*nsphere p*) {} (*λx. a*) *x* = *x* $[\urcorner]_{?G}$ *d*
then have *c* $[\urcorner]_{?G}$ *d* = $\mathbf{1}_{?H}$
using ∗ *c* **by** *blast*
then have *int* (*ord c*) *dvd d*
using *c eq1 int_pow_eq_id* **by** *auto*
then show *d* = *0*
by (*simp add*: ∗ *del*: *one_relative_homology_group*)
qed (*use* ∗ *eq1* **in** *force*)
qed

corollary *Brouwer_degree2_nonsurjective*:
⟦*continuous_map*(*nsphere p*) (*nsphere p*) *f*; *f* ‘ *topspace* (*nsphere p*) ≠ *topspace* (*nsphere p*)⟧
⟹ *Brouwer_degree2 p f* = *0*
by (*meson Brouwer_degree2_nullhomotopic nullhomotopic_nonsurjective_sphere_map*)

```
proposition Brouwer_degree2_reflection:
  Brouwer_degree2 p (λx i. if i = 0 then −x i else x i) = −1 (is Brouwer_degree2
_ ?r = −1)
proof (induction p)
  case 0
  let ?G = homology_group 0 (nsphere 0)
  let ?D = homology_group 0 (discrete_topology {()})
  interpret group ?G
    by simp
  define r where r ≡ λx::nat⇒real. λi. if i = 0 then −x i else x i
  then have [simp]: r ∘ r = id
    by force
  have cmr: continuous_map (nsphere 0) (nsphere 0) r
    by (simp add: r_def continuous_map_nsphere_reflection)
  have ∗: hom_induced 0 (nsphere 0) {} (nsphere 0) {} r c = inv?G c
    if c ∈ carrier(reduced_homology_group 0 (nsphere 0)) for c
  proof −
    have c: c ∈ carrier ?G
      and ceq: hom_induced 0 (nsphere 0) {} (discrete_topology {()}) {} (λx. ())
c = 1?D
      using that by (auto simp: carrier_reduced_homology_group kernel_def)
    define pp::nat⇒real where pp ≡ λi. if i = 0 then 1 else 0
    define nn::nat⇒real where nn ≡ λi. if i = 0 then −1 else 0
    have topn0: topspace(nsphere 0) = {pp,nn}
       by (auto simp: nsphere pp_def nn_def fun_eq_iff power2_eq_1_iff split:
if_split_asm)
    have t1_space (nsphere 0)
      unfolding nsphere
      apply (rule t1_space_subtopology)
      by (metis (full_types) open_fun_def t1_space t1_space_def)
    then have dtn0: discrete_topology {pp,nn} = nsphere 0
      using finite_t1_space_imp_discrete_topology [OF topn0] by auto
    have pp ≠ nn
      by (auto simp: pp_def nn_def fun_eq_iff)
    have [simp]: r pp = nn r nn = pp
      by (auto simp: r_def pp_def nn_def fun_eq_iff)
    have iso: (λ(a,b). hom_induced 0 (subtopology (nsphere 0) {pp}) {} (nsphere
0) {} id a
                ⊗?G hom_induced 0 (subtopology (nsphere 0) {nn}) {} (nsphere 0)
{} id b)
              ∈ iso (homology_group 0 (subtopology (nsphere 0) {pp}) ×× homol-
ogy_group 0 (subtopology (nsphere 0) {nn}))
                ?G (is ?f ∈ iso (?P ×× ?N) ?G)
      apply (rule homology_additivity_explicit)
      using dtn0 ‹pp ≠ nn› by (auto simp: discrete_topology_unique)
    then have fim: ?f ‘ carrier(?P ×× ?N) = carrier ?G
      by (simp add: iso_def bij_betw_def)
   obtain d d′ where d: d ∈ carrier ?P and d′: d′ ∈ carrier ?N and eqc: ?f(d,d′)
```

= *c*
using *c* **by** (*force simp flip*: *fim*)
let *?h* = *λxx. hom_induced 0* (*subtopology* (*nsphere 0*) {*xx*}) {} (*discrete_topology* {()}) {} (*λx.* ())
have *continuous_map* (*subtopology* (*nsphere 0*) {*nn*}) (*nsphere 0*) *r*
using *cmr continuous_map_from_subtopology* **by** *blast*
then have *retraction_map* (*subtopology* (*nsphere 0*) {*pp*}) (*subtopology* (*nsphere 0*) {*nn*}) *r*
apply (*simp add*: *retraction_map_def retraction_maps_def continuous_map_in_subtopology*)
using ‹*r nn* = *pp*› ‹*r pp* = *nn*› *cmr continuous_map_from_subtopology*
by *blast*
then have *carrier ?N* = (*hom_induced 0* (*subtopology* (*nsphere 0*) {*pp*}) {} (*subtopology* (*nsphere 0*) {*nn*}) {} *r*) ‘ *carrier ?P*
by (*rule surj_hom_induced_retraction_map*)
then obtain *e* **where** *e*: *e* ∈ *carrier ?P* **and** *eqd'*: *hom_induced 0* (*subtopology* (*nsphere 0*) {*pp*}) {} (*subtopology* (*nsphere 0*) {*nn*}) {} *r e* = *d'*
using *d'* **by** *auto*
have *section_map* (*subtopology* (*nsphere 0*) {*pp*}) (*discrete_topology* {()}) (*λx.* ())
by (*force simp*: *section_map_def retraction_maps_def topn0*)
then have *?h pp* ∈ *mon ?P ?D*
by (*rule mon_hom_induced_section_map*)
then have *one*: *x* = *one ?P*
if *?h pp x* = $\mathbf{1}_{?D}$ *x* ∈ *carrier ?P* **for** *x*
using *that* **by** (*simp add*: *mon_iff_hom_one*)
interpret *hpd*: *group_hom ?P ?D ?h pp*
using *hom_induced_empty_hom* **by** (*simp add*: *hom_induced_empty_hom group_hom_axioms_def group_hom_def*)
interpret *hgd*: *group_hom ?G ?D hom_induced 0* (*nsphere 0*) {} (*discrete_topology* {()}) {} (*λx.* ())
using *hom_induced_empty_hom* **by** (*simp add*: *hom_induced_empty_hom group_hom_axioms_def group_hom_def*)
interpret *hpg*: *group_hom ?P ?G hom_induced 0* (*subtopology* (*nsphere 0*) {*pp*}) {} (*nsphere 0*) {} *r*
using *hom_induced_empty_hom* **by** (*simp add*: *hom_induced_empty_hom group_hom_axioms_def group_hom_def*)
interpret *hgg*: *group_hom ?G ?G hom_induced 0* (*nsphere 0*) {} (*nsphere 0*) {} *r*
using *hom_induced_empty_hom* **by** (*simp add*: *hom_induced_empty_hom group_hom_axioms_def group_hom_def*)
have *?h pp d* =
(*hom_induced 0* (*nsphere 0*) {} (*discrete_topology* {()}) {} (*λx.* ())
∘ *hom_induced 0* (*subtopology* (*nsphere 0*) {*pp*}) {} (*nsphere 0*) {} *id*) *d*
by (*simp flip*: *hom_induced_compose_empty*)
moreover
have *?h pp* = *?h nn* ∘ *hom_induced 0* (*subtopology* (*nsphere 0*) {*pp*}) {} (*subtopology* (*nsphere 0*) {*nn*}) {} *r*
by (*simp add*: *cmr continuous_map_from_subtopology continuous_map_in_subtopology image_subset_iff flip*: *hom_induced_compose_empty*)

```
    then have ?h pp e =
              (hom_induced 0 (nsphere 0) {} (discrete_topology {()}) {} (λx. ())
               ∘ hom_induced 0 (subtopology (nsphere 0) {nn}) {} (nsphere 0) {}
id) d'
      by (simp flip: hom_induced_compose_empty eqd')
   ultimately have ?h pp (d ⊗?P e) = hom_induced 0 (nsphere 0) {} (discrete_topology
{()}) {} (λx. ()) (?f(d,d'))
      by (simp add: d e hom_induced_carrier)
    then have ?h pp (d ⊗?P e) = 1?D
      using ceq eqc by simp
    then have inv_p: inv?P d = e
    by (metis (no_types, lifting) Group.group_def d e group.inv_equality group.r_inv
group_relative_homology_group one monoid.m_closed)
     have cmr_pn: continuous_map (subtopology (nsphere 0) {pp}) (subtopology
(nsphere 0) {nn}) r
    by (simp add: cmr continuous_map_from_subtopology continuous_map_in_subtopology
image_subset_iff)
    then have hom_induced 0 (subtopology (nsphere 0) {pp}) {} (nsphere 0) {}
(id ∘ r) =
               hom_induced 0 (subtopology (nsphere 0) {nn}) {} (nsphere 0) {} id ∘
             hom_induced 0 (subtopology (nsphere 0) {pp}) {} (subtopology (nsphere
0) {nn}) {} r
      using hom_induced_compose_empty continuous_map_id_subt by blast
    then have inv?G hom_induced 0 (subtopology (nsphere 0) {pp}) {} (nsphere
0) {} r d =
                    hom_induced 0 (subtopology (nsphere 0) {nn}) {} (nsphere 0) {}
id d'
      apply (simp add: flip: inv_p eqd')
      using d hpg.hom_inv by auto
    then have c: c = (hom_induced 0 (subtopology (nsphere 0) {pp}) {} (nsphere
0) {} id d)
                        ⊗?G inv?G (hom_induced 0 (subtopology (nsphere 0) {pp}) {}
(nsphere 0) {} r d)
      by (simp flip: eqc)
    have hom_induced 0 (nsphere 0) {} (nsphere 0) {} r ∘
          hom_induced 0 (subtopology (nsphere 0) {pp}) {} (nsphere 0) {} id =
          hom_induced 0 (subtopology (nsphere 0) {pp}) {} (nsphere 0) {} r
    by (metis cmr comp_id continuous_map_id_subt hom_induced_compose_empty)
    moreover
    have hom_induced 0 (nsphere 0) {} (nsphere 0) {} r ∘
          hom_induced 0 (subtopology (nsphere 0) {pp}) {} (nsphere 0) {} r =
          hom_induced 0 (subtopology (nsphere 0) {pp}) {} (nsphere 0) {} id
    by (metis ‹r ∘ r = id› cmr continuous_map_from_subtopology hom_induced_compose_empty)
    ultimately show ?thesis
    by (metis inv_p c comp_def d e hgg.hom_inv hgg.hom_mult hom_induced_carrier
hpd.G.inv_inv hpg.hom_inv inv_mult_group)
  qed
  show ?case
    unfolding r_def [symmetric]
```

using *Brouwer_degree2_unique* [*OF cmr*]
by (*auto simp*: ∗ *group.int_pow_neg group.int_pow_1 reduced_homology_group_def intro*!: *Brouwer_degree2_unique* [*OF cmr*])
next
case (*Suc p*)
let *?G* = *reduced_homology_group* (*int p*) (*nsphere p*)
let *?G1* = *reduced_homology_group* (*1* + *int p*) (*nsphere* (*Suc p*))
obtain *f g* **where** *fg*: *group_isomorphisms ?G ?G1 f g*
and ∗: ∀ *c*∈*carrier ?G*.
hom_induced (*1* + *int p*) (*nsphere* (*Suc p*)) {} (*nsphere* (*Suc p*)) {} *?r* (*f c*) =
f (*hom_induced p* (*nsphere p*) {} (*nsphere p*) {} *?r c*)
using *reduced_homology_group_nsphere_step*
by (*meson group.iso_iff_group_isomorphisms group_reduced_homology_group*)
then have *eq*: *carrier ?G1* = *f* ‘ *carrier ?G*
by (*fastforce simp add*: *iso_iff dest*: *group_isomorphisms_imp_iso*)
interpret *group_hom ?G ?G1 f*
by (*meson fg group_hom_axioms_def group_hom_def group_isomorphisms_def group_reduced_homology_group*)
have *homf*: *f* ∈ *hom ?G ?G1*
using *fg group_isomorphisms_def* **by** *blast*
have *hom_induced* (*1* + *int p*) (*nsphere* (*Suc p*)) {} (*nsphere* (*Suc p*)) {} *?r* (*f y*) = *f y* $[^]_{?G1}$ (−*1*::*int*)
if *y* ∈ *carrier ?G* **for** *y*
by (*simp add*: *that* ∗ *Brouwer_degree2 Suc hom_int_pow*)
then show *?case*
by (*fastforce simp*: *eq intro*: *Brouwer_degree2_unique* [*OF continuous_map_nsphere_reflection*])
qed

end

0.4 Invariance of Domain

theory *Invariance_of_Domain*
imports *Brouwer_Degree HOL*−*Analysis.Continuous_Extension HOL*−*Analysis.Homeomorphism*

begin

0.4.1 Degree invariance mod 2 for map between pairs

theorem *Borsuk_odd_mapping_degree_step*:
assumes *cmf*: *continuous_map* (*nsphere n*) (*nsphere n*) *f*
and *f*: ⋀*u*. *u* ∈ *topspace*(*nsphere n*) ⟹ (*f* ∘ (λ*x i*. −*x i*)) *u* = ((λ*x i*. −*x i*) ∘ *f*) *u*
and *fim*: *f* ∈ (*topspace*(*nsphere*(*n* − *Suc 0*))) → *topspace*(*nsphere*(*n* − *Suc 0*))
shows *even* (*Brouwer_degree2 n f* − *Brouwer_degree2* (*n* − *Suc 0*) *f*)
proof (*cases n* = *0*)
case *False*
define *neg* **where** *neg* ≡ λ*x*::*nat*⇒*real*. λ*i*. −*x i*

```
  define upper where upper ≡ λn. {x::nat⇒real. x n ≥ 0}
  define lower where lower ≡ λn. {x::nat⇒real. x n ≤ 0}
  define equator where equator ≡ λn. {x::nat⇒real. x n = 0}
  define usphere where usphere ≡ λn. subtopology (nsphere n) (upper n)
  define lsphere where lsphere ≡ λn. subtopology (nsphere n) (lower n)
  have [simp]: neg x i = −x i for x i
    by (force simp: neg_def)
  have equator_upper: equator n ⊆ upper n
    by (force simp: equator_def upper_def)
  then have [simp]: id ∈ equator n → upper n
    by force
  have upper_usphere: subtopology (nsphere n) (upper n) = usphere n
    by (simp add: usphere_def)
  let ?rhgn = relative_homology_group n (nsphere n)
  let ?hi_ee = hom_induced n (nsphere n) (equator n) (nsphere n) (equator n)
  interpret GE: comm_group ?rhgn (equator n)
    by simp
  interpret HB: group_hom ?rhgn (equator n)
                      homology_group (int n − 1) (subtopology (nsphere n) (equator
n))
                        hom_boundary n (nsphere n) (equator n)
    by (simp add: group_hom_axioms_def group_hom_def hom_boundary_hom)
  interpret HIU: group_hom ?rhgn (equator n)
                         ?rhgn (upper n)
                          hom_induced n (nsphere n) (equator n) (nsphere n) (upper
n) id
    by (simp add: group_hom_axioms_def group_hom_def hom_induced_hom)
  have subt_eq: subtopology (nsphere n) {x. x n = 0} = nsphere (n − Suc 0)
    by (metis False Suc_pred le_zero_eq not_le subtopology_nsphere_equator)
  then have equ: subtopology (nsphere n) (equator n) = nsphere(n − Suc 0)
                 subtopology (lsphere n) (equator n) = nsphere(n − Suc 0)
                 subtopology (usphere n) (equator n) = nsphere(n − Suc 0)
     using False by (auto simp: lsphere_def usphere_def equator_def lower_def
upper_def
        subtopology_subtopology simp flip: Collect_conj_eq cong: rev_conj_cong)
  have cmr: continuous_map (nsphere(n − Suc 0)) (nsphere(n − Suc 0)) f
   by (metis cmf continuous_map_from_subtopology continuous_map_in_subtopology
equ(1)
        fim subtopology_restrict topspace_subtopology)
  have f x n = 0 if x ∈ topspace (nsphere n) x n = 0 for x
  proof −
    have x ∈ topspace (nsphere (n − Suc 0))
      by (simp add: that topspace_nsphere_minus1)
    moreover have topspace (nsphere n) ∩ {f. f n = 0} = topspace (nsphere (n
− Suc 0))
      by (metis subt_eq topspace_subtopology)
    ultimately show ?thesis
      using fim by auto
  qed
```

then have fimeq: f ∈ (topspace (nsphere n) ∩ equator n) → topspace (nsphere n) ∩ equator n
using fim cmf **by** (auto simp: equator_def continuous_map_def image_subset_iff)
have ⋀k. continuous_map (powertop_real UNIV) euclideanreal (λx. − x k)
by (metis UNIV_I continuous_map_product_projection continuous_map_minus)
then have cm_neg: continuous_map (nsphere m) (nsphere m) neg **for** m
by (force simp: nsphere continuous_map_in_subtopology neg_def continuous_map_componentwise_UNIV intro: continuous_map_from_subtopology)
then have cm_neg_lu: continuous_map (lsphere n) (usphere n) neg
by (auto simp: lsphere_def usphere_def lower_def upper_def continuous_map_from_subtopology continuous_map_in_subtopology)
have neg_in_top_iff: neg x ∈ topspace(nsphere m) ⟷ x ∈ topspace(nsphere m) **for** m x
by (simp add: nsphere_def neg_def topspace_Euclidean_space)
obtain z **where** zcarr: z ∈ carrier (reduced_homology_group (int n − 1) (nsphere (n − Suc 0)))
and zeq: subgroup_generated (reduced_homology_group (int n − 1) (nsphere (n − Suc 0))) {z}
= reduced_homology_group (int n − 1) (nsphere (n − Suc 0))
using cyclic_reduced_homology_group_nsphere [of int n − 1 n − Suc 0] **by** (auto simp: cyclic_group_def)
have hom_boundary n (subtopology (nsphere n) {x. x n ≤ 0}) {x. x n = 0}
∈ Group.iso (relative_homology_group n
(subtopology (nsphere n) {x. x n ≤ 0}) {x. x n = 0})
(reduced_homology_group (int n − 1) (nsphere (n − Suc 0)))
using iso_lower_hemisphere_reduced_homology_group [of int n − 1 n − Suc 0] False **by** simp
then obtain gp **where** g: group_isomorphisms
(relative_homology_group n (subtopology (nsphere n) {x. x n ≤ 0}) {x. x n = 0})
(reduced_homology_group (int n − 1) (nsphere (n − Suc 0)))
(hom_boundary n (subtopology (nsphere n) {x. x n ≤ 0}) {x. x n = 0})
gp
by (auto simp: group.iso_iff_group_isomorphisms)
then interpret gp: group_hom reduced_homology_group (int n − 1) (nsphere (n − Suc 0))
relative_homology_group n (subtopology (nsphere n) {x. x n ≤ 0}) {x. x n = 0} gp
by (simp add: group_hom_axioms_def group_hom_def group_isomorphisms_def)
obtain zp **where** zpcarr: zp ∈ carrier(relative_homology_group n (lsphere n) (equator n))
and zp_z: hom_boundary n (lsphere n) (equator n) zp = z
and zp_sg: subgroup_generated (relative_homology_group n (lsphere n) (equator n)) {zp}
= relative_homology_group n (lsphere n) (equator n)
proof
show gp z ∈ carrier (relative_homology_group n (lsphere n) (equator n))
hom_boundary n (lsphere n) (equator n) (gp z) = z

using *g zcarr* **by** (*auto simp*: *lsphere_def equator_def lower_def group_isomorphisms_def*)
have *giso*: *gp* ∈ *Group.iso* (*reduced_homology_group* (*int n* − *1*) (*nsphere* (*n* − *Suc 0*)))
(*relative_homology_group n* (*subtopology* (*nsphere n*) {*x*. *x n* ≤ *0*}) {*x*. *x n* = *0*})
by (*metis* (*mono_tags*, *lifting*) *g group_isomorphisms_imp_iso group_isomorphisms_sym*)
show *subgroup_generated* (*relative_homology_group n* (*lsphere n*) (*equator n*)) {*gp z*} =
relative_homology_group n (*lsphere n*) (*equator n*)
apply (*rule monoid.equality*)
using *giso gp.subgroup_generated_by_image* [*of* {*z*}] *zcarr*
by (*auto simp*: *lsphere_def equator_def lower_def zeq gp.iso_iff*)
qed
have *hb_iso*: *hom_boundary n* (*subtopology* (*nsphere n*) {*x*. *x n* ≥ *0*}) {*x*. *x n* = *0*}
∈ *iso* (*relative_homology_group n* (*subtopology* (*nsphere n*) {*x*. *x n* ≥ *0*}) {*x*. *x n* = *0*})
(*reduced_homology_group* (*int n* − *1*) (*nsphere* (*n* − *Suc 0*)))
using *iso_upper_hemisphere_reduced_homology_group* [*of int n* − *1 n* − *Suc 0*] *False* **by** *simp*
then obtain *gn* **where** *g*: *group_isomorphisms*
(*relative_homology_group n* (*subtopology* (*nsphere n*) {*x*. *x n* ≥ *0*}) {*x*. *x n* = *0*})
(*reduced_homology_group* (*int n* − *1*) (*nsphere* (*n* − *Suc 0*)))
(*hom_boundary n* (*subtopology* (*nsphere n*) {*x*. *x n* ≥ *0*}) {*x*. *x n* = *0*})
gn
by (*auto simp*: *group.iso_iff_group_isomorphisms*)
then interpret *gn*: *group_hom reduced_homology_group* (*int n* − *1*) (*nsphere* (*n* − *Suc 0*))
relative_homology_group n (*subtopology* (*nsphere n*) {*x*. *x n* ≥ *0*}) {*x*. *x n* = *0*} *gn*
by (*simp add*: *group_hom_axioms_def group_hom_def group_isomorphisms_def*)
obtain *zn* **where** *zncarr*: *zn* ∈ *carrier*(*relative_homology_group n* (*usphere n*) (*equator n*))
and *zn_z*: *hom_boundary n* (*usphere n*) (*equator n*) *zn* = *z*
and *zn_sg*: *subgroup_generated* (*relative_homology_group n* (*usphere n*) (*equator n*)) {*zn*}
= *relative_homology_group n* (*usphere n*) (*equator n*)
proof
show *gn z* ∈ *carrier* (*relative_homology_group n* (*usphere n*) (*equator n*))
hom_boundary n (*usphere n*) (*equator n*) (*gn z*) = *z*
using *g zcarr* **by** (*auto simp*: *usphere_def equator_def upper_def group_isomorphisms_def*)
have *giso*: *gn* ∈ *Group.iso* (*reduced_homology_group* (*int n* − *1*) (*nsphere* (*n* − *Suc 0*)))
(*relative_homology_group n* (*subtopology* (*nsphere n*) {*x*. *x n* ≥ *0*}) {*x*. *x n* = *0*})
by (*metis* (*mono_tags*, *lifting*) *g group_isomorphisms_imp_iso group_isomorphisms_sym*)
show *subgroup_generated* (*relative_homology_group n* (*usphere n*) (*equator n*))

{*gn z*} =
relative_homology_group n (*usphere n*) (*equator n*)
apply (*rule monoid.equality*)
using *giso gn.subgroup_generated_by_image* [*of* {*z*}] *zcarr*
by (*auto simp*: *usphere_def equator_def upper_def zeq gn.iso_iff*)
qed
let *?hi_lu* = *hom_induced n* (*lsphere n*) (*equator n*) (*nsphere n*) (*upper n*) *id*
interpret *gh_lu*: *group_hom relative_homology_group n* (*lsphere n*) (*equator n*) *?rhgn* (*upper n*) *?hi_lu*
by (*simp add*: *group_hom_axioms_def group_hom_def hom_induced_hom*)
interpret *gh_eef*: *group_hom ?rhgn* (*equator n*) *?rhgn* (*equator n*) *?hi_ee f*
by (*simp add*: *group_hom_axioms_def group_hom_def hom_induced_hom*)
define *wp* **where** *wp* ≡ *?hi_lu zp*
then have *wpcarr*: *wp* ∈ *carrier*(*?rhgn* (*upper n*))
by (*simp add*: *hom_induced_carrier*)
have *hom_induced n* (*nsphere n*) {} (*nsphere n*) {*x*. *x n* ≥ *0*} *id*
∈ *iso* (*reduced_homology_group n* (*nsphere n*))
(*?rhgn* {*x*. *x n* ≥ *0*})
using *iso_reduced_homology_group_upper_hemisphere* [*of n n n*] **by** *auto*
then have *carrier*(*?rhgn* {*x*. *x n* ≥ *0*})
⊆ (*hom_induced n* (*nsphere n*) {} (*nsphere n*) {*x*. *x n* ≥ *0*} *id*)
‘ *carrier*(*reduced_homology_group n* (*nsphere n*))
by (*simp add*: *iso_iff*)
then obtain *vp* **where** *vpcarr*: *vp* ∈ *carrier*(*reduced_homology_group n* (*nsphere n*))
and *eqwp*: *hom_induced n* (*nsphere n*) {} (*nsphere n*) (*upper n*) *id vp* = *wp*
using *wpcarr* **by** (*auto simp*: *upper_def*)
define *wn* **where** *wn* ≡ *hom_induced n* (*usphere n*) (*equator n*) (*nsphere n*) (*lower n*) *id zn*
then have *wncarr*: *wn* ∈ *carrier*(*?rhgn* (*lower n*))
by (*simp add*: *hom_induced_carrier*)
have *hom_induced n* (*nsphere n*) {} (*nsphere n*) {*x*. *x n* ≤ *0*} *id*
∈ *iso* (*reduced_homology_group n* (*nsphere n*))
(*?rhgn* {*x*. *x n* ≤ *0*})
using *iso_reduced_homology_group_lower_hemisphere* [*of n n n*] **by** *auto*
then have *carrier*(*?rhgn* {*x*. *x n* ≤ *0*})
⊆ (*hom_induced n* (*nsphere n*) {} (*nsphere n*) {*x*. *x n* ≤ *0*} *id*)
‘ *carrier*(*reduced_homology_group n* (*nsphere n*))
by (*simp add*: *iso_iff*)
then obtain *vn* **where** *vpcarr*: *vn* ∈ *carrier*(*reduced_homology_group n* (*nsphere n*))
and *eqwp*: *hom_induced n* (*nsphere n*) {} (*nsphere n*) (*lower n*) *id vn* = *wn*
using *wncarr* **by** (*auto simp*: *lower_def*)
define *up* **where** *up* ≡ *hom_induced n* (*lsphere n*) (*equator n*) (*nsphere n*) (*equator n*) *id zp*
then have *upcarr*: *up* ∈ *carrier*(*?rhgn* (*equator n*))
by (*simp add*: *hom_induced_carrier*)
define *un* **where** *un* ≡ *hom_induced n* (*usphere n*) (*equator n*) (*nsphere n*) (*equator n*) *id zn*

```
  then have uncarr: un ∈ carrier(?rhgn (equator n))
    by (simp add: hom_induced_carrier)
  have *: (λ(x, y).
            hom_induced n (lsphere n) (equator n) (nsphere n) (equator n) id x
          ⊗?rhgn (equator n)
            hom_induced n (usphere n) (equator n) (nsphere n) (equator n) id y)
        ∈ Group.iso
            (relative_homology_group n (lsphere n) (equator n) ××
             relative_homology_group n (usphere n) (equator n))
            (?rhgn (equator n))
  proof (rule conjunct1 [OF exact_sequence_sum_lemma [OF abelian_relative_homology_group]])
    show hom_induced n (lsphere n) (equator n) (nsphere n) (upper n) id
        ∈ Group.iso (relative_homology_group n (lsphere n) (equator n))
            (?rhgn (upper n))
      unfolding lsphere_def usphere_def equator_def lower_def upper_def
      using iso_relative_homology_group_lower_hemisphere by blast
    show hom_induced n (usphere n) (equator n) (nsphere n) (lower n) id
        ∈ Group.iso (relative_homology_group n (usphere n) (equator n))
            (?rhgn (lower n))
      unfolding lsphere_def usphere_def equator_def lower_def upper_def
      using iso_relative_homology_group_upper_hemisphere by blast
    show exact_seq
         ([?rhgn (lower n),
           ?rhgn (equator n),
           relative_homology_group n (lsphere n) (equator n)],
          [hom_induced n (nsphere n) (equator n) (nsphere n) (lower n) id,
           hom_induced n (lsphere n) (equator n) (nsphere n) (equator n) id])
      unfolding lsphere_def usphere_def equator_def lower_def upper_def
      by (rule homology_exactness_triple_3) force
    show exact_seq
         ([?rhgn (upper n),
           ?rhgn (equator n),
           relative_homology_group n (usphere n) (equator n)],
          [hom_induced n (nsphere n) (equator n) (nsphere n) (upper n) id,
           hom_induced n (usphere n) (equator n) (nsphere n) (equator n) id])
      unfolding lsphere_def usphere_def equator_def lower_def upper_def
      by (rule homology_exactness_triple_3) force
  next
    fix x
    assume x ∈ carrier (relative_homology_group n (lsphere n) (equator n))
    show hom_induced n (nsphere n) (equator n) (nsphere n) (upper n) id
         (hom_induced n (lsphere n) (equator n) (nsphere n) (equator n) id x) =
        hom_induced n (lsphere n) (equator n) (nsphere n) (upper n) id x
        by (simp add: hom_induced_compose' subset_iff lsphere_def usphere_def
equator_def lower_def upper_def)
  next
    fix x
    assume x ∈ carrier (relative_homology_group n (usphere n) (equator n))
    show hom_induced n (nsphere n) (equator n) (nsphere n) (lower n) id
```

```
        (hom_induced n (usphere n) (equator n) (nsphere n) (equator n) id x) =
       hom_induced n (usphere n) (equator n) (nsphere n) (lower n) id x
       by (simp add: hom_induced_compose' subset_iff lsphere_def usphere_def
equator_def lower_def upper_def)
  qed
  then have sb: carrier (?rhgn (equator n))
            ⊆ (λ(x, y).
            hom_induced n (lsphere n) (equator n) (nsphere n) (equator n) id x
          ⊗?rhgn (equator n)
            hom_induced n (usphere n) (equator n) (nsphere n) (equator n) id y)
               ‘ carrier (relative_homology_group n (lsphere n) (equator n) ××
             relative_homology_group n (usphere n) (equator n))
    by (simp add: iso_iff)
  obtain a b::int
    where up_ab: ?hi_ee f up
            = up [^]?rhgn (equator n) a⊗?rhgn (equator n) un [^]?rhgn (equator n) b
  proof −
    have hiupcarr: ?hi_ee f up ∈ carrier(?rhgn (equator n))
      by (simp add: hom_induced_carrier)
     obtain u v where u: u ∈ carrier (relative_homology_group n (lsphere n)
(equator n))
      and v: v ∈ carrier (relative_homology_group n (usphere n) (equator n))
      and eq: ?hi_ee f up =
                hom_induced n (lsphere n) (equator n) (nsphere n) (equator n) id u
                ⊗?rhgn (equator n)
                hom_induced n (usphere n) (equator n) (nsphere n) (equator n) id v
      using subsetD [OF sb hiupcarr] by auto
    have u ∈ carrier (subgroup_generated (relative_homology_group n (lsphere n)
(equator n)) {zp})
      by (simp_all add: u zp_sg)
   then obtain a::int where a: u = zp [^]relative_homology_group n (lsphere n) (equator n)
a
     by (metis group.carrier_subgroup_generated_by_singleton group_relative_homology_group
rangeE zpcarr)
    have ae: hom_induced n (lsphere n) (equator n) (nsphere n) (equator n) id
           (pow (relative_homology_group n (lsphere n) (equator n)) zp a)
        = pow (?rhgn (equator n)) (hom_induced n (lsphere n) (equator n) (nsphere
n) (equator n) id zp) a
     by (meson group_hom.hom_int_pow group_hom_axioms_def group_hom_def
group_relative_homology_group hom_induced zpcarr)
    have v ∈ carrier (subgroup_generated (relative_homology_group n (usphere n)
(equator n)) {zn})
      by (simp_all add: v zn_sg)
   then obtain b::int where b: v = zn [^]relative_homology_group n (usphere n) (equator n)
b
     by (metis group.carrier_subgroup_generated_by_singleton group_relative_homology_group
rangeE zncarr)
    have be: hom_induced n (usphere n) (equator n) (nsphere n) (equator n) id
          (zn [^]relative_homology_group n (usphere n) (equator n) b)
```

= *hom_induced n (usphere n) (equator n) (nsphere n) (equator n) id zn* [⌒]*relative_homology_group n (nsphere n) (equator n) b*
by (*meson group_hom.hom_int_pow group_hom_axioms_def group_hom_def group_relative_homology_group hom_induced zncarr*)
show *thesis*
proof
show *?hi_ee f up*
= *up* [⌒]*?rhgn (equator n) a* ⊗*?rhgn (equator n) un* [⌒]*?rhgn (equator n) b*
using *a ae b be eq local.up_def un_def* **by** *auto*
qed
qed
have (*hom_boundary n (nsphere n) (equator n)*
∘ *hom_induced n (lsphere n) (equator n) (nsphere n) (equator n) id*) *zp* = *z*
using *zp_z equ* **apply** (*simp add*: *lsphere_def naturality_hom_induced*)
by (*metis hom_boundary_carrier hom_induced_id*)
then have *up_z*: *hom_boundary n (nsphere n) (equator n) up* = *z*
by (*simp add*: *up_def*)
have (*hom_boundary n (nsphere n) (equator n)*
∘ *hom_induced n (usphere n) (equator n) (nsphere n) (equator n) id*) *zn* = *z*
using *zn_z equ* **apply** (*simp add*: *usphere_def naturality_hom_induced*)
by (*metis hom_boundary_carrier hom_induced_id*)
then have *un_z*: *hom_boundary n (nsphere n) (equator n) un* = *z*
by (*simp add*: *un_def*)
have *Bd_ab*: *Brouwer_degree2 (n − Suc 0) f* = *a* + *b*
proof (*rule Brouwer_degree2_unique_generator*; *use False int_ops* **in** *simp_all*)
show *continuous_map (nsphere (n − Suc 0)) (nsphere (n − Suc 0)) f*
using *cmr* **by** *auto*
show *subgroup_generated (reduced_homology_group (int n − 1) (nsphere (n − Suc 0))) {z}* =
reduced_homology_group (int n − 1) (nsphere (n − Suc 0))
using *zeq* **by** *blast*
have (*hom_induced (int n − 1) (nsphere (n − Suc 0)) {} (nsphere (n − Suc 0)) {} f*
∘ *hom_boundary n (nsphere n) (equator n)*) *up*
= (*hom_boundary n (nsphere n) (equator n)* ∘
?hi_ee f) *up*
using *naturality_hom_induced* [*OF cmf fimeq*, *of n*, *symmetric*]
by (*simp add*: *subtopology_restrict equ fun_eq_iff*)
also have ... = *hom_boundary n (nsphere n) (equator n)*
(*up* [⌒]*relative_homology_group n (nsphere n) (equator n)*
a ⊗*relative_homology_group n (nsphere n) (equator n)*
un [⌒]*relative_homology_group n (nsphere n) (equator n) b*)
by (*simp add*: *o_def up_ab*)
also have ... = *z* [⌒]*reduced_homology_group (int n − 1) (nsphere (n − Suc 0))* (*a* + *b*)
using *zcarr*
apply (*simp add*: *HB.hom_int_pow reduced_homology_group_def group.int_pow_subgroup_generated upcarr uncarr*)

```
    by (metis equ(1) group.int_pow_mult group_relative_homology_group hom_boundary_carrier
un_z up_z)
  finally show hom_induced (int n − 1) (nsphere (n − Suc 0)) {} (nsphere (n
− Suc 0)) {} f z =
        z [^]reduced_homology_group (int n − 1) (nsphere (n − Suc 0)) (a + b)
      by (simp add: up_z)
  qed
  define u where u ≡ up ⊗?rhgn (equator n) inv?rhgn (equator n) un
  have ucarr: u ∈ carrier (?rhgn (equator n))
    by (simp add: u_def uncarr upcarr)
  then have u [^]?rhgn (equator n) Brouwer_degree2 n f = u [^]?rhgn (equator n)
(a − b)
            ⟷ (GE.ord u) dvd a − b − Brouwer_degree2 n f
    by (simp add: GE.int_pow_eq)
  moreover
  have GE.ord u = 0
  proof (clarsimp simp add: GE.ord_eq_0 ucarr)
    fix d :: nat
    assume 0 < d
         and u [^]?rhgn (equator n) d = singular_relboundary_set n (nsphere n)
(equator n)
    then have hom_induced n (nsphere n) (equator n) (nsphere n) (upper n) id u
[^]?rhgn (upper n) d
              = 1?rhgn (upper n)
      by (metis HIU.hom_one HIU.hom_nat_pow one_relative_homology_group
ucarr)
    moreover
    have ?hi_lu
        = hom_induced n (nsphere n) (equator n) (nsphere n) (upper n) id ∘
          hom_induced n (lsphere n) (equator n) (nsphere n) (equator n) id
     by (simp add: lsphere_def image_subset_iff equator_upper flip: hom_induced_compose)
    then have p: wp = hom_induced n (nsphere n) (equator n) (nsphere n) (upper
n) id up
      by (simp add: local.up_def wp_def)
    have n: hom_induced n (nsphere n) (equator n) (nsphere n) (upper n) id un =
1?rhgn (upper n)
      using homology_exactness_triple_3 [OF equator_upper, of n nsphere n]
      using un_def zncarr by (auto simp: upper_usphere kernel_def)
    have hom_induced n (nsphere n) (equator n) (nsphere n) (upper n) id u = wp
      unfolding u_def
      using p n HIU.inv_one HIU.r_one uncarr upcarr by auto
    ultimately have (wp [^]?rhgn (upper n) d) = 1?rhgn (upper n)
      by simp
   moreover have infinite (carrier (subgroup_generated (?rhgn (upper n)) {wp}))
    proof −
      have ?rhgn (upper n) ≅ reduced_homology_group n (nsphere n)
        unfolding upper_def
        using iso_reduced_homology_group_upper_hemisphere [of n n n]
```

```
      by (blast intro: group.iso_sym group_reduced_homology_group is_isoI)
    also have ... ≅ integer_group
      by (simp add: reduced_homology_group_nsphere)
    finally have iso: ?rhgn (upper n) ≅ integer_group .
    have carrier (subgroup_generated (?rhgn (upper n)) {wp}) = carrier (?rhgn
(upper n))
    using gh_lu.subgroup_generated_by_image [of {zp}] zpcarr HIU.carrier_subgroup_generated_subset
        gh_lu.iso_iff iso_relative_homology_group_lower_hemisphere zp_sg
      by (auto simp: lower_def lsphere_def upper_def equator_def wp_def)
    then show ?thesis
      using infinite_UNIV_int iso_finite [OF iso] by simp
  qed
  ultimately show False
    using HIU.finite_cyclic_subgroup ‹0 < d› wpcarr by blast
 qed
 ultimately have iff: u [^]_?rhgn (equator n) Brouwer_degree2 n f = u [^]_?rhgn (equator n)
(a − b)
              ⟷ Brouwer_degree2 n f = a − b
  by auto
 have u [^]_?rhgn (equator n) Brouwer_degree2 n f = ?hi_ee f u
 proof −
  have ne: topspace (nsphere n) ∩ equator n ≠ {}
    using False equator_def in_topspace_nsphere by fastforce
  have eq1: hom_boundary n (nsphere n) (equator n) u
        = 1_reduced_homology_group (int n − 1) (subtopology (nsphere n) (equator n))
   using one_reduced_homology_group u_def un_z uncarr up_z upcarr by force
  then have uhom: u ∈ hom_induced n (nsphere n) {} (nsphere n) (equator n)
id `
                    carrier (reduced_homology_group (int n) (nsphere n))
   using homology_exactness_reduced_1 [OF ne, of n] eq1 ucarr by (auto simp:
kernel_def)
   then obtain v where vcarr: v ∈ carrier (reduced_homology_group (int n)
(nsphere n))
               and ueq: u = hom_induced n (nsphere n) {} (nsphere n) (equator
n) id v
    by blast
  interpret GH_hi: group_hom homology_group n (nsphere n)
                   ?rhgn (equator n)
                   hom_induced n (nsphere n) {} (nsphere n) (equator n) id
    by (simp add: group_hom_axioms_def group_hom_def hom_induced_hom)
  have poweq: pow (homology_group n (nsphere n)) x i = pow (reduced_homology_group
n (nsphere n)) x i
    for x and i::int
    by (simp add: False un_reduced_homology_group)
  have vcarr': v ∈ carrier (homology_group n (nsphere n))
    using carrier_reduced_homology_group_subset vcarr by blast
  have u [^]_?rhgn (equator n) Brouwer_degree2 n f
      = hom_induced n (nsphere n) {} (nsphere n) (equator n) f v
    using vcarr vcarr'
```

by (*simp add*: *ueq poweq hom_induced_compose′ cmf flip*: *GH_hi*.*hom_int_pow Brouwer_degree2*)
also have ... = *hom_induced n* (*nsphere n*) (*topspace*(*nsphere n*) ∩ *equator n*) (*nsphere n*) (*equator n*) *f*
(*hom_induced n* (*nsphere n*) {} (*nsphere n*) (*topspace*(*nsphere n*) ∩ *equator n*) *id v*)
using *fimeq* **by** (*simp add*: *hom_induced_compose′ cmf Pi_iff*)
also have ... = *?hi_ee f u*
by (*metis hom_induced inf*.*left_idem ueq*)
finally show *?thesis* **.**
qed
moreover
interpret *gh_een*: *group_hom ?rhgn* (*equator n*) *?rhgn* (*equator n*) *?hi_ee neg*
by (*simp add*: *group_hom_axioms_def group_hom_def hom_induced_hom*)
have *hi_up_eq_un*: *?hi_ee neg up* = *un* [^]$_{?rhgn\ (equator\ n)}$ *Brouwer_degree2* (*n* − *Suc 0*) *neg*
proof −
have *?hi_ee neg* (*hom_induced n* (*lsphere n*) (*equator n*) (*nsphere n*) (*equator n*) *id zp*)
= *hom_induced n* (*lsphere n*) (*equator n*) (*nsphere n*) (*equator n*) (*neg* ∘ *id*) *zp*
by (*intro hom_induced_compose′*) (*auto simp*: *lsphere_def equator_def cm_neg*)
also have ... = *hom_induced n* (*usphere n*) (*equator n*) (*nsphere n*) (*equator n*) *id*
(*hom_induced n* (*lsphere n*) (*equator n*) (*usphere n*) (*equator n*) *neg zp*)
by (*subst hom_induced_compose′* [*OF cm_neg_lu*]) (*auto simp*: *usphere_def equator_def*)
also have *hom_induced n* (*lsphere n*) (*equator n*) (*usphere n*) (*equator n*) *neg zp*
= *zn* [^]$_{relative\_homology\_group\ n\ (usphere\ n)\ (equator\ n)}$ *Brouwer_degree2* (*n* − *Suc 0*) *neg*
proof −
let *?hb* = *hom_boundary n* (*usphere n*) (*equator n*)
have *eq*: *subtopology* (*nsphere n*) {*x*. *x n* ≥ *0*} = *usphere n* ∧ {*x*. *x n* = *0*} = *equator n*
by (*auto simp*: *usphere_def upper_def equator_def*)
with *hb_iso* **have** *inj*: *inj_on* (*?hb*) (*carrier* (*relative_homology_group n* (*usphere n*) (*equator n*)))
by (*simp add*: *iso_iff*)
interpret *hb_hom*: *group_hom relative_homology_group n* (*usphere n*) (*equator n*)
reduced_homology_group (*int n* − *1*) (*nsphere* (*n* − *Suc 0*))
?hb
using *hb_iso iso_iff eq group_hom_axioms_def group_hom_def* **by** *fastforce*
show *?thesis*
proof (*rule inj_onD* [*OF inj*])
have ∗: *hom_induced* (*int n* − *1*) (*nsphere* (*n* − *Suc 0*)) {} (*nsphere* (*n* − *Suc 0*)) {} *neg z*

```
            = z [^]homology_group (int n − 1) (nsphere (n − Suc 0)) Brouwer_degree2
(n − Suc 0) neg
          using Brouwer_degree2 [of z n − Suc 0 neg] False zcarr
        by (simp add: int_ops group.int_pow_subgroup_generated reduced_homology_group_def)
        have ?hb ∘
              hom_induced n (lsphere n) (equator n) (usphere n) (equator n) neg
             = hom_induced (int n − 1) (nsphere (n − Suc 0)) {} (nsphere (n −
Suc 0)) {} neg ∘
              hom_boundary n (lsphere n) (equator n)
          apply (subst naturality_hom_induced [OF cm_neg_lu])
           apply (force simp: equator_def neg_def)
          by (simp add: equ)
        then have ?hb
                   (hom_induced n (lsphere n) (equator n) (usphere n) (equator n)
neg zp)
          = (z [^]homology_group (int n − 1) (nsphere (n − Suc 0)) Brouwer_degree2
(n − Suc 0) neg)
          by (metis ∗ comp_apply zp_z)
        also have . . . = ?hb (zn [^]relative_homology_group n (usphere n) (equator n)
          Brouwer_degree2 (n − Suc 0) neg)
        by (metis group.int_pow_subgroup_generated group_relative_homology_group
hb_hom.hom_int_pow reduced_homology_group_def zcarr zn_z zncarr)
          finally show ?hb (hom_induced n (lsphere n) (equator n) (usphere n)
(equator n) neg zp) =
        ?hb (zn [^]relative_homology_group n (usphere n) (equator n)
          Brouwer_degree2 (n − Suc 0) neg) by simp
      qed (auto simp: hom_induced_carrier group.int_pow_closed zncarr)
    qed
    finally show ?thesis
     by (metis (no_types, lifting) group_hom.hom_int_pow group_hom_axioms_def
group_hom_def group_relative_homology_group hom_induced local.up_def un_def
zncarr)
  qed
  have continuous_map (nsphere (n − Suc 0)) (nsphere (n − Suc 0)) neg
    using cm_neg by blast
  then have homeomorphic_map (nsphere (n − Suc 0)) (nsphere (n − Suc 0))
neg
    apply (auto simp: homeomorphic_map_maps homeomorphic_maps_def)
    apply (rule_tac x=neg in exI, auto)
    done
  then have Brouwer_degree2_21: Brouwer_degree2 (n − Suc 0) neg ^ 2 = 1
    using Brouwer_degree2_homeomorphic_map power2_eq_1_iff by force
  have hi_un_eq_up: ?hi_ee neg un = up [^]?rhgn (equator n) Brouwer_degree2
(n − Suc 0) neg (is ?f un = ?y)
  proof −
    have [simp]: neg ∘ neg = id
      by force
    have ?f (?f ?y) = ?y
```

apply (*subst hom_induced_compose′ [OF cm_neg _ cm_neg]*)
apply(*force simp*: *equator_def*)
apply (*simp add*: *upcarr hom_induced_id_gen*)
done
moreover have *?f ?y = un*
using *upcarr* **apply** (*simp only*: *gh_een.hom_int_pow hi_up_eq_un*)
by (*metis* (*no_types*, *lifting*) *Brouwer_degree2_21 GE.group_l_invI GE.l_inv_ex group.int_pow_1 group.int_pow_pow power2_eq_1_iff uncarr zmult_eq_1_iff*)
ultimately show *?f un = ?y*
by *simp*
qed
have *?hi_ee f un = un* [^]$_{?rhgn\ (equator\ n)}$ *a* ⊗$_{?rhgn\ (equator\ n)}$ *up* [^]$_{?rhgn\ (equator\ n)}$ *b*
proof −
let *?TE = topspace* (*nsphere n*) ∩ *equator n*
have *fneg*: (*f* ∘ *neg*) *x* = (*neg* ∘ *f*) *x* **if** *x* ∈ *topspace* (*nsphere n*) **for** *x*
using *f* [*OF that*] **by** (*force simp*: *neg_def*)
have *neg_im*: *neg* ∈ (*topspace* (*nsphere n*) ∩ *equator n*) → *topspace* (*nsphere n*) ∩ *equator n*
using *cm_neg continuous_map_image_subset_topspace equator_def*
by *fastforce*
have *1*: *hom_induced n* (*nsphere n*) *?TE* (*nsphere n*) *?TE f* ∘ *hom_induced n* (*nsphere n*) *?TE* (*nsphere n*) *?TE neg*
= *hom_induced n* (*nsphere n*) *?TE* (*nsphere n*) *?TE neg* ∘ *hom_induced n* (*nsphere n*) *?TE* (*nsphere n*) *?TE f*
using *neg_im fimeq cm_neg cmf fneg*
apply (*simp flip*: *hom_induced_compose del*: *hom_induced_restrict*)
using *fneg* **by** (*auto intro*: *hom_induced_eq*)
have (*un* [^]$_{?rhgn\ (equator\ n)}$ *a*) ⊗$_{?rhgn\ (equator\ n)}$ (*up* [^]$_{?rhgn\ (equator\ n)}$ *b*)
= *un* [^]$_{?rhgn\ (equator\ n)}$ (*Brouwer_degree2* (*n* − *1*) *neg* ∗ *a* ∗ *Brouwer_degree2* (*n* − *1*) *neg*)
⊗$_{?rhgn\ (equator\ n)}$
up [^]$_{?rhgn\ (equator\ n)}$ (*Brouwer_degree2* (*n* − *1*) *neg* ∗ *b* ∗ *Brouwer_degree2* (*n* − *1*) *neg*)
proof −
have *Brouwer_degree2* (*n* − *Suc 0*) *neg* = *1* ∨ *Brouwer_degree2* (*n* − *Suc 0*) *neg* = − *1*
using *Brouwer_degree2_21 power2_eq_1_iff* **by** *blast*
then show *?thesis*
by *fastforce*
qed
also have . . . = ((*un* [^]$_{?rhgn\ (equator\ n)}$ *Brouwer_degree2* (*n* − *1*) *neg*) [^]$_{?rhgn\ (equator\ n)}$ *a* ⊗$_{?rhgn\ (equator\ n)}$
(*up* [^]$_{?rhgn\ (equator\ n)}$ *Brouwer_degree2* (*n* − *1*) *neg*) [^]$_{?rhgn\ (equator\ n)}$ *b*) [^]$_{?rhgn\ (equator\ n)}$
Brouwer_degree2 (*n* − *1*) *neg*
by (*simp add*: *GE.int_pow_distrib GE.int_pow_pow uncarr upcarr*)
also have . . . = *?hi_ee neg* (*?hi_ee f up*) [^]$_{?rhgn\ (equator\ n)}$ *Brouwer_degree2*

(n − Suc 0) neg
by (simp add: gh_een.hom_int_pow hi_un_eq_up hi_up_eq_un uncarr up_ab upcarr)
finally have 2: (un [^]$_{?rhgn\ (equator\ n)}$ a) ⊗$_{?rhgn\ (equator\ n)}$ (up [^]$_{?rhgn\ (equator\ n)}$ b)
= ?hi_ee neg (?hi_ee f up) [^]$_{?rhgn\ (equator\ n)}$ Brouwer_degree2 (n − Suc 0) neg .
have un = ?hi_ee neg up [^]$_{?rhgn\ (equator\ n)}$ Brouwer_degree2 (n − Suc 0) neg
by (metis (no_types, opaque_lifting) Brouwer_degree2_21 GE.int_pow_1 GE.int_pow_pow hi_up_eq_un power2_eq_1_iff uncarr zmult_eq_1_iff)
moreover have ?hi_ee f ((?hi_ee neg up) [^]$_{?rhgn\ (equator\ n)}$ (Brouwer_degree2 (n − Suc 0) neg))
= un [^]$_{?rhgn\ (equator\ n)}$ a ⊗$_{?rhgn\ (equator\ n)}$ up [^]$_{?rhgn\ (equator\ n)}$ b
using 1 2 **by** (simp add: hom_induced_carrier gh_eef.hom_int_pow fun_eq_iff)
ultimately show ?thesis
by blast
qed
then have ?hi_ee f u = u [^]$_{?rhgn\ (equator\ n)}$ (a − b)
by (simp add: u_def upcarr uncarr up_ab GE.int_pow_diff GE.m_ac GE.int_pow_distrib GE.int_pow_inv GE.inv_mult_group)
ultimately
have Brouwer_degree2 n f = a − b
using iff **by** blast
with Bd_ab **show** ?thesis
by simp
qed simp

0.4.2 General Jordan-Brouwer separation theorem and invariance of dimension

proposition relative_homology_group_Euclidean_complement_step:
assumes closedin (Euclidean_space n) S
shows relative_homology_group p (Euclidean_space n) (topspace(Euclidean_space n) − S)
≅ relative_homology_group (p + k) (Euclidean_space (n+k)) (topspace(Euclidean_space (n+k)) − S)
proof −
have ∗: relative_homology_group p (Euclidean_space n) (topspace(Euclidean_space n) − S)
≅ relative_homology_group (p + 1) (Euclidean_space (Suc n)) (topspace(Euclidean_space (Suc n)) − {x ∈ S. x n = 0})
(**is** ?lhs ≅ ?rhs)
if clo: closedin (Euclidean_space (Suc n)) S **and** cong: ⋀x y. ⟦x ∈ S; ⋀i. i ≠ n ⟹ x i = y i⟧ ⟹ y ∈ S
for p n S
proof −

have *Ssub*: $S \subseteq$ *topspace* (*Euclidean_space* (*Suc n*))
by (*meson clo closedin_def*)
define *lo* **where** *lo* ≡ {*x* ∈ *topspace*(*Euclidean_space* (*Suc n*)). *x n* < (*if x* ∈ *S then 0 else 1*)}
define *hi* **where** *hi* = {*x* ∈ *topspace*(*Euclidean_space* (*Suc n*)). *x n* > (*if x* ∈ *S then 0 else −1*)}
have *lo_hi_Int*: *lo* ∩ *hi* = {*x* ∈ *topspace*(*Euclidean_space* (*Suc n*)) − *S*. *x n* ∈ {*−1*<..<*1*}}
by (*auto simp*: *hi_def lo_def*)
have *lo_hi_Un*: *lo* ∪ *hi* = *topspace*(*Euclidean_space* (*Suc n*)) − {*x* ∈ *S*. *x n* = *0*}
by (*auto simp*: *hi_def lo_def*)
define *ret* **where** *ret* ≡ λ*c*::*real*. λ*x i*. *if i* = *n then c else x i*
have *cm_ret*: *continuous_map* (*powertop_real UNIV*) (*powertop_real UNIV*) (*ret t*) **for** *t*
by (*auto simp*: *ret_def continuous_map_componentwise_UNIV intro*: *continuous_map_product_projection*)
let *?ST* = λ*t*. *subtopology* (*Euclidean_space* (*Suc n*)) {*x*. *x n* = *t*}
define *squashable* **where**
squashable ≡ λ*t S*. ∀ *x t'*. *x* ∈ *S* ∧ (*x n* ≤ *t'* ∧ *t'* ≤ *t* ∨ *t* ≤ *t'* ∧ *t'* ≤ *x n*) ⟶ *ret t' x* ∈ *S*
have *squashable*: *squashable t* (*topspace*(*Euclidean_space*(*Suc n*))) **for** *t*
by (*simp add*: *squashable_def topspace_Euclidean_space ret_def*)
have *squashableD*: ⟦*squashable t S*; *x* ∈ *S*; *x n* ≤ *t'* ∧ *t'* ≤ *t* ∨ *t* ≤ *t'* ∧ *t'* ≤ *x n*⟧ ⟹ *ret t' x* ∈ *S* **for** *x t' t S*
by (*auto simp*: *squashable_def*)
have *squashable 1 hi*
by (*force simp*: *squashable_def hi_def ret_def topspace_Euclidean_space intro*: *cong*)
have *squashable t UNIV* **for** *t*
by (*force simp*: *squashable_def hi_def ret_def topspace_Euclidean_space intro*: *cong*)
have *squashable_0_lohi*: *squashable 0* (*lo* ∩ *hi*)
using *Ssub*
by (*auto simp*: *squashable_def hi_def lo_def ret_def topspace_Euclidean_space intro*: *cong*)
have *rm_ret*: *retraction_maps* (*subtopology* (*Euclidean_space* (*Suc n*)) *U*)
(*subtopology* (*Euclidean_space* (*Suc n*)) {*x*. *x* ∈ *U* ∧ *x n* = *t*})
(*ret t*) *id*
if *squashable t U* **for** *t U*
unfolding *retraction_maps_def*
proof (*intro conjI ballI*)
show *continuous_map* (*subtopology* (*Euclidean_space* (*Suc n*)) *U*)
(*subtopology* (*Euclidean_space* (*Suc n*)) {*x* ∈ *U*. *x n* = *t*}) (*ret t*)
apply (*simp add*: *cm_ret continuous_map_in_subtopology continuous_map_from_subtopology Euclidean_space_def*)
using *that* **by** (*fastforce simp*: *squashable_def ret_def*)
next

show *continuous_map* (*subtopology* (*Euclidean_space* (*Suc n*)) {*x* ∈ *U*. *x n* = *t*})
(*subtopology* (*Euclidean_space* (*Suc n*)) *U*) *id*
using *continuous_map_in_subtopology* **by** *fastforce*
show *ret t* (*id x*) = *x*
if *x* ∈ *topspace* (*subtopology* (*Euclidean_space* (*Suc n*)) {*x* ∈ *U*. *x n* = *t*})
for *x*
using *that* **by** (*simp add*: *topspace_Euclidean_space ret_def fun_eq_iff*)
qed
have *cm_snd*: *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) (*subtopology* (*powertop_real UNIV*) *S*))
euclideanreal (λ*x*. *snd x k*) **for** *k*::*nat* **and** *S*
using *continuous_map_componentwise_UNIV continuous_map_into_fulltopology continuous_map_snd* **by** *fastforce*
have *cm_fstsnd*: *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) (*subtopology* (*powertop_real UNIV*) *S*))
euclideanreal (λ*x*. *fst x* ∗ *snd x k*) **for** *k*::*nat* **and** *S*
by (*intro continuous_intros continuous_map_into_fulltopology* [*OF continuous_map_fst*] *cm_snd*)
have *hw_sub*: *homotopic_with* (λ*k*. *k* ‘ *V* ⊆ *V*) (*subtopology* (*Euclidean_space* (*Suc n*)) *U*)
(*subtopology* (*Euclidean_space* (*Suc n*)) *U*) (*ret t*) *id*
if *squashable t U squashable t V* **for** *U V t*
unfolding *homotopic_with_def*
proof (*intro exI conjI allI ballI*)
define *h* **where** *h* ≡ λ(*z*,*x*). *ret* ((*1* − *z*) ∗ *t* + *z* ∗ *x n*) *x*
show (λ*x*. *h* (*u*, *x*)) ‘ *V* ⊆ *V* **if** *u* ∈ {*0..1*} **for** *u*
using *that* **unfolding** *h_def*
by *clarsimp* (*metis squashableD* [*OF* ‹*squashable t V*›] *convex_bound_le diff_ge_0_iff_ge eq_diff_eq′ le_cases less_eq_real_def segment_bound_lemma*)
have ⋀*x y i*. ⟦∀ *k*≥*Suc n*. *y k* = *0*; *Suc n* ≤ *i*⟧ ⟹ *ret* ((*1* − *x*) ∗ *t* + *x* ∗ *y n*) *y i* = *0*
by (*simp add*: *ret_def*)
then have *h* ∈ {*0..1*} × ({*x*. ∀ *i*≥*Suc n*. *x i* = *0*} ∩ *U*) → {*x*. ∀ *i*≥*Suc n*. *x i* = *0*} ∩ *U*
using *squashableD* [*OF* ‹*squashable t U*›] *segment_bound_lemma*
apply (*clarsimp simp*: *h_def Pi_iff*)
by (*metis convex_bound_le eq_diff_eq ge_iff_diff_ge_0 linorder_le_cases*)
moreover
have *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) (*subtopology* (*powertop_real UNIV*)
({*x*. ∀ *i*≥*Suc n*. *x i* = *0*} ∩ *U*))) (*powertop_real UNIV*) *h*
apply (*auto simp*: *h_def case_prod_unfold ret_def continuous_map_componentwise_UNIV*)
apply (*intro continuous_map_into_fulltopology* [*OF continuous_map_fst*] *cm_snd continuous_intros*)
by (*auto simp*: *cm_snd*)
ultimately show *continuous_map* (*prod_topology* (*top_of_set* {*0..1*}) (*subtopology* (*Euclidean_space* (*Suc n*)) *U*))
(*subtopology* (*Euclidean_space* (*Suc n*)) *U*) *h*

```
    by (simp add: continuous_map_in_subtopology Euclidean_space_def subtopology_subtopology)
  qed (auto simp: ret_def)
  have cs_hi: contractible_space(subtopology (Euclidean_space(Suc n)) hi)
  proof −
    have homotopic_with (λx. True) (?ST 1) (?ST 1) id (λx. (λi. if i = n then 1 else 0))
      apply (subst homotopic_with_sym)
      apply (simp add: homotopic_with)
      apply (rule_tac x=(λ(z,x) i. if i=n then 1 else z ∗ x i) in exI)
      apply (auto simp: Euclidean_space_def subtopology_subtopology continuous_map_in_subtopology case_prod_unfold continuous_map_componentwise_UNIV cm_fstsnd)
      done
    then have contractible_space (?ST 1)
      unfolding contractible_space_def by metis
    moreover have ?thesis = contractible_space (?ST 1)
    proof (intro deformation_retract_imp_homotopy_equivalent_space homotopy_equivalent_space_contractibility)
      have {x. ∀ i≥Suc n. x i = 0} ∩ {x ∈ hi. x n = 1} = {x. ∀ i≥Suc n. x i = 0} ∩ {x. x n = 1}
        by (auto simp: hi_def topspace_Euclidean_space)
      then have eq: subtopology (Euclidean_space (Suc n)) {x. x ∈ hi ∧ x n = 1} = ?ST 1
        by (simp add: Euclidean_space_def subtopology_subtopology)
      show homotopic_with (λx. True) (subtopology (Euclidean_space (Suc n)) hi) (subtopology (Euclidean_space (Suc n)) hi) (ret 1) id
        using hw_sub [OF ‹squashable 1 hi› ‹squashable 1 UNIV›] eq by simp
      show retraction_maps (subtopology (Euclidean_space (Suc n)) hi) (?ST 1) (ret 1) id
        using rm_ret [OF ‹squashable 1 hi›] eq by simp
    qed
    ultimately show ?thesis by metis
  qed
  have ?lhs ≅ relative_homology_group p (Euclidean_space (Suc n)) (lo ∩ hi)
 proof (rule group.iso_sym [OF _ deformation_retract_imp_isomorphic_relative_homology_groups])
    have {x. ∀ i≥Suc n. x i = 0} ∩ {x. x n = 0} = {x. ∀ i≥n. x i = (0::real)}
      by auto (metis le_less_Suc_eq not_le)
    then have ?ST 0 = Euclidean_space n
      by (simp add: Euclidean_space_def subtopology_subtopology)
    then show retraction_maps (Euclidean_space (Suc n)) (Euclidean_space n) (ret 0) id
      using rm_ret [OF ‹squashable 0 UNIV›] by auto
    then have ret 0 x ∈ topspace (Euclidean_space n)
      if x ∈ topspace (Euclidean_space (Suc n)) −1 < x n x n < 1 for x
    using that by (metis continuous_map_image_subset_topspace image_subset_iff retraction_maps_def)
    then show (ret 0) ∈ (lo ∩ hi) → topspace (Euclidean_space n) − S
      by (auto simp: local.cong ret_def hi_def lo_def)
```

```
      show homotopic_with (λh. h ‘ (lo ∩ hi) ⊆ lo ∩ hi) (Euclidean_space (Suc
n)) (Euclidean_space (Suc n)) (ret 0) id
        using hw_sub [OF squashable squashable_0_lohi] by simp
    qed (auto simp: lo_def hi_def Euclidean_space_def)
     also have ... ≅ relative_homology_group p (subtopology (Euclidean_space
(Suc n)) hi) (lo ∩ hi)
   proof (rule group.iso_sym [OF _ isomorphic_relative_homology_groups_inclusion_contractible])
      show contractible_space (subtopology (Euclidean_space (Suc n)) hi)
        by (simp add: cs_hi)
      show topspace (Euclidean_space (Suc n)) ∩ hi ≠ {}
        apply (simp add: hi_def topspace_Euclidean_space set_eq_iff)
        apply (rule_tac x=λi. if i = n then 1 else 0 in exI, auto)
        done
    qed auto
     also have ... ≅ relative_homology_group p (subtopology (Euclidean_space
(Suc n)) (lo ∪ hi)) lo
    proof −
      have oo: openin (Euclidean_space (Suc n)) {x ∈ topspace (Euclidean_space
(Suc n)). x n ∈ A}
        if open A for A
      proof (rule openin_continuous_map_preimage)
        show continuous_map (Euclidean_space (Suc n)) euclideanreal (λx. x n)
        proof −
          have ∀ n f. continuous_map (product_topology f UNIV) (f (n::nat)) (λf.
f n::real)
            by (simp add: continuous_map_product_projection)
          then show ?thesis
            using Euclidean_space_def continuous_map_from_subtopology
            by (metis (mono_tags))
        qed
      qed (auto intro: that)
      have openin (Euclidean_space(Suc n)) lo
        apply (simp add: openin_subopen [of _ lo])
        apply (simp add: lo_def, safe)
         apply (force intro: oo [of lessThan 0, simplified] open_Collect_less)
        apply (rule_tac x={x ∈ topspace(Euclidean_space(Suc n)). x n < 1}
                        ∩ (topspace(Euclidean_space(Suc n)) − S) in exI)
      using clo apply (force intro: oo [of lessThan 1, simplified] open_Collect_less)
        done
      moreover have openin (Euclidean_space(Suc n)) hi
        apply (simp add: openin_subopen [of _ hi])
        apply (simp add: hi_def, safe)
         apply (force intro: oo [of greaterThan 0, simplified] open_Collect_less)
        apply (rule_tac x={x ∈ topspace(Euclidean_space(Suc n)). x n > −1}
                            ∩ (topspace(Euclidean_space(Suc n)) − S) in exI)
      using clo apply (force intro: oo [of greaterThan (−1), simplified] open_Collect_less)
        done
      ultimately
      have ∗: subtopology (Euclidean_space (Suc n)) (lo ∪ hi) closure_of
```

(*topspace* (*subtopology* (*Euclidean_space* (*Suc n*)) (*lo* ∪ *hi*)) − *hi*)
⊆ *subtopology* (*Euclidean_space* (*Suc n*)) (*lo* ∪ *hi*) *interior_of lo*
by (*metis* (*no_types*, *lifting*) *Diff_idemp Diff_subset_conv Un_commute Un_upper2 closure_of_interior_of interior_of_closure_of interior_of_complement interior_of_eq lo_hi_Un openin_Un openin_open_subtopology topspace_subtopology_subset*)
have *eq*: ((*lo* ∪ *hi*) ∩ (*lo* ∪ *hi* − (*topspace* (*Euclidean_space* (*Suc n*)) ∩ (*lo* ∪ *hi*) − *hi*))) = *hi*
(*lo* − (*topspace* (*Euclidean_space* (*Suc n*)) ∩ (*lo* ∪ *hi*) − *hi*)) = *lo* ∩ *hi*
by (*auto simp*: *lo_def hi_def Euclidean_space_def*)
show *?thesis*
using *homology_excision_axiom* [*OF* ∗, *of lo* ∪ *hi p*]
by (*force simp*: *subtopology_subtopology eq is_iso_def*)
qed
also have . . . ≅ *relative_homology_group* (*p* + *1* − *1*) (*subtopology* (*Euclidean_space* (*Suc n*)) (*lo* ∪ *hi*)) *lo*
by *simp*
also have . . . ≅ *relative_homology_group* (*p* + *1*) (*Euclidean_space* (*Suc n*)) (*lo* ∪ *hi*)
proof (*rule group.iso_sym* [*OF _ isomorphic_relative_homology_groups_relboundary_contractible*])
have *proj*: *continuous_map* (*powertop_real UNIV*) *euclideanreal* (λ*f*. *f n*)
by (*metis UNIV_I continuous_map_product_projection*)
have *hilo*: ⋀*x*. *x* ∈ *hi* ⟹ (λ*i*. *if i* = *n then* − *x i else x i*) ∈ *lo*
⋀*x*. *x* ∈ *lo* ⟹ (λ*i*. *if i* = *n then* − *x i else x i*) ∈ *hi*
using *local.cong*
by (*auto simp*: *hi_def lo_def topspace_Euclidean_space split*: *if_split_asm*)
have *subtopology* (*Euclidean_space* (*Suc n*)) *hi homeomorphic_space subtopology* (*Euclidean_space* (*Suc n*)) *lo*
unfolding *homeomorphic_space_def*
apply (*rule_tac x*=λ*x i*. *if i* = *n then* −(*x i*) *else x i* **in** *exI*)+
using *proj*
apply (*auto simp*: *homeomorphic_maps_def Euclidean_space_def continuous_map_in_subtopology*
hilo continuous_map_componentwise_UNIV continuous_map_from_subtopology continuous_map_minus
intro: *continuous_map_from_subtopology continuous_map_product_projection*)
done
then have *contractible_space*(*subtopology* (*Euclidean_space*(*Suc n*)) *hi*)
⟷ *contractible_space* (*subtopology* (*Euclidean_space* (*Suc n*)) *lo*)
by (*rule homeomorphic_space_contractibility*)
then show *contractible_space* (*subtopology* (*Euclidean_space* (*Suc n*)) *lo*)
using *cs_hi* **by** *auto*
show *topspace* (*Euclidean_space* (*Suc n*)) ∩ *lo* ≠ {}
apply (*simp add*: *lo_def Euclidean_space_def set_eq_iff*)
apply (*rule_tac x*=λ*i*. *if i* = *n then* −*1 else 0* **in** *exI*, *auto*)
done
qed *auto*
also have . . . ≅ *?rhs*
by (*simp flip*: *lo_hi_Un*)
finally show *?thesis* **.**

qed
show *?thesis*
proof (*induction k*)
case (*Suc m*)
with *assms* **obtain** *T* **where** *cloT*: *closedin* (*powertop_real UNIV*) *T*
and *SeqT*: $S = T \cap \{x. \forall i \geq n. x\ i = 0\}$
by (*auto simp*: *Euclidean_space_def closedin_subtopology*)
then have *closedin* (*Euclidean_space* (*m* + *n*)) *S*
apply (*simp add*: *Euclidean_space_def closedin_subtopology*)
apply (*rule_tac x=T* ∩ *topspace*(*Euclidean_space n*) **in** *exI*)
using *closedin_Euclidean_space topspace_Euclidean_space* **by** *force*
moreover have *relative_homology_group p* (*Euclidean_space n*) (*topspace* (*Euclidean_space n*) − *S*)
≅ *relative_homology_group* (*p* + *1*) (*Euclidean_space* (*Suc n*)) (*topspace* (*Euclidean_space* (*Suc n*)) − *S*)
if *closedin* (*Euclidean_space n*) *S* **for** *p n*
proof −
define *S′* **where** *S′* ≡ {*x* ∈ *topspace*(*Euclidean_space*(*Suc n*)). (λ*i*. *if i* < *n then x i else 0*) ∈ *S*}
have *Ssub_n*: *S* ⊆ *topspace* (*Euclidean_space n*)
by (*meson that closedin_def*)
have *relative_homology_group p* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *S′*)
≅ *relative_homology_group* (*p* + *1*) (*Euclidean_space* (*Suc n*)) (*topspace*(*Euclidean_space* (*Suc n*)) − {*x* ∈ *S′*. *x n* = *0*})
proof (*rule* ∗)
have *cm*: *continuous_map* (*powertop_real UNIV*) *euclideanreal* (λ*f*. *f u*) **for** *u*
by (*metis UNIV_I continuous_map_product_projection*)
have *continuous_map* (*subtopology* (*powertop_real UNIV*) {*x*. ∀ *i*>*n*. *x i* = *0*}) *euclideanreal*
(λ*x*. *if k* ≤ *n then x k else 0*) **for** *k*
by (*simp add*: *continuous_map_from_subtopology* [*OF cm*])
moreover have ∀ *i*≥*n*. (*if i* < *n then x i else 0*) = *0*
if *x* ∈ *topspace* (*subtopology* (*powertop_real UNIV*) {*x*. ∀ *i*>*n*. *x i* = *0*}) **for** *x*
using *that* **by** *simp*
ultimately have *continuous_map* (*Euclidean_space* (*Suc n*)) (*Euclidean_space n*) (λ*x i*. *if i* < *n then x i else 0*)
by (*simp add*: *Euclidean_space_def continuous_map_in_subtopology continuous_map_componentwise_UNIV*
continuous_map_from_subtopology [*OF cm*] *image_subset_iff*)
then show *closedin* (*Euclidean_space* (*Suc n*)) *S′*
unfolding *S′_def* **using** *that* **by** (*rule closedin_continuous_map_preimage*)
next
fix *x y*
assume *xy*: ⋀*i*. *i* ≠ *n* ⟹ *x i* = *y i x* ∈ *S′*
then have (λ*i*. *if i* < *n then x i else 0*) = (λ*i*. *if i* < *n then y i else 0*)
by (*simp add*: *S′_def Euclidean_space_def fun_eq_iff*)

with *xy* **show** $y \in S'$
by (*simp add*: *S'_def Euclidean_space_def*)
qed
moreover
have *abs_eq*: ($\lambda i.$ *if* $i < n$ *then* $x\ i$ *else 0*) $= x$ **if** $\bigwedge i.\ i \geq n \Longrightarrow x\ i = 0$ **for** $x :: nat \Rightarrow real$ **and** n
using *that* **by** *auto*
then have *topspace* (*Euclidean_space n*) $-$ S' = *topspace* (*Euclidean_space n*) $-$ S
by (*simp add*: *S'_def Euclidean_space_def set_eq_iff cong*: *conj_cong*)
moreover
have *topspace* (*Euclidean_space* (*Suc n*)) $-$ $\{x \in S'.\ x\ n = 0\}$ = *topspace* (*Euclidean_space* (*Suc n*)) $-$ S
using *Ssub_n*
apply (*auto simp*: *S'_def subset_iff Euclidean_space_def set_eq_iff abs_eq cong*: *conj_cong*)
by (*metis abs_eq le_antisym not_less_eq_eq*)
ultimately show *?thesis*
by *simp*
qed
ultimately have *relative_homology_group* (*p* + *m*)(*Euclidean_space* (*m* + *n*))(*topspace* (*Euclidean_space* (*m* + *n*)) $-$ S)
$\cong$ *relative_homology_group* (*p* + *m* + *1*) (*Euclidean_space* (*Suc* (*m* + *n*))) (*topspace* (*Euclidean_space* (*Suc* (*m* + *n*))) $-$ S)
by (*metis* ‹*closedin* (*Euclidean_space* (*m* + *n*)) *S*›)
then show *?case*
using *Suc.IH iso_trans* **by** (*force simp*: *algebra_simps*)
qed (*simp add*: *iso_refl*)
qed

lemma *iso_Euclidean_complements_lemma1*:
assumes *S*: *closedin* (*Euclidean_space m*) *S* **and** *cmf*: *continuous_map*(*subtopology* (*Euclidean_space m*) *S*) (*Euclidean_space n*) *f*
obtains *g* **where** *continuous_map* (*Euclidean_space m*) (*Euclidean_space n*) *g*
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
proof $-$
have *cont*: *continuous_on* (*topspace* (*Euclidean_space m*) $\cap$ *S*) ($\lambda x.\ f\ x\ i$) **for** *i*
by (*metis* (*no_types*) *continuous_on_product_then_coordinatewise cm_Euclidean_space_iff_continuous_on cmf topspace_subtopology*)
have *f* ‘ (*topspace* (*Euclidean_space m*) $\cap$ *S*) $\subseteq$ *topspace* (*Euclidean_space n*)
using *cmf continuous_map_image_subset_topspace* **by** *fastforce*
then
have $\exists g.$ *continuous_on* (*topspace* (*Euclidean_space m*)) *g* $\wedge$ ($\forall x \in S.\ g\ x = f\ x\ i$) **for** *i*
using *S Tietze_unbounded* [*OF cont* [*of i*]]
by (*metis closedin_Euclidean_space_iff closedin_closed_Int topspace_subtopology topspace_subtopology_subset*)
then obtain *g* **where** *cmg*: $\bigwedge i.$ *continuous_map* (*Euclidean_space m*) *euclideanreal* (*g i*)

and *gf*: $\bigwedge i\ x.\ x \in S \Longrightarrow g\ i\ x = f\ x\ i$
unfolding *continuous_map_Euclidean_space_iff* **by** *metis*
let *?GG* = $\lambda x\ i.$ *if* $i < n$ *then* $g\ i\ x$ *else* *0*
show *thesis*
proof
show *continuous_map* (*Euclidean_space m*) (*Euclidean_space n*) *?GG*
unfolding *Euclidean_space_def* [*of n*]
by (*auto simp*: *continuous_map_in_subtopology continuous_map_componentwise cmg*)
show *?GG x* = *f x* **if** $x \in S$ **for** *x*
proof −
have $S \subseteq$ *topspace* (*Euclidean_space m*)
by (*meson S closedin_def*)
then have $f\ x \in$ *topspace* (*Euclidean_space n*)
using *cmf that* **unfolding** *continuous_map_def topspace_subtopology* **by** *blast*
then show *?thesis*
by (*force simp*: *topspace_Euclidean_space gf that*)
qed
qed
qed

lemma *iso_Euclidean_complements_lemma2*:
assumes *S*: *closedin* (*Euclidean_space m*) *S*
and *T*: *closedin* (*Euclidean_space n*) *T*
and *hom*: *homeomorphic_map* (*subtopology* (*Euclidean_space m*) *S*) (*subtopology* (*Euclidean_space n*) *T*) *f*
obtains *g* **where** *homeomorphic_map* (*prod_topology* (*Euclidean_space m*) (*Euclidean_space n*))
(*prod_topology* (*Euclidean_space n*) (*Euclidean_space m*)) *g*
$\bigwedge x.\ x \in S \Longrightarrow g(x,(\lambda i.\ 0)) = (f\ x,(\lambda i.\ 0))$
proof −
obtain *g* **where** *cmf*: *continuous_map* (*subtopology* (*Euclidean_space m*) *S*) (*subtopology* (*Euclidean_space n*) *T*) *f*
and *cmg*: *continuous_map* (*subtopology* (*Euclidean_space n*) *T*) (*subtopology* (*Euclidean_space m*) *S*) *g*
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g\ (f\ x) = x$
and *fg*: $\bigwedge y.\ y \in T \Longrightarrow f\ (g\ y) = y$
using *hom S T closedin_subset* **unfolding** *homeomorphic_map_maps homeomorphic_maps_def*
by *fastforce*
obtain *f'* **where** *cmf'*: *continuous_map* (*Euclidean_space m*) (*Euclidean_space n*) *f'*
and *f'f*: $\bigwedge x.\ x \in S \Longrightarrow f'\ x = f\ x$
using *iso_Euclidean_complements_lemma1 S cmf continuous_map_into_fulltopology* **by** *metis*
obtain *g'* **where** *cmg'*: *continuous_map* (*Euclidean_space n*) (*Euclidean_space*

m) *g′*
and *g′g*: ⋀*x*. *x* ∈ *T* ⟹ *g′ x* = *g x*
using *iso_Euclidean_complements_lemma1 T cmg continuous_map_into_fulltopology*
by *metis*
define *p* where *p* ≡ λ(*x*,*y*). (*x*,(λ*i*. *y i* + *f′ x i*))
define *p′* where *p′* ≡ λ(*x*,*y*). (*x*,(λ*i*. *y i* − *f′ x i*))
define *q* where *q* ≡ λ(*x*,*y*). (*x*,(λ*i*. *y i* + *g′ x i*))
define *q′* where *q′* ≡ λ(*x*,*y*). (*x*,(λ*i*. *y i* − *g′ x i*))
have *homeomorphic_maps* (*prod_topology* (*Euclidean_space m*) (*Euclidean_space n*))
(*prod_topology* (*Euclidean_space m*) (*Euclidean_space n*))
p p′
homeomorphic_maps (*prod_topology* (*Euclidean_space n*) (*Euclidean_space m*))
(*prod_topology* (*Euclidean_space n*) (*Euclidean_space m*))
q q′
homeomorphic_maps (*prod_topology* (*Euclidean_space m*) (*Euclidean_space n*))
(*prod_topology* (*Euclidean_space n*) (*Euclidean_space m*))
(λ(*x*,*y*). (*y*,*x*)) (λ(*x*,*y*). (*y*,*x*))
apply (*simp_all add*: *p_def p′_def q_def q′_def homeomorphic_maps_def continuous_map_pairwise*)
apply (*force simp*: *case_prod_unfold continuous_map_of_fst* [*unfolded o_def*] *cmf′ cmg′ intro*: *continuous_intros*)+
done
then have *homeomorphic_maps* (*prod_topology* (*Euclidean_space m*) (*Euclidean_space n*))
(*prod_topology* (*Euclidean_space n*) (*Euclidean_space m*))
(*q′* ∘ (λ(*x*,*y*). (*y*,*x*)) ∘ *p*) (*p′* ∘ ((λ(*x*,*y*). (*y*,*x*)) ∘ *q*))
using *homeomorphic_maps_compose homeomorphic_maps_sym* by (*metis* (*no_types*, *lifting*))
moreover
have ⋀*x*. *x* ∈ *S* ⟹ (*q′* ∘ (λ(*x*,*y*). (*y*,*x*)) ∘ *p*) (*x*, λ*i*. *0*) = (*f x*, λ*i*. *0*)
apply (*simp add*: *q′_def p_def f′f*)
apply (*simp add*: *fun_eq_iff*)
by (*metis S T closedin_subset g′g gf hom homeomorphic_imp_surjective_map image_eqI topspace_subtopology_subset*)
ultimately
show *thesis*
using *homeomorphic_map_maps that* by *blast*
qed

proposition *isomorphic_relative_homology_groups_Euclidean_complements*:
assumes *S*: *closedin* (*Euclidean_space n*) *S* **and** *T*: *closedin* (*Euclidean_space n*) *T*
and *hom*: (*subtopology* (*Euclidean_space n*) *S*) *homeomorphic_space* (*subtopology* (*Euclidean_space n*) *T*)
shows *relative_homology_group p* (*Euclidean_space n*) (*topspace*(*Euclidean_space*

n) − S)
  ≅ *relative_homology_group p* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *T*)
proof −
 have *subST*: *S* ⊆ *topspace*(*Euclidean_space n*) *T* ⊆ *topspace*(*Euclidean_space n*)
  by (*meson S T closedin_def*)+
 have *relative_homology_group p* (*Euclidean_space n*) (*topspace* (*Euclidean_space n*) − *S*)
  ≅ *relative_homology_group* (*p* + *int n*) (*Euclidean_space* (*n* + *n*)) (*topspace* (*Euclidean_space* (*n* + *n*)) − *S*)
  using *relative_homology_group_Euclidean_complement_step* [*OF S*] **by** *blast*
 moreover have *relative_homology_group p* (*Euclidean_space n*) (*topspace* (*Euclidean_space n*) − *T*)
  ≅ *relative_homology_group* (*p* + *int n*) (*Euclidean_space* (*n* + *n*)) (*topspace* (*Euclidean_space* (*n* + *n*)) − *T*)
  using *relative_homology_group_Euclidean_complement_step* [*OF T*] **by** *blast*
 moreover have *relative_homology_group* (*p* + *int n*) (*Euclidean_space* (*n* + *n*)) (*topspace* (*Euclidean_space* (*n* + *n*)) − *S*)
   ≅ *relative_homology_group* (*p* + *int n*) (*Euclidean_space* (*n* + *n*)) (*topspace* (*Euclidean_space* (*n* + *n*)) − *T*)
 proof −
  obtain *f* **where** *f*: *homeomorphic_map* (*subtopology* (*Euclidean_space n*) *S*)
     (*subtopology* (*Euclidean_space n*) *T*) *f*
   using *hom* **unfolding** *homeomorphic_space* **by** *blast*
  obtain *g* **where** *g*: *homeomorphic_map* (*prod_topology* (*Euclidean_space n*) (*Euclidean_space n*))
     (*prod_topology* (*Euclidean_space n*) (*Euclidean_space n*)) *g*
   and *gf*: ⋀*x*. *x* ∈ *S* ⟹ *g*(*x*,(λ*i*. *0*)) = (*f x*,(λ*i*. *0*))
   using *S T f iso_Euclidean_complements_lemma2* **by** *blast*
  define *h* **where** *h* ≡ λ*x*::*nat* ⇒*real*. ((λ*i*. *if i* < *n then x i else 0*), (λ*j*. *if j* < *n then x*(*n* + *j*) *else 0*))
  define *k* **where** *k* ≡ λ(*x*,*y*) *i*. *if i* < *2* ∗ *n then if i* < *n then x i else y*(*i* − *n*) *else* (*0*::*real*)
 have *hk*: *homeomorphic_maps* (*Euclidean_space*(*2* ∗ *n*)) (*prod_topology* (*Euclidean_space n*) (*Euclidean_space n*)) *h k*
   unfolding *homeomorphic_maps_def*
  proof *safe*
   show *continuous_map* (*Euclidean_space* (*2* ∗ *n*))
     (*prod_topology* (*Euclidean_space n*) (*Euclidean_space n*)) *h*
   apply (*simp add*: *h_def continuous_map_pairwise o_def continuous_map_componentwise_Euclidean_space*)
    unfolding *Euclidean_space_def*
    by (*metis* (*mono_tags*) *UNIV_I continuous_map_from_subtopology continuous_map_product_projection*)
   have *continuous_map* (*prod_topology* (*Euclidean_space n*) (*Euclidean_space n*)) *euclideanreal* (λ*p*. *fst p i*) **for** *i*
    using *Euclidean_space_def continuous_map_into_fulltopology continuous_map_fst* **by** *fastforce*

moreover
have *continuous_map* (*prod_topology* (*Euclidean_space* *n*) (*Euclidean_space* *n*)) *euclideanreal* (λ*p*. *snd* *p* (*i* − *n*)) **for** *i*
using *Euclidean_space_def continuous_map_into_fulltopology continuous_map_snd* **by** *fastforce*
ultimately
show *continuous_map* (*prod_topology* (*Euclidean_space* *n*) (*Euclidean_space* *n*))
(*Euclidean_space* (*2* ∗ *n*)) *k*
by (*simp add*: *k_def continuous_map_pairwise o_def continuous_map_componentwise_Euclidean_space case_prod_unfold*)
qed (*auto simp*: *k_def h_def fun_eq_iff topspace_Euclidean_space*)
define *kgh* **where** *kgh* ≡ *k* ∘ *g* ∘ *h*
let *?i* = *hom_induced* (*p* + *n*) (*Euclidean_space*(*2* ∗ *n*)) (*topspace*(*Euclidean_space*(*2* ∗ *n*)) − *S*)
(*Euclidean_space*(*2* ∗ *n*)) (*topspace*(*Euclidean_space*(*2* ∗ *n*)) − *T*) *kgh*
have *?i* ∈ *iso* (*relative_homology_group* (*p* + *int* *n*) (*Euclidean_space* (*2* ∗ *n*))
(*topspace* (*Euclidean_space* (*2* ∗ *n*)) − *S*))
(*relative_homology_group* (*p* + *int* *n*) (*Euclidean_space* (*2* ∗ *n*))
(*topspace* (*Euclidean_space* (*2* ∗ *n*)) − *T*))
proof (*rule homeomorphic_map_relative_homology_iso*)
show *hm*: *homeomorphic_map* (*Euclidean_space* (*2* ∗ *n*)) (*Euclidean_space* (*2* ∗ *n*)) *kgh*
unfolding *kgh_def* **by** (*meson hk g homeomorphic_map_maps homeomorphic_maps_compose homeomorphic_maps_sym*)
have *Teq*: *T* = *f* ‘ *S*
using *f homeomorphic_imp_surjective_map subST*(*1*) *subST*(*2*) *topspace_subtopology_subset* **by** *blast*
have *khf*: ⋀*x*. *x* ∈ *S* ⟹ *k*(*h*(*f* *x*)) = *f* *x*
by (*metis* (*no_types*, *lifting*) *Teq hk homeomorphic_maps_def image_subset_iff le_add1 mult_2 subST*(*2*) *subsetD subset_Euclidean_space*)
have *gh*: *g*(*h* *x*) = *h*(*f* *x*) **if** *x* ∈ *S* **for** *x*
proof −
have [*simp*]: (λ*i*. *if i* < *n then x i else 0*) = *x*
using *subST*(*1*) *that topspace_Euclidean_space* **by** (*auto simp*: *fun_eq_iff*)
have *f* *x* ∈ *topspace*(*Euclidean_space* *n*)
using *Teq subST*(*2*) *that* **by** *blast*
moreover have (λ*j*. *if j* < *n then x* (*n* + *j*) *else 0*) = (λ*j*. *0*::*real*)
using *Euclidean_space_def subST*(*1*) *that* **by** *force*
ultimately show *?thesis*
by (*simp add*: *topspace_Euclidean_space h_def gf* ‹*x* ∈ *S*› *fun_eq_iff*)
qed
have ∗: ⟦*S* ⊆ *U*; *T* ⊆ *U*; *kgh* ‘ *U* = *U*; *inj_on kgh U*; *kgh* ‘ *S* = *T*⟧ ⟹ *kgh* ‘ (*U* − *S*) = *U* − *T* **for** *U*
unfolding *inj_on_def set_eq_iff* **by** *blast*
show *kgh* ‘ (*topspace* (*Euclidean_space* (*2* ∗ *n*)) − *S*) = *topspace* (*Euclidean_space* (*2* ∗ *n*)) − *T*
proof (*rule* ∗)

```
      show kgh ` topspace (Euclidean_space (2 * n)) = topspace (Euclidean_space
(2 * n))
          by (simp add: hm homeomorphic_imp_surjective_map)
        show inj_on kgh (topspace (Euclidean_space (2 * n)))
          using hm homeomorphic_map_def by auto
        show kgh ` S = T
          by (simp add: Teq kgh_def gh khf)
      qed (use subST topspace_Euclidean_space in ‹fastforce+›)
    qed auto
    then show ?thesis
      by (simp add: is_isoI mult_2)
  qed
  ultimately show ?thesis
    by (meson group.iso_sym iso_trans group_relative_homology_group)
qed

lemma lemma_iod:
  assumes S ⊆ T S ≠ {} and Tsub: T ⊆ topspace(Euclidean_space n)
      and S: ⋀a b u. ⟦a ∈ S; b ∈ T; 0 < u; u < 1⟧ ⟹ (λi. (1 − u) * a i + u *
b i) ∈ S
    shows path_connectedin (Euclidean_space n) T
proof −
  obtain a where a ∈ S
    using assms by blast
  have path_component_of (subtopology (Euclidean_space n) T) a b if b ∈ T for
b
    unfolding path_component_of_def
  proof (intro exI conjI)
    have [simp]: ∀i≥n. a i = 0
      using Tsub ‹a ∈ S› assms(1) topspace_Euclidean_space by auto
    have [simp]: ∀i≥n. b i = 0
      using Tsub that topspace_Euclidean_space by auto
    have inT: (λi. (1 − x) * a i + x * b i) ∈ T if 0 ≤ x x ≤ 1 for x
    proof (cases x = 0 ∨ x = 1)
      case True
      with ‹a ∈ S› ‹b ∈ T› ‹S ⊆ T› show ?thesis
        by force
    next
      case False
      then show ?thesis
        using subsetD [OF ‹S ⊆ T› S] ‹a ∈ S› ‹b ∈ T› that by auto
    qed
    have continuous_on {0..1} (λx. (1 − x) * a k + x * b k) for k
      by (intro continuous_intros)
    then show pathin (subtopology (Euclidean_space n) T) (λt i. (1 − t) * a i +
t * b i)
     apply (simp add: Euclidean_space_def subtopology_subtopology pathin_subtopology)
      apply (simp add: pathin_def continuous_map_componentwise_UNIV inT)
      done
```

qed *auto*
then have *path_connected_space* (*subtopology* (*Euclidean_space n*) *T*)
by (*metis Tsub path_component_of_equiv path_connected_space_iff_path_component topspace_subtopology_subset*)
then show *?thesis*
by (*simp add*: *Tsub path_connectedin_def*)
qed

lemma *invariance_of_dimension_closedin_Euclidean_space*:
assumes *closedin* (*Euclidean_space n*) *S*
shows *subtopology* (*Euclidean_space n*) *S homeomorphic_space Euclidean_space n*
$\longleftrightarrow$ *S* = *topspace*(*Euclidean_space n*)
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
have *Ssub*: *S* ⊆ *topspace* (*Euclidean_space n*)
by (*meson assms closedin_def*)
moreover have *False* **if** *a* ∉ *S* **and** *a* ∈ *topspace* (*Euclidean_space n*) **for** *a*
proof −
have *cl_n*: *closedin* (*Euclidean_space* (*Suc n*)) (*topspace*(*Euclidean_space n*))
using *Euclidean_space_def closedin_Euclidean_space closedin_subtopology* **by** *fastforce*
then have *sub*: *subtopology* (*Euclidean_space*(*Suc n*)) (*topspace*(*Euclidean_space n*)) = *Euclidean_space n*
by (*metis* (*no_types*, *lifting*) *Euclidean_space_def closedin_subset subtopology_subtopology topspace_Euclidean_space topspace_subtopology topspace_subtopology_subset*)
then have *cl_S*: *closedin* (*Euclidean_space*(*Suc n*)) *S*
using *cl_n assms closedin_closed_subtopology* **by** *fastforce*
have *sub_SucS*: *subtopology* (*Euclidean_space* (*Suc n*)) *S* = *subtopology* (*Euclidean_space n*) *S*
by (*metis Ssub sub subtopology_subtopology topspace_subtopology topspace_subtopology_subset*)
have *non0*: {*y*. ∃ *x*::*nat*⇒*real*. (∀ *i*≥*Suc n*. *x i* = *0*) ∧ (∃ *i*≥*n*. *x i* ≠ *0*) ∧ *y* = *x n*} = −{*0*}
proof *safe*
show *False* **if** ∀ *i*≥*Suc n*. *f i* = *0 0* = *f n n* ≤ *i f i* ≠ *0* **for** *f*::*nat*⇒*real* **and** *i*
by (*metis that le_antisym not_less_eq_eq*)
show ∃ *f*::*nat*⇒*real*. (∀ *i*≥*Suc n*. *f i* = *0*) ∧ (∃ *i*≥*n*. *f i* ≠ *0*) ∧ *a* = *f n* **if** *a* ≠ *0* **for** *a*
by (*rule_tac x*=(λ*i*. *0*)(*n*:= *a*) **in** *exI*) (*force simp*: *that*)
qed
have *homology_group 0* (*subtopology* (*Euclidean_space* (*Suc n*)) (*topspace* (*Euclidean_space* (*Suc n*)) − *S*))
≅ *homology_group 0* (*subtopology* (*Euclidean_space* (*Suc n*)) (*topspace* (*Euclidean_space* (*Suc n*)) − *topspace* (*Euclidean_space n*)))
proof (*rule isomorphic_relative_contractible_space_imp_homology_groups*)
show (*topspace* (*Euclidean_space* (*Suc n*)) − *S* = {}) =
(*topspace* (*Euclidean_space* (*Suc n*)) − *topspace* (*Euclidean_space n*) =

```
{})
        using cl_n closedin_subset that by auto
    next
      fix p
      show relative_homology_group p (Euclidean_space (Suc n))
         (topspace (Euclidean_space (Suc n)) − S) ≅
        relative_homology_group p (Euclidean_space (Suc n))
         (topspace (Euclidean_space (Suc n)) − topspace (Euclidean_space n))
      by (simp add: L sub_SucS cl_S cl_n isomorphic_relative_homology_groups_Euclidean_complements
sub)
    qed (auto simp: L)
    moreover
    have continuous_map (powertop_real UNIV) euclideanreal (λx. x n)
      by (metis (no_types) UNIV_I continuous_map_product_projection)
   then have cm: continuous_map (subtopology (Euclidean_space (Suc n)) (topspace
(Euclidean_space (Suc n)) − topspace (Euclidean_space n)))
                                  euclideanreal (λx. x n)
      by (simp add: Euclidean_space_def continuous_map_from_subtopology)
    have False if path_connected_space
                      (subtopology (Euclidean_space (Suc n))
                    (topspace (Euclidean_space (Suc n)) − topspace (Euclidean_space
n)))
        using path_connectedin_continuous_map_image [OF cm that [unfolded
path_connectedin_topspace [symmetric]]]
            bounded_path_connected_Compl_real [of {0}]
        by (simp add: topspace_Euclidean_space image_def Bex_def non0 flip:
path_connectedin_topspace)
    moreover
    have eq: T = T ∩ {x. x n ≤ 0} ∪ T ∩ {x. x n ≥ 0} for T :: (nat ⇒ real) set
      by auto
    have path_connectedin (Euclidean_space (Suc n)) (topspace (Euclidean_space
(Suc n)) − S)
    proof (subst eq, rule path_connectedin_Un)
    have topspace(Euclidean_space(Suc n)) ∩ {x. x n = 0} = topspace(Euclidean_space
n)
        apply (auto simp: topspace_Euclidean_space)
        by (metis Suc_leI inf.absorb_iff2 inf.orderE leI)
      let ?S = topspace(Euclidean_space(Suc n)) ∩ {x. x n < 0}
      show path_connectedin (Euclidean_space (Suc n))
            ((topspace (Euclidean_space (Suc n)) − S) ∩ {x. x n ≤ 0})
      proof (rule lemma_iod)
        show ?S ⊆ (topspace (Euclidean_space (Suc n)) − S) ∩ {x. x n ≤ 0}
          using Ssub topspace_Euclidean_space by auto
        show ?S ≠ {}
          apply (simp add: topspace_Euclidean_space set_eq_iff)
          apply (rule_tac x=(λi. 0)(n:= −1) in exI)
          apply auto
          done
        fix a b and u::real
```

```
        assume
          a ∈ ?S 0 < u u < 1
          b ∈ (topspace (Euclidean_space (Suc n)) − S) ∩ {x. x n ≤ 0}
        then show (λi. (1 − u) ∗ a i + u ∗ b i) ∈ ?S
        by (simp add: topspace_Euclidean_space add_neg_nonpos less_eq_real_def
mult_less_0_iff)
      qed (simp add: topspace_Euclidean_space subset_iff)
      let ?T = topspace(Euclidean_space(Suc n)) ∩ {x. x n > 0}
      show path_connectedin (Euclidean_space (Suc n))
              ((topspace (Euclidean_space (Suc n)) − S) ∩ {x. 0 ≤ x n})
      proof (rule lemma_iod)
        show ?T ⊆ (topspace (Euclidean_space (Suc n)) − S) ∩ {x. 0 ≤ x n}
          using Ssub topspace_Euclidean_space by auto
        show ?T ≠ {}
          apply (simp add: topspace_Euclidean_space set_eq_iff)
          apply (rule_tac x=(λi. 0)(n:= 1) in exI)
          apply auto
          done
        fix a b and u::real
        assume  a ∈ ?T 0 < u u < 1 b ∈ (topspace (Euclidean_space (Suc n)) −
S) ∩ {x. 0 ≤ x n}
        then show (λi. (1 − u) ∗ a i + u ∗ b i) ∈ ?T
          by (simp add: topspace_Euclidean_space add_pos_nonneg)
      qed (simp add: topspace_Euclidean_space subset_iff)
      show (topspace (Euclidean_space (Suc n)) − S) ∩ {x. x n ≤ 0} ∩
            ((topspace (Euclidean_space (Suc n)) − S) ∩ {x. 0 ≤ x n}) ≠ {}
        using that
        apply (auto simp: Set.set_eq_iff topspace_Euclidean_space)
        by (metis Suc_leD order_refl)
    qed
    then have path_connected_space (subtopology (Euclidean_space (Suc n))
                                    (topspace (Euclidean_space (Suc n)) − S))
     apply (simp add: path_connectedin_subtopology flip: path_connectedin_topspace)
      by (metis Int_Diff inf_idem)
    ultimately
    show ?thesis
      using isomorphic_homology_imp_path_connectedness by blast
  qed
  ultimately show ?rhs
    by blast
qed (simp add: homeomorphic_space_refl)
```

```
lemma isomorphic_homology_groups_Euclidean_complements:
  assumes closedin (Euclidean_space n) S closedin (Euclidean_space n) T
          (subtopology (Euclidean_space n) S) homeomorphic_space (subtopology
(Euclidean_space n) T)
 shows homology_group p (subtopology (Euclidean_space n) (topspace(Euclidean_space
```

n) − *S*))
≅ *homology_group p* (*subtopology* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *T*))
proof (*rule isomorphic_relative_contractible_space_imp_homology_groups*)
show *topspace* (*Euclidean_space n*) − *S* ⊆ *topspace* (*Euclidean_space n*)
using *assms homeomorphic_space_sym invariance_of_dimension_closedin_Euclidean_space subtopology_superset* **by** *fastforce*
show *topspace* (*Euclidean_space n*) − *T* ⊆ *topspace* (*Euclidean_space n*)
using *assms invariance_of_dimension_closedin_Euclidean_space subtopology_superset* **by** *force*
show (*topspace* (*Euclidean_space n*) − *S* = {}) = (*topspace* (*Euclidean_space n*) − *T* = {})
by (*metis Diff_eq_empty_iff assms closedin_subset homeomorphic_space_sym invariance_of_dimension_closedin_Euclidean_space subset_antisym subtopology_topspace*)
show *relative_homology_group p* (*Euclidean_space n*) (*topspace* (*Euclidean_space n*) − *S*) ≅
relative_homology_group p (*Euclidean_space n*) (*topspace* (*Euclidean_space n*) − *T*) **for** *p*
using *assms isomorphic_relative_homology_groups_Euclidean_complements* **by** *blast*
qed *auto*

lemma *eqpoll_path_components_Euclidean_complements*:
assumes *closedin* (*Euclidean_space n*) *S closedin* (*Euclidean_space n*) *T*
(*subtopology* (*Euclidean_space n*) *S*) *homeomorphic_space* (*subtopology* (*Euclidean_space n*) *T*)
shows *path_components_of*
(*subtopology* (*Euclidean_space n*)
(*topspace*(*Euclidean_space n*) − *S*))
≈ *path_components_of*
(*subtopology* (*Euclidean_space n*)
(*topspace*(*Euclidean_space n*) − *T*))
by (*simp add*: *assms isomorphic_homology_groups_Euclidean_complements isomorphic_homology_imp_path_components*)

lemma *path_connectedin_Euclidean_complements*:
assumes *closedin* (*Euclidean_space n*) *S closedin* (*Euclidean_space n*) *T*
(*subtopology* (*Euclidean_space n*) *S*) *homeomorphic_space* (*subtopology* (*Euclidean_space n*) *T*)
shows *path_connectedin* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *S*)
⟷ *path_connectedin* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *T*)
by (*meson Diff_subset assms isomorphic_homology_groups_Euclidean_complements isomorphic_homology_imp_path_connectedness path_connectedin_def*)

lemma *eqpoll_connected_components_Euclidean_complements*:
assumes *S*: *closedin* (*Euclidean_space n*) *S* **and** *T*: *closedin* (*Euclidean_space n*) *T*

and *ST*: (*subtopology* (*Euclidean_space n*) *S*) *homeomorphic_space* (*subtopology* (*Euclidean_space n*) *T*)
shows *connected_components_of*
(*subtopology* (*Euclidean_space n*)
(*topspace*(*Euclidean_space n*) − *S*))
≈ *connected_components_of*
(*subtopology* (*Euclidean_space n*)
(*topspace*(*Euclidean_space n*) − *T*))
using *eqpoll_path_components_Euclidean_complements* [*OF assms*]
by (*metis S T closedin_def locally_path_connected_Euclidean_space locally_path_connected_space_op path_components_eq_connected_components_of*)

lemma *connected_in_Euclidean_complements*:
assumes *closedin* (*Euclidean_space n*) *S closedin* (*Euclidean_space n*) *T*
(*subtopology* (*Euclidean_space n*) *S*) *homeomorphic_space* (*subtopology* (*Euclidean_space n*) *T*)
shows *connectedin* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *S*)
⟷ *connectedin* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *T*)
apply (*simp add*: *connectedin_def connected_space_iff_components_subset_singleton subset_singleton_iff_lepoll*)
using *eqpoll_connected_components_Euclidean_complements* [*OF assms*]
by (*meson eqpoll_sym lepoll_trans1*)

theorem *invariance_of_dimension_Euclidean_space*:
Euclidean_space m homeomorphic_space Euclidean_space n ⟷ *m* = *n*
proof (*cases m n rule*: *linorder_cases*)
case *less*
then have ∗: *topspace* (*Euclidean_space m*) ⊆ *topspace* (*Euclidean_space n*)
by (*meson le_cases not_le subset_Euclidean_space*)
then have *Euclidean_space m* = *subtopology* (*Euclidean_space n*) (*topspace*(*Euclidean_space m*))
by (*simp add*: *Euclidean_space_def inf.absorb_iff2 subtopology_subtopology*)
then show *?thesis*
by (*metis* (*no_types*, *lifting*) ∗ *Euclidean_space_def closedin_Euclidean_space closedin_closed_subtopology eq_iff invariance_of_dimension_closedin_Euclidean_space subset_Euclidean_space topspace_Euclidean_space*)
next
case *equal*
then show *?thesis*
by (*simp add*: *homeomorphic_space_refl*)
next
case *greater*
then have ∗: *topspace* (*Euclidean_space n*) ⊆ *topspace* (*Euclidean_space m*)
by (*meson le_cases not_le subset_Euclidean_space*)
then have *Euclidean_space n* = *subtopology* (*Euclidean_space m*) (*topspace*(*Euclidean_space n*))
by (*simp add*: *Euclidean_space_def inf.absorb_iff2 subtopology_subtopology*)
then show *?thesis*

by (*metis* (*no_types*, *lifting*) ∗ *Euclidean_space_def closedin_Euclidean_space closedin_closed_subtopology eq_iff homeomorphic_space_sym invariance_of_dimension_closedin_Euclidean_spac subset_Euclidean_space topspace_Euclidean_space*)
qed

lemma *biglemma*:
assumes *n* ≠ *0* **and** *S*: *compactin* (*Euclidean_space n*) *S*
and *cmh*: *continuous_map* (*subtopology* (*Euclidean_space n*) *S*) (*Euclidean_space n*) *h*
and *inj_on h S*
shows *path_connectedin* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *h* ‘ *S*)
⟷ *path_connectedin* (*Euclidean_space n*) (*topspace*(*Euclidean_space n*) − *S*)
proof (*rule path_connectedin_Euclidean_complements*)
have *hS_sub*: *h* ‘ *S* ⊆ *topspace*(*Euclidean_space n*)
by (*metis* (*no_types*) *S cmh compactin_subspace continuous_map_image_subset_topspace topspace_subtopology_subset*)
show *clo_S*: *closedin* (*Euclidean_space n*) *S*
using *assms* **by** (*simp add*: *continuous_map_in_subtopology Hausdorff_Euclidean_space compactin_imp_closedin*)
show *clo_hS*: *closedin* (*Euclidean_space n*) (*h* ‘ *S*)
using *Hausdorff_Euclidean_space S cmh compactin_absolute compactin_imp_closedin image_compactin* **by** *blast*
have *homeomorphic_map* (*subtopology* (*Euclidean_space n*) *S*) (*subtopology* (*Euclidean_space n*) (*h* ‘ *S*)) *h*
proof (*rule continuous_imp_homeomorphic_map*)
show *compact_space* (*subtopology* (*Euclidean_space n*) *S*)
by (*simp add*: *S compact_space_subtopology*)
show *Hausdorff_space* (*subtopology* (*Euclidean_space n*) (*h* ‘ *S*))
using *hS_sub*
by (*simp add*: *Hausdorff_Euclidean_space Hausdorff_space_subtopology*)
show *continuous_map* (*subtopology* (*Euclidean_space n*) *S*) (*subtopology* (*Euclidean_space n*) (*h* ‘ *S*)) *h*
using *cmh continuous_map_in_subtopology* **by** *fastforce*
show *h* ‘ *topspace* (*subtopology* (*Euclidean_space n*) *S*) = *topspace* (*subtopology* (*Euclidean_space n*) (*h* ‘ *S*))
using *clo_hS clo_S closedin_subset* **by** *auto*
show *inj_on h* (*topspace* (*subtopology* (*Euclidean_space n*) *S*))
by (*metis* ‹*inj_on h S*› *clo_S closedin_def topspace_subtopology_subset*)
qed
then show *subtopology* (*Euclidean_space n*) (*h* ‘ *S*) *homeomorphic_space subtopology* (*Euclidean_space n*) *S*
using *homeomorphic_space homeomorphic_space_sym* **by** *blast*
qed

lemma *lemmaIOD*:
assumes
$\exists T.\ T \in U \wedge c \subseteq T$ $\exists T.\ T \in U \wedge d \subseteq T$ $\bigcup U = c \cup d$ $\bigwedge T.\ T \in U \Longrightarrow T \neq \{\}$
pairwise disjnt U $\sim(\exists T.\ U \subseteq \{T\})$
shows $c \in U$
using *assms*
apply *safe*
subgoal for C' D'
proof (*cases* $C'=D'$)
show $c \in U$
if UU: $\bigcup\ U = c \cup d$
and U: $\bigwedge T.\ T \in U \Longrightarrow T \neq \{\}$ *disjoint U* **and** $\nexists T.\ U \subseteq \{T\}$ $c \subseteq C'$ $D' \in U$ $d \subseteq D'$ $C' = D'$
proof −
have $c \cup d = D'$
using *Union_upper sup_mono UU that*(*5*) *that*(*6*) *that*(*7*) *that*(*8*) **by** *auto*
then have $\bigcup U = D'$
by (*simp add*: *UU*)
with *U* **have** $U = \{D'\}$
by (*metis* (*no_types*, *lifting*) *disjnt_Union1 disjnt_self_iff_empty insertCI pairwiseD subset_iff that*(*4*) *that*(*6*))
then show *?thesis*
using *that*(*4*) **by** *auto*
qed
show $c \in U$
if $\bigcup\ U = c \cup d$*disjoint U* $C' \in U$ $c \subseteq C'$$D' \in U$ $d \subseteq D'$ $C' \neq D'$
proof −
have $C' \cap D' = \{\}$
using ‹*disjoint U*› ‹$C' \in U$› ‹$D' \in U$› ‹$C' \neq D'$›**unfolding** *disjnt_iff pairwise_def*
by *blast*
then show *?thesis*
using *subset_antisym that*(*1*) ‹$C' \in U$› ‹$c \subseteq C'$› ‹$d \subseteq D'$› **by** *fastforce*
qed
qed
done

theorem *invariance_of_domain_Euclidean_space*:
assumes *U*: *openin* (*Euclidean_space n*) *U*
and *cmf*: *continuous_map* (*subtopology* (*Euclidean_space n*) *U*) (*Euclidean_space n*) *f*
and *inj_on f U*
shows *openin* (*Euclidean_space n*) (*f* ‘ *U*) (**is** *openin ?E* (*f* ‘ *U*))
proof (*cases* $n = 0$)
case *True*

```
  have [simp]: Euclidean_space 0 = discrete_topology {λi. 0}
   by (auto simp: subtopology_eq_discrete_topology_sing topspace_Euclidean_space)
  show ?thesis
    using cmf True U by auto
next
  case False
  define enorm where enorm ≡ λx. sqrt(∑ i<n. x i ^ 2)
  have enorm_if [simp]: enorm (λi. if i = k then d else 0) = (if k < n then |d|
else 0) for k d
    using ‹n ≠ 0› by (auto simp: enorm_def power2_eq_square if_distrib [of λx.
x * _] cong: if_cong)
  define zero::nat⇒real where zero ≡ λi. 0
  have zero_in [simp]: zero ∈ topspace ?E
    using False by (simp add: zero_def topspace_Euclidean_space)
  have enorm_eq_0 [simp]: enorm x = 0 ⟷ x = zero
    if x ∈ topspace(Euclidean_space n) for x
    using that unfolding zero_def enorm_def
   apply (simp add: sum_nonneg_eq_0_iff fun_eq_iff topspace_Euclidean_space)
    using le_less_linear by blast
  have [simp]: enorm zero = 0
    by (simp add: zero_def enorm_def)
  have cm_enorm: continuous_map ?E euclideanreal enorm
    unfolding enorm_def
  proof (intro continuous_intros)
    show continuous_map ?E euclideanreal (λx. x i)
      if i ∈ {..<n} for i
     using that by (auto simp: Euclidean_space_def intro: continuous_map_product_projection
continuous_map_from_subtopology)
  qed auto
  have enorm_ge0: 0 ≤ enorm x for x
    by (auto simp: enorm_def sum_nonneg)
  have le_enorm: |x i| ≤ enorm x if i < n for i x
  proof -
    have |x i| ≤ sqrt (∑ k∈{i}. (x k)^2)
      by auto
    also have ... ≤ sqrt (∑ k<n. (x k)^2)
      by (rule real_sqrt_le_mono [OF sum_mono2]) (use that in auto)
    finally show ?thesis
      by (simp add: enorm_def)
  qed
  define B where B ≡ λr. {x ∈ topspace ?E. enorm x < r}
  define C where C ≡ λr. {x ∈ topspace ?E. enorm x ≤ r}
  define S where S ≡ λr. {x ∈ topspace ?E. enorm x = r}
  have BC: B r ⊆ C r and SC: S r ⊆ C r and disjSB: disjnt (S r) (B r) and
eqC: B r ∪ S r = C r for r
    by (auto simp: B_def C_def S_def disjnt_def)
  consider n = 1 | n ≥ 2
    using False by linarith
  then have **: openin ?E (h ` (B r))
```

if *r* > *0* **and** *cmh*: *continuous_map*(*subtopology ?E* (*C r*)) *?E h* **and** *injh*: *inj_on h* (*C r*) **for** *r h*
proof *cases*
case *1*
define *e* :: [*real*,*nat*]⇒*real* **where** *e* ≡ λ*x i*. *if i* = *0 then x else 0*
define *e′* :: (*nat*⇒*real*)⇒*real* **where** *e′* ≡ λ*x*. *x 0*
have *continuous_map euclidean euclideanreal* (λ*f*. *f* (*0*::*nat*))
by *auto*
then have *continuous_map* (*subtopology* (*powertop_real UNIV*) {*f*. ∀ *n*≥*Suc 0*. *f n* = *0*}) *euclideanreal* (λ*f*. *f 0*)
by (*metis* (*mono_tags*) *continuous_map_from_subtopology euclidean_product_topology*)
then have *hom_ee′*: *homeomorphic_maps euclideanreal* (*Euclidean_space 1*) *e e′*
by (*auto simp*: *homeomorphic_maps_def e_def e′_def continuous_map_in_subtopology Euclidean_space_def*)
have *eBr*: *e* ‘ {−*r*<..<*r*} = *B r*
unfolding *B_def e_def C_def*
by(*force simp*: *1 topspace_Euclidean_space enorm_def power2_eq_square if_distrib* [*of* λ*x*. *x* ∗ _] *cong*: *if_cong*)
have *in_Cr*: ⋀*x*. ⟦−*r* < *x*; *x* < *r*⟧ ⟹ (λ*i*. *if i* = *0 then x else 0*) ∈ *C r*
using ‹*n* ≠ *0*› **by** (*auto simp*: *C_def topspace_Euclidean_space*)
have *inj*: *inj_on* (*e′* ∘ *h* ∘ *e*) {− *r*<..<*r*}
proof (*clarsimp simp*: *inj_on_def e_def e′_def*)
show (*x*::*real*) = *y*
if *f*: *h* (λ*i*. *if i* = *0 then x else 0*) *0* = *h* (λ*i*. *if i* = *0 then y else 0*) *0*
and −*r* < *x x* < *r* −*r* < *y y* < *r*
for *x y* :: *real*
proof −
have *x*: (λ*i*. *if i* = *0 then x else 0*) ∈ *C r* **and** *y*: (λ*i*. *if i* = *0 then y else 0*) ∈ *C r*
by (*blast intro*: *inj_onD* [*OF* ‹*inj_on h* (*C r*)›] *that in_Cr*)+
have *continuous_map* (*subtopology* (*Euclidean_space* (*Suc 0*)) (*C r*)) (*Euclidean_space* (*Suc 0*)) *h*
using *cmh* **by** (*simp add*: *1*)
then have *h* ‘ ({*x*. ∀ *i*≥*Suc 0*. *x i* = *0*} ∩ *C r*) ⊆ {*x*. ∀ *i*≥*Suc 0*. *x i* = *0*}
by (*force simp*: *Euclidean_space_def subtopology_subtopology continuous_map_def*)
have *h* (λ*i*. *if i* = *0 then x else 0*) *j* = *h* (λ*i*. *if i* = *0 then y else 0*) *j* **for** *j*
proof (*cases j*)
case (*Suc j′*)
have *h* ‘ ({*x*. ∀ *i*≥*Suc 0*. *x i* = *0*} ∩ *C r*) ⊆ {*x*. ∀ *i*≥*Suc 0*. *x i* = *0*}
using *continuous_map_image_subset_topspace* [*OF cmh*]
by (*simp add*: *1 Euclidean_space_def subtopology_subtopology*)
with *Suc f x y* **show** *?thesis*
by (*simp add*: *1 image_subset_iff*)
qed (*use f* **in** *blast*)
then have (λ*i*. *if i* = *0 then x else 0*) = (λ*i*::*nat*. *if i* = *0 then y else 0*)
by (*blast intro*: *inj_onD* [*OF* ‹*inj_on h* (*C r*)›] *that in_Cr*)
then show *?thesis*

```
      by (simp add: fun_eq_iff) presburger
    qed
  qed
  have hom_e': homeomorphic_map (Euclidean_space 1) euclideanreal e'
    using hom_ee' homeomorphic_maps_map by blast
  have openin (Euclidean_space n) (h ` e ` {- r<..<r})
    unfolding 1
  proof (subst homeomorphic_map_openness [OF hom_e', symmetric])
    show hesub: h ` e ` {- r<..<r} ⊆ topspace (Euclidean_space 1)
    using 1 C_def ‹⋀r. B r ⊆ C r› cmh continuous_map_image_subset_topspace eBr by fastforce
    have cont: continuous_on {- r<..<r} (e' ∘ h ∘ e)
    proof (intro continuous_on_compose)
      have ⋀i. continuous_on {- r<..<r} (λx. if i = 0 then x else 0)
        by (auto simp: continuous_on_topological)
      then show continuous_on {- r<..<r} e
       by (force simp: e_def intro: continuous_on_coordinatewise_then_product)
      have subCr: e ` {- r<..<r} ⊆ topspace (subtopology ?E (C r))
        by (auto simp: eBr ‹⋀r. B r ⊆ C r›) (auto simp: B_def)
      with cmh show continuous_on (e ` {- r<..<r}) h
     by (meson cm_Euclidean_space_iff_continuous_on continuous_on_subset)
      have continuous_on (topspace ?E) e'
        by (metis 1 continuous_map_Euclidean_space_iff hom_ee' homeomorphic_maps_def)
      then show continuous_on (h ` e ` {- r<..<r}) e'
        using hesub by (simp add: 1 e'_def continuous_on_subset)
    qed
    show openin euclideanreal (e' ` h ` e ` {- r<..<r})
        using injective_eq_1d_open_map_UNIV [OF cont] inj by (simp add: image_image is_interval_1)
  qed
  then show ?thesis
    by (simp flip: eBr)
next
  case 2
  have cloC: ⋀r. closedin (Euclidean_space n) (C r)
    unfolding C_def
   by (rule closedin_continuous_map_preimage [OF cm_enorm, of concl: {.._}, simplified])
  have cloS: ⋀r. closedin (Euclidean_space n) (S r)
    unfolding S_def
   by (rule closedin_continuous_map_preimage [OF cm_enorm, of concl: {_}, simplified])
  have C_subset: C r ⊆ UNIV →E {- |r|..|r|}
    using le_enorm ‹r > 0›
    apply (auto simp: C_def topspace_Euclidean_space abs_le_iff)
   apply (metis add.inverse_neutral le_cases less_minus_iff not_le order_trans)
    by (metis enorm_ge0 not_le order.trans)
  have compactinC: compactin (Euclidean_space n) (C r)
```

unfolding *Euclidean_space_def compactin_subtopology*
proof
show *compactin* (*powertop_real UNIV*) (*C r*)
proof (*rule closed_compactin* [*OF _ C_subset*])
show *closedin* (*powertop_real UNIV*) (*C r*)
by (*metis Euclidean_space_def cloC closedin_Euclidean_space closedin_closed_subtopology topspace_Euclidean_space*)
qed (*simp add*: *compactin_PiE*)
qed (*auto simp*: *C_def topspace_Euclidean_space*)
have *compactinS*: *compactin* (*Euclidean_space n*) (*S r*)
unfolding *Euclidean_space_def compactin_subtopology*
proof
show *compactin* (*powertop_real UNIV*) (*S r*)
proof (*rule closed_compactin*)
show *S r* ⊆ *UNIV* $\rightarrow_E$ {− |*r*|..|*r*|}
using *C_subset* ‹⋀*r*. *S r* ⊆ *C r*› **by** *blast*
show *closedin* (*powertop_real UNIV*) (*S r*)
by (*metis Euclidean_space_def cloS closedin_Euclidean_space closedin_closed_subtopology topspace_Euclidean_space*)
qed (*simp add*: *compactin_PiE*)
qed (*auto simp*: *S_def topspace_Euclidean_space*)
have *h_if_B*: ⋀*y*. *y* ∈ *B r* ⟹ *h y* ∈ *topspace ?E*
using *B_def* ‹⋀*r*. *B r* ∪ *S r* = *C r*› *cmh continuous_map_image_subset_topspace* **by** *fastforce*
have *com_hSr*: *compactin* (*Euclidean_space n*) (*h* ‘ *S r*)
by (*meson* ‹⋀*r*. *S r* ⊆ *C r*› *cmh compactinS compactin_subtopology image_compactin*)
have *ope_comp_hSr*: *openin* (*Euclidean_space n*) (*topspace* (*Euclidean_space n*) − *h* ‘ *S r*)
proof (*rule openin_diff*)
show *closedin* (*Euclidean_space n*) (*h* ‘ *S r*)
using *Hausdorff_Euclidean_space com_hSr compactin_imp_closedin* **by** *blast*
qed *auto*
have *h_pcs*: *h* ‘ (*B r*) ∈ *path_components_of* (*subtopology ?E* (*topspace ?E* − *h* ‘ (*S r*)))
proof (*rule lemmaIOD*)
have *pc_interval*: *path_connectedin* (*Euclidean_space n*) {*x* ∈ *topspace*(*Euclidean_space n*). *enorm x* ∈ *T*}
if *T*: *is_interval T* **for** *T*
proof −
define *mul* :: [*real*, *nat* ⇒ *real*, *nat*] ⇒ *real* **where** *mul* ≡ λ*a x i*. *a* ∗ *x i*
let *?neg* = *mul* (−*1*)
have *neg_neg* [*simp*]: *?neg* (*?neg x*) = *x* **for** *x*
by (*simp add*: *mul_def*)
have *enorm_mul* [*simp*]: *enorm*(*mul a x*) = *abs a* ∗ *enorm x* **for** *a x*
by (*simp add*: *enorm_def mul_def power_mult_distrib*) (*metis real_sqrt_abs real_sqrt_mult sum_distrib_left*)
have *mul_in_top*: *mul a x* ∈ *topspace ?E*

```
      if x ∈ topspace ?E for a x
    using mul_def that topspace_Euclidean_space by auto
  have neg_in_S: ?neg x ∈ S r
      if x ∈ S r for x r
   using that topspace_Euclidean_space S_def by simp (simp add: mul_def)
  have ∗: path_connectedin ?E (S d)
    if d ≥ 0 for d
  proof (cases d = 0)
    let ?ES = subtopology ?E (S d)
    case False
    then have d > 0
      using that by linarith
    moreover have path_connected_space ?ES
      unfolding path_connected_space_iff_path_component
    proof clarify
      have ∗∗: path_component_of ?ES x y
        if x: x ∈ topspace ?ES and y: y ∈ topspace ?ES x ≠ ?neg y for x y
      proof −
        show ?thesis
          unfolding path_component_of_def pathin_def S_def
        proof (intro exI conjI)
          let ?g = (λx. mul (d / enorm x) x) ∘ (λt i. (1 − t) ∗ x i + t ∗ y i)
          show continuous_map (top_of_set {0::real..1}) (subtopology ?E {x
∈ topspace ?E. enorm x = d}) ?g
          proof (rule continuous_map_compose)
            let ?Y = subtopology ?E (− {zero})
            have ∗∗: False
              if eq0: ⋀j. (1 − r) ∗ x j + r ∗ y j = 0
                and ne: x i ≠ − y i
                and d: enorm x = d enorm y = d
                and r: 0 ≤ r r ≤ 1
              for i r
            proof −
              have mul (1−r) x = ?neg (mul r y)
                using eq0 by (simp add: mul_def fun_eq_iff algebra_simps)
              then have enorm (mul (1−r) x) = enorm (?neg (mul r y))
                by metis
              with r have (1−r) ∗ enorm x = r ∗ enorm y
                by simp
              then have r12: r = 1/2
                using ‹d ≠ 0› d by auto
              show ?thesis
               using ne eq0 [of i] unfolding r12 by (simp add: algebra_simps)
            qed
           show continuous_map (top_of_set {0..1}) ?Y (λt i. (1 − t) ∗ x i
+ t ∗ y i)
              using x y
                   unfolding continuous_map_componentwise_UNIV Eu-
clidean_space_def continuous_map_in_subtopology
```

apply (*intro conjI allI continuous_intros*)
apply (*auto simp*: *zero_def mul_def S_def Euclidean_space_def fun_eq_iff*)
using ∗∗ **by** *blast*
have *cm_enorm′*: *continuous_map* (*subtopology* (*powertop_real UNIV*) *A*) *euclideanreal enorm* **for** *A*
unfolding *enorm_def* **by** (*intro continuous_intros*) *auto*
have *continuous_map ?Y* (*subtopology ?E* {*x*. *enorm x* = *d*}) (λ*x*. *mul* (*d* / *enorm x*) *x*)
unfolding *continuous_map_in_subtopology*
proof (*intro conjI*)
show *continuous_map ?Y* (*Euclidean_space n*) (λ*x*. *mul* (*d* / *enorm x*) *x*)
unfolding *continuous_map_in_subtopology Euclidean_space_def mul_def zero_def subtopology_subtopology continuous_map_componentwise_UNIV*
proof (*intro conjI allI cm_enorm′ continuous_intros*)
show *enorm x* ≠ *0*
if *x* ∈ *topspace* (*subtopology* (*powertop_real UNIV*) ({*x*. ∀ *i*≥*n*. *x i* = *0*} ∩ − {λ*i*. *0*})) **for** *x*
using *that* **by** *simp* (*metis abs_le_zero_iff le_enorm not_less*)
qed *auto*
qed (*use* ‹*d* > *0*› *enorm_ge0* **in** *auto*)
moreover have *subtopology ?E* {*x* ∈ *topspace ?E*. *enorm x* = *d*} = *subtopology ?E* {*x*. *enorm x* = *d*}
by (*simp add*: *subtopology_restrict Collect_conj_eq*)
ultimately show *continuous_map ?Y* (*subtopology* (*Euclidean_space n*) {*x* ∈ *topspace* (*Euclidean_space n*). *enorm x* = *d*}) (λ*x*. *mul* (*d* / *enorm x*) *x*)
by *metis*
qed
show *?g* (*0*::*real*) = *x ?g* (*1*::*real*) = *y*
using *that* **by** (*auto simp*: *S_def zero_def mul_def fun_eq_iff*)
qed
qed
obtain *a b* **where** *a*: *a* ∈ *topspace ?ES* **and** *b*: *b* ∈ *topspace ?ES*
and *a* ≠ *b* **and** *negab*: *?neg a* ≠ *b*
proof
let *?v* = λ*j i*::*nat*. *if i* = *j then d else 0*
show *?v 0* ∈ *topspace* (*subtopology ?E* (*S d*)) *?v 1* ∈ *topspace* (*subtopology ?E* (*S d*))
using ‹*n* ≥ *2*› ‹*d* ≥ *0*› **by** (*auto simp*: *S_def topspace_Euclidean_space*)
show *?v 0* ≠ *?v 1 ?neg* (*?v 0*) ≠ (*?v 1*)
using ‹*d* > *0*› **by** (*auto simp*: *mul_def fun_eq_iff*)
qed
show *path_component_of ?ES x y*
if *x*: *x* ∈ *topspace ?ES* **and** *y*: *y* ∈ *topspace ?ES*
for *x y*
proof −
have *path_component_of ?ES x* (*?neg x*)
proof −

```
            have path_component_of ?ES x a
                  by (metis (no_types, opaque_lifting) ** a b ‹a ≠ b› negab
path_component_of_trans path_component_of_sym x)
            moreover
          have pa_ab: path_component_of ?ES a b using ** a b negab neg_neg
by blast
            then have path_component_of ?ES a (?neg x)
         by (metis ** ‹a ≠ b› cloS closedin_def neg_in_S path_component_of_equiv
topspace_subtopology_subset x)
            ultimately show ?thesis
              by (meson path_component_of_trans)
          qed
          then show ?thesis
            using ** x y by force
        qed
      qed
      ultimately show ?thesis
        by (simp add: cloS closedin_subset path_connectedin_def)
    qed (simp add: S_def cong: conj_cong)
     have path_component_of (subtopology ?E {x ∈ topspace ?E. enorm x ∈
T}) x y
      if enorm x = a x ∈ topspace ?E enorm x ∈ T enorm y = b y ∈ topspace
?E enorm y ∈ T
      for x y a b
      using that
      proof (induction a b arbitrary: x y rule: linorder_less_wlog)
        case (less a b)
        then have a ≥ 0
          using enorm_ge0 by blast
        with less.hyps have b > 0
          by linarith
        show ?case
        proof (rule path_component_of_trans)
          have y'_ts: mul (a / b) y ∈ topspace ?E
            using ‹y ∈ topspace ?E› mul_in_top by blast
          moreover have enorm (mul (a / b) y) = a
            unfolding enorm_mul using ‹0 < b› ‹0 ≤ a› less.prems by simp
          ultimately have y'_S: mul (a / b) y ∈ S a
            using S_def by blast
          have x ∈ S a
            using S_def less.prems by blast
          with ‹x ∈ topspace ?E› y'_ts y'_S
          have path_component_of (subtopology ?E (S a)) x (mul (a / b) y)
          by (metis * [OF ‹a ≥ 0›] path_connected_space_iff_path_component
path_connectedin_def topspace_subtopology_subset)
          moreover
         have {f ∈ topspace ?E. enorm f = a} ⊆ {f ∈ topspace ?E. enorm f ∈
T}
            using ‹enorm x = a› ‹enorm x ∈ T› by force
```

ultimately
show *path_component_of* (*subtopology ?E* {*x*. *x* ∈ *topspace ?E* ∧ *enorm x* ∈ *T*}) *x* (*mul* (*a* / *b*) *y*)
by (*simp add*: *S_def path_component_of_mono*)
have *pathin ?E* (λ*t*. *mul* (((*1* − *t*) ∗ *b* + *t* ∗ *a*) / *b*) *y*)
using ‹*b* > *0*› ‹*y* ∈ *topspace ?E*›
unfolding *pathin_def Euclidean_space_def mul_def continuous_map_in_subtopology continuous_map_componentwise_UNIV*
by (*intro allI conjI continuous_intros*) *auto*
moreover have *mul* (((*1* − *t*) ∗ *b* + *t* ∗ *a*) / *b*) *y* ∈ *topspace ?E*
if *t* ∈ {*0*..*1*} **for** *t*
using ‹*y* ∈ *topspace ?E*› *mul_in_top* **by** *blast*
moreover have *enorm* (*mul* (((*1* − *t*) ∗ *b* + *t* ∗ *a*) / *b*) *y*) ∈ *T*
if *t* ∈ {*0*..*1*} **for** *t*
proof −
have *a* ∈ *T* *b* ∈ *T*
using *less.prems* **by** *auto*
then have |(*1* − *t*) ∗ *b* + *t* ∗ *a*| ∈ *T*
proof (*rule mem_is_interval_1_I* [*OF T*])
show *a* ≤ |(*1* − *t*) ∗ *b* + *t* ∗ *a*|
using *that* ‹*a* ≥ *0*› *less.hyps segment_bound_lemma* **by** *auto*
show |(*1* − *t*) ∗ *b* + *t* ∗ *a*| ≤ *b*
using *that* ‹*a* ≥ *0*› *less.hyps* **by** (*auto intro*: *convex_bound_le*)
qed
then show *?thesis*
unfolding *enorm_mul* ‹*enorm y* = *b*› **using** *that* ‹*b* > *0*› **by** *simp*
qed
ultimately have *pa*: *pathin* (*subtopology ?E* {*x* ∈ *topspace ?E*. *enorm x* ∈ *T*})
(λ*t*. *mul* (((*1* − *t*) ∗ *b* + *t* ∗ *a*) / *b*) *y*)
by (*auto simp*: *pathin_subtopology*)
have *ex_pathin*: ∃ *g*. *pathin* (*subtopology ?E* {*x* ∈ *topspace ?E*. *enorm x* ∈ *T*}) *g* ∧
g 0 = *y* ∧ *g 1* = *mul* (*a* / *b*) *y*
apply (*rule_tac x*=λ*t*. *mul* (((*1* − *t*) ∗ *b* + *t* ∗ *a*) / *b*) *y* **in** *exI*)
using ‹*b* > *0*› *pa* **by** (*auto simp*: *mul_def*)
show *path_component_of* (*subtopology ?E* {*x*. *x* ∈ *topspace ?E* ∧ *enorm x* ∈ *T*}) (*mul* (*a* / *b*) *y*) *y*
by (*rule path_component_of_sym*) (*simp add*: *path_component_of_def ex_pathin*)
qed
next
case (*refl a*)
then have *pc*: *path_component_of* (*subtopology ?E* (*S* (*enorm u*))) *u v*
if *u* ∈ *topspace ?E* ∩ *S* (*enorm x*) *v* ∈ *topspace ?E* ∩ *S* (*enorm u*) **for** *u v*
using ∗ [*of a*] *enorm_ge0 that*
by (*auto simp*: *path_connectedin_def path_connected_space_iff_path_component S_def*)

```
            have sub: {u ∈ topspace ?E. enorm u = enorm x} ⊆ {u ∈ topspace ?E.
enorm u ∈ T}
              using ‹enorm x ∈ T› by auto
            show ?case
              using pc [of x y] refl by (auto simp: S_def path_component_of_mono
[OF _ sub])
          next
            case (sym a b)
            then show ?case
              by (blast intro: path_component_of_sym)
          qed
        then show ?thesis
         by (simp add: path_connectedin_def path_connected_space_iff_path_component)
      qed
      have h ‘ S r ⊆ topspace ?E
        by (meson SC cmh compact_imp_compactin_subtopology compactinS com-
pactin_subset_topspace image_compactin)
      moreover
      have ¬ compact_space ?E
        by (metis compact_Euclidean_space ‹n ≠ 0›)
      then have ¬ compactin ?E (topspace ?E)
        by (simp add: compact_space_def topspace_Euclidean_space)
      then have h ‘ S r ≠ topspace ?E
        using com_hSr by auto
      ultimately have top_hSr_ne: topspace (subtopology ?E (topspace ?E − h ‘
S r)) ≠ {}
        by auto
      show pc1: ∃ T. T ∈ path_components_of (subtopology ?E (topspace ?E − h
‘ S r)) ∧ h ‘ B r ⊆ T
      proof (rule exists_path_component_of_superset [OF _ top_hSr_ne])
        have path_connectedin ?E (h ‘ B r)
        proof (rule path_connectedin_continuous_map_image)
          show continuous_map (subtopology ?E (C r)) ?E h
            by (simp add: cmh)
          have path_connectedin ?E (B r)
              using pc_interval[of {..<r}] is_interval_convex_1 unfolding B_def
by auto
            then show path_connectedin (subtopology ?E (C r)) (B r)
              by (simp add: path_connectedin_subtopology BC)
          qed
          moreover have h ‘ B r ⊆ topspace ?E − h ‘ S r
            apply (auto simp: h_if_B)
            by (metis BC SC disjSB disjnt_iff inj_onD [OF injh] subsetD)
         ultimately show path_connectedin (subtopology ?E (topspace ?E − h ‘ S
r)) (h ‘ B r)
          by (simp add: path_connectedin_subtopology)
      qed metis
      show ∃ T. T ∈ path_components_of (subtopology ?E (topspace ?E − h ‘ S
r)) ∧ topspace ?E − h ‘ (C r) ⊆ T
```

proof (*rule exists_path_component_of_superset* [*OF* _ *top_hSr_ne*])
have *eq*: *topspace ?E* − {*x* ∈ *topspace ?E*. *enorm x* ≤ *r*} = {*x* ∈ *topspace ?E*. *r* < *enorm x*}
by *auto*
have *path_connectedin ?E* (*topspace ?E* − *C r*)
using *pc_interval*[*of* {*r*<..}] *is_interval_convex_1* **unfolding** *C_def eq* **by** *auto*
then have *path_connectedin ?E* (*topspace ?E* − *h* ‘ *C r*)
by (*metis biglemma* [*OF* ‹*n* ≠ *0*› *compactinC cmh injh*])
then show *path_connectedin* (*subtopology ?E* (*topspace ?E* − *h* ‘ *S r*)) (*topspace ?E* − *h* ‘ *C r*)
by (*simp add*: *Diff_mono SC image_mono path_connectedin_subtopology*)
qed *metis*
have *topspace ?E* ∩ (*topspace ?E* − *h* ‘ *S r*) = *h* ‘ *B r* ∪ (*topspace ?E* − *h* ‘ *C r*) (**is** *?lhs* = *?rhs*)
proof
show *?lhs* ⊆ *?rhs*
using ‹⋀*r*. *B r* ∪ *S r* = *C r*› **by** *auto*
have *h* ‘ *B r* ∩ *h* ‘ *S r* = {}
by (*metis Diff_triv* ‹⋀*r*. *B r* ∪ *S r* = *C r*› ‹⋀*r*. *disjnt* (*S r*) (*B r*)› *disjnt_def inf_commute inj_on_Un injh*)
then show *?rhs* ⊆ *?lhs*
using *path_components_of_subset pc1* ‹⋀*r*. *B r* ∪ *S r* = *C r*›
by (*fastforce simp add*: *h_if_B*)
qed
then show ⋃ (*path_components_of* (*subtopology ?E* (*topspace ?E* − *h* ‘ *S r*))) = *h* ‘ *B r* ∪ (*topspace ?E* − *h* ‘ (*C r*))
by (*simp add*: *Union_path_components_of*)
show *T* ≠ {}
if *T* ∈ *path_components_of* (*subtopology ?E* (*topspace ?E* − *h* ‘ *S r*)) **for** *T*
using *that* **by** (*simp add*: *nonempty_path_components_of*)
show *disjoint* (*path_components_of* (*subtopology ?E* (*topspace ?E* − *h* ‘ *S r*)))
by (*simp add*: *pairwise_disjoint_path_components_of*)
have ¬ *path_connectedin ?E* (*topspace ?E* − *h* ‘ *S r*)
proof (*subst biglemma* [*OF* ‹*n* ≠ *0*› *compactinS*])
show *continuous_map* (*subtopology ?E* (*S r*)) *?E h*
by (*metis Un_commute Un_upper1 cmh continuous_map_from_subtopology_mono eqC*)
show *inj_on h* (*S r*)
using *SC inj_on_subset injh* **by** *blast*
show ¬ *path_connectedin ?E* (*topspace ?E* − *S r*)
proof
have *topspace ?E* − *S r* = {*x* ∈ *topspace ?E*. *enorm x* ≠ *r*}
by (*auto simp*: *S_def*)
moreover have *enorm* ‘ {*x* ∈ *topspace ?E*. *enorm x* ≠ *r*} = {*0*..} − {*r*}
proof
have ∃*x*. *x* ∈ *topspace ?E* ∧ *enorm x* ≠ *r* ∧ *d* = *enorm x*
if *d* ≠ *r d* ≥ *0* **for** *d*

proof (*intro exI conjI*)
  show (λi. *if* $i = 0$ *then* d *else* 0) $\in$ *topspace ?E*
    using ‹$n \neq 0$› **by** (*auto simp*: *Euclidean_space_def*)
  show *enorm* (λi. *if* $i = 0$ *then* d *else* 0) $\neq r$ $d =$ *enorm* (λi. *if* $i = 0$ *then* d *else* 0)
    using ‹$n \neq 0$› *that* **by** *simp_all*
qed
then show $\{0..\} - \{r\} \subseteq$ *enorm* ‘ $\{x \in$ *topspace ?E*. *enorm* $x \neq r\}$
  by (*auto simp*: *image_def*)
qed (*auto simp*: *enorm_ge0*)
ultimately have *non_r*: *enorm* ‘ (*topspace ?E* $-$ *S r*) $= \{0..\} - \{r\}$
  by *simp*
have $\exists x \geq 0$. $x \neq r \wedge r \leq x$
  by (*metis gt_ex le_cases not_le order_trans*)
then have $\neg$ *is_interval* ($\{0..\} - \{r\}$)
  unfolding *is_interval_1*
  using ‹$r > 0$› **by** (*auto simp*: *Bex_def*)
then show *False*
  if *path_connectedin ?E* (*topspace ?E* $-$ *S r*)
  using *path_connectedin_continuous_map_image* [*OF cm_enorm that*] **by** (*simp add*: *is_interval_path_connected_1 non_r*)
qed
qed
then have $\neg$ *path_connected_space* (*subtopology ?E* (*topspace ?E* $-$ *h* ‘ *S r*))
  by (*simp add*: *path_connectedin_def*)
then show $\nexists T$. *path_components_of* (*subtopology ?E* (*topspace ?E* $-$ *h* ‘ *S r*)) $\subseteq \{T\}$
  by (*simp add*: *path_components_of_subset_singleton*)
qed
moreover have *openin ?E A*
  if $A \in$ *path_components_of* (*subtopology ?E* (*topspace ?E* $-$ *h* ‘ (*S r*))) **for** *A*
  using *locally_path_connected_Euclidean_space* [*of n*] *that ope_comp_hSr*
  by (*simp add*: *locally_path_connected_space_open_path_components*)
ultimately show *?thesis* **by** *metis*
qed
have $\exists T$. *openin ?E T* $\wedge f\ x \in T \wedge T \subseteq f$ ‘ U
  if $x \in U$ **for** x
proof $-$
  have x: $x \in$ *topspace ?E*
    by (*meson U in_mono openin_subset that*)
  obtain V **where** V: *openin* (*powertop_real UNIV*) V **and** *Ueq*: $U = V \cap \{x.\ \forall i \geq n.\ x\ i = 0\}$
    using U **by** (*auto simp*: *openin_subtopology Euclidean_space_def*)
  with ‹$x \in U$› **have** $x \in V$ **by** *blast*
  then obtain T **where** *Tfin*: *finite* $\{i.\ T\ i \neq UNIV\}$ **and** *Topen*: $\bigwedge i$. *open* (T i)
    and *Tx*: $x \in Pi_E$ *UNIV T* **and** *TV*: Pi_E *UNIV T* $\subseteq V$
    using V **by** (*force simp*: *openin_product_topology_alt*)

```
  have ∃ e>0. ∀ x'. |x' − x i| < e ⟶ x' ∈ T i for i
    using Topen [of i] Tx by (auto simp: open_real)
  then obtain β where B0: ⋀i. β i > 0 and BT: ⋀i x'. |x' − x i| < β i ⟹
x' ∈ T i
    by metis
  define r where r ≡ Min (insert 1 (β ‘ {i. T i ≠ UNIV}))
  have r > 0
    by (simp add: B0 Tfin r_def)
  have inU: y ∈ U
    if y: y ∈ topspace ?E and yxr: ⋀i. i<n ⟹ |y i − x i| < r for y
  proof −
    have y i ∈ T i for i
    proof (cases T i = UNIV)
      show y i ∈ T i if T i ≠ UNIV
      proof (cases i < n)
        case True
        then show ?thesis
          using yxr [OF True] that by (simp add: r_def BT Tfin)
      next
        case False
        then show ?thesis
         using B0 Ueq ‹x ∈ U› topspace_Euclidean_space y by (force intro: BT)
      qed
    qed auto
    with TV have y ∈ V by auto
    then show ?thesis
      using that by (auto simp: Ueq topspace_Euclidean_space)
  qed
  have xinU: (λi. x i + y i) ∈ U if y ∈ C(r/2) for y
  proof (rule inU)
    have y: y ∈ topspace ?E
      using C_def that by blast
    show (λi. x i + y i) ∈ topspace ?E
      using x y by (simp add: topspace_Euclidean_space)
    have enorm y ≤ r/2
      using that by (simp add: C_def)
    then show |x i + y i − x i| < r if i < n for i
      using le_enorm enorm_ge0 that ‹0 < r› leI order_trans by fastforce
  qed
  show ?thesis
  proof (intro exI conjI)
    show openin ?E ((f ∘ (λy i. x i + y i)) ‘ B (r/2))
    proof (rule **)
       have continuous_map (subtopology ?E (C(r/2))) (subtopology ?E U) (λy
i. x i + y i)
         by (auto simp: xinU continuous_map_in_subtopology
             intro!: continuous_intros continuous_map_Euclidean_space_add x)
       then show continuous_map (subtopology ?E (C(r/2))) ?E (f ∘ (λy i. x i
+ y i))
```

```
        by (rule continuous_map_compose) (simp add: cmf)
      show inj_on (f ∘ (λy i. x i + y i)) (C(r/2))
       proof (clarsimp simp add: inj_on_def C_def topspace_Euclidean_space
simp del: divide_const_simps)
        show y' = y
          if ey: enorm y ≤ r / 2 and ey': enorm y' ≤ r / 2
            and y0: ∀i≥n. y i = 0 and y'0: ∀i≥n. y' i = 0
            and feq: f (λi. x i + y' i) = f (λi. x i + y i)
          for y' y :: nat ⇒ real
        proof -
          have (λi. x i + y i) ∈ U
          proof (rule inU)
            show (λi. x i + y i) ∈ topspace ?E
              using topspace_Euclidean_space x y0 by auto
            show |x i + y i - x i| < r if i < n for i
              using ey le_enorm [of _ y] ‹r > 0› that by fastforce
          qed
          moreover have (λi. x i + y' i) ∈ U
          proof (rule inU)
            show (λi. x i + y' i) ∈ topspace ?E
              using topspace_Euclidean_space x y'0 by auto
            show |x i + y' i - x i| < r if i < n for i
              using ey' le_enorm [of _ y'] ‹r > 0› that by fastforce
          qed
          ultimately have (λi. x i + y' i) = (λi. x i + y i)
            using feq by (meson ‹inj_on f U› inj_on_def)
          then show ?thesis
            by (auto simp: fun_eq_iff)
        qed
      qed
    qed (simp add: ‹0 < r›)
    have x ∈ (λy i. x i + y i) ` B (r / 2)
    proof
      show x = (λi. x i + zero i)
        by (simp add: zero_def)
    qed (auto simp: B_def ‹r > 0›)
    then show f x ∈ (f ∘ (λy i. x i + y i)) ` B (r/2)
      by (metis image_comp image_eqI)
    show (f ∘ (λy i. x i + y i)) ` B (r/2) ⊆ f ` U
      using ‹⋀r. B r ⊆ C r› xinU by fastforce
  qed
 qed
 then show ?thesis
   using openin_subopen by force
qed


corollary invariance_of_domain_Euclidean_space_embedding_map:
  assumes openin (Euclidean_space n) U
```

and *cmf*: *continuous_map*(*subtopology* (*Euclidean_space n*) *U*) (*Euclidean_space n*) *f*
and *inj_on f U*
shows *embedding_map*(*subtopology* (*Euclidean_space n*) *U*) (*Euclidean_space n*) *f*
proof (*rule injective_open_imp_embedding_map* [*OF cmf*])
show *open_map* (*subtopology* (*Euclidean_space n*) *U*) (*Euclidean_space n*) *f*
unfolding *open_map_def*
by (*meson assms continuous_map_from_subtopology_mono inj_on_subset invariance_of_domain_Euclidean_space openin_imp_subset openin_trans_full*)
show *inj_on f* (*topspace* (*subtopology* (*Euclidean_space n*) *U*))
using *assms openin_subset topspace_subtopology_subset* **by** *fastforce*
qed

corollary *invariance_of_domain_Euclidean_space_gen*:
assumes $n \leq m$ **and** *U*: *openin* (*Euclidean_space m*) *U*
and *cmf*: *continuous_map*(*subtopology* (*Euclidean_space m*) *U*) (*Euclidean_space n*) *f*
and *inj_on f U*
shows *openin* (*Euclidean_space n*) (*f ‘ U*)
proof −
have ∗: *Euclidean_space n* = *subtopology* (*Euclidean_space m*) (*topspace*(*Euclidean_space n*))
by (*metis Euclidean_space_def* ‹$n \leq m$› *inf.absorb_iff2 subset_Euclidean_space subtopology_subtopology topspace_Euclidean_space*)
then have *openin* (*Euclidean_space m*) (*f ‘ U*)
by (*metis* ∗ *U assms*(*4*) *cmf continuous_map_in_subtopology invariance_of_domain_Euclidean_spac*
moreover have $U \subseteq$ *topspace* (*subtopology* (*Euclidean_space m*) *U*)
by (*metis U inf.absorb_iff2 openin_subset openin_subtopology openin_topspace*)
ultimately show *?thesis*
by (*metis* ∗ *cmf continuous_map_image_subset_topspace dual_order.antisym openin_imp_subset openin_topspace subset_openin_subtopology*)
qed

corollary *invariance_of_domain_Euclidean_space_embedding_map_gen*:
assumes $n \leq m$ **and** *U*: *openin* (*Euclidean_space m*) *U*
and *cmf*: *continuous_map*(*subtopology* (*Euclidean_space m*) *U*) (*Euclidean_space n*) *f*
and *inj_on f U*
shows *embedding_map*(*subtopology* (*Euclidean_space m*) *U*) (*Euclidean_space n*) *f*
proof (*rule injective_open_imp_embedding_map* [*OF cmf*])
show *open_map* (*subtopology* (*Euclidean_space m*) *U*) (*Euclidean_space n*) *f*
by (*meson U* ‹$n \leq m$› ‹*inj_on f U*› *cmf continuous_map_from_subtopology_mono invariance_of_domain_Euclidean_space_gen open_map_def openin_open_subtopology inj_on_subset*)
show *inj_on f* (*topspace* (*subtopology* (*Euclidean_space m*) *U*))
using *assms openin_subset topspace_subtopology_subset* **by** *fastforce*
qed

0.4.3 Relating two variants of Euclidean space, one within product topology.

proposition *homeomorphic_maps_Euclidean_space_euclidean_gen_OLD*:
fixes *B* :: *'n::euclidean_space set*
assumes *finite B independent B* **and** *orth*: *pairwise orthogonal B* **and** *n*: *card B = n*
obtains *f g* **where** *homeomorphic_maps* (*Euclidean_space n*) (*top_of_set* (*span B*)) *f g*
proof −
note *representation_basis* [*OF* ‹*independent B*›, *simp*]
obtain *b* **where** *injb*: *inj_on b* {..<*n*} **and** *beq*: *b* ‘ {..<*n*} = *B*
using *finite_imp_nat_seg_image_inj_on* [*OF* ‹*finite B*›]
by (*metis n card_Collect_less_nat card_image lessThan_def*)
then have *biB*: $\bigwedge i.\ i < n \Longrightarrow b\ i \in B$
by *force*
have *repr*: $\bigwedge v.\ v \in span\ B \Longrightarrow (\sum i<n.\ representation\ B\ v\ (b\ i) *_R\ b\ i) = v$
using *real_vector.sum_representation_eq* [*OF* ‹*independent B*› _ ‹*finite B*›]
by (*metis* (*no_types, lifting*) *injb beq order_refl sum.reindex_cong*)
let $?f = \lambda x.\ \sum i<n.\ x\ i *_R\ b\ i$
let $?g = \lambda v\ i.$ *if* $i < n$ *then representation B v* (*b i*) *else 0*
show *thesis*
proof
show *homeomorphic_maps* (*Euclidean_space n*) (*top_of_set* (*span B*)) *?f ?g*
unfolding *homeomorphic_maps_def*
proof (*intro conjI*)
have ∗: *continuous_map euclidean* (*top_of_set* (*span B*)) *?f*
by (*metis* (*mono_tags*) *biB continuous_map_span_sum lessThan_iff*)
show *continuous_map* (*Euclidean_space n*) (*top_of_set* (*span B*)) *?f*
unfolding *Euclidean_space_def*
by (*rule continuous_map_from_subtopology*) (*simp add*: *euclidean_product_topology* ∗)
show *continuous_map* (*top_of_set* (*span B*)) (*Euclidean_space n*) *?g*
unfolding *Euclidean_space_def*
by (*auto simp*: *continuous_map_in_subtopology continuous_map_componentwise_UNIV continuous_on_representation* ‹*independent B*› *biB orth pairwise_orthogonal_imp_finite*)
have [*simp*]: $\bigwedge x\ i.\ i<n \Longrightarrow x\ i *_R\ b\ i \in span\ B$
by (*simp add*: *biB span_base span_scale*)
have *representation B* (*?f x*) (*b j*) = *x j*
if *0*: $\forall i\geq n.\ x\ i = (0::real)$ **and** $j < n$ **for** *x j*
proof −
have *representation B* (*?f x*) (*b j*) = $(\sum i<n.\ representation\ B\ (x\ i *_R\ b\ i)\ (b\ j))$
by (*subst real_vector.representation_sum*) (*auto simp add*: ‹*independent B*›)
also have ... = $(\sum i<n.\ x\ i * representation\ B\ (b\ i)\ (b\ j))$
by (*simp add*: *assms*(*2*) *biB representation_scale span_base*)
also have ... = $(\sum i<n.$ *if* $b\ j = b\ i$ *then* $x\ i$ *else* $0)$
by (*simp add*: *biB if_distrib cong*: *if_cong*)
also have ... = *x j*

```
        using that inj_on_eq_iff [OF injb] by auto
      finally show ?thesis .
    qed
    then show ∀x∈topspace (Euclidean_space n). ?g (?f x) = x
      by (auto simp: Euclidean_space_def)
    show ∀y∈topspace (top_of_set (span B)). ?f (?g y) = y
      using repr by (auto simp: Euclidean_space_def)
  qed
 qed
qed

proposition homeomorphic_maps_Euclidean_space_euclidean_gen:
 fixes B :: 'n::euclidean_space set
 assumes independent B and orth: pairwise orthogonal B and n: card B = n
   and 1: ⋀u. u ∈ B ⟹ norm u = 1
 obtains f g where homeomorphic_maps (Euclidean_space n) (top_of_set (span B)) f g
   and ⋀x. x ∈ topspace (Euclidean_space n) ⟹ (norm (f x))^2 = (∑i<n. (x i)^2)
proof −
 note representation_basis [OF ‹independent B›, simp]
 have finite B
   using ‹independent B› finiteI_independent by metis
 obtain b where injb: inj_on b {..<n} and beq: b ` {..<n} = B
   using finite_imp_nat_seg_image_inj_on [OF ‹finite B›]
   by (metis n card_Collect_less_nat card_image lessThan_def)
 then have biB: ⋀i. i < n ⟹ b i ∈ B
   by force
 have 0 ∉ B
   using ‹independent B› dependent_zero by blast
 have [simp]: b i · b j = (if j = i then 1 else 0)
   if i < n j < n for i j
 proof (cases i = j)
   case True
   with 1 that show ?thesis
     by (auto simp: norm_eq_sqrt_inner biB)
 next
   case False
   then have b i ≠ b j
     by (meson inj_onD injb lessThan_iff that)
   then show ?thesis
     using orth by (auto simp: orthogonal_def pairwise_def norm_eq_sqrt_inner that biB)
 qed
 have [simp]: ⋀x i. i<n ⟹ x i *R b i ∈ span B
   by (simp add: biB span_base span_scale)
 have repr: ⋀v. v ∈ span B ⟹ (∑i<n. representation B v (b i) *R b i) = v
   using real_vector.sum_representation_eq [OF ‹independent B› _ ‹finite B›]
   by (metis (no_types, lifting) injb beq order_refl sum.reindex_cong)
```

```
    define f where f ≡ λx. ∑ i<n. x i *R b i
    define g where g ≡ λv i. if i < n then representation B v (b i) else 0
  show thesis
  proof
    show homeomorphic_maps (Euclidean_space n) (top_of_set (span B)) f g
      unfolding homeomorphic_maps_def
    proof (intro conjI)
      have *: continuous_map euclidean (top_of_set (span B)) f
        unfolding f_def
        by (rule continuous_map_span_sum) (use biB ‹0 ∉ B› in auto)
      show continuous_map (Euclidean_space n) (top_of_set (span B)) f
        unfolding Euclidean_space_def
      by (rule continuous_map_from_subtopology) (simp add: euclidean_product_topology
*)
      show continuous_map (top_of_set (span B)) (Euclidean_space n) g
        unfolding Euclidean_space_def g_def
      by (auto simp: continuous_map_in_subtopology continuous_map_componentwise_UNIV
continuous_on_representation ‹independent B› biB orth pairwise_orthogonal_imp_finite)
      have representation B (f x) (b j) = x j
        if 0: ∀ i≥n. x i = (0::real) and j < n for x j
      proof −
        have representation B (f x) (b j) = (∑ i<n. representation B (x i *R b i)
(b j))
          unfolding f_def
           by (subst real_vector.representation_sum) (auto simp add: ‹independent
B›)
        also have ... = (∑ i<n. x i * representation B (b i) (b j))
          by (simp add: ‹independent B› biB representation_scale span_base)
        also have ... = (∑ i<n. if b j = b i then x i else 0)
          by (simp add: biB if_distrib cong: if_cong)
        also have ... = x j
          using that inj_on_eq_iff [OF injb] by auto
        finally show ?thesis .
      qed
      then show ∀ x∈topspace (Euclidean_space n). g (f x) = x
        by (auto simp: Euclidean_space_def f_def g_def)
      show ∀ y∈topspace (top_of_set (span B)). f (g y) = y
        using repr by (auto simp: Euclidean_space_def f_def g_def)
    qed
   show normeq: (norm (f x))^2 = (∑ i<n. (x i)^2) if x ∈ topspace (Euclidean_space
n) for x
      unfolding f_def  dot_square_norm [symmetric]
     by (simp add: power2_eq_square inner_sum_left inner_sum_right if_distrib
biB cong: if_cong)
  qed
qed

corollary homeomorphic_maps_Euclidean_space_euclidean:
  obtains f :: (nat ⇒ real) ⇒ 'n::euclidean_space and g
```

where *homeomorphic_maps* (*Euclidean_space* (*DIM*(*'n*))) *euclidean f g*
by (*force intro*: *homeomorphic_maps_Euclidean_space_euclidean_gen* [*OF independent_Basis orthogonal_Basis refl norm_Basis*])

lemma *homeomorphic_maps_nsphere_euclidean_sphere*:
fixes *B* :: *'n::euclidean_space set*
assumes *B*: *independent B* **and** *orth*: *pairwise orthogonal B* **and** *n*: *card B* = *n* **and** *n* ≠ *0*
and *1*: ⋀*u*. *u* ∈ *B* ⟹ *norm u* = *1*
obtains *f* :: (*nat* ⇒ *real*) ⇒ *'n::euclidean_space* **and** *g*
where *homeomorphic_maps* (*nsphere*(*n* − *1*)) (*top_of_set* (*sphere 0 1* ∩ *span B*)) *f g*
proof −
have *finite B*
using ‹*independent B*› *finiteI_independent* **by** *metis*
obtain *f g* **where** *fg*: *homeomorphic_maps* (*Euclidean_space n*) (*top_of_set* (*span B*)) *f g*
and *normf*: ⋀*x*. *x* ∈ *topspace* (*Euclidean_space n*) ⟹ (*norm* (*f x*))^2 = (∑ *i*<*n*. (*x i*)^2)
using *homeomorphic_maps_Euclidean_space_euclidean_gen* [*OF B orth n 1*]
by *blast*
obtain *b* **where** *injb*: *inj_on b* {..<*n*} **and** *beq*: *b* ‘ {..<*n*} = *B*
using *finite_imp_nat_seg_image_inj_on* [*OF* ‹*finite B*›]
by (*metis n card_Collect_less_nat card_image lessThan_def*)
then have *biB*: ⋀*i*. *i* < *n* ⟹ *b i* ∈ *B*
by *force*
have [*simp*]: ⋀*i*. *i* < *n* ⟹ *b i* ≠ *0*
using ‹*independent B*› *biB dependent_zero* **by** *fastforce*
have [*simp*]: *b i* · *b j* = (*if j* = *i then* (*norm* (*b i*))^2 *else 0*)
if *i* < *n j* < *n* **for** *i j*
proof (*cases i* = *j*)
case *False*
then have *b i* ≠ *b j*
by (*meson inj_onD injb lessThan_iff that*)
then show *?thesis*
using *orth* **by** (*auto simp*: *orthogonal_def pairwise_def norm_eq_sqrt_inner that biB*)
qed (*auto simp*: *norm_eq_sqrt_inner*)
have [*simp*]: *Suc* (*n* − *Suc 0*) = *n*
using *Suc_pred* ‹*n* ≠ *0*› **by** *blast*
then have [*simp*]: {..*card B* − *Suc 0*} = {..<*card B*}
using *n* **by** *fastforce*
show *thesis*
proof
have *1*: *norm* (*f x*) = *1*
if (∑ *i*<*card B*. (*x i*)^2) = (*1::real*) *x* ∈ *topspace* (*Euclidean_space n*) **for** *x*
proof −
have *norm* (*f x*)^*2* = *1*
using *normf that* **by** (*simp add*: *n*)

with *that* **show** *?thesis*
by (*simp add*: *power2_eq_imp_eq*)
qed
have *homeomorphic_maps* (*nsphere* (*n* − *1*)) (*top_of_set* (*span B* ∩ *sphere 0 1*)) *f g*
unfolding *nsphere_def subtopology_subtopology* [*symmetric*]
proof (*rule homeomorphic_maps_subtopologies_alt*)
show *homeomorphic_maps* (*Euclidean_space* (*Suc* (*n* − *1*))) (*top_of_set* (*span B*)) *f g*
using *fg* **by** (*force simp add*:)
show *f* ‘ (*topspace* (*Euclidean_space* (*Suc* (*n* − *1*))) ∩ {*x*. ($\sum$ *i*≤*n* − *1*. (*x i*)^2) = *1*}) ⊆ *sphere 0 1*
using *n* **by** (*auto simp*: *image_subset_iff Euclidean_space_def 1*)
have ($\sum$ *i*≤*n* − *Suc 0*. (*g u i*)^2) = *1*
if *u* ∈ *span B* **and** *norm* (*u*::′*n*) = *1* **for** *u*
proof −
obtain *v* **where** [*simp*]: *u* = *f v v* ∈ *topspace* (*Euclidean_space n*)
using *fg* **unfolding** *homeomorphic_maps_map subset_iff*
by (*metis* ‹*u* ∈ *span B*› *homeomorphic_imp_surjective_map image_eqI topspace_euclidean_subtopology*)
then have [*simp*]: *g* (*f v*) = *v*
by (*meson fg homeomorphic_maps_map*)
have *fv21*: *norm* (*f v*) ^ *2* = *1*
using *that* **by** *simp*
show *?thesis*
using *that normf fv21* ‹*v* ∈ *topspace* (*Euclidean_space n*)› *n* **by** *force*
qed
then show *g* ‘ (*topspace* (*top_of_set* (*span B*)) ∩ *sphere 0 1*) ⊆ {*x*. ($\sum$ *i*≤*n* − *1*. (*x i*)^2) = *1*}
by *auto*
qed
then show *homeomorphic_maps* (*nsphere*(*n* − *1*)) (*top_of_set* (*sphere 0 1* ∩ *span B*)) *f g*
by (*simp add*: *inf_commute*)
qed
qed

0.4.4 Invariance of dimension and domain

lemma *homeomorphic_maps_iff_homeomorphism* [*simp*]:
homeomorphic_maps (*top_of_set S*) (*top_of_set T*) *f g* ⟷ *homeomorphism S T f g*
by (*force simp*: *Pi_iff homeomorphic_maps_def homeomorphism_def*)

lemma *homeomorphic_space_iff_homeomorphic* [*simp*]:
(*top_of_set S*) *homeomorphic_space* (*top_of_set T*) ⟷ *S homeomorphic T*
by (*simp add*: *homeomorphic_def homeomorphic_space_def*)

lemma *homeomorphic_subspace_Euclidean_space*:

fixes S :: 'a::euclidean_space set
assumes subspace S
shows top_of_set S homeomorphic_space Euclidean_space n ⟷ dim S = n
proof −
obtain B **where** B: B ⊆ S independent B span B = S card B = dim S
and orth: pairwise orthogonal B **and** 1: ⋀x. x ∈ B ⟹ norm x = 1
by (metis assms orthonormal_basis_subspace)
then have finite B
by (simp add: pairwise_orthogonal_imp_finite)
have top_of_set S homeomorphic_space top_of_set (span B)
unfolding homeomorphic_space_iff_homeomorphic
by (auto simp: assms B intro: homeomorphic_subspaces)
also have … homeomorphic_space Euclidean_space (dim S)
unfolding homeomorphic_space_def
using homeomorphic_maps_Euclidean_space_euclidean_gen [OF ‹independent B› orth] homeomorphic_maps_sym 1 B
by metis
finally have top_of_set S homeomorphic_space Euclidean_space (dim S) **.**
then show ?thesis
using homeomorphic_space_sym homeomorphic_space_trans invariance_of_dimension_Euclidean_
by blast
qed

lemma homeomorphic_subspace_Euclidean_space_dim:
fixes S :: 'a::euclidean_space set
assumes subspace S
shows top_of_set S homeomorphic_space Euclidean_space (dim S)
by (simp add: homeomorphic_subspace_Euclidean_space assms)

lemma homeomorphic_subspaces_eq:
fixes S T:: 'a::euclidean_space set
assumes subspace S subspace T
shows S homeomorphic T ⟷ dim S = dim T
proof
show dim S = dim T
if S homeomorphic T
proof −
have Euclidean_space (dim S) homeomorphic_space top_of_set S
using ‹subspace S› homeomorphic_space_sym homeomorphic_subspace_Euclidean_space_dim **by** blast
also have … homeomorphic_space top_of_set T
by (simp add: that)
also have … homeomorphic_space Euclidean_space (dim T)
by (simp add: homeomorphic_subspace_Euclidean_space assms)
finally have Euclidean_space (dim S) homeomorphic_space Euclidean_space (dim T) **.**
then show ?thesis
by (simp add: invariance_of_dimension_Euclidean_space)
qed

```
next
  show S homeomorphic T
    if dim S = dim T
    by (metis that assms homeomorphic_subspaces)
qed

lemma homeomorphic_affine_Euclidean_space:
  assumes affine S
  shows top_of_set S homeomorphic_space Euclidean_space n ⟷ aff_dim S =
n
   (is ?X homeomorphic_space ?E ⟷ aff_dim S = n)
proof (cases S = {})
  case True
  with assms show ?thesis
    using homeomorphic_empty_space nontrivial_Euclidean_space by fastforce
next
  case False
  then obtain a where a ∈ S
    by force
  have (?X homeomorphic_space ?E)
       = (top_of_set (image (λx. −a + x) S) homeomorphic_space ?E)
  proof
    show top_of_set ((+) (− a) ‘ S) homeomorphic_space ?E
      if ?X homeomorphic_space ?E
      using that
     by (meson homeomorphic_space_iff_homeomorphic homeomorphic_space_sym
homeomorphic_space_trans homeomorphic_translation)
    show ?X homeomorphic_space ?E
      if top_of_set ((+) (− a) ‘ S) homeomorphic_space ?E
      using that
     by (meson homeomorphic_space_iff_homeomorphic homeomorphic_space_trans
homeomorphic_translation)
  qed
  also have ... ⟷ aff_dim S = n
   by (metis ‹a ∈ S› aff_dim_eq_dim affine_diffs_subspace affine_hull_eq assms
homeomorphic_subspace_Euclidean_space of_nat_eq_iff)
  finally show ?thesis .
qed


corollary invariance_of_domain_subspaces:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes ope: openin (top_of_set U) S
      and subspace U subspace V and VU: dim V ≤ dim U
      and contf: continuous_on S f and fim: f ∈ S → V
      and injf: inj_on f S
    shows openin (top_of_set V) (f ‘ S)
proof −
  have S ⊆ U
```

```
    using openin_imp_subset [OF ope] .
  have Uhom: top_of_set U homeomorphic_space Euclidean_space (dim U)
   and Vhom: top_of_set V homeomorphic_space Euclidean_space (dim V)
    by (simp_all add: assms homeomorphic_subspace_Euclidean_space_dim)
 then obtain φ φ' where hom: homeomorphic_maps (top_of_set U) (Euclidean_space
(dim U)) φ φ'
    by (auto simp: homeomorphic_space_def)
  obtain ψ ψ' where ψ: homeomorphic_map (top_of_set V) (Euclidean_space
(dim V)) ψ
             and ψ'ψ: ∀x∈V. ψ' (ψ x) = x
   using Vhom by (auto simp: homeomorphic_space_def homeomorphic_maps_map)
  have ((ψ ∘ f ∘ φ') o φ) ` S = (ψ o f) ` S
  proof (rule image_cong [OF refl])
    show (ψ ∘ f ∘ φ' ∘ φ) x = (ψ ∘ f) x if x ∈ S for x
      using that unfolding o_def
     by (metis ‹S ⊆ U› hom homeomorphic_maps_map in_mono topspace_euclidean_subtopology)
  qed
  moreover
  have openin (Euclidean_space (dim V)) ((ψ ∘ f ∘ φ') ` φ ` S)
  proof (rule invariance_of_domain_Euclidean_space_gen [OF VU])
    show openin (Euclidean_space (dim U)) (φ ` S)
      using homeomorphic_map_openness_eq hom homeomorphic_maps_map ope
by blast
   show continuous_map (subtopology (Euclidean_space (dim U)) (φ ` S)) (Euclidean_space
(dim V)) (ψ ∘ f ∘ φ')
    proof (intro continuous_map_compose)
      have continuous_on ({x. ∀i≥dim U. x i = 0} ∩ φ ` S) φ'
        if continuous_on {x. ∀i≥dim U. x i = 0} φ'
        using that by (force elim: continuous_on_subset)
      moreover have φ' ∈ ({x. ∀i≥dim U. x i = 0} ∩ φ ` S) → S
        if ∀x∈U. φ' (φ x) = x
        using that ‹S ⊆ U› by fastforce
      ultimately show continuous_map (subtopology (Euclidean_space (dim U))
(φ ` S)) (top_of_set S) φ'
        using hom unfolding homeomorphic_maps_def
      by (simp add: Euclidean_space_def subtopology_subtopology euclidean_product_topology)
      show continuous_map (top_of_set S) (top_of_set V) f
        by (simp add: contf fim)
      show continuous_map (top_of_set V) (Euclidean_space (dim V)) ψ
        by (simp add: ψ homeomorphic_imp_continuous_map)
    qed
    show inj_on (ψ ∘ f ∘ φ') (φ ` S)
      using injf hom ‹S ⊆ U› ψ'ψ fim
     by (simp add: inj_on_def homeomorphic_maps_map Pi_iff) (metis subsetD)
  qed
  ultimately have openin (Euclidean_space (dim V)) (ψ ` f ` S)
    by (simp add: image_comp)
  with fim show ?thesis
    by (auto simp: homeomorphic_map_openness_eq [OF ψ])
```

qed

lemma *invariance_of_domain*:
  fixes *f* :: *'a ⇒ 'a::euclidean_space*
  assumes *continuous_on S f open S inj_on f S* **shows** *open(f ' S)*
  using *invariance_of_domain_subspaces* [*of UNIV S UNIV*] *assms* **by** (*force simp add*:)

corollary *invariance_of_dimension_subspaces*:
  fixes *f* :: *'a::euclidean_space ⇒ 'b::euclidean_space*
  assumes *ope*: *openin* (*top_of_set U*) *S*
    and *subspace U subspace V*
    and *contf*: *continuous_on S f* **and** *fim*: *f ' S ⊆ V*
    and *injf*: *inj_on f S* **and** *S ≠ {}*
  shows *dim U ≤ dim V*
proof −
  have *False* **if** *dim V < dim U*
  proof −
    obtain *T* **where** *subspace T T ⊆ U dim T = dim V*
      using *choose_subspace_of_subspace* [*of dim V U*]
      by (*metis* ‹*dim V < dim U*› *assms*(*2*) *order.strict_implies_order span_eq_iff*)
    then have *V homeomorphic T*
      by (*simp add*: ‹*subspace V*› *homeomorphic_subspaces*)
    then obtain *h k* **where** *homhk*: *homeomorphism V T h k*
      using *homeomorphic_def* **by** *blast*
    have *continuous_on S* (*h ∘ f*)
      by (*meson contf continuous_on_compose continuous_on_subset fim homeomorphism_cont1 homhk*)
    moreover have (*h ∘ f*) *' S ⊆ U*
      using ‹*T ⊆ U*› *fim homeomorphism_image1 homhk* **by** *fastforce*
    moreover have *inj_on* (*h ∘ f*) *S*
      apply (*clarsimp simp*: *inj_on_def*)
      by (*metis fim homeomorphism_apply1 homhk image_subset_iff inj_onD injf*)
    ultimately have *ope_hf*: *openin* (*top_of_set U*) ((*h ∘ f*) *' S*)
      using *invariance_of_domain_subspaces* [*OF ope* ‹*subspace U*› ‹*subspace U*›] **by** *blast*
    have (*h ∘ f*) *' S ⊆ T*
      using *fim homeomorphism_image1 homhk* **by** *fastforce*
    then have *dim* ((*h ∘ f*) *' S*) *≤ dim T*
      by (*rule dim_subset*)
    also have *dim* ((*h ∘ f*) *' S*) = *dim U*
      using ‹*S ≠ {}*› ‹*subspace U*›
      by (*blast intro*: *dim_openin ope_hf*)
    finally show *False*
      using ‹*dim V < dim U*› ‹*dim T = dim V*› **by** *simp*
  qed
  then show *?thesis*
    using *not_less* **by** *blast*
qed

corollary *invariance_of_domain_affine_sets*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V aff_dim V* ≤ *aff_dim U*
and *contf*: *continuous_on S f* **and** *fim*: *f* ‘ *S* ⊆ *V*
and *injf*: *inj_on f S*
shows *openin* (*top_of_set V*) (*f* ‘ *S*)
proof (*cases S* = {})
case *False*
obtain *a b* **where** *a* ∈ *S a* ∈ *U b* ∈ *V*
using *False fim ope openin_contains_cball* **by** *fastforce*
have *openin* (*top_of_set* ((+) (− *b*) ‘ *V*)) (((+) (− *b*) ∘ *f* ∘ (+) *a*) ‘ (+) (− *a*) ‘ *S*)
proof (*rule invariance_of_domain_subspaces*)
show *openin* (*top_of_set* ((+) (− *a*) ‘ *U*)) ((+) (− *a*) ‘ *S*)
by (*metis ope homeomorphism_imp_open_map homeomorphism_translation translation_galois*)
show *subspace* ((+) (− *a*) ‘ *U*)
by (*simp add*: ‹*a* ∈ *U*› *affine_diffs_subspace_subtract* ‹*affine U*› *cong*: *image_cong_simp*)
show *subspace* ((+) (− *b*) ‘ *V*)
by (*simp add*: ‹*b* ∈ *V*› *affine_diffs_subspace_subtract* ‹*affine V*› *cong*: *image_cong_simp*)
show *dim* ((+) (− *b*) ‘ *V*) ≤ *dim* ((+) (− *a*) ‘ *U*)
by (*metis* ‹*a* ∈ *U*› ‹*b* ∈ *V*› *aff_dim_eq_dim affine_hull_eq aff of_nat_le_iff*)
show *continuous_on* ((+) (− *a*) ‘ *S*) ((+) (− *b*) ∘ *f* ∘ (+) *a*)
by (*metis contf continuous_on_compose homeomorphism_cont2 homeomorphism_translation translation_galois*)
show ((+) (− *b*) ∘ *f* ∘ (+) *a*) ∈ (+) (− *a*) ‘ *S* → (+) (− *b*) ‘ *V*
using *fim* **by** *auto*
show *inj_on* ((+) (− *b*) ∘ *f* ∘ (+) *a*) ((+) (− *a*) ‘ *S*)
by (*auto simp*: *inj_on_def*) (*meson inj_onD injf*)
qed
then show *?thesis*
by (*metis* (*no_types*, *lifting*) *homeomorphism_imp_open_map homeomorphism_translation image_comp translation_galois*)
qed *auto*

corollary *invariance_of_dimension_affine_sets*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V*
and *contf*: *continuous_on S f* **and** *fim*: *f* ‘ *S* ⊆ *V*
and *injf*: *inj_on f S* **and** *S* ≠ {}
shows *aff_dim U* ≤ *aff_dim V*
proof −
obtain *a b* **where** *a* ∈ *S a* ∈ *U b* ∈ *V*
using ‹*S* ≠ {}› *fim ope openin_contains_cball* **by** *fastforce*

```
  have dim ((+) (- a) ` U) ≤ dim ((+) (- b) ` V)
  proof (rule invariance_of_dimension_subspaces)
    show openin (top_of_set ((+) (- a) ` U)) ((+) (- a) ` S)
      by (metis ope homeomorphism_imp_open_map homeomorphism_translation
translation_galois)
    show subspace ((+) (- a) ` U)
        by (simp add: ‹a ∈ U› affine_diffs_subspace_subtract ‹affine U› cong:
image_cong_simp)
    show subspace ((+) (- b) ` V)
        by (simp add: ‹b ∈ V› affine_diffs_subspace_subtract ‹affine V› cong:
image_cong_simp)
    show continuous_on ((+) (- a) ` S) ((+) (- b) ∘ f ∘ (+) a)
      by (metis contf continuous_on_compose homeomorphism_cont2 homeomor-
phism_translation translation_galois)
    show ((+) (- b) ∘ f ∘ (+) a) ` (+) (- a) ` S ⊆ (+) (- b) ` V
      using fim by auto
    show inj_on ((+) (- b) ∘ f ∘ (+) a) ((+) (- a) ` S)
      by (auto simp: inj_on_def) (meson inj_onD injf)
  qed (use ‹S ≠ {}› in auto)
  then show ?thesis
   by (metis ‹a ∈ U› ‹b ∈ V› aff_dim_eq_dim affine_hull_eq aff of_nat_le_iff)
qed

corollary invariance_of_dimension:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on S f and open S
      and injf: inj_on f S and S ≠ {}
    shows DIM('a) ≤ DIM('b)
  using invariance_of_dimension_subspaces [of UNIV S UNIV f] assms
  by auto

corollary continuous_injective_image_subspace_dim_le:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes subspace S subspace T
      and contf: continuous_on S f and fim: f ` S ⊆ T
      and injf: inj_on f S
    shows dim S ≤ dim T
  apply (rule invariance_of_dimension_subspaces [of S S _ f])
  using assms by (auto simp: subspace_affine)

lemma invariance_of_dimension_convex_domain:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes convex S
      and contf: continuous_on S f and fim: f ` S ⊆ affine hull T
      and injf: inj_on f S
    shows aff_dim S ≤ aff_dim T
proof (cases S = {})
  case True
  then show ?thesis by (simp add: aff_dim_geq)
```

next
  case *False*
  have *aff_dim* (*affine hull S*) ≤ *aff_dim* (*affine hull T*)
  proof (*rule invariance_of_dimension_affine_sets*)
    show *openin* (*top_of_set* (*affine hull S*)) (*rel_interior S*)
      by (*simp add*: *openin_rel_interior*)
    show *continuous_on* (*rel_interior S*) *f*
      using *contf continuous_on_subset rel_interior_subset* **by** *blast*
    show *f* ‘ *rel_interior S* ⊆ *affine hull T*
      using *fim rel_interior_subset* **by** *blast*
    show *inj_on f* (*rel_interior S*)
      using *inj_on_subset injf rel_interior_subset* **by** *blast*
    show *rel_interior S* ≠ {}
      by (*simp add*: *False* ‹*convex S*› *rel_interior_eq_empty*)
  qed *auto*
  then show *?thesis*
    by *simp*
qed

lemma *homeomorphic_convex_sets_le*:
  assumes *convex S S homeomorphic T*
  shows *aff_dim S* ≤ *aff_dim T*
proof −
  obtain *h k* **where** *homhk*: *homeomorphism S T h k*
    using *homeomorphic_def assms* **by** *blast*
  show *?thesis*
  proof (*rule invariance_of_dimension_convex_domain* [*OF* ‹*convex S*›])
    show *continuous_on S h*
      using *homeomorphism_def homhk* **by** *blast*
    show *h* ‘ *S* ⊆ *affine hull T*
      by (*metis homeomorphism_def homhk hull_subset*)
    show *inj_on h S*
      by (*meson homeomorphism_apply1 homhk inj_on_inverseI*)
  qed
qed

lemma *homeomorphic_convex_sets*:
  assumes *convex S convex T S homeomorphic T*
  shows *aff_dim S* = *aff_dim T*
 by (*meson assms dual_order.antisym homeomorphic_convex_sets_le homeomorphic_sym*)

lemma *homeomorphic_convex_compact_sets_eq*:
  assumes *convex S compact S convex T compact T*
  shows *S homeomorphic T* ⟷ *aff_dim S* = *aff_dim T*
 by (*meson assms homeomorphic_convex_compact_sets homeomorphic_convex_sets*)

lemma *invariance_of_domain_gen*:
  fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*

```
  assumes open S continuous_on S f inj_on f S DIM('b) ≤ DIM('a)
    shows open(f ` S)
  using invariance_of_domain_subspaces [of UNIV S UNIV f] assms by auto

lemma injective_into_1d_imp_open_map_UNIV:
  fixes f :: 'a::euclidean_space ⇒ real
  assumes open T continuous_on S f inj_on f S T ⊆ S
    shows open (f ` T)
  apply (rule invariance_of_domain_gen [OF ‹open T›])
  using assms apply (auto simp: elim: continuous_on_subset inj_on_subset)
  done

lemma continuous_on_inverse_open:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes open S continuous_on S f DIM('b) ≤ DIM('a) and gf: ⋀x. x ∈ S ⟹
g(f x) = x
    shows continuous_on (f ` S) g
proof (clarsimp simp add: continuous_openin_preimage_eq)
  fix T :: 'a set
  assume open T
  have eq: f ` S ∩ g -` T = f ` (S ∩ T)
    by (auto simp: gf)
  have openin (top_of_set (f ` S)) (f ` (S ∩ T))
  proof (rule open_openin_trans [OF invariance_of_domain_gen])
    show inj_on f S
      using inj_on_inverseI gf by auto
    show open (f ` (S ∩ T))
     by (meson ‹inj_on f S› ‹open T› assms(1-3) continuous_on_subset inf_le1
inj_on_subset invariance_of_domain_gen open_Int)
  qed (use assms in auto)
  then show openin (top_of_set (f ` S)) (f ` S ∩ g -` T)
    by (simp add: eq)
qed

lemma invariance_of_domain_homeomorphism:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes open S continuous_on S f DIM('b) ≤ DIM('a) inj_on f S
  obtains g where homeomorphism S (f ` S) f g
proof
  show homeomorphism S (f ` S) f (inv_into S f)
    by (simp add: assms continuous_on_inverse_open homeomorphism_def)
qed

corollary invariance_of_domain_homeomorphic:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes open S continuous_on S f DIM('b) ≤ DIM('a) inj_on f S
  shows S homeomorphic (f ` S)
  using invariance_of_domain_homeomorphism [OF assms]
  by (meson homeomorphic_def)
```

```
lemma continuous_image_subset_interior:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes continuous_on S f inj_on f S DIM('b) ≤ DIM('a)
  shows f ` (interior S) ⊆ interior(f ` S)
proof (rule interior_maximal)
  show f ` interior S ⊆ f ` S
    by (simp add: image_mono interior_subset)
  show open (f ` interior S)
    using assms
     by (auto simp: inj_on_subset interior_subset continuous_on_subset invari-
ance_of_domain_gen)
qed

lemma homeomorphic_interiors_same_dimension:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T and dimeq: DIM('a) = DIM('b)
  shows (interior S) homeomorphic (interior T)
  using assms [unfolded homeomorphic_minimal]
  unfolding homeomorphic_def
proof (clarify elim!: ex_forward)
  fix f g
  assume S: ∀ x∈S. f x ∈ T ∧ g (f x) = x and T: ∀ y∈T. g y ∈ S ∧ f (g y) = y
     and contf: continuous_on S f and contg: continuous_on T g
  then have fST: f ` S = T and gTS: g ` T = S and inj_on f S inj_on g T
    by (auto simp: inj_on_def intro: rev_image_eqI) metis+
  have fim: f ` interior S ⊆ interior T
     using continuous_image_subset_interior [OF contf ‹inj_on f S›] dimeq fST
by simp
  have gim: g ` interior T ⊆ interior S
    using continuous_image_subset_interior [OF contg ‹inj_on g T›] dimeq gTS
by simp
  show homeomorphism (interior S) (interior T) f g
    unfolding homeomorphism_def
  proof (intro conjI ballI)
    show ⋀x. x ∈ interior S ⟹ g (f x) = x
      by (meson ‹∀ x∈S. f x ∈ T ∧ g (f x) = x› subsetD interior_subset)
    have interior T ⊆ f ` interior S
    proof
      fix x assume x ∈ interior T
      then have g x ∈ interior S
        using gim by blast
      then show x ∈ f ` interior S
        by (metis T ‹x ∈ interior T› image_iff interior_subset subsetCE)
    qed
    then show f ` interior S = interior T
      using fim by blast
    show continuous_on (interior S) f
      by (metis interior_subset continuous_on_subset contf)
```

```
    show ⋀y. y ∈ interior T ⟹ f (g y) = y
      by (meson T subsetD interior_subset)
    have interior S ⊆ g ‘ interior T
    proof
      fix x assume x ∈ interior S
      then have f x ∈ interior T
        using fim by blast
      then show x ∈ g ‘ interior T
        by (metis S ‹x ∈ interior S› image_iff interior_subset subsetCE)
    qed
    then show g ‘ interior T = interior S
      using gim by blast
    show continuous_on (interior T) g
      by (metis interior_subset continuous_on_subset contg)
  qed
qed

lemma homeomorphic_open_imp_same_dimension:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T open S S ≠ {} open T T ≠ {}
  shows DIM('a) = DIM('b)
    using assms
    apply (simp add: homeomorphic_minimal)
    apply (rule order_antisym; metis inj_onI invariance_of_dimension)
    done

proposition homeomorphic_interiors:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T interior S = {} ⟷ interior T = {}
    shows (interior S) homeomorphic (interior T)
proof (cases interior T = {})
  case True
  with assms show ?thesis by auto
next
  case False
  then have DIM('a) = DIM('b)
    using assms
    apply (simp add: homeomorphic_minimal)
   apply (rule order_antisym; metis continuous_on_subset inj_onI inj_on_subset
interior_subset invariance_of_dimension open_interior)
    done
  then show ?thesis
   by (rule homeomorphic_interiors_same_dimension [OF ‹S homeomorphic T›])
qed

lemma homeomorphic_frontiers_same_dimension:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T closed S closed T and dimeq: DIM('a) = DIM('b)
  shows (frontier S) homeomorphic (frontier T)
```

```
  using assms [unfolded homeomorphic_minimal]
  unfolding homeomorphic_def
proof (clarify elim!: ex_forward)
  fix f g
  assume S: ∀x∈S. f x ∈ T ∧ g (f x) = x and T: ∀y∈T. g y ∈ S ∧ f (g y) = y
     and contf: continuous_on S f and contg: continuous_on T g
  then have fST: f ` S = T and gTS: g ` T = S and inj_on f S inj_on g T
    by (auto simp: inj_on_def intro: rev_image_eqI) metis+
  have g ` interior T ⊆ interior S
    using continuous_image_subset_interior [OF contg ‹inj_on g T›] dimeq gTS
by simp
  then have fim: f ` frontier S ⊆ frontier T
    apply (simp add: frontier_def)
    using continuous_image_subset_interior assms(2) assms(3) S by auto
  have f ` interior S ⊆ interior T
     using continuous_image_subset_interior [OF contf ‹inj_on f S›] dimeq fST
by simp
  then have gim: g ` frontier T ⊆ frontier S
    apply (simp add: frontier_def)
    using continuous_image_subset_interior T assms(2) assms(3) by auto
  show homeomorphism (frontier S) (frontier T) f g
    unfolding homeomorphism_def
  proof (intro conjI ballI)
    show gf: ⋀x. x ∈ frontier S ⟹ g (f x) = x
      by (simp add: S assms(2) frontier_def)
    show fg: ⋀y. y ∈ frontier T ⟹ f (g y) = y
      by (simp add: T assms(3) frontier_def)
    have frontier T ⊆ f ` frontier S
    proof
      fix x assume x ∈ frontier T
      then have g x ∈ frontier S
        using gim by blast
      then show x ∈ f ` frontier S
        by (metis fg ‹x ∈ frontier T› imageI)
    qed
    then show f ` frontier S = frontier T
      using fim by blast
    show continuous_on (frontier S) f
     by (metis Diff_subset assms(2) closure_eq contf continuous_on_subset fron-
tier_def)
    have frontier S ⊆ g ` frontier T
    proof
      fix x assume x ∈ frontier S
      then have f x ∈ frontier T
        using fim by blast
      then show x ∈ g ` frontier T
        by (metis gf ‹x ∈ frontier S› imageI)
    qed
    then show g ` frontier T = frontier S
```

```
      using gim by blast
    show continuous_on (frontier T) g
      by (metis Diff_subset assms(3) closure_closed contg continuous_on_subset
frontier_def)
  qed
qed

lemma homeomorphic_frontiers:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T closed S closed T
          interior S = {} ⟷ interior T = {}
    shows (frontier S) homeomorphic (frontier T)
proof (cases interior T = {})
  case True
  then show ?thesis
    by (metis Diff_empty assms closure_eq frontier_def)
next
  case False
  show ?thesis
    apply (rule homeomorphic_frontiers_same_dimension)
       apply (simp_all add: assms)
   using False assms homeomorphic_interiors homeomorphic_open_imp_same_dimension
by blast
qed

lemma continuous_image_subset_rel_interior:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on S f and injf: inj_on f S and fim: f ` S ⊆ T
      and TS: aff_dim T ≤ aff_dim S
  shows f ` (rel_interior S) ⊆ rel_interior(f ` S)
proof (rule rel_interior_maximal)
  show f ` rel_interior S ⊆ f ` S
    by(simp add: image_mono rel_interior_subset)
  show openin (top_of_set (affine hull f ` S)) (f ` rel_interior S)
  proof (rule invariance_of_domain_affine_sets)
    show openin (top_of_set (affine hull S)) (rel_interior S)
      by (simp add: openin_rel_interior)
    show aff_dim (affine hull f ` S) ≤ aff_dim (affine hull S)
      by (metis aff_dim_affine_hull aff_dim_subset fim TS order_trans)
    show f ` rel_interior S ⊆ affine hull f ` S
      by (meson ‹f ` rel_interior S ⊆ f ` S› hull_subset order_trans)
    show continuous_on (rel_interior S) f
      using contf continuous_on_subset rel_interior_subset by blast
    show inj_on f (rel_interior S)
      using inj_on_subset injf rel_interior_subset by blast
  qed auto
qed

lemma homeomorphic_rel_interiors_same_dimension:
```

```
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T and aff: aff_dim S = aff_dim T
  shows (rel_interior S) homeomorphic (rel_interior T)
  using assms [unfolded homeomorphic_minimal]
  unfolding homeomorphic_def
proof (clarify elim!: ex_forward)
  fix f g
  assume S: ∀x∈S. f x ∈ T ∧ g (f x) = x and T: ∀y∈T. g y ∈ S ∧ f (g y) = y
    and contf: continuous_on S f and contg: continuous_on T g
  then have fST: f ` S = T and gTS: g ` T = S and inj_on f S inj_on g T
    by (auto simp: inj_on_def intro: rev_image_eqI) metis+
  have fim: f ` rel_interior S ⊆ rel_interior T
    by (metis ‹inj_on f S› aff contf continuous_image_subset_rel_interior fST
order_refl)
  have gim: g ` rel_interior T ⊆ rel_interior S
    by (metis ‹inj_on g T› aff contg continuous_image_subset_rel_interior gTS
order_refl)
  show homeomorphism (rel_interior S) (rel_interior T) f g
    unfolding homeomorphism_def
  proof (intro conjI ballI)
    show gf: ⋀x. x ∈ rel_interior S ⟹ g (f x) = x
      using S rel_interior_subset by blast
    show fg: ⋀y. y ∈ rel_interior T ⟹ f (g y) = y
      using T mem_rel_interior_ball by blast
    have rel_interior T ⊆ f ` rel_interior S
    proof
      fix x assume x ∈ rel_interior T
      then have g x ∈ rel_interior S
        using gim by blast
      then show x ∈ f ` rel_interior S
        by (metis fg ‹x ∈ rel_interior T› imageI)
    qed
    moreover have f ` rel_interior S ⊆ rel_interior T
      by (metis ‹inj_on f S› aff contf continuous_image_subset_rel_interior fST
order_refl)
    ultimately show f ` rel_interior S = rel_interior T
      by blast
    show continuous_on (rel_interior S) f
      using contf continuous_on_subset rel_interior_subset by blast
    have rel_interior S ⊆ g ` rel_interior T
    proof
      fix x assume x ∈ rel_interior S
      then have f x ∈ rel_interior T
        using fim by blast
      then show x ∈ g ` rel_interior T
        by (metis gf ‹x ∈ rel_interior S› imageI)
    qed
    then show g ` rel_interior T = rel_interior S
      using gim by blast
```

```
    show continuous_on (rel_interior T) g
      using contg continuous_on_subset rel_interior_subset by blast
  qed
qed

lemma homeomorphic_rel_interiors:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T rel_interior S = {} ⟷ rel_interior T = {}
    shows (rel_interior S) homeomorphic (rel_interior T)
proof (cases rel_interior T = {})
  case True
  with assms show ?thesis by auto
next
  case False
  obtain f g
    where S: ∀x∈S. f x ∈ T ∧ g (f x) = x and T: ∀y∈T. g y ∈ S ∧ f (g y) = y
      and contf: continuous_on S f and contg: continuous_on T g
    using  assms [unfolded homeomorphic_minimal] by auto
  have aff_dim (affine hull S) ≤ aff_dim (affine hull T)
    apply (rule invariance_of_dimension_affine_sets [of _ rel_interior S _ f])
          apply (simp_all add: openin_rel_interior False assms)
    using contf continuous_on_subset rel_interior_subset apply blast
     apply (meson S hull_subset image_subsetI rel_interior_subset rev_subsetD)
    apply (metis S inj_on_inverseI inj_on_subset rel_interior_subset)
    done
  moreover have aff_dim (affine hull T) ≤ aff_dim (affine hull S)
    apply (rule invariance_of_dimension_affine_sets [of _ rel_interior T _ g])
          apply (simp_all add: openin_rel_interior False assms)
    using contg continuous_on_subset rel_interior_subset apply blast
     apply (meson T hull_subset image_subsetI rel_interior_subset rev_subsetD)
    apply (metis T inj_on_inverseI inj_on_subset rel_interior_subset)
    done
  ultimately have aff_dim S = aff_dim T by force
  then show ?thesis
    by (rule homeomorphic_rel_interiors_same_dimension [OF ‹S homeomorphic
T›])
qed


lemma homeomorphic_rel_boundaries_same_dimension:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T and aff: aff_dim S = aff_dim T
  shows (S − rel_interior S) homeomorphic (T − rel_interior T)
  using assms [unfolded homeomorphic_minimal]
  unfolding homeomorphic_def
proof (clarify elim!: ex_forward)
  fix f g
  assume S: ∀x∈S. f x ∈ T ∧ g (f x) = x and T: ∀y∈T. g y ∈ S ∧ f (g y) = y
     and contf: continuous_on S f and contg: continuous_on T g
```

```
  then have fST: f ` S = T and gTS: g ` T = S and inj_on f S inj_on g T
    by (auto simp: inj_on_def intro: rev_image_eqI) metis+
  have fim: f ` rel_interior S ⊆ rel_interior T
     by (metis ‹inj_on f S› aff contf continuous_image_subset_rel_interior fST
order_refl)
  have gim: g ` rel_interior T ⊆ rel_interior S
    by (metis ‹inj_on g T› aff contg continuous_image_subset_rel_interior gTS
order_refl)
  show homeomorphism (S − rel_interior S) (T − rel_interior T) f g
    unfolding homeomorphism_def
  proof (intro conjI ballI)
    show gf: ⋀x. x ∈ S − rel_interior S ⟹ g (f x) = x
      using S rel_interior_subset by blast
    show fg: ⋀y. y ∈ T − rel_interior T ⟹ f (g y) = y
      using T mem_rel_interior_ball by blast
    show f ` (S − rel_interior S) = T − rel_interior T
      using S fST fim gim by auto
    show continuous_on (S − rel_interior S) f
      using contf continuous_on_subset rel_interior_subset by blast
    show g ` (T − rel_interior T) = S − rel_interior S
      using T gTS gim fim by auto
    show continuous_on (T − rel_interior T) g
      using contg continuous_on_subset rel_interior_subset by blast
  qed
qed

lemma homeomorphic_rel_boundaries:
  fixes S :: 'a::euclidean_space set and T :: 'b::euclidean_space set
  assumes S homeomorphic T rel_interior S = {} ⟷ rel_interior T = {}
    shows (S − rel_interior S) homeomorphic (T − rel_interior T)
proof (cases rel_interior T = {})
  case True
  with assms show ?thesis by auto
next
  case False
  obtain f g
    where S: ∀x∈S. f x ∈ T ∧ g (f x) = x and T: ∀y∈T. g y ∈ S ∧ f (g y) = y
      and contf: continuous_on S f and contg: continuous_on T g
    using  assms [unfolded homeomorphic_minimal] by auto
  have aff_dim (affine hull S) ≤ aff_dim (affine hull T)
    apply (rule invariance_of_dimension_affine_sets [of _ rel_interior S _ f])
         apply (simp_all add: openin_rel_interior False assms)
    using contf continuous_on_subset rel_interior_subset apply blast
     apply (meson S hull_subset image_subsetI rel_interior_subset rev_subsetD)
    apply (metis S inj_on_inverseI inj_on_subset rel_interior_subset)
    done
  moreover have aff_dim (affine hull T) ≤ aff_dim (affine hull S)
    apply (rule invariance_of_dimension_affine_sets [of _ rel_interior T _ g])
         apply (simp_all add: openin_rel_interior False assms)
```

```
    using contg continuous_on_subset rel_interior_subset apply blast
     apply (meson T hull_subset image_subsetI rel_interior_subset rev_subsetD)
    apply (metis T inj_on_inverseI inj_on_subset rel_interior_subset)
    done
  ultimately have aff_dim S = aff_dim T by force
  then show ?thesis
    by (rule homeomorphic_rel_boundaries_same_dimension [OF ‹S homeomor-
phic T›])
qed

proposition uniformly_continuous_homeomorphism_UNIV_trivial:
  fixes f :: 'a::euclidean_space ⇒ 'a
   assumes contf: uniformly_continuous_on S f and hom: homeomorphism S
UNIV f g
  shows S = UNIV
proof (cases S = {})
  case True
  then show ?thesis
    by (metis UNIV_I hom empty_iff homeomorphism_def image_eqI)
next
  case False
  have inj g
    by (metis UNIV_I hom homeomorphism_apply2 injI)
  then have open (g ` UNIV)
    by (blast intro: invariance_of_domain hom homeomorphism_cont2)
  then have open S
    using hom homeomorphism_image2 by blast
  moreover have complete S
    unfolding complete_def
  proof clarify
    fix σ
    assume σ: ∀ n. σ n ∈ S and Cauchy σ
    have Cauchy (f o σ)
      using uniformly_continuous_imp_Cauchy_continuous ‹Cauchy σ› σ contf
      unfolding Cauchy_continuous_on_def by blast
    then obtain l where (f ∘ σ) ⟶ l
      by (auto simp: convergent_eq_Cauchy [symmetric])
    show ∃ l∈S. σ ⟶ l
    proof
      show g l ∈ S
        using hom homeomorphism_image2 by blast
      have (g ∘ (f ∘ σ)) ⟶ g l
          by (meson UNIV_I ‹(f ∘ σ) ⟶ l› continuous_on_sequentially hom
homeomorphism_cont2)
      then show σ ⟶ g l
      proof -
        have ∀ n. σ n = (g ∘ (f ∘ σ)) n
          by (metis (no_types) σ comp_eq_dest_lhs hom homeomorphism_apply1)
        then show ?thesis
```

```
        by (metis (no_types) LIMSEQ_iff ‹(g ∘ (f ∘ σ)) ⟶ g l›)
      qed
    qed
  qed
  then have closed S
    by (simp add: complete_eq_closed)
  ultimately show ?thesis
    using clopen [of S] False  by simp
qed

proposition invariance_of_domain_sphere_affine_set_gen:
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on S f and injf: inj_on f S and fim: f ` S ⊆ T
      and U: bounded U convex U
      and affine T and affTU: aff_dim T < aff_dim U
      and ope: openin (top_of_set (rel_frontier U)) S
   shows openin (top_of_set T) (f ` S)
proof (cases rel_frontier U = {})
  case True
  then show ?thesis
    using ope openin_subset by force
next
  case False
  obtain b c where b: b ∈ rel_frontier U and c: c ∈ rel_frontier U and b ≠ c
    using ‹bounded U› rel_frontier_not_sing [of U] subset_singletonD False  by
fastforce
  obtain V :: 'a set where affine V and affV: aff_dim V = aff_dim U − 1
  proof (rule choose_affine_subset [OF affine_UNIV])
    show − 1 ≤ aff_dim U − 1
       by (metis aff_dim_empty aff_dim_geq aff_dim_negative_iff affTU diff_0
diff_right_mono not_le)
    show aff_dim U − 1 ≤ aff_dim (UNIV::'a set)
      by (metis aff_dim_UNIV aff_dim_le_DIM le_cases not_le zle_diff1_eq)
  qed auto
  have SU: S ⊆ rel_frontier U
    using ope openin_imp_subset by auto
  have homb: rel_frontier U − {b} homeomorphic V
   and homc: rel_frontier U − {c} homeomorphic V
    using homeomorphic_punctured_sphere_affine_gen [of U _ V]
    by (simp_all add: ‹affine V› affV U b c)
  then obtain g h j k
          where gh: homeomorphism (rel_frontier U − {b}) V g h
            and jk: homeomorphism (rel_frontier U − {c}) V j k
    by (auto simp: homeomorphic_def)
  with SU have hgsub: (h ` g ` (S − {b})) ⊆ S and kjsub: (k ` j ` (S − {c})) ⊆ S
    by (simp_all add: homeomorphism_def subset_eq)
  have [simp]: aff_dim T ≤ aff_dim V
    by (simp add: affTU affV)
  have openin (top_of_set T) ((f ∘ h) ` g ` (S − {b}))
```

proof (*rule invariance_of_domain_affine_sets* [*OF* _ ‹*affine V*›])
show *openin* (*top_of_set V*) (*g* ‘ (*S* − {*b*}))
apply (*rule homeomorphism_imp_open_map* [*OF gh*])
by (*meson Diff_mono Diff_subset SU ope openin_delete openin_subset_trans order_refl*)
show *continuous_on* (*g* ‘ (*S* − {*b*})) (*f* ∘ *h*)
apply (*rule continuous_on_compose*)
apply (*meson Diff_mono SU homeomorphism_def homeomorphism_of_subsets gh set_eq_subset*)
using *contf continuous_on_subset hgsub* **by** *blast*
show *inj_on* (*f* ∘ *h*) (*g* ‘ (*S* − {*b*}))
using *kjsub*
apply (*clarsimp simp add*: *inj_on_def*)
by (*metis SU b homeomorphism_def inj_onD injf insert_Diff insert_iff gh rev_subsetD*)
show (*f* ∘ *h*) ‘ *g* ‘ (*S* − {*b*}) ⊆ *T*
by (*metis fim image_comp image_mono hgsub subset_trans*)
qed (*auto simp*: *assms*)
moreover
have *openin* (*top_of_set T*) ((*f* ∘ *k*) ‘ *j* ‘ (*S* − {*c*}))
proof (*rule invariance_of_domain_affine_sets* [*OF* _ ‹*affine V*›])
show *openin* (*top_of_set V*) (*j* ‘ (*S* − {*c*}))
apply (*rule homeomorphism_imp_open_map* [*OF jk*])
by (*meson Diff_mono Diff_subset SU ope openin_delete openin_subset_trans order_refl*)
show *continuous_on* (*j* ‘ (*S* − {*c*})) (*f* ∘ *k*)
apply (*rule continuous_on_compose*)
apply (*meson Diff_mono SU homeomorphism_def homeomorphism_of_subsets jk set_eq_subset*)
using *contf continuous_on_subset kjsub* **by** *blast*
show *inj_on* (*f* ∘ *k*) (*j* ‘ (*S* − {*c*}))
using *kjsub*
apply (*clarsimp simp add*: *inj_on_def*)
by (*metis SU c homeomorphism_def inj_onD injf insert_Diff insert_iff jk rev_subsetD*)
show (*f* ∘ *k*) ‘ *j* ‘ (*S* − {*c*}) ⊆ *T*
by (*metis fim image_comp image_mono kjsub subset_trans*)
qed (*auto simp*: *assms*)
ultimately have *openin* (*top_of_set T*) ((*f* ∘ *h*) ‘ *g* ‘ (*S* − {*b*}) ∪ ((*f* ∘ *k*) ‘ *j* ‘ (*S* − {*c*})))
by (*rule openin_Un*)
moreover have (*f* ∘ *h*) ‘ *g* ‘ (*S* − {*b*}) = *f* ‘ (*S* − {*b*})
proof −
have *h* ‘ *g* ‘ (*S* − {*b*}) = (*S* − {*b*})
proof
show *h* ‘ *g* ‘ (*S* − {*b*}) ⊆ *S* − {*b*}
using *homeomorphism_apply1* [*OF gh*] *SU*
by (*fastforce simp add*: *image_iff image_subset_iff*)
show *S* − {*b*} ⊆ *h* ‘ *g* ‘ (*S* − {*b*})

using *SU gh homeomorphism_apply1* [*of* ‹(*rel_frontier U* − {*b*})› *V g h*]
by (*auto simp add*: *image_iff*) (*metis DiffI singletonD subsetD*)
qed
then show *?thesis*
by (*metis image_comp*)
qed
moreover have (*f* ∘ *k*) ‘ *j* ‘ (*S* − {*c*}) = *f* ‘ (*S* − {*c*})
proof −
have *k* ‘ *j* ‘ (*S* − {*c*}) = (*S* − {*c*})
proof
show *k* ‘ *j* ‘ (*S* − {*c*}) ⊆ *S* − {*c*}
using *homeomorphism_apply1* [*OF jk*] *SU*
by (*fastforce simp add*: *image_iff image_subset_iff*)
show *S* − {*c*} ⊆ *k* ‘ *j* ‘ (*S* − {*c*})
using *SU jk homeomorphism_apply1* [*of* ‹(*rel_frontier U* − {*c*})› *V j k*]
by (*auto simp add*: *image_iff*) (*metis DiffI singletonD subsetD*)
qed
then show *?thesis*
by (*metis image_comp*)
qed
moreover have *f* ‘ (*S* − {*b*}) ∪ *f* ‘ (*S* − {*c*}) = *f* ‘ (*S*)
using ‹*b* ≠ *c*› **by** *blast*
ultimately show *?thesis*
by *simp*
qed

lemma *invariance_of_domain_sphere_affine_set*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*euclidean_space*
assumes *contf*: *continuous_on S f* **and** *injf*: *inj_on f S* **and** *fim*: *f* ‘ *S* ⊆ *T*
and *r* ≠ *0 affine T* **and** *affTU*: *aff_dim T* < *DIM*(′*a*)
and *ope*: *openin* (*top_of_set* (*sphere a r*)) *S*
shows *openin* (*top_of_set T*) (*f* ‘ *S*)
proof (*cases sphere a r* = {})
case *True*
then show *?thesis*
using *ope openin_subset* **by** *force*
next
case *False*
show *?thesis*
proof (*rule invariance_of_domain_sphere_affine_set_gen* [*OF contf injf fim bounded_cball convex_cball* ‹*affine T*›])
show *aff_dim T* < *aff_dim* (*cball a r*)
by (*metis False affTU aff_dim_cball assms*(*4*) *linorder_cases sphere_empty*)
show *openin* (*top_of_set* (*rel_frontier* (*cball a r*))) *S*
by (*simp add*: ‹*r* ≠ *0*› *ope*)
qed
qed

lemma *no_embedding_sphere_lowdim*:

```
  fixes f :: 'a::euclidean_space ⇒ 'b::euclidean_space
  assumes contf: continuous_on (sphere a r) f and injf: inj_on f (sphere a r)
and r > 0
   shows DIM('a) ≤ DIM('b)
proof -
  have False if DIM('a) > DIM('b)
  proof -
    have compact (f ` sphere a r)
      using compact_continuous_image
      by (simp add: compact_continuous_image contf)
    then have ¬ open (f ` sphere a r)
      using compact_open
     by (metis assms(3) image_is_empty not_less_iff_gr_or_eq sphere_eq_empty)
    then show False
     using invariance_of_domain_sphere_affine_set [OF contf injf subset_UNIV]
‹r > 0›
     by (metis aff_dim_UNIV affine_UNIV less_irrefl of_nat_less_iff open_openin
openin_subtopology_self subtopology_UNIV that)
  qed
  then show ?thesis
    using not_less by blast
qed

lemma empty_interior_lowdim_gen:
  fixes S :: 'N::euclidean_space set and T :: 'M::euclidean_space set
  assumes dim: DIM('M) < DIM('N) and ST: S homeomorphic T
  shows interior S = {}
proof -
  obtain h :: 'M ⇒ 'N where linear h ⋀x. norm(h x) = norm x
   by (rule isometry_subset_subspace [OF subspace_UNIV subspace_UNIV, where
?'a = 'M and ?'b = 'N])
      (use dim in auto)
  then have inj h
    by (metis linear_inj_iff_eq_0 norm_eq_zero)
  then have h ` T homeomorphic T
    using ‹linear h› homeomorphic_sym linear_homeomorphic_image by blast
  then have interior (h ` T) homeomorphic interior S
    using homeomorphic_interiors_same_dimension
    by (metis ST homeomorphic_sym homeomorphic_trans)
  moreover
  have interior (range h) = {}
    by (simp add: ‹inj h› ‹linear h› dim dim_image_eq empty_interior_lowdim)
  then have interior (h ` T) = {}
    by (metis image_mono interior_mono subset_empty top_greatest)
  ultimately show ?thesis
    by simp
qed

lemma empty_interior_lowdim_gen_le:
```

fixes *S* :: *'N*::*euclidean_space set* **and** *T* :: *'M*::*euclidean_space set*
assumes *DIM('M)* ≤ *DIM('N)* *interior T* = {} *S homeomorphic T*
shows *interior S* = {}
by (*metis assms empty_interior_lowdim_gen homeomorphic_empty(1) homeomorphic_interiors_same_dimension less_le*)

lemma *homeomorphic_affine_sets_eq*:
fixes *S* :: *'a*::*euclidean_space set* **and** *T* :: *'b*::*euclidean_space set*
assumes *affine S affine T*
shows *S homeomorphic T* ⟷ *aff_dim S* = *aff_dim T*
proof (*cases S* = {} ∨ *T* = {})
case *True*
then show *?thesis*
using *assms homeomorphic_affine_sets* **by** *force*
next
case *False*
then obtain *a b* **where** *a* ∈ *S b* ∈ *T*
by *blast*
then have *subspace* ((+) (− *a*) ‘ *S*) *subspace* ((+) (− *b*) ‘ *T*)
using *affine_diffs_subspace assms* **by** *blast+*
then show *?thesis*
by (*metis affine_imp_convex assms homeomorphic_affine_sets homeomorphic_convex_sets*)
qed

lemma *homeomorphic_hyperplanes_eq*:
fixes *a* :: *'M*::*euclidean_space* **and** *c* :: *'N*::*euclidean_space*
assumes *a* ≠ *0 c* ≠ *0*
shows ({*x*. *a* · *x* = *b*} *homeomorphic* {*x*. *c* · *x* = *d*} ⟷ *DIM('M)* = *DIM('N)*) (**is** *?lhs* = *?rhs*)
proof −
have (*DIM('M)* − *Suc 0* = *DIM('N)* − *Suc 0*) ⟷ (*DIM('M)* = *DIM('N)*)
by *auto* (*metis DIM_positive Suc_pred*)
then show *?thesis*
using *assms* **by** (*simp add*: *homeomorphic_affine_sets_eq affine_hyperplane*)
qed

end
theory *Homology*
imports *Invariance_of_Domain*
begin

end