Computational Algebra

January 18, 2026

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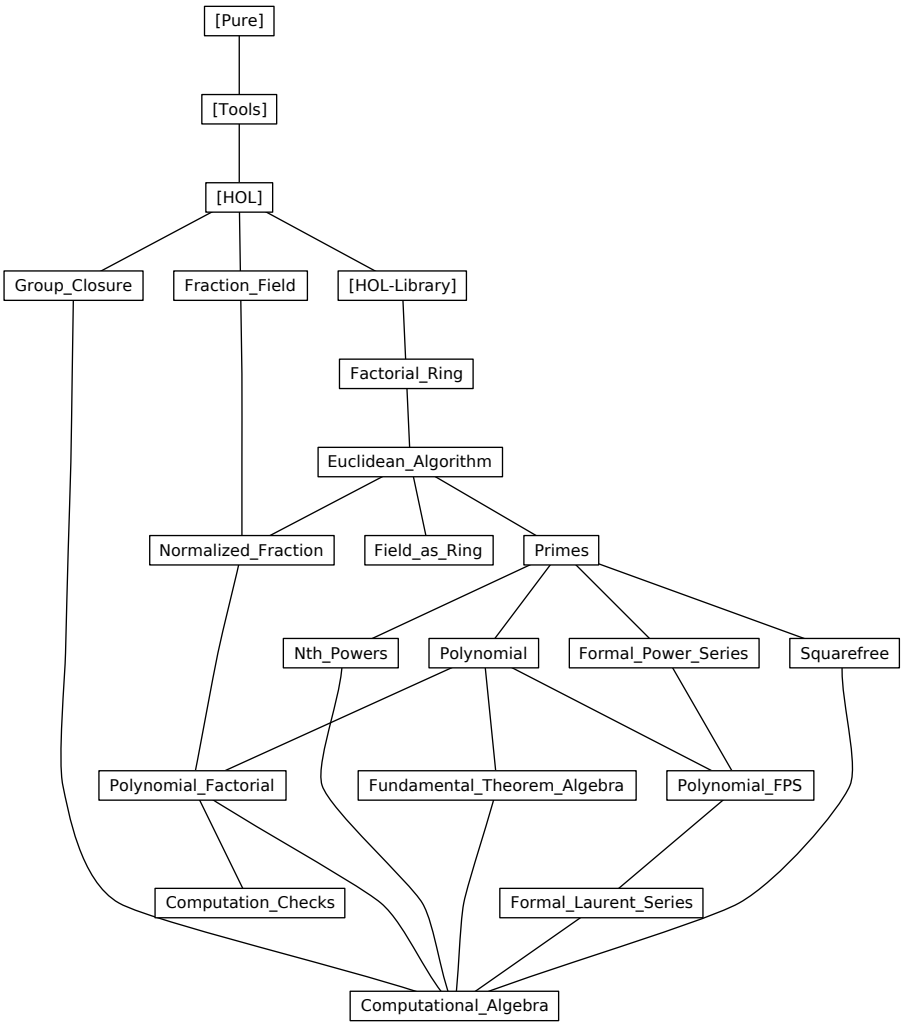

1 Factorial (semi)rings

theory *Factorial-Ring*
imports
Main
HOL−Library.Multiset
begin

unbundle *multiset.lifting*

1.1 Irreducible and prime elements

context *comm-semiring-1*
begin

definition *irreducible* :: *'a* ⇒ *bool* **where**
irreducible p ⟷ *p* ≠ *0* ∧ ¬*p dvd 1* ∧ (∀ *a b*. *p* = *a* ∗ *b* ⟶ *a dvd 1* ∨ *b dvd 1*)

lemma *not-irreducible-zero* [*simp*]: ¬*irreducible 0*
by (*simp add*: *irreducible-def*)

lemma *irreducible-not-unit*: *irreducible p* ⟹ ¬*p dvd 1*
by (*simp add*: *irreducible-def*)

lemma *not-irreducible-one* [*simp*]: ¬*irreducible 1*
by (*simp add*: *irreducible-def*)

lemma *irreducibleI*:
p ≠ *0* ⟹ ¬*p dvd 1* ⟹ (⋀*a b*. *p* = *a* ∗ *b* ⟹ *a dvd 1* ∨ *b dvd 1*) ⟹ *irreducible p*
by (*simp add*: *irreducible-def*)

lemma *irreducibleD*: *irreducible p* ⟹ *p* = *a* ∗ *b* ⟹ *a dvd 1* ∨ *b dvd 1*
by (*simp add*: *irreducible-def*)

lemma *irreducible-mono*:
assumes *irr*: *irreducible b* **and** *a dvd b* ¬*a dvd 1*
shows *irreducible a*
proof (*rule irreducibleI*)
fix *c d* **assume** *a* = *c* ∗ *d*
from *assms* **obtain** *k* **where** [*simp*]: *b* = *a* ∗ *k* **by** *auto*
from ‹*a* = *c* ∗ *d*› **have** *b* = *c* ∗ *d* ∗ *k*
by *simp*
hence *c dvd 1* ∨ (*d* ∗ *k*) *dvd 1*
using *irreducibleD*[*OF irr*, *of c d* ∗ *k*] **by** (*auto simp*: *mult.assoc*)
thus *c dvd 1* ∨ *d dvd 1*
by *auto*
qed (*use assms* **in** ‹*auto simp*: *irreducible-def*›)

lemma *irreducible-multD*:

```
  assumes l: irreducible (a*b)
  shows a dvd 1 ∧ irreducible b ∨ b dvd 1 ∧ irreducible a
proof−
  have *: irreducible b if l: irreducible (a*b) and a: a dvd 1 for a b :: 'a
  proof (rule irreducibleI)
    show ¬(b dvd 1)
    proof
      assume b dvd 1
      hence a * b dvd 1 * 1
        using ‹a dvd 1› by (intro mult-dvd-mono) auto
      with l show False
        by (auto simp: irreducible-def)
    qed
  next
    fix x y assume b = x * y
    have a * x dvd 1 ∨ y dvd 1
      using l by (rule irreducibleD) (use ‹b = x * y› in ‹auto simp: mult-ac›)
    thus x dvd 1 ∨ y dvd 1
      by auto
  qed (use l a in auto)

  from irreducibleD[OF assms refl] have a dvd 1 ∨ b dvd 1
    by (auto simp: irreducible-def)
  with *[of a b] *[of b a] l show ?thesis
    by (auto simp: mult.commute)
qed

lemma irreducible-power-iff [simp]:
  irreducible (p ^ n) ⟷ irreducible p ∧ n = 1
proof
  assume *: irreducible (p ^ n)
  have irreducible p
    using * by (induction n) (auto dest!: irreducible-multD)
  hence [simp]: ¬p dvd 1
    using * by (auto simp: irreducible-def)

  consider n = 0 | n = 1 | n > 1
    by linarith
  thus irreducible p ∧ n = 1
  proof cases
    assume n > 1
    hence p ^ n = p * p ^ (n − 1)
      by (cases n) auto
    with * ‹¬ p dvd 1› have p ^ (n − 1) dvd 1
      using irreducible-multD[of p p ^ (n − 1)] by auto
    with ‹¬p dvd 1› and ‹n > 1› have False
      by (meson dvd-power dvd-trans zero-less-diff)
    thus ?thesis ..
  qed (use * in auto)
```

qed *auto*

definition *prime-elem* :: *'a* ⇒ *bool* **where**
prime-elem p ⟷ *p* ≠ *0* ∧ ¬*p dvd 1* ∧ (∀ *a b*. *p dvd* (*a* ∗ *b*) ⟶ *p dvd a* ∨ *p dvd b*)

lemma *not-prime-elem-zero* [*simp*]: ¬*prime-elem 0*
by (*simp add*: *prime-elem-def*)

lemma *prime-elem-not-unit*: *prime-elem p* ⟹ ¬*p dvd 1*
by (*simp add*: *prime-elem-def*)

lemma *prime-elemI*:
p ≠ *0* ⟹ ¬*p dvd 1* ⟹ (⋀*a b*. *p dvd* (*a* ∗ *b*) ⟹ *p dvd a* ∨ *p dvd b*) ⟹ *prime-elem p*
by (*simp add*: *prime-elem-def*)

lemma *prime-elem-dvd-multD*:
prime-elem p ⟹ *p dvd* (*a* ∗ *b*) ⟹ *p dvd a* ∨ *p dvd b*
by (*simp add*: *prime-elem-def*)

lemma *prime-elem-dvd-mult-iff*:
prime-elem p ⟹ *p dvd* (*a* ∗ *b*) ⟷ *p dvd a* ∨ *p dvd b*
by (*auto simp*: *prime-elem-def*)

lemma *not-prime-elem-one* [*simp*]:
¬ *prime-elem 1*
by (*auto dest*: *prime-elem-not-unit*)

lemma *prime-elem-not-zeroI*:
assumes *prime-elem p*
shows *p* ≠ *0*
using *assms* **by** (*auto intro*: *ccontr*)

lemma *prime-elem-dvd-power*:
prime-elem p ⟹ *p dvd x* ^ *n* ⟹ *p dvd x*
by (*induction n*) (*auto dest*: *prime-elem-dvd-multD intro*: *dvd-trans*[*of - 1*])

lemma *prime-elem-dvd-power-iff*:
prime-elem p ⟹ *n* > *0* ⟹ *p dvd x* ^ *n* ⟷ *p dvd x*
by (*auto dest*: *prime-elem-dvd-power intro*: *dvd-trans*)

lemma *prime-elem-imp-nonzero* [*simp*]:
ASSUMPTION (*prime-elem x*) ⟹ *x* ≠ *0*
unfolding *ASSUMPTION-def* **by** (*rule prime-elem-not-zeroI*)

lemma *prime-elem-imp-not-one* [*simp*]:
ASSUMPTION (*prime-elem x*) ⟹ *x* ≠ *1*

unfolding *ASSUMPTION-def* **by** *auto*

end

lemma (**in** *normalization-semidom*) *irreducible-cong*:
assumes *normalize a* = *normalize b*
shows *irreducible a* ⟷ *irreducible b*
proof (*cases a* = *0* ∨ *a dvd 1*)
case *True*
hence ¬*irreducible a* **by** (*auto simp*: *irreducible-def*)
from *True* **have** *normalize a* = *0* ∨ *normalize a dvd 1*
by *auto*
also note *assms*
finally have *b* = *0* ∨ *b dvd 1* **by** *simp*
hence ¬*irreducible b* **by** (*auto simp*: *irreducible-def*)
with ‹¬*irreducible a*› **show** *?thesis* **by** *simp*
next
case *False*
hence *b*: *b* ≠ *0* ¬*is-unit b* **using** *assms*
by (*auto simp*: *is-unit-normalize*[*of b*])
show *?thesis*
proof
assume *irreducible a*
thus *irreducible b*
by (*rule irreducible-mono*) (*use assms False b* **in** ‹*auto dest*: *associatedD2*›)
next
assume *irreducible b*
thus *irreducible a*
by (*rule irreducible-mono*) (*use assms False b* **in** ‹*auto dest*: *associatedD1*›)
qed
qed

lemma (**in** *normalization-semidom*) *associatedE1*:
assumes *normalize a* = *normalize b*
obtains *u* **where** *is-unit u a* = *u* ∗ *b*
proof (*cases a* = *0*)
case [*simp*]: *False*
from *assms* **have** [*simp*]: *b* ≠ *0* **by** *auto*
show *?thesis*
proof (*rule that*)
show *is-unit* (*unit-factor a div unit-factor b*)
by *auto*
have *unit-factor a div unit-factor b* ∗ *b* = *unit-factor a* ∗ (*b div unit-factor b*)
using ‹*b* ≠ *0*› *unit-div-commute unit-div-mult-swap unit-factor-is-unit* **by** *metis*
also have *b div unit-factor b* = *normalize b* **by** *simp*
finally show *a* = *unit-factor a div unit-factor b* ∗ *b*
by (*metis assms unit-factor-mult-normalize*)

qed
next
case [*simp*]: *True*
hence [*simp*]: *b = 0*
using *assms*[*symmetric*] **by** *auto*
show *?thesis*
by (*intro that*[*of 1*]) *auto*
qed

lemma (**in** *normalization-semidom*) *associatedE2*:
assumes *normalize a = normalize b*
obtains *u* **where** *is-unit u b = u * a*
proof −
from *assms* **have** *normalize b = normalize a*
by *simp*
then obtain *u* **where** *is-unit u b = u * a*
by (*elim associatedE1*)
thus *?thesis* **using** *that* **by** *blast*
qed

lemma (**in** *normalization-semidom*) *normalize-power-normalize*:
normalize (*normalize x ^ n*) = *normalize* (*x ^ n*)
proof (*induction n*)
case (*Suc n*)
have *normalize* (*normalize x ^ Suc n*) = *normalize* (*x * normalize* (*normalize x ^ n*))
by *simp*
also note *Suc.IH*
finally show *?case* **by** *simp*
qed *auto*

context *algebraic-semidom*
begin

lemma *prime-elem-imp-irreducible*:
assumes *prime-elem p*
shows *irreducible p*
proof (*rule irreducibleI*)
fix *a b*
assume *p-eq*: *p = a * b*
with *assms* **have** *nz*: *a ≠ 0 b ≠ 0* **by** *auto*
from *p-eq* **have** *p dvd a * b* **by** *simp*
with ‹*prime-elem p*› **have** *p dvd a ∨ p dvd b* **by** (*rule prime-elem-dvd-multD*)
with ‹*p = a * b*› **have** *a * b dvd 1 * b ∨ a * b dvd a * 1* **by** *auto*
thus *a dvd 1 ∨ b dvd 1*
by (*simp only*: *dvd-times-left-cancel-iff*[*OF nz*(*1*)] *dvd-times-right-cancel-iff*[*OF nz*(*2*)])

qed (*insert assms*, *simp-all add*: *prime-elem-def*)

lemma (**in** *algebraic-semidom*) *unit-imp-no-irreducible-divisors*:
assumes *is-unit x irreducible p*
shows *¬p dvd x*
proof (*rule notI*)
assume *p dvd x*
with ‹*is-unit x*› **have** *is-unit p*
by (*auto intro*: *dvd-trans*)
with ‹*irreducible p*› **show** *False*
by (*simp add*: *irreducible-not-unit*)
qed

lemma *unit-imp-no-prime-divisors*:
assumes *is-unit x prime-elem p*
shows *¬p dvd x*
using *unit-imp-no-irreducible-divisors*[*OF assms*(*1*) *prime-elem-imp-irreducible*[*OF assms*(*2*)]] **.**

lemma *prime-elem-mono*:
assumes *prime-elem p ¬q dvd 1 q dvd p*
shows *prime-elem q*
proof −
from ‹*q dvd p*› **obtain** *r* **where** *r*: *p* = *q* ∗ *r* **by** (*elim dvdE*)
hence *p dvd q* ∗ *r* **by** *simp*
with ‹*prime-elem p*› **have** *p dvd q* ∨ *p dvd r* **by** (*rule prime-elem-dvd-multD*)
hence *p dvd q*
proof
assume *p dvd r*
then obtain *s* **where** *s*: *r* = *p* ∗ *s* **by** (*elim dvdE*)
from *r* **have** *p* ∗ *1* = *p* ∗ (*q* ∗ *s*) **by** (*subst* (*asm*) *s*) (*simp add*: *mult-ac*)
with ‹*prime-elem p*› **have** *q dvd 1*
by (*subst* (*asm*) *mult-cancel-left*) *auto*
with ‹*¬q dvd 1*› **show** *?thesis* **by** *contradiction*
qed

show *?thesis*
proof (*rule prime-elemI*)
fix *a b* **assume** *q dvd* (*a* ∗ *b*)
with ‹*p dvd q*› **have** *p dvd* (*a* ∗ *b*) **by** (*rule dvd-trans*)
with ‹*prime-elem p*› **have** *p dvd a* ∨ *p dvd b* **by** (*rule prime-elem-dvd-multD*)
with ‹*q dvd p*› **show** *q dvd a* ∨ *q dvd b* **by** (*blast intro*: *dvd-trans*)
qed (*insert assms*, *auto*)
qed

lemma *irreducibleD′*:
assumes *irreducible a b dvd a*
shows *a dvd b* ∨ *is-unit b*
proof −

from *assms* **obtain** *c* **where** *c*: $a = b * c$ **by** (*elim dvdE*)
from *irreducibleD*[*OF assms*(*1*) *this*] **have** *is-unit b* ∨ *is-unit c* **.**
thus *?thesis* **by** (*auto simp*: *c mult-unit-dvd-iff*)
qed

lemma *irreducibleI'*:
assumes $a \neq 0$ ¬*is-unit a* $\bigwedge b.\ b\ dvd\ a \Longrightarrow a\ dvd\ b \vee$ *is-unit b*
shows *irreducible a*
proof (*rule irreducibleI*)
fix *b c* **assume** *a-eq*: $a = b * c$
hence *a dvd b* ∨ *is-unit b* **by** (*intro assms*) *simp-all*
thus *is-unit b* ∨ *is-unit c*
proof
assume *a dvd b*
hence $b * c\ dvd\ b * 1$ **by** (*simp add*: *a-eq*)
moreover from ‹$a \neq 0$› *a-eq* **have** $b \neq 0$ **by** *auto*
ultimately show *?thesis* **by** (*subst* (*asm*) *dvd-times-left-cancel-iff*) *auto*
qed *blast*
qed (*simp-all add*: *assms*(*1*,*2*))

lemma *irreducible-altdef*:
irreducible x ⟷ $x \neq 0 \wedge$ ¬*is-unit x* ∧ (∀ *b*. *b dvd x* ⟶ *x dvd b* ∨ *is-unit b*)
using *irreducibleI'*[*of x*] *irreducibleD'*[*of x*] *irreducible-not-unit*[*of x*] **by** *auto*

lemma *prime-elem-multD*:
assumes *prime-elem* (*a* ∗ *b*)
shows *is-unit a* ∨ *is-unit b*
proof −
from *assms* **have** $a \neq 0\ b \neq 0$ **by** (*auto dest*!: *prime-elem-not-zeroI*)
moreover from *assms prime-elem-dvd-multD* [*of a* ∗ *b*] **have** $a * b\ dvd\ a \vee a *$
b dvd b
by *auto*
ultimately show *?thesis*
using *dvd-times-left-cancel-iff* [*of a b 1*]
dvd-times-right-cancel-iff [*of b a 1*]
by *auto*
qed

lemma *prime-elemD2*:
assumes *prime-elem p* **and** *a dvd p* **and** ¬ *is-unit a*
shows *p dvd a*
proof −
from ‹*a dvd p*› **obtain** *b* **where** $p = a * b$ **..**
with ‹*prime-elem p*› *prime-elem-multD* ‹¬ *is-unit a*› **have** *is-unit b* **by** *auto*
with ‹$p = a * b$› **show** *?thesis*
by (*auto simp add*: *mult-unit-dvd-iff*)
qed

lemma *prime-elem-dvd-prod-msetE*:

```
  assumes prime-elem p
  assumes dvd: p dvd prod-mset A
  obtains a where a ∈# A and p dvd a
proof −
  from dvd have ∃ a. a ∈# A ∧ p dvd a
  proof (induct A)
    case empty then show ?case
    using ‹prime-elem p› by (simp add: prime-elem-not-unit)
  next
    case (add a A)
    then have p dvd a ∗ prod-mset A by simp
    with ‹prime-elem p› consider (A) p dvd prod-mset A | (B) p dvd a
      by (blast dest: prime-elem-dvd-multD)
    then show ?case proof cases
      case B then show ?thesis by auto
    next
      case A
      with add.hyps obtain b where b ∈# A p dvd b
        by auto
      then show ?thesis by auto
    qed
  qed
  with that show thesis by blast

qed

context
begin

lemma prime-elem-powerD:
  assumes prime-elem (p ^ n)
  shows   prime-elem p ∧ n = 1
proof (cases n)
  case (Suc m)
  note assms
  also from Suc have p ^ n = p ∗ p^m by simp
  finally have is-unit p ∨ is-unit (p^m) by (rule prime-elem-multD)
 moreover from assms have ¬is-unit p by (simp add: prime-elem-def is-unit-power-iff)
  ultimately have is-unit (p ^ m) by simp
  with ‹¬is-unit p› have m = 0 by (simp add: is-unit-power-iff)
  with Suc assms show ?thesis by simp
qed (insert assms, simp-all)

lemma prime-elem-power-iff:
  prime-elem (p ^ n) ⟷ prime-elem p ∧ n = 1
  by (auto dest: prime-elem-powerD)

end
```

lemma *irreducible-mult-unit-left*:
  is-unit a $\Longrightarrow$ *irreducible* (*a* ∗ *p*) $\longleftrightarrow$ *irreducible p*
  by (*auto simp*: *irreducible-altdef mult.commute*[*of a*] *is-unit-mult-iff*
      mult-unit-dvd-iff dvd-mult-unit-iff)

lemma *prime-elem-mult-unit-left*:
  is-unit a $\Longrightarrow$ *prime-elem* (*a* ∗ *p*) $\longleftrightarrow$ *prime-elem p*
 by (*auto simp*: *prime-elem-def mult.commute*[*of a*] *is-unit-mult-iff mult-unit-dvd-iff*)

lemma *prime-elem-dvd-cases*:
  assumes *pk*: *p*∗*k dvd m*∗*n* **and** *p*: *prime-elem p*
  shows (∃ *x*. *k dvd x*∗*n* ∧ *m* = *p*∗*x*) ∨ (∃ *y*. *k dvd m*∗*y* ∧ *n* = *p*∗*y*)
proof −
  have *p dvd m*∗*n* **using** *dvd-mult-left pk* **by** *blast*
  then consider *p dvd m* | *p dvd n*
    using *p prime-elem-dvd-mult-iff* **by** *blast*
  then show *?thesis*
  proof *cases*
    case *1* **then obtain** *a* **where** *m* = *p* ∗ *a* **by** (*metis dvd-mult-div-cancel*)
      then have ∃ *x*. *k dvd x* ∗ *n* ∧ *m* = *p* ∗ *x*
        using *p pk* **by** (*auto simp*: *mult.assoc*)
    then show *?thesis* **..**
  next
    case *2* **then obtain** *b* **where** *n* = *p* ∗ *b* **by** (*metis dvd-mult-div-cancel*)
    with *p pk* **have** ∃ *y*. *k dvd m*∗*y* ∧ *n* = *p*∗*y*
    by (*metis dvd-mult-right dvd-times-left-cancel-iff mult.left-commute mult-zero-left*)
    then show *?thesis* **..**
  qed
qed

lemma *prime-elem-power-dvd-prod*:
  assumes *pc*: *p*^*c dvd m*∗*n* **and** *p*: *prime-elem p*
  shows ∃ *a b*. *a*+*b* = *c* ∧ *p*^*a dvd m* ∧ *p*^*b dvd n*
using *pc*
proof (*induct c arbitrary*: *m n*)
  case *0* **show** *?case* **by** *simp*
next
  case (*Suc c*)
  consider *x* **where** *p*^*c dvd x*∗*n m* = *p*∗*x* | *y* **where** *p*^*c dvd m*∗*y n* = *p*∗*y*
    using *prime-elem-dvd-cases* [*of* - *p*^*c*, *OF* - *p*] *Suc.prems* **by** *force*
  then show *?case*
  proof *cases*
    case (*1 x*)
    with *Suc.hyps*[*of x n*] **obtain** *a b* **where** *a* + *b* = *c* ∧ *p* ^ *a dvd x* ∧ *p* ^ *b dvd*
n **by** *blast*
    with *1* **have** *Suc a* + *b* = *Suc c* ∧ *p* ^ *Suc a dvd m* ∧ *p* ^ *b dvd n*
      by (*auto intro*: *mult-dvd-mono*)
    thus *?thesis* **by** *blast*
  next

```
    case (2 y)
    with Suc.hyps[of m y] obtain a b where a + b = c ∧ p ^ a dvd m ∧ p ^ b
dvd y by blast
    with 2 have a + Suc b = Suc c ∧ p ^ a dvd m ∧ p ^ Suc b dvd n
      by (auto intro: mult-dvd-mono)
    with Suc.hyps [of m y] show ∃ a b. a + b = Suc c ∧ p ^ a dvd m ∧ p ^ b dvd
n
      by blast
  qed
qed

lemma prime-elem-power-dvd-cases:
  assumes p ^ c dvd m * n and a + b = Suc c and prime-elem p
  shows p ^ a dvd m ∨ p ^ b dvd n
proof −
  from assms obtain r s
    where r + s = c ∧ p ^ r dvd m ∧ p ^ s dvd n
    by (blast dest: prime-elem-power-dvd-prod)
  moreover with assms have
    a ≤ r ∨ b ≤ s by arith
  ultimately show ?thesis by (auto intro: power-le-dvd)
qed

lemma prime-elem-not-unit' [simp]:
  ASSUMPTION (prime-elem x) ⟹ ¬is-unit x
  unfolding ASSUMPTION-def by (rule prime-elem-not-unit)

lemma prime-elem-dvd-power-iff:
  assumes prime-elem p
  shows p dvd a ^ n ⟷ p dvd a ∧ n > 0
 using assms by (induct n) (auto dest: prime-elem-not-unit prime-elem-dvd-multD)

lemma prime-power-dvd-multD:
  assumes prime-elem p
  assumes p ^ n dvd a * b and n > 0 and ¬ p dvd a
  shows p ^ n dvd b
  using ‹p ^ n dvd a * b› and ‹n > 0›
proof (induct n arbitrary: b)
  case 0 then show ?case by simp
next
  case (Suc n) show ?case
  proof (cases n = 0)
    case True with Suc ‹prime-elem p› ‹¬ p dvd a› show ?thesis
      by (simp add: prime-elem-dvd-mult-iff)
  next
    case False then have n > 0 by simp
    from ‹prime-elem p› have p ≠ 0 by auto
    from Suc.prems have *: p * p ^ n dvd a * b
      by simp
```

```
    then have p dvd a * b
      by (rule dvd-mult-left)
    with Suc ‹prime-elem p› ‹¬ p dvd a› have p dvd b
      by (simp add: prime-elem-dvd-mult-iff)
    moreover define c where c = b div p
    ultimately have b: b = p * c by simp
    with * have p * p ^ n dvd p * (a * c)
      by (simp add: ac-simps)
    with ‹p ≠ 0› have p ^ n dvd a * c
      by simp
    with Suc.hyps ‹n > 0› have p ^ n dvd c
      by blast
    with ‹p ≠ 0› show ?thesis
      by (simp add: b)
  qed
qed

end
```

1.2 Generalized primes: normalized prime elements

```
context normalization-semidom
begin

lemma irreducible-normalized-divisors:
  assumes irreducible x y dvd x normalize y = y
  shows   y = 1 ∨ y = normalize x
proof −
  from assms have is-unit y ∨ x dvd y by (auto simp: irreducible-altdef)
  thus ?thesis
  proof (elim disjE)
    assume is-unit y
    hence normalize y = 1 by (simp add: is-unit-normalize)
    with assms show ?thesis by simp
  next
    assume x dvd y
    with ‹y dvd x› have normalize y = normalize x by (rule associatedI)
    with assms show ?thesis by simp
  qed
qed

lemma irreducible-normalize-iff [simp]: irreducible (normalize x) = irreducible x
  using irreducible-mult-unit-left[of 1 div unit-factor x x]
  by (cases x = 0) (simp-all add: unit-div-commute)

lemma prime-elem-normalize-iff [simp]: prime-elem (normalize x) = prime-elem
x
  using prime-elem-mult-unit-left[of 1 div unit-factor x x]
  by (cases x = 0) (simp-all add: unit-div-commute)
```

lemma *prime-elem-associated*:
  assumes *prime-elem p* **and** *prime-elem q* **and** *q dvd p*
  shows *normalize q = normalize p*
using ‹*q dvd p*› **proof** (*rule associatedI*)
  from ‹*prime-elem q*› **have** ¬ *is-unit q*
    by (*auto simp add*: *prime-elem-not-unit*)
  with ‹*prime-elem p*› ‹*q dvd p*› **show** *p dvd q*
    by (*blast intro*: *prime-elemD2*)
qed

definition *prime* :: *'a* ⇒ *bool* **where**
  prime p ⟷ *prime-elem p* ∧ *normalize p = p*

lemma *not-prime-0* [*simp*]: ¬*prime 0* **by** (*simp add*: *prime-def*)

lemma *not-prime-unit*: *is-unit x* ⟹ ¬*prime x*
  using *prime-elem-not-unit*[*of x*] **by** (*auto simp add*: *prime-def*)

lemma *not-prime-1* [*simp*]: ¬*prime 1* **by** (*simp add*: *not-prime-unit*)

lemma *primeI*: *prime-elem x* ⟹ *normalize x = x* ⟹ *prime x*
  by (*simp add*: *prime-def*)

lemma *prime-imp-prime-elem* [*dest*]: *prime p* ⟹ *prime-elem p*
  by (*simp add*: *prime-def*)

lemma *normalize-prime*: *prime p* ⟹ *normalize p = p*
  by (*simp add*: *prime-def*)

lemma *prime-normalize-iff* [*simp*]: *prime* (*normalize p*) ⟷ *prime-elem p*
  by (*auto simp add*: *prime-def*)

lemma *prime-power-iff*:
  prime (*p* ^ *n*) ⟷ *prime p* ∧ *n = 1*
  by (*auto simp*: *prime-def prime-elem-power-iff*)

lemma *prime-imp-nonzero* [*simp*]:
  ASSUMPTION (*prime x*) ⟹ *x* ≠ *0*
  unfolding *ASSUMPTION-def prime-def* **by** *auto*

lemma *prime-imp-not-one* [*simp*]:
  ASSUMPTION (*prime x*) ⟹ *x* ≠ *1*
  unfolding *ASSUMPTION-def* **by** *auto*

lemma *prime-not-unit'* [*simp*]:
  ASSUMPTION (*prime x*) ⟹ ¬*is-unit x*
  unfolding *ASSUMPTION-def prime-def* **by** *auto*

lemma *prime-normalize'* [*simp*]: *ASSUMPTION* (*prime x*) ⟹ *normalize x* = *x*
unfolding *ASSUMPTION-def prime-def* **by** *simp*

lemma *unit-factor-prime*: *prime x* ⟹ *unit-factor x* = *1*
using *unit-factor-normalize*[*of x*] **unfolding** *prime-def* **by** *auto*

lemma *unit-factor-prime'* [*simp*]: *ASSUMPTION* (*prime x*) ⟹ *unit-factor x* = *1*
unfolding *ASSUMPTION-def* **by** (*rule unit-factor-prime*)

lemma *prime-imp-prime-elem'* [*simp*]: *ASSUMPTION* (*prime x*) ⟹ *prime-elem x*
by (*simp add*: *prime-def ASSUMPTION-def*)

lemma *prime-dvd-multD*: *prime p* ⟹ *p dvd a* ∗ *b* ⟹ *p dvd a* ∨ *p dvd b*
by (*intro prime-elem-dvd-multD*) *simp-all*

lemma *prime-dvd-mult-iff*: *prime p* ⟹ *p dvd a* ∗ *b* ⟷ *p dvd a* ∨ *p dvd b*
by (*auto dest*: *prime-dvd-multD*)

lemma *prime-dvd-power*:
prime p ⟹ *p dvd x* ^ *n* ⟹ *p dvd x*
by (*auto dest*!: *prime-elem-dvd-power simp*: *prime-def*)

lemma *prime-dvd-power-iff*:
prime p ⟹ *n* > *0* ⟹ *p dvd x* ^ *n* ⟷ *p dvd x*
by (*subst prime-elem-dvd-power-iff*) *simp-all*

lemma *prime-dvd-prod-mset-iff*: *prime p* ⟹ *p dvd prod-mset A* ⟷ (∃ *x*. *x* ∈# *A* ∧ *p dvd x*)
by (*induction A*) (*simp-all add*: *prime-elem-dvd-mult-iff prime-imp-prime-elem*, *blast*+)

lemma *prime-dvd-prod-iff*: *finite A* ⟹ *prime p* ⟹ *p dvd prod f A* ⟷ (∃ *x*∈*A*. *p dvd f x*)
by (*auto simp*: *prime-dvd-prod-mset-iff prod-unfold-prod-mset*)

lemma *primes-dvd-imp-eq*:
assumes *prime p prime q p dvd q*
shows *p* = *q*
proof −
from *assms* **have** *irreducible q* **by** (*simp add*: *prime-elem-imp-irreducible prime-def*)
from *irreducibleD'*[*OF this* ‹*p dvd q*›] *assms* **have** *q dvd p* **by** *simp*
with ‹*p dvd q*› **have** *normalize p* = *normalize q* **by** (*rule associatedI*)
with *assms* **show** *p* = *q* **by** *simp*
qed

lemma *prime-dvd-prod-mset-primes-iff*:
assumes *prime p* ⋀*q*. *q* ∈# *A* ⟹ *prime q*

shows *p dvd prod-mset A* ⟷ *p* ∈# *A*
proof −
from *assms(1)* **have** *p dvd prod-mset A* ⟷ (∃ *x*. *x* ∈# *A* ∧ *p dvd x*) **by** (*rule prime-dvd-prod-mset-iff*)
also from *assms* **have** ... ⟷ *p* ∈# *A* **by** (*auto dest*: *primes-dvd-imp-eq*)
finally show *?thesis* **.**
qed

lemma *prod-mset-primes-dvd-imp-subset*:
assumes *prod-mset A dvd prod-mset B* ⋀*p*. *p* ∈# *A* ⟹ *prime p* ⋀*p*. *p* ∈# *B* ⟹ *prime p*
shows *A* ⊆# *B*
using *assms*
proof (*induction A arbitrary*: *B*)
case *empty*
thus *?case* **by** *simp*
next
case (*add p A B*)
hence *p*: *prime p* **by** *simp*
define *B′* **where** *B′* = *B* − {#*p*#}
from *add.prems* **have** *p dvd prod-mset B* **by** (*simp add*: *dvd-mult-left*)
with *add.prems* **have** *p* ∈# *B*
by (*subst* (*asm*) (*2*) *prime-dvd-prod-mset-primes-iff*) *simp-all*
hence *B*: *B* = *B′* + {#*p*#} **by** (*simp add*: *B′-def*)
from *add.prems p* **have** *A* ⊆# *B′* **by** (*intro add.IH*) (*simp-all add*: *B*)
thus *?case* **by** (*simp add*: *B*)
qed

lemma *prod-mset-dvd-prod-mset-primes-iff*:
assumes ⋀*x*. *x* ∈# *A* ⟹ *prime x* ⋀*x*. *x* ∈# *B* ⟹ *prime x*
shows *prod-mset A dvd prod-mset B* ⟷ *A* ⊆# *B*
using *assms* **by** (*auto intro*: *prod-mset-subset-imp-dvd prod-mset-primes-dvd-imp-subset*)

lemma *is-unit-prod-mset-primes-iff*:
assumes ⋀*x*. *x* ∈# *A* ⟹ *prime x*
shows *is-unit* (*prod-mset A*) ⟷ *A* = {#}
by (*auto simp add*: *is-unit-prod-mset-iff*)
(*meson all-not-in-conv assms not-prime-unit set-mset-eq-empty-iff*)

lemma *prod-mset-primes-irreducible-imp-prime*:
assumes *irred*: *irreducible* (*prod-mset A*)
assumes *A*: ⋀*x*. *x* ∈# *A* ⟹ *prime x*
assumes *B*: ⋀*x*. *x* ∈# *B* ⟹ *prime x*
assumes *C*: ⋀*x*. *x* ∈# *C* ⟹ *prime x*
assumes *dvd*: *prod-mset A dvd prod-mset B* ∗ *prod-mset C*
shows *prod-mset A dvd prod-mset B* ∨ *prod-mset A dvd prod-mset C*
proof −
from *dvd* **have** *prod-mset A dvd prod-mset* (*B* + *C*)
by *simp*

with *A B C* **have** *subset*: $A \subseteq\# B + C$
by (*subst* (*asm*) *prod-mset-dvd-prod-mset-primes-iff*) *auto*
define *A1* **and** *A2* **where** $A1 = A \cap\# B$ **and** $A2 = A - A1$
have $A = A1 + A2$ **unfolding** *A1-def A2-def*
by (*rule sym*, *intro subset-mset.add-diff-inverse*) *simp-all*
from *subset* **have** $A1 \subseteq\# B$ $A2 \subseteq\# C$
by (*auto simp*: *A1-def A2-def Multiset.subset-eq-diff-conv Multiset.union-commute*)
from ‹$A = A1 + A2$› **have** *prod-mset A* = *prod-mset A1* $*$ *prod-mset A2* **by** *simp*
from *irred* **and** *this* **have** *is-unit* (*prod-mset A1*) $\vee$ *is-unit* (*prod-mset A2*)
by (*rule irreducibleD*)
with *A* **have** $A1 = \{\#\} \vee A2 = \{\#\}$ **unfolding** *A1-def A2-def*
by (*subst* (*asm*) (*1 2*) *is-unit-prod-mset-primes-iff*) (*auto dest*: *Multiset.in-diffD*)
with *dvd* ‹$A = A1 + A2$› ‹$A1 \subseteq\# B$› ‹$A2 \subseteq\# C$› **show** *?thesis*
by (*auto intro*: *prod-mset-subset-imp-dvd*)
qed

lemma *prod-mset-primes-finite-divisor-powers*:
assumes *A*: $\bigwedge x.\ x \in\# A \Longrightarrow prime\ x$
assumes *B*: $\bigwedge x.\ x \in\# B \Longrightarrow prime\ x$
assumes $A \neq \{\#\}$
shows *finite* $\{n.\ prod\text{-}mset\ A \mathbin{\hat{}} n\ dvd\ prod\text{-}mset\ B\}$
proof −
from ‹$A \neq \{\#\}$› **obtain** *x* **where** *x*: $x \in\# A$ **by** *blast*
define *m* **where** $m = count\ B\ x$
have $\{n.\ prod\text{-}mset\ A \mathbin{\hat{}} n\ dvd\ prod\text{-}mset\ B\} \subseteq \{..m\}$
proof *safe*
fix *n* **assume** *dvd*: *prod-mset A* $\hat{}$ *n dvd prod-mset B*
from *x* **have** *x* $\hat{}$ *n dvd prod-mset A* $\hat{}$ *n* **by** (*intro dvd-power-same dvd-prod-mset*)
also note *dvd*
also have $x \mathbin{\hat{}} n = prod\text{-}mset\ (replicate\text{-}mset\ n\ x)$ **by** *simp*
finally have *replicate-mset n x* $\subseteq\#$ *B*
by (*rule prod-mset-primes-dvd-imp-subset*) (*insert A B x*, *simp-all split*: *if-splits*)
thus $n \leq m$ **by** (*simp add*: *count-le-replicate-mset-subset-eq m-def*)
qed
moreover have *finite* $\{..m\}$ **by** *simp*
ultimately show *?thesis* **by** (*rule finite-subset*)
qed

end

1.3 In a semiring with GCD, each irreducible element is a prime element

context *semiring-gcd*
begin

lemma *irreducible-imp-prime-elem-gcd*:

assumes *irreducible x*
shows *prime-elem x*
proof (*rule prime-elemI*)
fix *a b* **assume** *x dvd a * b*
from *dvd-productE*[*OF this*] **obtain** *y z* **where** *yz*: *x = y * z y dvd a z dvd b* **.**
from ‹*irreducible x*› **and** ‹*x = y * z*› **have** *is-unit y ∨ is-unit z* **by** (*rule irreducibleD*)
with *yz* **show** *x dvd a ∨ x dvd b*
by (*auto simp*: *mult-unit-dvd-iff mult-unit-dvd-iff'*)
qed (*insert assms*, *auto simp*: *irreducible-not-unit*)

lemma *prime-elem-imp-coprime*:
assumes *prime-elem p ¬p dvd n*
shows *coprime p n*
proof (*rule coprimeI*)
fix *d* **assume** *d dvd p d dvd n*
show *is-unit d*
proof (*rule ccontr*)
assume *¬is-unit d*
from ‹*prime-elem p*› **and** ‹*d dvd p*› **and** *this* **have** *p dvd d*
by (*rule prime-elemD2*)
from *this* **and** ‹*d dvd n*› **have** *p dvd n* **by** (*rule dvd-trans*)
with ‹*¬p dvd n*› **show** *False* **by** *contradiction*
qed
qed

lemma *prime-imp-coprime*:
assumes *prime p ¬p dvd n*
shows *coprime p n*
using *assms* **by** (*simp add*: *prime-elem-imp-coprime*)

lemma *prime-elem-imp-power-coprime*:
prime-elem p ⟹ ¬ p dvd a ⟹ coprime a (p ^ m)
by (*cases m > 0*) (*auto dest*: *prime-elem-imp-coprime simp add*: *ac-simps*)

lemma *prime-imp-power-coprime*:
prime p ⟹ ¬ p dvd a ⟹ coprime a (p ^ m)
by (*rule prime-elem-imp-power-coprime*) *simp-all*

lemma *prime-elem-divprod-pow*:
assumes *p*: *prime-elem p* **and** *ab*: *coprime a b* **and** *pab*: *p^n dvd a * b*
shows *p^n dvd a ∨ p^n dvd b*
using *assms*
proof −
from *p* **have** *¬ is-unit p*
by *simp*
with *ab p* **have** *¬ p dvd a ∨ ¬ p dvd b*
using *not-coprimeI* **by** *blast*
with *p* **have** *coprime (p ^ n) a ∨ coprime (p ^ n) b*

by (*auto dest*: *prime-elem-imp-power-coprime simp add*: *ac-simps*)
with *pab* **show** *?thesis*
by (*auto simp add*: *coprime-dvd-mult-left-iff coprime-dvd-mult-right-iff*)
qed

lemma *primes-coprime*:
prime p $\Longrightarrow$ *prime q* $\Longrightarrow$ $p \neq q$ $\Longrightarrow$ *coprime p q*
using *prime-imp-coprime primes-dvd-imp-eq* **by** *blast*

end

1.4 Factorial semirings: algebraic structures with unique prime factorizations

class *factorial-semiring* = *normalization-semidom* +
assumes *prime-factorization-exists*:
$x \neq 0 \Longrightarrow \exists A.\ (\forall x.\ x \in\# A \longrightarrow \textit{prime-elem}\ x) \wedge \textit{normalize}\ (\textit{prod-mset}\ A) = \textit{normalize}\ x$

Alternative characterization

lemma (**in** *normalization-semidom*) *factorial-semiring-altI-aux*:
assumes *finite-divisors*: $\bigwedge x.\ x \neq 0 \Longrightarrow \textit{finite}\ \{y.\ y\ \textit{dvd}\ x \wedge \textit{normalize}\ y = y\}$
assumes *irreducible-imp-prime-elem*: $\bigwedge x.\ \textit{irreducible}\ x \Longrightarrow \textit{prime-elem}\ x$
assumes $x \neq 0$
shows $\exists A.\ (\forall x.\ x \in\# A \longrightarrow \textit{prime-elem}\ x) \wedge \textit{normalize}\ (\textit{prod-mset}\ A) = \textit{normalize}\ x$
using ‹$x \neq 0$›
proof (*induction card* $\{b.\ b\ \textit{dvd}\ x \wedge \textit{normalize}\ b = b\}$ *arbitrary*: *x rule*: *less-induct*)
case (*less a*)
let *?fctrs* = $\lambda a.\ \{b.\ b\ \textit{dvd}\ a \wedge \textit{normalize}\ b = b\}$
show *?case*
proof (*cases is-unit a*)
case *True*
thus *?thesis* **by** (*intro exI*[*of* - {#}]) (*auto simp*: *is-unit-normalize*)
next
case *False*
show *?thesis*
proof (*cases* $\exists b.\ b\ \textit{dvd}\ a \wedge \neg \textit{is-unit}\ b \wedge \neg a\ \textit{dvd}\ b$)
case *False*
with ‹$\neg$*is-unit a*› *less.prems* **have** *irreducible a* **by** (*auto simp*: *irreducible-altdef*)
hence *prime-elem a* **by** (*rule irreducible-imp-prime-elem*)
thus *?thesis* **by** (*intro exI*[*of* - {#*normalize a*#}]) *auto*
next
case *True*
then obtain *b* **where** *b*: *b dvd a* $\neg$ *is-unit b* $\neg$ *a dvd b* **by** *auto*
from *b* **have** *?fctrs b* $\subseteq$ *?fctrs a* **by** (*auto intro*: *dvd-trans*)
moreover from *b* **have** *normalize a* $\notin$ *?fctrs b normalize a* $\in$ *?fctrs a* **by** *simp-all*
hence *?fctrs b* $\neq$ *?fctrs a* **by** *blast*

```
      ultimately have ?fctrs b ⊂ ?fctrs a by (subst subset-not-subset-eq) blast
      with finite-divisors[OF ‹a ≠ 0›] have card (?fctrs b) < card (?fctrs a)
        by (rule psubset-card-mono)
      moreover from ‹a ≠ 0› b have b ≠ 0 by auto
     ultimately have ∃ A. (∀ x. x ∈# A ⟶ prime-elem x) ∧ normalize (prod-mset
A) = normalize b
        by (intro less) auto
     then obtain A where A: (∀ x. x ∈# A ⟶ prime-elem x) ∧ normalize (∏ #
A) = normalize b
        by auto

      define c where c = a div b
      from b have c: a = b * c by (simp add: c-def)
      from less.prems c have c ≠ 0 by auto
      from b c have ?fctrs c ⊆ ?fctrs a by (auto intro: dvd-trans)
      moreover have normalize a ∉ ?fctrs c
      proof safe
        assume normalize a dvd c
        hence b * c dvd 1 * c by (simp add: c)
        hence b dvd 1 by (subst (asm) dvd-times-right-cancel-iff) fact+
        with b show False by simp
      qed
      with ‹normalize a ∈ ?fctrs a› have ?fctrs a ≠ ?fctrs c by blast
      ultimately have ?fctrs c ⊂ ?fctrs a by (subst subset-not-subset-eq) blast
      with finite-divisors[OF ‹a ≠ 0›] have card (?fctrs c) < card (?fctrs a)
        by (rule psubset-card-mono)
        with ‹c ≠ 0› have ∃ A. (∀ x. x ∈# A ⟶ prime-elem x) ∧ normalize
(prod-mset A) = normalize c
        by (intro less) auto
     then obtain B where B: (∀ x. x ∈# B ⟶ prime-elem x) ∧ normalize (∏ #
B) = normalize c
        by auto

      show ?thesis
      proof (rule exI[of - A + B]; safe)
        have normalize (prod-mset (A + B)) =
              normalize (normalize (prod-mset A) * normalize (prod-mset B))
          by simp
        also have ... = normalize (b * c)
          by (simp only: A B) auto
        also have b * c = a
          using c by simp
        finally show normalize (prod-mset (A + B)) = normalize a .
      next
      qed (use A B in auto)
    qed
  qed
qed
```

lemma *factorial-semiring-altI*:
assumes *finite-divisors*: $\bigwedge x{::}'a.\ x \neq 0 \Longrightarrow$ *finite* $\{y.\ y$ *dvd* $x \wedge$ *normalize* $y = y\}$
assumes *irreducible-imp-prime*: $\bigwedge x{::}'a.$ *irreducible* $x \Longrightarrow$ *prime-elem* x
shows *OFCLASS*($'a ::$ *normalization-semidom*, *factorial-semiring-class*)
by *intro-classes* (*rule factorial-semiring-altI-aux*[*OF assms*])

Properties

context *factorial-semiring*
begin

lemma *prime-factorization-exists'*:
assumes $x \neq 0$
obtains A **where** $\bigwedge x.\ x \in\# A \Longrightarrow$ *prime* x *normalize* (*prod-mset* A) = *normalize* x
proof −
from *prime-factorization-exists*[*OF assms*] **obtain** A
where A: $\bigwedge x.\ x \in\# A \Longrightarrow$ *prime-elem* x *normalize* (*prod-mset* A) = *normalize* x **by** *blast*
define A' **where** $A' =$ *image-mset normalize* A
have *normalize* (*prod-mset* A') = *normalize* (*prod-mset* A)
by (*simp add*: A'*-def normalize-prod-mset-normalize*)
also note $A(2)$
finally have *normalize* (*prod-mset* A') = *normalize* x **by** *simp*
moreover from $A(1)$ **have** $\forall x.\ x \in\# A' \longrightarrow$ *prime* x **by** (*auto simp*: *prime-def* A'*-def*)
ultimately show *?thesis* **by** (*intro that*[*of* A']) *blast*
qed

lemma *irreducible-imp-prime-elem*:
assumes *irreducible* x
shows *prime-elem* x
proof (*rule prime-elemI*)
fix a b **assume** *dvd*: x *dvd* $a * b$
from *assms* **have** $x \neq 0$ **by** *auto*
show x *dvd* $a \vee x$ *dvd* b
proof (*cases* $a = 0 \vee b = 0$)
case *False*
hence $a \neq 0$ $b \neq 0$ **by** *blast*+
note $nz = \langle x \neq 0 \rangle$ *this*
from nz[*THEN prime-factorization-exists'*] **obtain** A B C
where *ABC*:
$\bigwedge z.\ z \in\# A \Longrightarrow$ *prime* z
normalize $(\prod_{\#} A)$ = *normalize* x
$\bigwedge z.\ z \in\# B \Longrightarrow$ *prime* z
normalize $(\prod_{\#} B)$ = *normalize* a
$\bigwedge z.\ z \in\# C \Longrightarrow$ *prime* z
normalize $(\prod_{\#} C)$ = *normalize* b
by *this blast*

```
    have irreducible (prod-mset A)
      by (subst irreducible-cong[OF ABC(2)]) fact
    moreover have normalize (prod-mset A) dvd
                  normalize (normalize (prod-mset B) * normalize (prod-mset C))
      unfolding ABC using dvd by simp
    hence prod-mset A dvd prod-mset B * prod-mset C
      unfolding normalize-mult-normalize-left normalize-mult-normalize-right by
simp
    ultimately have prod-mset A dvd prod-mset B ∨ prod-mset A dvd prod-mset
C
      by (intro prod-mset-primes-irreducible-imp-prime) (use ABC in auto)
    hence normalize (prod-mset A) dvd normalize (prod-mset B) ∨
          normalize (prod-mset A) dvd normalize (prod-mset C) by simp
    thus ?thesis unfolding ABC by simp
  qed auto
qed (use assms in ‹simp-all add: irreducible-def›)

lemma finite-divisor-powers:
  assumes y ≠ 0 ¬is-unit x
  shows   finite {n. x ^ n dvd y}
proof (cases x = 0)
  case True
  with assms have {n. x ^ n dvd y} = {0} by (auto simp: power-0-left)
  thus ?thesis by simp
next
  case False
  note nz = this ‹y ≠ 0›
  from nz[THEN prime-factorization-exists'] obtain A B
    where AB:
      ⋀z. z ∈# A ⟹ prime z
      normalize (∏# A) = normalize x
      ⋀z. z ∈# B ⟹ prime z
      normalize (∏# B) = normalize y
    by this blast

  from AB assms have A ≠ {#} by (auto simp: normalize-1-iff)
  from AB(2,4) prod-mset-primes-finite-divisor-powers [of A B, OF AB(1,3) this]
    have finite {n. prod-mset A ^ n dvd prod-mset B} by simp
  also have {n. prod-mset A ^ n dvd prod-mset B} =
            {n. normalize (normalize (prod-mset A) ^ n) dvd normalize (prod-mset
B)}
    unfolding normalize-power-normalize by simp
  also have ... = {n. x ^ n dvd y}
    unfolding AB unfolding normalize-power-normalize by simp
  finally show ?thesis .
qed

lemma finite-prime-divisors:
```

assumes $x \neq 0$
shows *finite* $\{p.\ prime\ p \wedge p\ dvd\ x\}$
proof −
from *prime-factorization-exists'*[*OF assms*] **obtain** A
where A: $\bigwedge z.\ z \in\# A \Longrightarrow prime\ z\ normalize\ (\prod_{\#} A) = normalize\ x$ **by** *this blast*
have $\{p.\ prime\ p \wedge p\ dvd\ x\} \subseteq set\text{-}mset\ A$
proof *safe*
fix p **assume** p: *prime* p **and** *dvd*: $p\ dvd\ x$
from *dvd* **have** $p\ dvd\ normalize\ x$ **by** *simp*
also from A **have** $normalize\ x = normalize\ (prod\text{-}mset\ A)$ **by** *simp*
finally have $p\ dvd\ prod\text{-}mset\ A$
by *simp*
thus $p \in\# A$ **using** $p\ A$
by (*subst* (*asm*) *prime-dvd-prod-mset-primes-iff*)
qed
moreover have *finite* (*set-mset* A) **by** *simp*
ultimately show *?thesis* **by** (*rule finite-subset*)
qed

lemma *infinite-unit-divisor-powers*:
assumes $y \neq 0$
assumes *is-unit* x
shows *infinite* $\{n.\ x\hat{}n\ dvd\ y\}$
proof −
from ‹*is-unit* x› **have** *is-unit* $(x\hat{}n)$ **for** n
using *is-unit-power-iff* **by** *auto*
hence $x\hat{}n\ dvd\ y$ **for** n
by *auto*
hence $\{n.\ x\hat{}n\ dvd\ y\} = UNIV$
by *auto*
thus *?thesis*
by *auto*
qed

corollary *is-unit-iff-infinite-divisor-powers*:
assumes $y \neq 0$
shows *is-unit* $x \longleftrightarrow$ *infinite* $\{n.\ x\hat{}n\ dvd\ y\}$
using *infinite-unit-divisor-powers finite-divisor-powers assms* **by** *auto*

lemma *prime-elem-iff-irreducible*: $prime\text{-}elem\ x \longleftrightarrow irreducible\ x$
by (*blast intro*: *irreducible-imp-prime-elem prime-elem-imp-irreducible*)

lemma *prime-divisor-exists*:
assumes $a \neq 0$ $\neg$*is-unit* a
shows $\exists b.\ b\ dvd\ a \wedge prime\ b$
proof −
from *prime-factorization-exists'*[*OF assms*(*1*)]
obtain A **where** A: $\bigwedge z.\ z \in\# A \Longrightarrow prime\ z\ normalize\ (\prod_{\#} A) = normalize\ a$

```
    by this blast
  with assms have A ≠ {#} by auto
  then obtain x where x ∈# A by blast
  with A(1) have ∗: x dvd normalize (prod-mset A) prime x
    by (auto simp: dvd-prod-mset)
  hence x dvd a by (simp add: A(2))
  with ∗ show ?thesis by blast
qed

lemma prime-divisors-induct [case-names zero unit factor]:
  assumes P 0 ⋀x. is-unit x ⟹ P x ⋀p x. prime p ⟹ P x ⟹ P (p ∗ x)
  shows   P x
proof (cases x = 0)
  case False
  from prime-factorization-exists'[OF this]
  obtain A where A: ⋀z. z ∈# A ⟹ prime z normalize (∏# A) = normalize x
    by this blast
  from A obtain u where u: is-unit u x = u ∗ prod-mset A
    by (elim associatedE2)

  from A(1) have P (u ∗ prod-mset A)
  proof (induction A)
    case (add p A)
    from add.prems have prime p by simp
     moreover from add.prems have P (u ∗ prod-mset A) by (intro add.IH)
simp-all
    ultimately have P (p ∗ (u ∗ prod-mset A)) by (rule assms(3))
    thus ?case by (simp add: mult-ac)
  qed (simp-all add: assms False u)
  with A u show ?thesis by simp
qed (simp-all add: assms(1))

lemma no-prime-divisors-imp-unit:
  assumes a ≠ 0 ⋀b. b dvd a ⟹ normalize b = b ⟹ ¬ prime-elem b
  shows is-unit a
proof (rule ccontr)
  assume ¬is-unit a
  from prime-divisor-exists[OF assms(1) this] obtain b where b dvd a prime b
by auto
  with assms(2)[of b] show False by (simp add: prime-def)
qed

lemma prime-divisorE:
  assumes a ≠ 0 and ¬ is-unit a
  obtains p where prime p and p dvd a
  using assms no-prime-divisors-imp-unit unfolding prime-def by blast

definition multiplicity :: 'a ⇒ 'a ⇒ nat where
  multiplicity p x = (if finite {n. p ^ n dvd x} then Max {n. p ^ n dvd x} else 0)
```

lemma *multiplicity-dvd*: $p \mathbin{\hat{}} multiplicity\ p\ x\ dvd\ x$
proof (*cases finite* $\{n.\ p \mathbin{\hat{}} n\ dvd\ x\}$)
  case *True*
  hence $multiplicity\ p\ x = Max\ \{n.\ p \mathbin{\hat{}} n\ dvd\ x\}$
    by (*simp add*: *multiplicity-def*)
  also have $\ldots \in \{n.\ p \mathbin{\hat{}} n\ dvd\ x\}$
    by (*rule Max-in*) (*auto intro*!: *True exI*[*of - 0*::*nat*])
  finally show *?thesis* **by** *simp*
qed (*simp add*: *multiplicity-def*)

lemma *multiplicity-dvd'*: $n \leq multiplicity\ p\ x \Longrightarrow p \mathbin{\hat{}} n\ dvd\ x$
  by (*rule dvd-trans*[*OF le-imp-power-dvd multiplicity-dvd*])

context
  fixes $x\ p :: 'a$
  assumes *xp*: $x \neq 0$ $\neg$*is-unit p*
begin

lemma *multiplicity-eq-Max*: $multiplicity\ p\ x = Max\ \{n.\ p \mathbin{\hat{}} n\ dvd\ x\}$
  using *finite-divisor-powers*[*OF xp*] **by** (*simp add*: *multiplicity-def*)

lemma *multiplicity-geI*:
  assumes $p \mathbin{\hat{}} n\ dvd\ x$
  shows $multiplicity\ p\ x \geq n$
proof −
  from *assms* **have** $n \leq Max\ \{n.\ p \mathbin{\hat{}} n\ dvd\ x\}$
    by (*intro Max-ge finite-divisor-powers xp*) *simp-all*
  thus *?thesis* **by** (*subst multiplicity-eq-Max*)
qed

lemma *multiplicity-lessI*:
  assumes $\neg p \mathbin{\hat{}} n\ dvd\ x$
  shows $multiplicity\ p\ x < n$
proof (*rule ccontr*)
  assume $\neg(n > multiplicity\ p\ x)$
  hence $p \mathbin{\hat{}} n\ dvd\ x$ **by** (*intro multiplicity-dvd'*) *simp*
  with *assms* **show** *False* **by** *contradiction*
qed

lemma *power-dvd-iff-le-multiplicity*:
  $p \mathbin{\hat{}} n\ dvd\ x \longleftrightarrow n \leq multiplicity\ p\ x$
  using *multiplicity-geI*[*of n*] *multiplicity-lessI*[*of n*] **by** (*cases* $p \mathbin{\hat{}} n\ dvd\ x$) *auto*

lemma *multiplicity-eq-zero-iff*:
  shows $multiplicity\ p\ x = 0 \longleftrightarrow \neg p\ dvd\ x$
  using *power-dvd-iff-le-multiplicity*[*of 1*] **by** *auto*

lemma *multiplicity-gt-zero-iff*:

shows *multiplicity p x > 0 ⟷ p dvd x*
using *power-dvd-iff-le-multiplicity*[*of 1*] **by** *auto*

lemma *multiplicity-decompose*:
¬*p dvd* (*x div p ^ multiplicity p x*)
proof
assume ∗: *p dvd x div p ^ multiplicity p x*
have *x* = *x div p ^ multiplicity p x* ∗ (*p ^ multiplicity p x*)
using *multiplicity-dvd*[*of p x*] **by** *simp*
also from ∗ **have** *x div p ^ multiplicity p x* = (*x div p ^ multiplicity p x div p*)
∗ *p* **by** *simp*
also have *x div p ^ multiplicity p x div p* ∗ *p* ∗ *p ^ multiplicity p x* =
x div p ^ multiplicity p x div p ∗ *p ^ Suc* (*multiplicity p x*)
by (*simp add*: *mult-assoc*)
also have *p ^ Suc* (*multiplicity p x*) *dvd* . . . **by** (*rule dvd-triv-right*)
finally show *False* **by** (*subst* (*asm*) *power-dvd-iff-le-multiplicity*) *simp*
qed

lemma *multiplicity-decompose′*:
obtains *y* **where** *x* = *p ^ multiplicity p x* ∗ *y* ¬*p dvd y*
using *that*[*of x div p ^ multiplicity p x*]
by (*simp add*: *multiplicity-decompose multiplicity-dvd*)

end

lemma *multiplicity-zero* [*simp*]: *multiplicity p 0* = *0*
by (*simp add*: *multiplicity-def*)

lemma *prime-elem-multiplicity-eq-zero-iff*:
prime-elem p ⟹ *x* ≠ *0* ⟹ *multiplicity p x* = *0* ⟷ ¬*p dvd x*
by (*rule multiplicity-eq-zero-iff*) *simp-all*

lemma *prime-multiplicity-other*:
assumes *prime p prime q p* ≠ *q*
shows *multiplicity p q* = *0*
using *assms* **by** (*subst prime-elem-multiplicity-eq-zero-iff*) (*auto dest*: *primes-dvd-imp-eq*)

lemma *prime-multiplicity-gt-zero-iff*:
prime-elem p ⟹ *x* ≠ *0* ⟹ *multiplicity p x* > *0* ⟷ *p dvd x*
by (*rule multiplicity-gt-zero-iff*) *simp-all*

lemma *multiplicity-unit-left*: *is-unit p* ⟹ *multiplicity p x* = *0*
by (*simp add*: *multiplicity-def is-unit-power-iff unit-imp-dvd*)

lemma *multiplicity-unit-right*:
assumes *is-unit x*
shows *multiplicity p x* = *0*
proof (*cases is-unit p* ∨ *x* = *0*)
case *False*

```
  with multiplicity-lessI[of x p 1] this assms
    show ?thesis by (auto dest: dvd-unit-imp-unit)
qed (auto simp: multiplicity-unit-left)

lemma multiplicity-one [simp]: multiplicity p 1 = 0
  by (rule multiplicity-unit-right) simp-all

lemma multiplicity-eqI:
  assumes p ^ n dvd x ¬p ^ Suc n dvd x
  shows   multiplicity p x = n
proof −
  consider x = 0 | is-unit p | x ≠ 0 ¬is-unit p by blast
  thus ?thesis
  proof cases
    assume xp: x ≠ 0 ¬is-unit p
    from xp assms(1) have multiplicity p x ≥ n by (intro multiplicity-geI)
    moreover from assms(2) xp have multiplicity p x < Suc n by (intro multi-
plicity-lessI)
    ultimately show ?thesis by simp
  next
    assume is-unit p
    hence is-unit (p ^ Suc n) by (simp add: is-unit-power-iff del: power-Suc)
    hence p ^ Suc n dvd x by (rule unit-imp-dvd)
    with ‹¬p ^ Suc n dvd x› show ?thesis by contradiction
  qed (insert assms, simp-all)
qed


context
  fixes x p :: 'a
  assumes xp: x ≠ 0 ¬is-unit p
begin

lemma multiplicity-times-same:
  assumes p ≠ 0
  shows   multiplicity p (p * x) = Suc (multiplicity p x)
proof (rule multiplicity-eqI)
  show p ^ Suc (multiplicity p x) dvd p * x
    by (auto intro!: mult-dvd-mono multiplicity-dvd)
  from xp assms show ¬ p ^ Suc (Suc (multiplicity p x)) dvd p * x
    using power-dvd-iff-le-multiplicity[OF xp, of Suc (multiplicity p x)] by simp
qed

end

lemma multiplicity-same-power': multiplicity p (p ^ n) = (if p = 0 ∨ is-unit p
then 0 else n)
proof −
  consider p = 0 | is-unit p |p ≠ 0 ¬is-unit p by blast
```

```
  thus ?thesis
  proof cases
    assume p ≠ 0 ¬is-unit p
    thus ?thesis by (induction n) (simp-all add: multiplicity-times-same)
  qed (simp-all add: power-0-left multiplicity-unit-left)
qed

lemma multiplicity-same-power:
  p ≠ 0 ⟹ ¬is-unit p ⟹ multiplicity p (p ^ n) = n
  by (simp add: multiplicity-same-power')

lemma multiplicity-prime-elem-times-other:
  assumes prime-elem p ¬p dvd q
  shows   multiplicity p (q * x) = multiplicity p x
proof (cases x = 0)
  case False
  show ?thesis
  proof (rule multiplicity-eqI)
    have 1 * p ^ multiplicity p x dvd q * x
      by (intro mult-dvd-mono multiplicity-dvd) simp-all
    thus p ^ multiplicity p x dvd q * x by simp
  next
    define n where n = multiplicity p x
    from assms have ¬is-unit p by simp
    from multiplicity-decompose'[OF False this]
    obtain y where y [folded n-def]: x = p ^ multiplicity p x * y ¬ p dvd y .
    from y have p ^ Suc n dvd q * x ⟷ p ^ n * p dvd p ^ n * (q * y) by (simp
add: mult-ac)
    also from assms have ... ⟷ p dvd q * y by simp
    also have ... ⟷ p dvd q ∨ p dvd y by (rule prime-elem-dvd-mult-iff) fact+
    also from assms y have ... ⟷ False by simp
    finally show ¬(p ^ Suc n dvd q * x) by blast
  qed
qed simp-all

lemma multiplicity-self:
  assumes p ≠ 0 ¬is-unit p
  shows   multiplicity p p = 1
proof -
  from assms have multiplicity p p = Max {n. p ^ n dvd p}
    by (simp add: multiplicity-eq-Max)
  also from assms have p ^ n dvd p ⟷ n ≤ 1 for n
    using dvd-power-iff[of p n 1] by auto
  hence {n. p ^ n dvd p} = {..1} by auto
  also have ... = {0,1} by auto
  finally show ?thesis by simp
qed

lemma multiplicity-times-unit-left:
```

assumes *is-unit c*
shows *multiplicity* $(c * p)$ x = *multiplicity p x*
proof −
from *assms* **have** $\{n.\ (c * p) \mathbin{\hat{}} n\ dvd\ x\} = \{n.\ p \mathbin{\hat{}} n\ dvd\ x\}$
by (*subst mult.commute*) (*simp add*: *mult-unit-dvd-iff power-mult-distrib is-unit-power-iff*)
thus *?thesis* **by** (*simp add*: *multiplicity-def*)
qed

lemma *multiplicity-times-unit-right*:
assumes *is-unit c*
shows *multiplicity p* $(c * x)$ = *multiplicity p x*
proof −
from *assms* **have** $\{n.\ p \mathbin{\hat{}} n\ dvd\ c * x\} = \{n.\ p \mathbin{\hat{}} n\ dvd\ x\}$
by (*subst mult.commute*) (*simp add*: *dvd-mult-unit-iff*)
thus *?thesis* **by** (*simp add*: *multiplicity-def*)
qed

lemma *multiplicity-normalize-left* [*simp*]:
multiplicity (*normalize p*) *x* = *multiplicity p x*
proof (*cases p* = *0*)
case [*simp*]: *False*
have *normalize p* = (*1 div unit-factor p*) ∗ *p*
by (*simp add*: *unit-div-commute is-unit-unit-factor*)
also have *multiplicity* ... *x* = *multiplicity p x*
by (*rule multiplicity-times-unit-left*) (*simp add*: *is-unit-unit-factor*)
finally show *?thesis* **.**
qed *simp-all*

lemma *multiplicity-normalize-right* [*simp*]:
multiplicity p (*normalize x*) = *multiplicity p x*
proof (*cases x* = *0*)
case [*simp*]: *False*
have *normalize x* = (*1 div unit-factor x*) ∗ *x*
by (*simp add*: *unit-div-commute is-unit-unit-factor*)
also have *multiplicity p* ... = *multiplicity p x*
by (*rule multiplicity-times-unit-right*) (*simp add*: *is-unit-unit-factor*)
finally show *?thesis* **.**
qed *simp-all*

lemma *multiplicity-prime* [*simp*]: *prime-elem p* ⟹ *multiplicity p p* = *1*
by (*rule multiplicity-self*) *auto*

lemma *multiplicity-prime-power* [*simp*]: *prime-elem p* ⟹ *multiplicity p* $(p \mathbin{\hat{}} n)$
= *n*
by (*subst multiplicity-same-power′*) *auto*

lift-definition *prime-factorization* :: *′a* ⇒ *′a multiset* **is**
λ*x p*. *if prime p then multiplicity p x else 0*
proof −

```
  fix x :: 'a
  show finite {p. 0 < (if prime p then multiplicity p x else 0)} (is finite ?A)
  proof (cases x = 0)
    case False
    from False have ?A ⊆ {p. prime p ∧ p dvd x}
      by (auto simp: multiplicity-gt-zero-iff)
    moreover from False have finite {p. prime p ∧ p dvd x}
      by (rule finite-prime-divisors)
    ultimately show ?thesis by (rule finite-subset)
  qed simp-all
qed

abbreviation prime-factors :: 'a ⇒ 'a set where
  prime-factors a ≡ set-mset (prime-factorization a)

lemma count-prime-factorization-nonprime:
  ¬prime p ⟹ count (prime-factorization x) p = 0
  by transfer simp

lemma count-prime-factorization-prime:
  prime p ⟹ count (prime-factorization x) p = multiplicity p x
  by transfer simp

lemma count-prime-factorization:
  count (prime-factorization x) p = (if prime p then multiplicity p x else 0)
  by transfer simp

lemma dvd-imp-multiplicity-le:
  assumes a dvd b b ≠ 0
  shows   multiplicity p a ≤ multiplicity p b
proof (cases is-unit p)
  case False
  with assms show ?thesis
   by (intro multiplicity-geI ) (auto intro: dvd-trans[OF multiplicity-dvd' assms(1)])
qed (insert assms, auto simp: multiplicity-unit-left)

lemma prime-power-inj:
  assumes prime a a ^ m = a ^ n
  shows   m = n
proof −
  have multiplicity a (a ^ m) = multiplicity a (a ^ n) by (simp only: assms)
 thus ?thesis using assms by (subst (asm) (1 2) multiplicity-prime-power) simp-all
qed

lemma prime-power-inj':
  assumes prime p prime q
  assumes p ^ m = q ^ n m > 0 n > 0
  shows   p = q m = n
proof −
```

```
  from assms have p ^ 1 dvd p ^ m by (intro le-imp-power-dvd) simp
  also have p ^ m = q ^ n by fact
  finally have p dvd q ^ n by simp
  with assms have p dvd q using prime-dvd-power[of p q] by simp
  with assms show p = q by (simp add: primes-dvd-imp-eq)
  with assms show m = n by (simp add: prime-power-inj)
qed

lemma prime-power-eq-one-iff [simp]: prime p ⟹ p ^ n = 1 ⟷ n = 0
  using prime-power-inj[of p n 0] by auto

lemma one-eq-prime-power-iff [simp]: prime p ⟹ 1 = p ^ n ⟷ n = 0
  using prime-power-inj[of p 0 n] by auto

lemma prime-power-inj'':
  assumes prime p prime q
  shows   p ^ m = q ^ n ⟷ (m = 0 ∧ n = 0) ∨ (p = q ∧ m = n)
  using assms
  by (cases m = 0; cases n = 0)
     (auto dest: prime-power-inj'[OF assms])

lemma prime-factorization-0 [simp]: prime-factorization 0 = {#}
  by (simp add: multiset-eq-iff count-prime-factorization)

lemma prime-factorization-empty-iff:
  prime-factorization x = {#} ⟷ x = 0 ∨ is-unit x
proof
  assume ∗: prime-factorization x = {#}
  {
    assume x: x ≠ 0 ¬is-unit x
    {
      fix p assume p: prime p
      have count (prime-factorization x) p = 0 by (simp add: ∗)
      also from p have count (prime-factorization x) p = multiplicity p x
        by (rule count-prime-factorization-prime)
         also from x p have ... = 0 ⟷ ¬p dvd x by (simp add: multiplic-
ity-eq-zero-iff)
      finally have ¬p dvd x .
    }
    with prime-divisor-exists[OF x] have False by blast
  }
  thus x = 0 ∨ is-unit x by blast
next
  assume x = 0 ∨ is-unit x
  thus prime-factorization x = {#}
  proof
    assume x: is-unit x
    {
      fix p assume p: prime p
```

```
    from p x have multiplicity p x = 0
      by (subst multiplicity-eq-zero-iff)
         (auto simp: multiplicity-eq-zero-iff dest: unit-imp-no-prime-divisors)
  }
  thus ?thesis by (simp add: multiset-eq-iff count-prime-factorization)
 qed simp-all
qed

lemma prime-factorization-unit:
 assumes is-unit x
 shows   prime-factorization x = {#}
proof (rule multiset-eqI)
 fix p :: 'a
 show count (prime-factorization x) p = count {#} p
 proof (cases prime p)
  case True
  with assms have multiplicity p x = 0
   by (subst multiplicity-eq-zero-iff)
      (auto simp: multiplicity-eq-zero-iff dest: unit-imp-no-prime-divisors)
  with True show ?thesis by (simp add: count-prime-factorization-prime)
 qed (simp-all add: count-prime-factorization-nonprime)
qed

lemma prime-factorization-1 [simp]: prime-factorization 1 = {#}
 by (simp add: prime-factorization-unit)

lemma prime-factorization-times-prime:
 assumes x ≠ 0 prime p
 shows   prime-factorization (p * x) = {#p#} + prime-factorization x
proof (rule multiset-eqI)
 fix q :: 'a
 consider ¬prime q | p = q | prime q p ≠ q by blast
 thus count (prime-factorization (p * x)) q = count ({#p#} + prime-factorization
x) q
 proof cases
  assume q: prime q p ≠ q
  with assms primes-dvd-imp-eq[of q p] have ¬q dvd p by auto
  with q assms show ?thesis
   by (simp add: multiplicity-prime-elem-times-other count-prime-factorization)
 qed (insert assms, auto simp: count-prime-factorization multiplicity-times-same)
qed

lemma prod-mset-prime-factorization-weak:
 assumes x ≠ 0
 shows   normalize (prod-mset (prime-factorization x)) = normalize x
 using assms
proof (induction x rule: prime-divisors-induct)
 case (factor p x)
 have normalize (prod-mset (prime-factorization (p * x))) =
```

```
        normalize (p * normalize (prod-mset (prime-factorization x)))
    using factor.prems factor.hyps by (simp add: prime-factorization-times-prime)
  also have normalize (prod-mset (prime-factorization x)) = normalize x
    by (rule factor.IH) (use factor in auto)
  finally show ?case by simp
qed (auto simp: prime-factorization-unit is-unit-normalize)

lemma in-prime-factors-iff:
  p ∈ prime-factors x ⟷ x ≠ 0 ∧ p dvd x ∧ prime p
proof −
  have p ∈ prime-factors x ⟷ count (prime-factorization x) p > 0 by simp
  also have ... ⟷ x ≠ 0 ∧ p dvd x ∧ prime p
   by (subst count-prime-factorization, cases x = 0)
      (auto simp: multiplicity-eq-zero-iff multiplicity-gt-zero-iff)
  finally show ?thesis .
qed

lemma in-prime-factors-imp-prime [intro]:
  p ∈ prime-factors x ⟹ prime p
  by (simp add: in-prime-factors-iff)

lemma in-prime-factors-imp-dvd [dest]:
  p ∈ prime-factors x ⟹ p dvd x
  by (simp add: in-prime-factors-iff)

lemma prime-factorsI:
  x ≠ 0 ⟹ prime p ⟹ p dvd x ⟹ p ∈ prime-factors x
  by (auto simp: in-prime-factors-iff)

lemma prime-factors-dvd:
  x ≠ 0 ⟹ prime-factors x = {p. prime p ∧ p dvd x}
  by (auto intro: prime-factorsI)

lemma prime-factors-multiplicity:
  prime-factors n = {p. prime p ∧ multiplicity p n > 0}
 by (cases n = 0) (auto simp add: prime-factors-dvd prime-multiplicity-gt-zero-iff)

lemma prime-factorization-prime:
  assumes prime p
  shows   prime-factorization p = {#p#}
proof (rule multiset-eqI)
  fix q :: 'a
  consider ¬prime q | q = p | prime q q ≠ p by blast
  thus count (prime-factorization p) q = count {#p#} q
    by cases (insert assms, auto dest: primes-dvd-imp-eq
          simp: count-prime-factorization multiplicity-self multiplicity-eq-zero-iff)
qed

lemma prime-factorization-prod-mset-primes:
```

assumes $\bigwedge p.\ p \in\# A \Longrightarrow prime\ p$
shows *prime-factorization* (*prod-mset A*) = *A*
using *assms*
proof (*induction A*)
case (*add p A*)
from *add.prems*[*of 0*] **have** *0* $\notin\#$ *A* **by** *auto*
hence *prod-mset A* $\neq$ *0* **by** *auto*
with *add* **show** *?case*
by (*simp-all add*: *mult-ac prime-factorization-times-prime Multiset.union-commute*)
qed *simp-all*

lemma *prime-factorization-cong*:
normalize x = *normalize y* $\Longrightarrow$ *prime-factorization x* = *prime-factorization y*
by (*simp add*: *multiset-eq-iff count-prime-factorization*
multiplicity-normalize-right [*of - x*, *symmetric*]
multiplicity-normalize-right [*of - y*, *symmetric*]
del: *multiplicity-normalize-right*)

lemma *prime-factorization-unique*:
assumes *x* $\neq$ *0 y* $\neq$ *0*
shows *prime-factorization x* = *prime-factorization y* $\longleftrightarrow$ *normalize x* = *normalize y*
proof
assume *prime-factorization x* = *prime-factorization y*
hence *prod-mset* (*prime-factorization x*) = *prod-mset* (*prime-factorization y*) **by** *simp*
hence *normalize* (*prod-mset* (*prime-factorization x*)) =
normalize (*prod-mset* (*prime-factorization y*))
by (*simp only*:)
with *assms* **show** *normalize x* = *normalize y*
by (*simp add*: *prod-mset-prime-factorization-weak*)
qed (*rule prime-factorization-cong*)

lemma *prime-factorization-normalize* [*simp*]:
prime-factorization (*normalize x*) = *prime-factorization x*
by (*cases x* = *0*, *simp*, *subst prime-factorization-unique*) *auto*

lemma *prime-factorization-eqI-strong*:
assumes $\bigwedge p.\ p \in\# P \Longrightarrow prime\ p$ *prod-mset P* = *n*
shows *prime-factorization n* = *P*
using *prime-factorization-prod-mset-primes*[*of P*] *assms* **by** *simp*

lemma *prime-factorization-eqI*:
assumes $\bigwedge p.\ p \in\# P \Longrightarrow prime\ p$ *normalize* (*prod-mset P*) = *normalize n*
shows *prime-factorization n* = *P*
proof −
have *P* = *prime-factorization* (*normalize* (*prod-mset P*))
using *prime-factorization-prod-mset-primes*[*of P*] *assms*(*1*) **by** *simp*
with *assms*(*2*) **show** *?thesis* **by** *simp*

qed

lemma *prime-factorization-mult*:
assumes $x \neq 0$ $y \neq 0$
shows *prime-factorization* $(x * y) =$ *prime-factorization* $x +$ *prime-factorization* y
proof −
have *normalize* (*prod-mset* (*prime-factorization* x) $*$ *prod-mset* (*prime-factorization* y)) =
normalize (*normalize* (*prod-mset* (*prime-factorization* x)) $*$
normalize (*prod-mset* (*prime-factorization* y)))
by (*simp only*: *normalize-mult-normalize-left normalize-mult-normalize-right*)
also have $\ldots =$ *normalize* $(x * y)$
by (*subst* (*1 2*) *prod-mset-prime-factorization-weak*) (*use assms* **in** *auto*)
finally show *?thesis*
by (*intro prime-factorization-eqI*) *auto*
qed

lemma *prime-factorization-prod*:
assumes *finite* A $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq 0$
shows *prime-factorization* (*prod* f A) = $(\sum n{\in}A.$ *prime-factorization* $(f\ n))$
using *assms* **by** (*induction A rule*: *finite-induct*)
(*auto simp*: *Sup-multiset-empty prime-factorization-mult*)

lemma *prime-elem-multiplicity-mult-distrib*:
assumes *prime-elem* p $x \neq 0$ $y \neq 0$
shows *multiplicity* p $(x * y) =$ *multiplicity* p $x +$ *multiplicity* p y
proof −
have *multiplicity* p $(x * y) =$ *count* (*prime-factorization* $(x * y)$) (*normalize* p)
by (*subst count-prime-factorization-prime*) (*simp-all add*: *assms*)
also from *assms*
have *prime-factorization* $(x * y) =$ *prime-factorization* $x +$ *prime-factorization* y
by (*intro prime-factorization-mult*)
also have *count* $\ldots$ (*normalize* p) =
count (*prime-factorization* x) (*normalize* p) + *count* (*prime-factorization* y) (*normalize* p)
by *simp*
also have $\ldots =$ *multiplicity* p $x +$ *multiplicity* p y
by (*subst* (*1 2*) *count-prime-factorization-prime*) (*simp-all add*: *assms*)
finally show *?thesis* **.**
qed

lemma *prime-elem-multiplicity-prod-mset-distrib*:
assumes *prime-elem* p $0 \notin\# A$
shows *multiplicity* p (*prod-mset* A) = *sum-mset* (*image-mset* (*multiplicity* p) A)
using *assms* **by** (*induction A*) (*auto simp*: *prime-elem-multiplicity-mult-distrib*)

lemma *prime-elem-multiplicity-power-distrib*:
assumes *prime-elem p x* ≠ *0*
shows *multiplicity p* (*x* ^ *n*) = *n* ∗ *multiplicity p x*
using *assms prime-elem-multiplicity-prod-mset-distrib* [*of p replicate-mset n x*]
by *simp*

lemma *prime-elem-multiplicity-prod-distrib*:
assumes *prime-elem p 0* ∉ *f* ‘ *A finite A*
shows *multiplicity p* (*prod f A*) = (∑ *x*∈*A*. *multiplicity p* (*f x*))
proof −
have *multiplicity p* (*prod f A*) = (∑ *x*∈#*mset-set A*. *multiplicity p* (*f x*))
using *assms* **by** (*subst prod-unfold-prod-mset*)
(*simp-all add*: *prime-elem-multiplicity-prod-mset-distrib sum-unfold-sum-mset*
multiset.map-comp o-def)
also from ‹*finite A*› **have** ... = (∑ *x*∈*A*. *multiplicity p* (*f x*))
by (*induction A rule*: *finite-induct*) *simp-all*
finally show *?thesis* **.**
qed

lemma *multiplicity-distinct-prime-power*:
prime p ⟹ *prime q* ⟹ *p* ≠ *q* ⟹ *multiplicity p* (*q* ^ *n*) = *0*
by (*subst prime-elem-multiplicity-power-distrib*) (*auto simp*: *prime-multiplicity-other*)

lemma *prime-factorization-prime-power*:
prime p ⟹ *prime-factorization* (*p* ^ *n*) = *replicate-mset n p*
by (*induction n*)
(*simp-all add*: *prime-factorization-mult prime-factorization-prime Multiset.union-commute*)

lemma *prime-factorization-subset-iff-dvd*:
assumes [*simp*]: *x* ≠ *0 y* ≠ *0*
shows *prime-factorization x* ⊆# *prime-factorization y* ⟷ *x dvd y*
proof −
have *x dvd y* ⟷
normalize (*prod-mset* (*prime-factorization x*)) *dvd normalize* (*prod-mset* (*prime-factorization y*))
using *assms* **by** (*subst* (*1 2*) *prod-mset-prime-factorization-weak*) *auto*
also have ... ⟷ *prime-factorization x* ⊆# *prime-factorization y*
by (*auto intro*!: *prod-mset-primes-dvd-imp-subset prod-mset-subset-imp-dvd*)
finally show *?thesis* **..**
qed

lemma *prime-factorization-subset-imp-dvd*:
x ≠ *0* ⟹ (*prime-factorization x* ⊆# *prime-factorization y*) ⟹ *x dvd y*
by (*cases y* = *0*) (*simp-all add*: *prime-factorization-subset-iff-dvd*)

lemma *prime-factorization-divide*:
assumes *b dvd a*
shows *prime-factorization* (*a div b*) = *prime-factorization a* − *prime-factorization b*

proof (*cases a = 0*)
case [*simp*]: *False*
from *assms* **have** [*simp*]: *b ≠ 0* **by** *auto*
have *prime-factorization ((a div b) * b) = prime-factorization (a div b) + prime-factorization b*
by (*intro prime-factorization-mult*) (*insert assms*, *auto elim*!: *dvdE*)
with *assms* **show** *?thesis* **by** *simp*
qed *simp-all*

lemma *zero-not-in-prime-factors* [*simp*]: *0 ∉ prime-factors x*
by (*auto dest*: *in-prime-factors-imp-prime*)

lemma *prime-prime-factors*:
prime p ⟹ prime-factors p = {p}
by (*drule prime-factorization-prime*) *simp*

lemma *prime-factors-product*:
*x ≠ 0 ⟹ y ≠ 0 ⟹ prime-factors (x * y) = prime-factors x ∪ prime-factors y*
by (*simp add*: *prime-factorization-mult*)

lemma *dvd-prime-factors* [*intro*]:
y ≠ 0 ⟹ x dvd y ⟹ prime-factors x ⊆ prime-factors y
by (*intro set-mset-mono*, *subst prime-factorization-subset-iff-dvd*) *auto*

lemma *multiplicity-le-imp-dvd*:
assumes *x ≠ 0* ⋀*p*. *prime p ⟹ multiplicity p x ≤ multiplicity p y*
shows *x dvd y*
proof (*cases y = 0*)
case *False*
from *assms this* **have** *prime-factorization x ⊆# prime-factorization y*
by (*intro mset-subset-eqI*) (*auto simp*: *count-prime-factorization*)
with *assms False* **show** *?thesis* **by** (*subst* (*asm*) *prime-factorization-subset-iff-dvd*)
qed *auto*

lemma *dvd-multiplicity-eq*:
x ≠ 0 ⟹ y ≠ 0 ⟹ x dvd y ⟷ (∀ *p*. *multiplicity p x ≤ multiplicity p y*)
by (*auto intro*: *dvd-imp-multiplicity-le multiplicity-le-imp-dvd*)

lemma *multiplicity-eq-imp-eq*:
assumes *x ≠ 0 y ≠ 0*
assumes ⋀*p*. *prime p ⟹ multiplicity p x = multiplicity p y*
shows *normalize x = normalize y*
using *assms* **by** (*intro associatedI multiplicity-le-imp-dvd*) *simp-all*

lemma *prime-factorization-unique'*:
assumes ∀ *p* ∈# *M*. *prime p* ∀ *p* ∈# *N*. *prime p* (∏ *i* ∈# *M*. *i*) = (∏ *i* ∈# *N*. *i*)
shows *M = N*

proof −
have *prime-factorization* $(\prod i \in\# M.\ i) =$ *prime-factorization* $(\prod i \in\# N.\ i)$
by (*simp only*: *assms*)
also from *assms* **have** *prime-factorization* $(\prod i \in\# M.\ i) = M$
by (*subst prime-factorization-prod-mset-primes*) *simp-all*
also from *assms* **have** *prime-factorization* $(\prod i \in\# N.\ i) = N$
by (*subst prime-factorization-prod-mset-primes*) *simp-all*
finally show *?thesis* **.**
qed

lemma *prime-factorization-unique''*:
assumes $\forall p \in\# M.$ *prime* p $\forall p \in\# N.$ *prime* p *normalize* $(\prod i \in\# M.\ i) =$ *normalize* $(\prod i \in\# N.\ i)$
shows $M = N$
proof −
have *prime-factorization* (*normalize* $(\prod i \in\# M.\ i)) =$
prime-factorization (*normalize* $(\prod i \in\# N.\ i)$)
by (*simp only*: *assms*)
also from *assms* **have** *prime-factorization* (*normalize* $(\prod i \in\# M.\ i)) = M$
by (*subst prime-factorization-normalize*, *subst prime-factorization-prod-mset-primes*) *simp-all*
also from *assms* **have** *prime-factorization* (*normalize* $(\prod i \in\# N.\ i)) = N$
by (*subst prime-factorization-normalize*, *subst prime-factorization-prod-mset-primes*) *simp-all*
finally show *?thesis* **.**
qed

lemma *multiplicity-cong*:
$(\bigwedge r.\ p \mathbin{\hat{}} r$ *dvd* $a \longleftrightarrow p \mathbin{\hat{}} r$ *dvd* $b) \Longrightarrow$ *multiplicity* $p\ a =$ *multiplicity* $p\ b$
by (*simp add*: *multiplicity-def*)

lemma *not-dvd-imp-multiplicity-0*:
assumes $\neg p$ *dvd* x
shows *multiplicity* $p\ x = 0$
proof −
from *assms* **have** *multiplicity* $p\ x < 1$
by (*intro multiplicity-lessI*) *auto*
thus *?thesis* **by** *simp*
qed

lemma *multiplicity-zero-left* [*simp*]: *multiplicity* $0\ x = 0$
by (*cases* $x = 0$) (*auto intro*: *not-dvd-imp-multiplicity-0*)

lemma *inj-on-Prod-primes*:
assumes $\bigwedge P\ p.\ P \in A \Longrightarrow p \in P \Longrightarrow$ *prime* p
assumes $\bigwedge P.\ P \in A \Longrightarrow$ *finite* P
shows *inj-on Prod A*
proof (*rule inj-onI*)
fix $P\ Q$ **assume** PQ: $P \in A\ Q \in A\ \prod P = \prod Q$

with *prime-factorization-unique'*[*of mset-set P mset-set Q*] *assms*[*of P*] *assms*[*of Q*]
have *mset-set P* = *mset-set Q* **by** (*auto simp*: *prod-unfold-prod-mset*)
with *assms*[*of P*] *assms*[*of Q*] *PQ* **show** *P* = *Q* **by** *simp*
qed

lemma *divides-primepow-weak*:
assumes *prime p* **and** *a dvd p* ^ *n*
obtains *m* **where** *m* ≤ *n* **and** *normalize a* = *normalize* (*p* ^ *m*)
proof −
from *assms* **have** *a* ≠ *0*
by *auto*
with *assms*
have *normalize* (*prod-mset* (*prime-factorization a*)) *dvd*
normalize (*prod-mset* (*prime-factorization* (*p* ^ *n*)))
by (*subst* (*1 2*) *prod-mset-prime-factorization-weak*) *auto*
then have *prime-factorization a* ⊆# *prime-factorization* (*p* ^ *n*)
by (*simp add*: *in-prime-factors-imp-prime prod-mset-dvd-prod-mset-primes-iff*)
with *assms* **have** *prime-factorization a* ⊆# *replicate-mset n p*
by (*simp add*: *prime-factorization-prime-power*)
then obtain *m* **where** *m* ≤ *n* **and** *prime-factorization a* = *replicate-mset m p*
by (*rule msubseteq-replicate-msetE*)
then have ∗: *normalize* (*prod-mset* (*prime-factorization a*)) =
normalize (*prod-mset* (*replicate-mset m p*)) **by** *metis*
also have *normalize* (*prod-mset* (*prime-factorization a*)) = *normalize a*
using ‹*a* ≠ *0*› **by** (*simp add*: *prod-mset-prime-factorization-weak*)
also have *prod-mset* (*replicate-mset m p*) = *p* ^ *m*
by *simp*
finally show *?thesis* **using** ‹*m* ≤ *n*›
by (*intro that*[*of m*])
qed

lemma *divide-out-primepow-ex*:
assumes *n* ≠ *0* ∃*p*∈*prime-factors n*. *P p*
obtains *p k n'* **where** *P p prime p p dvd n* ¬*p dvd n' k* > *0 n* = *p* ^ *k* ∗ *n'*
proof −
from *assms* **obtain** *p* **where** *p*: *P p prime p p dvd n*
by *auto*
define *k* **where** *k* = *multiplicity p n*
define *n'* **where** *n'* = *n div p* ^ *k*
have *n'*: *n* = *p* ^ *k* ∗ *n'* ¬*p dvd n'*
using *assms p multiplicity-decompose*[*of n p*]
by (*auto simp*: *n'-def k-def multiplicity-dvd*)
from *n' p* **have** *k* > *0* **by** (*intro Nat.gr0I*) *auto*
with *n' p that*[*of p n' k*] **show** *?thesis* **by** *auto*
qed

lemma *divide-out-primepow*:
assumes *n* ≠ *0* ¬*is-unit n*

obtains $p\ k\ n'$ **where** *prime p p dvd n* $\neg$*p dvd n'* $k > 0\ n = p\ \widehat{}\ k * n'$
using *divide-out-primepow-ex*[*OF assms*(*1*), *of* λ*-. True*] *prime-divisor-exists*[*OF assms*] *assms*
prime-factorsI **by** *metis*

1.5 GCD and LCM computation with unique factorizations

definition *gcd-factorial a b* = (*if a* = *0 then normalize b*
else if b = *0 then normalize a*
else normalize (*prod-mset* (*prime-factorization a* $\cap\#$ *prime-factorization b*)))

definition *lcm-factorial a b* = (*if a* = *0* $\vee$ *b* = *0 then 0*
else normalize (*prod-mset* (*prime-factorization a* $\cup\#$ *prime-factorization b*)))

definition *Gcd-factorial A* =
(*if A* $\subseteq$ {*0*} *then 0 else normalize* (*prod-mset* (*Inf* (*prime-factorization* ' (*A* $-$ {*0*})))))

definition *Lcm-factorial A* =
(*if A* = {} *then 1*
else if 0 $\notin$ *A* $\wedge$ *subset-mset.bdd-above* (*prime-factorization* ' (*A* $-$ {*0*})) *then*
normalize (*prod-mset* (*Sup* (*prime-factorization* ' *A*)))
else
0)

lemma *prime-factorization-gcd-factorial*:
assumes [*simp*]: *a* $\neq$ *0 b* $\neq$ *0*
shows *prime-factorization* (*gcd-factorial a b*) = *prime-factorization a* $\cap\#$ *prime-factorization b*
proof $-$
have *prime-factorization* (*gcd-factorial a b*) =
prime-factorization (*prod-mset* (*prime-factorization a* $\cap\#$ *prime-factorization b*))
by (*simp add*: *gcd-factorial-def*)
also have ... = *prime-factorization a* $\cap\#$ *prime-factorization b*
by (*subst prime-factorization-prod-mset-primes*) *auto*
finally show *?thesis* **.**
qed

lemma *prime-factorization-lcm-factorial*:
assumes [*simp*]: *a* $\neq$ *0 b* $\neq$ *0*
shows *prime-factorization* (*lcm-factorial a b*) = *prime-factorization a* $\cup\#$ *prime-factorization b*
proof $-$
have *prime-factorization* (*lcm-factorial a b*) =
prime-factorization (*prod-mset* (*prime-factorization a* $\cup\#$ *prime-factorization b*))
by (*simp add*: *lcm-factorial-def*)
also have ... = *prime-factorization a* $\cup\#$ *prime-factorization b*

by (*subst prime-factorization-prod-mset-primes*) *auto*
finally show *?thesis* **.**
qed

lemma *prime-factorization-Gcd-factorial*:
assumes $\neg A \subseteq \{0\}$
shows *prime-factorization* (*Gcd-factorial A*) = *Inf* (*prime-factorization* ‘ (*A* − {*0*}))
proof −
from *assms* **obtain** *x* **where** *x*: $x \in A - \{0\}$ **by** *auto*
hence *Inf* (*prime-factorization* ‘ (*A* − {*0*})) ⊆# *prime-factorization x*
by (*intro subset-mset.cInf-lower*) *simp-all*
hence ∀ *y*. *y* ∈# *Inf* (*prime-factorization* ‘ (*A* − {*0*})) ⟶ *y* ∈ *prime-factors x*
by (*auto dest*: *mset-subset-eqD*)
with *in-prime-factors-imp-prime*[*of* - *x*]
have ∀ *p*. *p* ∈# *Inf* (*prime-factorization* ‘ (*A* − {*0*})) ⟶ *prime p* **by** *blast*
with *assms* **show** *?thesis*
by (*simp add*: *Gcd-factorial-def prime-factorization-prod-mset-primes*)
qed

lemma *prime-factorization-Lcm-factorial*:
assumes *0* ∉ *A subset-mset.bdd-above* (*prime-factorization* ‘ *A*)
shows *prime-factorization* (*Lcm-factorial A*) = *Sup* (*prime-factorization* ‘ *A*)
proof (*cases A* = {})
case *True*
hence *prime-factorization* ‘ *A* = {} **by** *auto*
also have *Sup* ... = {#} **by** (*simp add*: *Sup-multiset-empty*)
finally show *?thesis* **by** (*simp add*: *Lcm-factorial-def*)
next
case *False*
have ∀ *y*. *y* ∈# *Sup* (*prime-factorization* ‘ *A*) ⟶ *prime y*
by (*auto simp*: *in-Sup-multiset-iff assms*)
with *assms False* **show** *?thesis*
by (*simp add*: *Lcm-factorial-def prime-factorization-prod-mset-primes*)
qed

lemma *gcd-factorial-commute*: *gcd-factorial a b* = *gcd-factorial b a*
by (*simp add*: *gcd-factorial-def multiset-inter-commute*)

lemma *gcd-factorial-dvd1*: *gcd-factorial a b dvd a*
proof (*cases a* = *0* ∨ *b* = *0*)
case *False*
hence *gcd-factorial a b* ≠ *0* **by** (*auto simp*: *gcd-factorial-def*)
with *False* **show** *?thesis*
by (*subst prime-factorization-subset-iff-dvd* [*symmetric*])
(*auto simp*: *prime-factorization-gcd-factorial*)
qed (*auto simp*: *gcd-factorial-def*)

lemma *gcd-factorial-dvd2*: *gcd-factorial a b dvd b*

by (*subst gcd-factorial-commute*) (*rule gcd-factorial-dvd1*)

lemma *normalize-gcd-factorial* [*simp*]: *normalize* (*gcd-factorial a b*) = *gcd-factorial a b*
by (*simp add*: *gcd-factorial-def*)

lemma *normalize-lcm-factorial* [*simp*]: *normalize* (*lcm-factorial a b*) = *lcm-factorial a b*
by (*simp add*: *lcm-factorial-def*)

lemma *gcd-factorial-greatest*: *c dvd gcd-factorial a b* **if** *c dvd a c dvd b* **for** *a b c*
proof (*cases a* = *0* ∨ *b* = *0*)
case *False*
with *that* **have** [*simp*]: *c* ≠ *0* **by** *auto*
let *?p* = *prime-factorization*
from *that False* **have** *?p c* ⊆# *?p a ?p c* ⊆# *?p b*
by (*simp-all add*: *prime-factorization-subset-iff-dvd*)
hence *prime-factorization c* ⊆#
prime-factorization (*prod-mset* (*prime-factorization a* ∩# *prime-factorization b*))
using *False* **by** (*subst prime-factorization-prod-mset-primes*) *auto*
with *False* **show** *?thesis*
by (*auto simp*: *gcd-factorial-def prime-factorization-subset-iff-dvd* [*symmetric*])
qed (*auto simp*: *gcd-factorial-def that*)

lemma *lcm-factorial-gcd-factorial*:
lcm-factorial a b = *normalize* (*a* ∗ *b div gcd-factorial a b*) **for** *a b*
proof (*cases a* = *0* ∨ *b* = *0*)
case *False*
let *?p* = *prime-factorization*
have *1*: *normalize x* ∗ *normalize y dvd z* ⟷ *x* ∗ *y dvd z* **for** *x y z* :: *'a*
proof −
have *normalize* (*normalize x* ∗ *normalize y*) *dvd z* ⟷ *x* ∗ *y dvd z*
unfolding *normalize-mult-normalize-left normalize-mult-normalize-right* **by** *simp*
thus *?thesis* **unfolding** *normalize-dvd-iff* **by** *simp*
qed

have *?p* (*a* ∗ *b*) = (*?p a* ∪# *?p b*) + (*?p a* ∩# *?p b*)
using *False* **by** (*subst prime-factorization-mult*) (*auto intro*!: *multiset-eqI*)
hence *normalize* (*prod-mset* (*?p* (*a* ∗ *b*))) =
normalize (*prod-mset* ((*?p a* ∪# *?p b*) + (*?p a* ∩# *?p b*)))
by (*simp only*:)
hence ∗: *normalize* (*a* ∗ *b*) = *normalize* (*lcm-factorial a b* ∗ *gcd-factorial a b*)
using *False*
by (*subst* (*asm*) *prod-mset-prime-factorization-weak*)
(*auto simp*: *lcm-factorial-def gcd-factorial-def*)

have [*simp*]: *gcd-factorial a b dvd a* ∗ *b lcm-factorial a b dvd a* ∗ *b*

```
    using associatedD2[OF *] by auto
  from False have [simp]: gcd-factorial a b ≠ 0 lcm-factorial a b ≠ 0
    by (auto simp: gcd-factorial-def lcm-factorial-def)

  show ?thesis
    by (rule associated-eqI)
       (use * in ‹auto simp: dvd-div-iff-mult div-dvd-iff-mult dest: associatedD1
associatedD2›)
qed (auto simp: lcm-factorial-def)

lemma normalize-Gcd-factorial:
  normalize (Gcd-factorial A) = Gcd-factorial A
  by (simp add: Gcd-factorial-def)

lemma Gcd-factorial-eq-0-iff:
  Gcd-factorial A = 0 ⟷ A ⊆ {0}
  by (auto simp: Gcd-factorial-def in-Inf-multiset-iff split: if-splits)

lemma Gcd-factorial-dvd:
  assumes x ∈ A
  shows   Gcd-factorial A dvd x
proof (cases x = 0)
  case False
 with assms have prime-factorization (Gcd-factorial A) = Inf (prime-factorization
‘ (A − {0}))
    by (intro prime-factorization-Gcd-factorial) auto
  also from False assms have ... ⊆# prime-factorization x
    by (intro subset-mset.cInf-lower) auto
  finally show ?thesis
    by (subst (asm) prime-factorization-subset-iff-dvd)
       (insert assms False, auto simp: Gcd-factorial-eq-0-iff)
qed simp-all

lemma Gcd-factorial-greatest:
  assumes ⋀y. y ∈ A ⟹ x dvd y
  shows   x dvd Gcd-factorial A
proof (cases A ⊆ {0})
  case False
  from False obtain y where y ∈ A y ≠ 0 by auto
  with assms[of y] have nz: x ≠ 0 by auto
  from nz assms have prime-factorization x ⊆# prime-factorization y if y ∈ A −
{0} for y
    using that by (subst prime-factorization-subset-iff-dvd) auto
 with False have prime-factorization x ⊆# Inf (prime-factorization ‘ (A − {0}))
    by (intro subset-mset.cInf-greatest) auto
  also from False have ... = prime-factorization (Gcd-factorial A)
    by (rule prime-factorization-Gcd-factorial [symmetric])
  finally show ?thesis
    by (subst (asm) prime-factorization-subset-iff-dvd)
```

```
    (insert nz False, auto simp: Gcd-factorial-eq-0-iff)
qed (simp-all add: Gcd-factorial-def)

lemma normalize-Lcm-factorial:
  normalize (Lcm-factorial A) = Lcm-factorial A
  by (simp add: Lcm-factorial-def)

lemma Lcm-factorial-eq-0-iff:
  Lcm-factorial A = 0 ⟷ 0 ∈ A ∨ ¬subset-mset.bdd-above (prime-factorization
‘ A)
  by (auto simp: Lcm-factorial-def in-Sup-multiset-iff)

lemma dvd-Lcm-factorial:
  assumes x ∈ A
  shows   x dvd Lcm-factorial A
proof (cases 0 ∉ A ∧ subset-mset.bdd-above (prime-factorization ‘ A))
  case True
  with assms have [simp]: 0 ∉ A x ≠ 0 A ≠ {} by auto
  from assms True have prime-factorization x ⊆# Sup (prime-factorization ‘ A)
    by (intro subset-mset.cSup-upper) auto
  also have ... = prime-factorization (Lcm-factorial A)
    by (rule prime-factorization-Lcm-factorial [symmetric]) (insert True, simp-all)
  finally show ?thesis
    by (subst (asm) prime-factorization-subset-iff-dvd)
      (insert True, auto simp: Lcm-factorial-eq-0-iff)
qed (insert assms, auto simp: Lcm-factorial-def)

lemma Lcm-factorial-least:
  assumes ⋀y. y ∈ A ⟹ y dvd x
  shows   Lcm-factorial A dvd x
proof −
  consider A = {} | 0 ∈ A | x = 0 | A ≠ {} 0 ∉ A x ≠ 0 by blast
  thus ?thesis
  proof cases
    assume ∗: A ≠ {} 0 ∉ A x ≠ 0
    hence nz: x ≠ 0 if x ∈ A for x using that by auto
    from ∗ have bdd: subset-mset.bdd-above (prime-factorization ‘ A)
      by (intro subset-mset.bdd-aboveI[of - prime-factorization x])
        (auto simp: prime-factorization-subset-iff-dvd nz dest: assms)
    have prime-factorization (Lcm-factorial A) = Sup (prime-factorization ‘ A)
      by (rule prime-factorization-Lcm-factorial) fact+
    also from ∗ have ... ⊆# prime-factorization x
      by (intro subset-mset.cSup-least)
        (auto simp: prime-factorization-subset-iff-dvd nz dest: assms)
    finally show ?thesis
      by (subst (asm) prime-factorization-subset-iff-dvd)
        (insert ∗ bdd, auto simp: Lcm-factorial-eq-0-iff)
  qed (auto simp: Lcm-factorial-def dest: assms)
qed
```

lemmas *gcd-lcm-factorial* =
gcd-factorial-dvd1 gcd-factorial-dvd2 gcd-factorial-greatest
normalize-gcd-factorial lcm-factorial-gcd-factorial
normalize-Gcd-factorial Gcd-factorial-dvd Gcd-factorial-greatest
normalize-Lcm-factorial dvd-Lcm-factorial Lcm-factorial-least

end

class *factorial-semiring-gcd* = *factorial-semiring* + *gcd* + *Gcd* +
assumes *gcd-eq-gcd-factorial*: *gcd a b* = *gcd-factorial a b*
and *lcm-eq-lcm-factorial*: *lcm a b* = *lcm-factorial a b*
and *Gcd-eq-Gcd-factorial*: *Gcd A* = *Gcd-factorial A*
and *Lcm-eq-Lcm-factorial*: *Lcm A* = *Lcm-factorial A*
begin

lemma *prime-factorization-gcd*:
assumes [*simp*]: $a \neq 0$ $b \neq 0$
shows *prime-factorization* (*gcd a b*) = *prime-factorization a* ∩# *prime-factorization b*
by (*simp add*: *gcd-eq-gcd-factorial prime-factorization-gcd-factorial*)

lemma *prime-factorization-lcm*:
assumes [*simp*]: $a \neq 0$ $b \neq 0$
shows *prime-factorization* (*lcm a b*) = *prime-factorization a* ∪# *prime-factorization b*
by (*simp add*: *lcm-eq-lcm-factorial prime-factorization-lcm-factorial*)

lemma *prime-factorization-Gcd*:
assumes *Gcd A* ≠ 0
shows *prime-factorization* (*Gcd A*) = *Inf* (*prime-factorization* ‘ (*A* − {*0*}))
using *assms*
by (*simp add*: *prime-factorization-Gcd-factorial Gcd-eq-Gcd-factorial Gcd-factorial-eq-0-iff*)

lemma *prime-factorization-Lcm*:
assumes *Lcm A* ≠ 0
shows *prime-factorization* (*Lcm A*) = *Sup* (*prime-factorization* ‘ *A*)
using *assms*
by (*simp add*: *prime-factorization-Lcm-factorial Lcm-eq-Lcm-factorial Lcm-factorial-eq-0-iff*)

lemma *prime-factors-gcd* [*simp*]:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *prime-factors* (*gcd a b*) =
prime-factors a ∩ *prime-factors b*
by (*subst prime-factorization-gcd*) *auto*

lemma *prime-factors-lcm* [*simp*]:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow$ *prime-factors* (*lcm a b*) =
prime-factors a ∪ *prime-factors b*
by (*subst prime-factorization-lcm*) *auto*

subclass *semiring-gcd*
by (*standard*, *unfold gcd-eq-gcd-factorial lcm-eq-lcm-factorial*)
(*rule gcd-lcm-factorial*; *assumption*)+

subclass *semiring-Gcd*
by (*standard*, *unfold Gcd-eq-Gcd-factorial Lcm-eq-Lcm-factorial*)
(*rule gcd-lcm-factorial*; *assumption*)+

lemma
assumes $x \neq 0$ $y \neq 0$
shows *gcd-eq-factorial'*:
$gcd\ x\ y = normalize\ (\prod p \in prime\text{-}factors\ x \cap prime\text{-}factors\ y.$
$p\ \widehat{}\ min\ (multiplicity\ p\ x)\ (multiplicity\ p\ y))$ (**is** - = *?rhs1*)
and *lcm-eq-factorial'*:
$lcm\ x\ y = normalize\ (\prod p \in prime\text{-}factors\ x \cup prime\text{-}factors\ y.$
$p\ \widehat{}\ max\ (multiplicity\ p\ x)\ (multiplicity\ p\ y))$ (**is** - = *?rhs2*)
proof −
have *gcd x y* = *gcd-factorial x y* **by** (*rule gcd-eq-gcd-factorial*)
also have ... = *?rhs1*
by (*auto simp*: *gcd-factorial-def assms prod-mset-multiplicity*
count-prime-factorization-prime
intro!: *arg-cong*[*of* - - *normalize*] *dest*: *in-prime-factors-imp-prime intro*!: *prod.cong*)
finally show *gcd x y* = *?rhs1* **.**
have *lcm x y* = *lcm-factorial x y* **by** (*rule lcm-eq-lcm-factorial*)
also have ... = *?rhs2*
by (*auto simp*: *lcm-factorial-def assms prod-mset-multiplicity*
count-prime-factorization-prime intro!: *arg-cong*[*of* - - *normalize*]
dest: *in-prime-factors-imp-prime intro*!: *prod.cong*)
finally show *lcm x y* = *?rhs2* **.**
qed

lemma
assumes $x \neq 0$ $y \neq 0$ *prime p*
shows *multiplicity-gcd*: *multiplicity p* (*gcd x y*) = *min* (*multiplicity p x*) (*multiplicity p y*)
and *multiplicity-lcm*: *multiplicity p* (*lcm x y*) = *max* (*multiplicity p x*) (*multiplicity p y*)
proof −
have *gcd x y* = *gcd-factorial x y* **by** (*rule gcd-eq-gcd-factorial*)
also from *assms* **have** *multiplicity p* ... = *min* (*multiplicity p x*) (*multiplicity p y*)
by (*simp add*: *count-prime-factorization-prime* [*symmetric*] *prime-factorization-gcd-factorial*)
finally show *multiplicity p* (*gcd x y*) = *min* (*multiplicity p x*) (*multiplicity p y*) **.**
have *lcm x y* = *lcm-factorial x y* **by** (*rule lcm-eq-lcm-factorial*)
also from *assms* **have** *multiplicity p* ... = *max* (*multiplicity p x*) (*multiplicity p y*)
by (*simp add*: *count-prime-factorization-prime* [*symmetric*] *prime-factorization-lcm-factorial*)

finally show *multiplicity p (lcm x y) = max (multiplicity p x) (multiplicity p y)*
.
qed

lemma *gcd-lcm-distrib*:
gcd x (lcm y z) = lcm (gcd x y) (gcd x z)
proof (*cases x = 0 ∨ y = 0 ∨ z = 0*)
case *True*
thus *?thesis*
by (*auto simp*: *lcm-proj1-if-dvd lcm-proj2-if-dvd*)
next
case *False*
hence *normalize (gcd x (lcm y z)) = normalize (lcm (gcd x y) (gcd x z))*
by (*intro associatedI prime-factorization-subset-imp-dvd*)
(*auto simp*: *lcm-eq-0-iff prime-factorization-gcd prime-factorization-lcm*
subset-mset.inf-sup-distrib1)
thus *?thesis* **by** *simp*
qed

lemma *lcm-gcd-distrib*:
lcm x (gcd y z) = gcd (lcm x y) (lcm x z)
proof (*cases x = 0 ∨ y = 0 ∨ z = 0*)
case *True*
thus *?thesis*
by (*auto simp*: *lcm-proj1-if-dvd lcm-proj2-if-dvd*)
next
case *False*
hence *normalize (lcm x (gcd y z)) = normalize (gcd (lcm x y) (lcm x z))*
by (*intro associatedI prime-factorization-subset-imp-dvd*)
(*auto simp*: *lcm-eq-0-iff prime-factorization-gcd prime-factorization-lcm*
subset-mset.sup-inf-distrib1)
thus *?thesis* **by** *simp*
qed

end

class *factorial-ring-gcd = factorial-semiring-gcd + idom*
begin

subclass *ring-gcd* **..**

subclass *idom-divide* **..**

end

class *factorial-semiring-multiplicative =*
factorial-semiring + normalization-semidom-multiplicative
begin

lemma *normalize-prod-mset-primes*:
(⋀*p*. *p* ∈# *A* ⟹ *prime p*) ⟹ *normalize* (*prod-mset A*) = *prod-mset A*
proof (*induction A*)
case (*add p A*)
hence *prime p* **by** *simp*
hence *normalize p* = *p* **by** *simp*
with *add* **show** *?case* **by** (*simp add*: *normalize-mult*)
qed *simp-all*

lemma *prod-mset-prime-factorization*:
assumes *x* ≠ *0*
shows *prod-mset* (*prime-factorization x*) = *normalize x*
using *assms*
by (*induction x rule*: *prime-divisors-induct*)
(*simp-all add*: *prime-factorization-unit prime-factorization-times-prime*
is-unit-normalize normalize-mult)

lemma *prime-decomposition*: *unit-factor x* ∗ *prod-mset* (*prime-factorization x*) =
x
by (*cases x* = *0*) (*simp-all add*: *prod-mset-prime-factorization*)

lemma *prod-prime-factors*:
assumes *x* ≠ *0*
shows (∏ *p* ∈ *prime-factors x*. *p* ^ *multiplicity p x*) = *normalize x*
proof −
have *normalize x* = *prod-mset* (*prime-factorization x*)
by (*simp add*: *prod-mset-prime-factorization assms*)
also have ... = (∏ *p* ∈ *prime-factors x*. *p* ^ *count* (*prime-factorization x*) *p*)
by (*subst prod-mset-multiplicity*) *simp-all*
also have ... = (∏ *p* ∈ *prime-factors x*. *p* ^ *multiplicity p x*)
by (*intro prod.cong*)
(*simp-all add*: *assms count-prime-factorization-prime in-prime-factors-imp-prime*)
finally show *?thesis* **..**
qed

lemma *prime-factorization-unique″*:
assumes *S-eq*: *S* = {*p*. *0* < *f p*}
and *finite S*
and *S*: ∀ *p*∈*S*. *prime p normalize n* = (∏ *p*∈*S*. *p* ^ *f p*)
shows *S* = *prime-factors n* ∧ (∀ *p*. *prime p* ⟶ *f p* = *multiplicity p n*)
proof
define *A* **where** *A* = *Abs-multiset f*
from ‹*finite S*› *S*(*1*) **have** (∏ *p*∈*S*. *p* ^ *f p*) ≠ *0* **by** *auto*
with *S*(*2*) **have** *nz*: *n* ≠ *0* **by** *auto*
from *S-eq* ‹*finite S*› **have** *count-A*: *count A* = *f*
unfolding *A-def* **by** (*subst multiset.Abs-multiset-inverse*) *simp-all*
from *S-eq count-A* **have** *set-mset-A*: *set-mset A* = *S*
by (*simp only*: *set-mset-def*)

```
  from S(2) have normalize n = (∏ p∈S. p ^ f p) .
 also have ... = prod-mset A by (simp add: prod-mset-multiplicity S-eq set-mset-A
count-A)
  also from nz have normalize n = prod-mset (prime-factorization n)
    by (simp add: prod-mset-prime-factorization)
  finally have prime-factorization (prod-mset A) =
                 prime-factorization (prod-mset (prime-factorization n)) by simp
  also from S(1) have prime-factorization (prod-mset A) = A
    by (intro prime-factorization-prod-mset-primes) (auto simp: set-mset-A)
 also have prime-factorization (prod-mset (prime-factorization n)) = prime-factorization
n
    by (intro prime-factorization-prod-mset-primes) auto
  finally show S = prime-factors n by (simp add: set-mset-A [symmetric])

  show (∀ p. prime p ⟶ f p = multiplicity p n)
  proof safe
    fix p :: 'a assume p: prime p
    have multiplicity p n = multiplicity p (normalize n) by simp
    also have normalize n = prod-mset A
      by (simp add: prod-mset-multiplicity S-eq set-mset-A count-A S)
    also from p set-mset-A S(1)
    have multiplicity p ... = sum-mset (image-mset (multiplicity p) A)
      by (intro prime-elem-multiplicity-prod-mset-distrib) auto
    also from S(1) p
    have image-mset (multiplicity p) A = image-mset (λq. if p = q then 1 else 0)
A
     by (intro image-mset-cong) (auto simp: set-mset-A multiplicity-self prime-multiplicity-other)
    also have sum-mset ... = f p
      by (simp add: semiring-1-class.sum-mset-delta' count-A)
    finally show f p = multiplicity p n ..
  qed
qed

lemma divides-primepow:
  assumes prime p and a dvd p ^ n
  obtains m where m ≤ n and normalize a = p ^ m
  using divides-primepow-weak[OF assms] that assms
  by (auto simp add: normalize-power)

lemma Ex-other-prime-factor:
  assumes n ≠ 0 and ¬(∃ k. normalize n = p ^ k) prime p
  shows  ∃ q∈prime-factors n. q ≠ p
proof (rule ccontr)
  assume *: ¬(∃ q∈prime-factors n. q ≠ p)
  have normalize n = (∏ p∈prime-factors n. p ^ multiplicity p n)
    using assms(1) by (intro prod-prime-factors [symmetric]) auto
  also from * have ... = (∏ p∈{p}. p ^ multiplicity p n)
   using assms(3) by (intro prod.mono-neutral-left) (auto simp: prime-factors-multiplicity)
  finally have normalize n = p ^ multiplicity p n by auto
```

with *assms* **show** *False* **by** *auto*
qed

Now a string of results due to Maya Kdzioka

lemma *multiplicity-dvd-iff-dvd*:
assumes $x \neq 0$
shows $p\hat{}k$ *dvd* $x \longleftrightarrow p\hat{}k$ *dvd* $p\hat{}$*multiplicity* p x
proof (*cases is-unit p*)
case *True*
then have *is-unit* $(p\hat{}k)$
using *is-unit-power-iff* **by** *simp*
hence $p\hat{}k$ *dvd* x
by *auto*
moreover from ‹*is-unit p*› **have** $p\hat{}k$ *dvd* $p\hat{}$*multiplicity* p x
using *multiplicity-unit-left is-unit-power-iff* **by** *simp*
ultimately show *?thesis* **by** *simp*
next
case *False*
show *?thesis*
proof (*cases* $p = 0$)
case *True*
then have $p\hat{}$*multiplicity* p $x = 1$
by *simp*
moreover have $p\hat{}k$ *dvd* $x \Longrightarrow k = 0$
proof (*rule ccontr*)
assume $p\hat{}k$ *dvd* x **and** $k \neq 0$
with ‹$p = 0$› **have** $p\hat{}k = 0$ **by** *auto*
with ‹$p\hat{}k$ *dvd* x› **have** *0 dvd x* **by** *auto*
hence $x = 0$ **by** *auto*
with ‹$x \neq 0$› **show** *False* **by** *auto*
qed
ultimately show *?thesis*
by (*auto simp add*: *is-unit-power-iff* ‹¬ *is-unit p*›)
next
case *False*
with ‹$x \neq 0$› ‹¬ *is-unit p*› **show** *?thesis*
by (*simp add*: *power-dvd-iff-le-multiplicity dvd-power-iff multiplicity-same-power*)
qed
qed

lemma *multiplicity-decomposeI*:
assumes $x = p\hat{}k * x'$ **and** ¬ p *dvd* x' **and** $p \neq 0$
shows *multiplicity* p $x = k$
using *assms local.multiplicity-eqI local.power-Suc2* **by** *force*

lemma *multiplicity-sum-lt*:
assumes *multiplicity* p $a <$ *multiplicity* p b $a \neq 0$ $b \neq 0$
shows *multiplicity* p $(a + b) =$ *multiplicity* p a
proof −

let *?vp* = *multiplicity p*
have *unit*: ¬ *is-unit p*
proof
assume *is-unit p*
then have *?vp a* = *0* **and** *?vp b* = *0* **using** *multiplicity-unit-left* **by** *auto*
with *assms* **show** *False* **by** *auto*
qed

from *multiplicity-decompose'* **obtain** *a'* **where** *a'*: *a* = *p*^*?vp a* ∗ *a'* ¬ *p dvd a'*
using *unit assms* **by** *metis*
from *multiplicity-decompose'* **obtain** *b'* **where** *b'*: *b* = *p*^*?vp b* ∗ *b'*
using *unit assms* **by** *metis*

show *?vp* (*a* + *b*) = *?vp a*
proof (*rule multiplicity-decomposeI*)
let *?k* = *?vp b* − *?vp a*
from *assms* **have** *k*: *?k* > *0* **by** *simp*
with *b'* **have** *b* = *p*^*?vp a* ∗ *p*^*?k* ∗ *b'*
by (*simp flip*: *power-add*)
with *a'* **show** ∗: *a* + *b* = *p*^*?vp a* ∗ (*a'* + *p*^*?k* ∗ *b'*)
by (*simp add*: *ac-simps distrib-left*)
moreover show ¬ *p dvd a'* + *p*^*?k* ∗ *b'*
using *a' k dvd-add-left-iff* **by** *auto*
show *p* ≠ *0* **using** *assms* **by** *auto*
qed
qed

corollary *multiplicity-sum-min*:
assumes *multiplicity p a* ≠ *multiplicity p b a* ≠ *0 b* ≠ *0*
shows *multiplicity p* (*a* + *b*) = *min* (*multiplicity p a*) (*multiplicity p b*)
proof −
let *?vp* = *multiplicity p*
from *assms* **have** *?vp a* < *?vp b* ∨ *?vp a* > *?vp b*
by *auto*
then show *?thesis*
by (*metis assms multiplicity-sum-lt min.commute add-commute min.strict-order-iff*)

qed

end

lifting-update *multiset.lifting*
lifting-forget *multiset.lifting*

end

2 Abstract euclidean algorithm in euclidean (semi)rings

theory *Euclidean-Algorithm*

imports *Factorial-Ring*
begin

2.1 Generic construction of the (simple) euclidean algorithm

class *normalization-euclidean-semiring* = *euclidean-semiring* + *normalization-semidom*
begin

lemma *euclidean-size-normalize* [*simp*]:
euclidean-size (*normalize a*) = *euclidean-size a*
proof (*cases a* = *0*)
case *True*
then show *?thesis*
by *simp*
next
case [*simp*]: *False*
have *euclidean-size* (*normalize a*) ≤ *euclidean-size* (*normalize a* ∗ *unit-factor a*)
by (*rule size-mult-mono*) *simp*
moreover have *euclidean-size a* ≤ *euclidean-size* (*a* ∗ (*1 div unit-factor a*))
by (*rule size-mult-mono*) *simp*
ultimately show *?thesis*
by *simp*
qed

context
begin

qualified function *gcd* :: *'a* ⇒ *'a* ⇒ *'a*
where *gcd a b* = (*if b* = *0 then normalize a else gcd b* (*a mod b*))
by *pat-completeness simp*
termination
by (*relation measure* (*euclidean-size* ∘ *snd*)) (*simp-all add*: *mod-size-less*)

declare *gcd.simps* [*simp del*]

lemma *eucl-induct* [*case-names zero mod*]:
assumes *H1*: ⋀*b*. *P b 0*
and *H2*: ⋀*a b*. *b* ≠ *0* ⟹ *P b* (*a mod b*) ⟹ *P a b*
shows *P a b*
proof (*induct a b rule*: *gcd.induct*)
case (*1 a b*)
show *?case*
proof (*cases b* = *0*)
case *True* **then show** *P a b* **by** *simp* (*rule H1*)
next
case *False*
then have *P b* (*a mod b*)
by (*rule 1.hyps*)
with ‹*b* ≠ *0*› **show** *P a b*

```
    by (blast intro: H2)
  qed
qed

qualified lemma gcd-0:
  gcd a 0 = normalize a
  by (simp add: gcd.simps [of a 0])

qualified lemma gcd-mod:
  a ≠ 0 ⟹ gcd a (b mod a) = gcd b a
  by (simp add: gcd.simps [of b 0] gcd.simps [of b a])

qualified definition lcm :: 'a ⇒ 'a ⇒ 'a
  where lcm a b = normalize (a * b div gcd a b)

qualified definition Lcm :: 'a set ⇒ 'a — Somewhat complicated definition of
Lcm that has the advantage of working for infinite sets as well
  where
  [code del]: Lcm A = (if ∃ l. l ≠ 0 ∧ (∀ a∈A. a dvd l) then
     let l = SOME l. l ≠ 0 ∧ (∀ a∈A. a dvd l) ∧ euclidean-size l =
       (LEAST n. ∃ l. l ≠ 0 ∧ (∀ a∈A. a dvd l) ∧ euclidean-size l = n)
       in normalize l
      else 0)

qualified definition Gcd :: 'a set ⇒ 'a
  where [code del]: Gcd A = Lcm {d. ∀ a∈A. d dvd a}

lemma semiring-gcd:
  class.semiring-gcd one zero times gcd lcm
    divide plus minus unit-factor normalize
proof
  show gcd a b dvd a
    and gcd a b dvd b for a b
    by (induct a b rule: eucl-induct)
      (simp-all add: local.gcd-0 local.gcd-mod dvd-mod-iff)
next
  show c dvd a ⟹ c dvd b ⟹ c dvd gcd a b for a b c
  proof (induct a b rule: eucl-induct)
    case (zero a) from ‹c dvd a› show ?case
      by (rule dvd-trans) (simp add: local.gcd-0)
  next
    case (mod a b)
    then show ?case
      by (simp add: local.gcd-mod dvd-mod-iff)
  qed
next
  show normalize (gcd a b) = gcd a b for a b
    by (induct a b rule: eucl-induct)
      (simp-all add: local.gcd-0 local.gcd-mod)
```

next
show *lcm a b = normalize (a * b div gcd a b)* **for** *a b*
by (*fact local.lcm-def*)
qed

interpretation *semiring-gcd one zero times gcd lcm*
divide plus minus unit-factor normalize
by (*fact semiring-gcd*)

lemma *semiring-Gcd*:
class.semiring-Gcd one zero times gcd lcm Gcd Lcm
divide plus minus unit-factor normalize
proof −
show *?thesis*
proof
have (∀ *a*∈*A*. *a dvd Lcm A*) ∧ (∀ *b*. (∀ *a*∈*A*. *a dvd b*) ⟶ *Lcm A dvd b*) **for** *A*
proof (*cases* ∃ *l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*))
case *False*
then have *Lcm A = 0*
by (*auto simp add*: *local.Lcm-def*)
with *False* **show** *?thesis*
by *auto*
next
case *True*
then obtain l_0 **where** l_0*-props*: l_0 ≠ *0* ∀ *a*∈*A*. *a dvd* l_0 **by** *blast*
define *n* **where** *n* = (*LEAST n*. ∃ *l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*) ∧ *euclidean-size l* = *n*)
define *l* **where** *l* = (*SOME l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*) ∧ *euclidean-size l* = *n*)
have ∃ *l*. *l* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd l*) ∧ *euclidean-size l* = *n*
apply (*subst n-def*)
apply (*rule LeastI* [*of* - *euclidean-size* l_0])
apply (*rule exI* [*of* - l_0])
apply (*simp add*: l_0*-props*)
done
from *someI-ex* [*OF this*] **have** *l* ≠ *0* **and** ∀ *a*∈*A*. *a dvd l*
and *euclidean-size l* = *n*
unfolding *l-def* **by** *simp-all*
{
fix *l′* **assume** ∀ *a*∈*A*. *a dvd l′*
with ‹∀ *a*∈*A*. *a dvd l*› **have** ∀ *a*∈*A*. *a dvd gcd l l′*
by (*auto intro*: *gcd-greatest*)
moreover from ‹*l* ≠ *0*› **have** *gcd l l′* ≠ *0*
by *simp*
ultimately have ∃ *b*. *b* ≠ *0* ∧ (∀ *a*∈*A*. *a dvd b*) ∧
euclidean-size b = euclidean-size (*gcd l l′*)
by (*intro exI* [*of* - *gcd l l′*], *auto*)
then have *euclidean-size* (*gcd l l′*) ≥ *n*
by (*subst n-def*) (*rule Least-le*)

```
      moreover have euclidean-size (gcd l l') ≤ n
      proof −
        have gcd l l' dvd l
          by simp
        then obtain a where l = gcd l l' * a ..
        with ‹l ≠ 0› have a ≠ 0
          by auto
        hence euclidean-size (gcd l l') ≤ euclidean-size (gcd l l' * a)
          by (rule size-mult-mono)
        also have gcd l l' * a = l using ‹l = gcd l l' * a› ..
        also note ‹euclidean-size l = n›
        finally show euclidean-size (gcd l l') ≤ n .
      qed
      ultimately have *: euclidean-size l = euclidean-size (gcd l l')
        by (intro le-antisym, simp-all add: ‹euclidean-size l = n›)
      from ‹l ≠ 0› have l dvd gcd l l'
        by (rule dvd-euclidean-size-eq-imp-dvd) (auto simp add: *)
      hence l dvd l' by (rule dvd-trans [OF - gcd-dvd2])
    }
    with ‹∀a∈A. a dvd l› and ‹l ≠ 0›
      have (∀a∈A. a dvd normalize l) ∧
        (∀l'. (∀a∈A. a dvd l') ⟶ normalize l dvd l')
      by auto
    also from True have normalize l = Lcm A
      by (simp add: local.Lcm-def Let-def n-def l-def)
    finally show ?thesis .
  qed
  then show dvd-Lcm: a ∈ A ⟹ a dvd Lcm A
    and Lcm-least: (⋀a. a ∈ A ⟹ a dvd b) ⟹ Lcm A dvd b for A and a b
    by auto
  show a ∈ A ⟹ Gcd A dvd a for A and a
    by (auto simp add: local.Gcd-def intro: Lcm-least)
  show (⋀a. a ∈ A ⟹ b dvd a) ⟹ b dvd Gcd A for A and b
    by (auto simp add: local.Gcd-def intro: dvd-Lcm)
  show [simp]: normalize (Lcm A) = Lcm A for A
    by (simp add: local.Lcm-def)
  show normalize (Gcd A) = Gcd A for A
    by (simp add: local.Gcd-def)
 qed
qed

end

interpretation semiring-Gcd one zero times
   Euclidean-Algorithm.gcd Euclidean-Algorithm.lcm Euclidean-Algorithm.Gcd Eu-
clidean-Algorithm.Lcm
   divide plus minus unit-factor normalize
 by (fact semiring-Gcd)
```

subclass *factorial-semiring*
proof −
show *class.factorial-semiring divide plus minus zero times one unit-factor normalize*
proof (*standard*, *rule factorial-semiring-altI-aux*) — FIXME rule
fix *x* **assume** *x* ≠ *0*
thus *finite* {*p*. *p dvd x* ∧ *normalize p* = *p*}
proof (*induction euclidean-size x arbitrary*: *x rule*: *less-induct*)
case (*less x*)
show *?case*
proof (*cases* ∃ *y*. *y dvd x* ∧ ¬*x dvd y* ∧ ¬*is-unit y*)
case *False*
have {*p*. *p dvd x* ∧ *normalize p* = *p*} ⊆ {*1*, *normalize x*}
proof
fix *p* **assume** *p*: *p* ∈ {*p*. *p dvd x* ∧ *normalize p* = *p*}
with *False* **have** *is-unit p* ∨ *x dvd p* **by** *blast*
thus *p* ∈ {*1*, *normalize x*}
proof (*elim disjE*)
assume *is-unit p*
hence *normalize p* = *1* **by** (*simp add*: *is-unit-normalize*)
with *p* **show** *?thesis* **by** *simp*
next
assume *x dvd p*
with *p* **have** *normalize p* = *normalize x* **by** (*intro associatedI*) *simp-all*
with *p* **show** *?thesis* **by** *simp*
qed
qed
moreover have *finite* ... **by** *simp*
ultimately show *?thesis* **by** (*rule finite-subset*)
next
case *True*
then obtain *y* **where** *y*: *y dvd x* ¬*x dvd y* ¬*is-unit y* **by** *blast*
define *z* **where** *z* = *x div y*
let *?fctrs* = λ*x*. {*p*. *p dvd x* ∧ *normalize p* = *p*}
from *y* **have** *x*: *x* = *y* ∗ *z* **by** (*simp add*: *z-def*)
with *less.prems* **have** *y* ≠ *0 z* ≠ *0* **by** *auto*
have *normalized-factors-product*:
{*p*. *p dvd a* ∗ *b* ∧ *normalize p* = *p*} ⊆
(λ(*x*,*y*). *normalize* (*x* ∗ *y*)) ‘ ({*p*. *p dvd a* ∧ *normalize p* = *p*} × {*p*. *p dvd b* ∧ *normalize p* = *p*})
for *a b*
proof *safe*
fix *p* **assume** *p*: *p dvd a* ∗ *b normalize p* = *p*
from *p*(*1*) **obtain** *x y* **where** *xy*: *p* = *x* ∗ *y x dvd a y dvd b*
by (*rule dvd-productE*)
define *x′ y′* **where** *x′* = *normalize x* **and** *y′* = *normalize y*
have *p* = *normalize* (*x′* ∗ *y′*)
using *p* **by** (*simp add*: *xy x′-def y′-def*)
moreover have *x′ dvd a* ∧ *normalize x′* = *x′* **and** *y′ dvd b* ∧ *normalize*

```
y' = y'
          using xy by (auto simp: x'-def y'-def)
        ultimately show p ∈ (λ(x, y). normalize (x * y)) `
            ({p. p dvd a ∧ normalize p = p} × {p. p dvd b ∧ normalize p = p})
by fast
      qed
      from x y have ¬is-unit z by (auto simp: mult-unit-dvd-iff)
      have ?fctrs x ⊆ (λ(p,p'). normalize (p * p')) ` (?fctrs y × ?fctrs z)
        by (subst x) (rule normalized-factors-product)
      moreover have ¬y * z dvd y * 1 ¬y * z dvd 1 * z
        by (subst dvd-times-left-cancel-iff dvd-times-right-cancel-iff; fact)+
      hence finite ((λ(p,p'). normalize (p * p')) ` (?fctrs y × ?fctrs z))
      by (intro finite-imageI finite-cartesian-product less dvd-proper-imp-size-less)
          (auto simp: x)
      ultimately show ?thesis by (rule finite-subset)
    qed
  qed
 next
  fix p
  assume irreducible p
  then show prime-elem p
    by (rule irreducible-imp-prime-elem-gcd)
 qed
qed

lemma Gcd-eucl-set [code]:
 Euclidean-Algorithm.Gcd (set xs) = fold Euclidean-Algorithm.gcd xs 0
 by (fact Gcd-set-eq-fold)

lemma Lcm-eucl-set [code]:
 Euclidean-Algorithm.Lcm (set xs) = fold Euclidean-Algorithm.lcm xs 1
 by (fact Lcm-set-eq-fold)

end

lemma prime-elem-int-abs-iff [simp]:
 fixes p :: int
 shows prime-elem |p| ⟷ prime-elem p
 using prime-elem-normalize-iff [of p] by simp

lemma prime-elem-int-minus-iff [simp]:
 fixes p :: int
 shows prime-elem (− p) ⟷ prime-elem p
 using prime-elem-normalize-iff [of − p] by simp

lemma prime-int-iff:
 fixes p :: int
 shows prime p ⟷ p > 0 ∧ prime-elem p
 by (auto simp add: prime-def dest: prime-elem-not-zeroI)
```

2.2 The (simple) euclidean algorithm as gcd computation

class *euclidean-semiring-gcd* = *normalization-euclidean-semiring* + *gcd* + *Gcd* +
  assumes *gcd-eucl*: *Euclidean-Algorithm.gcd* = *GCD.gcd*
    and *lcm-eucl*: *Euclidean-Algorithm.lcm* = *GCD.lcm*
  assumes *Gcd-eucl*: *Euclidean-Algorithm.Gcd* = *GCD.Gcd*
    and *Lcm-eucl*: *Euclidean-Algorithm.Lcm* = *GCD.Lcm*
begin

subclass *semiring-gcd*
  unfolding *gcd-eucl* [*symmetric*] *lcm-eucl* [*symmetric*]
  by (*fact semiring-gcd*)

subclass *semiring-Gcd*
  unfolding *gcd-eucl* [*symmetric*] *lcm-eucl* [*symmetric*]
    Gcd-eucl [*symmetric*] *Lcm-eucl* [*symmetric*]
  by (*fact semiring-Gcd*)

subclass *factorial-semiring-gcd*
proof
  show *gcd a b* = *gcd-factorial a b* **for** *a b*
    apply (*rule sym*)
    apply (*rule gcdI*)
      apply (*fact gcd-lcm-factorial*)+
    done
  then show *lcm a b* = *lcm-factorial a b* **for** *a b*
    by (*simp add*: *lcm-factorial-gcd-factorial lcm-gcd*)
  show *Gcd A* = *Gcd-factorial A* **for** *A*
    apply (*rule sym*)
    apply (*rule GcdI*)
      apply (*fact gcd-lcm-factorial*)+
    done
  show *Lcm A* = *Lcm-factorial A* **for** *A*
    apply (*rule sym*)
    apply (*rule LcmI*)
      apply (*fact gcd-lcm-factorial*)+
    done
qed

lemma *gcd-mod-right* [*simp*]:
  a ≠ *0* ⟹ *gcd a* (*b mod a*) = *gcd a b*
  unfolding *gcd.commute* [*of a b*]
  by (*simp add*: *gcd-eucl* [*symmetric*] *local.gcd-mod*)

lemma *gcd-mod-left* [*simp*]:
  b ≠ *0* ⟹ *gcd* (*a mod b*) *b* = *gcd a b*
  by (*drule gcd-mod-right* [*of* - *a*]) (*simp add*: *gcd.commute*)

lemma *euclidean-size-gcd-le1* [*simp*]:
  assumes *a* ≠ *0*

shows *euclidean-size* (*gcd a b*) ≤ *euclidean-size a*
proof −
from *gcd-dvd1* **obtain** *c* **where** *A*: *a* = *gcd a b* ∗ *c* **..**
with *assms* **have** *c* ≠ *0*
by *auto*
moreover from *this*
have *euclidean-size* (*gcd a b*) ≤ *euclidean-size* (*gcd a b* ∗ *c*)
by (*rule size-mult-mono*)
with *A* **show** *?thesis*
by *simp*
qed

lemma *euclidean-size-gcd-le2* [*simp*]:
b ≠ *0* ⟹ *euclidean-size* (*gcd a b*) ≤ *euclidean-size b*
by (*subst gcd.commute*, *rule euclidean-size-gcd-le1*)

lemma *euclidean-size-gcd-less1*:
assumes *a* ≠ *0* **and** ¬ *a dvd b*
shows *euclidean-size* (*gcd a b*) < *euclidean-size a*
proof (*rule ccontr*)
assume ¬*euclidean-size* (*gcd a b*) < *euclidean-size a*
with ‹*a* ≠ *0*› **have** *A*: *euclidean-size* (*gcd a b*) = *euclidean-size a*
by (*intro le-antisym*, *simp-all*)
have *a dvd gcd a b*
by (*rule dvd-euclidean-size-eq-imp-dvd*) (*simp-all add*: *assms A*)
hence *a dvd b* **using** *dvd-gcdD2* **by** *blast*
with ‹¬ *a dvd b*› **show** *False* **by** *contradiction*
qed

lemma *euclidean-size-gcd-less2*:
assumes *b* ≠ *0* **and** ¬ *b dvd a*
shows *euclidean-size* (*gcd a b*) < *euclidean-size b*
using *assms* **by** (*subst gcd.commute*, *rule euclidean-size-gcd-less1*)

lemma *euclidean-size-lcm-le1*:
assumes *a* ≠ *0* **and** *b* ≠ *0*
shows *euclidean-size a* ≤ *euclidean-size* (*lcm a b*)
proof −
have *a dvd lcm a b* **by** (*rule dvd-lcm1*)
then obtain *c* **where** *A*: *lcm a b* = *a* ∗ *c* **..**
with ‹*a* ≠ *0*› **and** ‹*b* ≠ *0*› **have** *c* ≠ *0* **by** (*auto simp*: *lcm-eq-0-iff*)
then show *?thesis* **by** (*subst A*, *intro size-mult-mono*)
qed

lemma *euclidean-size-lcm-le2*:
a ≠ *0* ⟹ *b* ≠ *0* ⟹ *euclidean-size b* ≤ *euclidean-size* (*lcm a b*)
using *euclidean-size-lcm-le1* [*of b a*] **by** (*simp add*: *ac-simps*)

lemma *euclidean-size-lcm-less1*:

```
  assumes b ≠ 0 and ¬ b dvd a
  shows euclidean-size a < euclidean-size (lcm a b)
proof (rule ccontr)
  from assms have a ≠ 0 by auto
  assume ¬euclidean-size a < euclidean-size (lcm a b)
  with ‹a ≠ 0› and ‹b ≠ 0› have euclidean-size (lcm a b) = euclidean-size a
    by (intro le-antisym, simp, intro euclidean-size-lcm-le1)
  with assms have lcm a b dvd a
    by (rule-tac dvd-euclidean-size-eq-imp-dvd) (auto simp: lcm-eq-0-iff)
  hence b dvd a by (rule lcm-dvdD2)
  with ‹¬b dvd a› show False by contradiction
qed

lemma euclidean-size-lcm-less2:
  assumes a ≠ 0 and ¬ a dvd b
  shows euclidean-size b < euclidean-size (lcm a b)
  using assms euclidean-size-lcm-less1 [of a b] by (simp add: ac-simps)

end

lemma factorial-euclidean-semiring-gcdI:
 OFCLASS('a::{factorial-semiring-gcd, normalization-euclidean-semiring}, euclidean-semiring-gcd-class)
proof
  interpret semiring-Gcd 1 0 times
    Euclidean-Algorithm.gcd Euclidean-Algorithm.lcm
    Euclidean-Algorithm.Gcd Euclidean-Algorithm.Lcm
    divide plus minus unit-factor normalize
    rewrites dvd.dvd (∗) = Rings.dvd
    by (fact semiring-Gcd) (simp add: dvd.dvd-def dvd-def fun-eq-iff)
  show [simp]: Euclidean-Algorithm.gcd = (gcd :: 'a ⇒ -)
  proof (rule ext)+
    fix a b :: 'a
    show Euclidean-Algorithm.gcd a b = gcd a b
    proof (induct a b rule: eucl-induct)
      case zero
      then show ?case
        by simp
    next
      case (mod a b)
      moreover have gcd b (a mod b) = gcd b a
        using GCD.gcd-add-mult [of b a div b a mod b, symmetric]
          by (simp add: div-mult-mod-eq)
      ultimately show ?case
        by (simp add: Euclidean-Algorithm.gcd-mod ac-simps)
    qed
  qed
  show [simp]: Euclidean-Algorithm.Lcm = (Lcm :: 'a set ⇒ -)
    by (auto intro!: Lcm-eqI GCD.dvd-Lcm GCD.Lcm-least)
  show Euclidean-Algorithm.lcm = (lcm :: 'a ⇒ -)
```

by (*simp add*: *fun-eq-iff Euclidean-Algorithm.lcm-def semiring-gcd-class.lcm-gcd*)
show *Euclidean-Algorithm.Gcd* = (*Gcd* :: *'a set* ⇒ -)
by (*simp add*: *fun-eq-iff Euclidean-Algorithm.Gcd-def semiring-Gcd-class.Gcd-Lcm*)
qed

2.3 The extended euclidean algorithm

class *euclidean-ring-gcd* = *euclidean-semiring-gcd* + *idom*
begin

subclass *euclidean-ring* **..**
subclass *ring-gcd* **..**
subclass *factorial-ring-gcd* **..**

function *euclid-ext-aux* :: *'a* ⇒ *'a* ⇒ *'a* ⇒ *'a* ⇒ *'a* ⇒ *'a* ⇒ (*'a* × *'a*) × *'a*
where *euclid-ext-aux s' s t' t r' r* = (
if r = *0 then let c* = *1 div unit-factor r' in* ((*s'* ∗ *c*, *t'* ∗ *c*), *normalize r'*)
else let q = *r' div r*
in euclid-ext-aux s (*s'* − *q* ∗ *s*) *t* (*t'* − *q* ∗ *t*) *r* (*r' mod r*))
by *auto*
termination
by (*relation measure* (λ(-, -, -, -, -, *b*). *euclidean-size b*))
(*simp-all add*: *mod-size-less*)

abbreviation (*input*) *euclid-ext* :: *'a* ⇒ *'a* ⇒ (*'a* × *'a*) × *'a*
where *euclid-ext* ≡ *euclid-ext-aux 1 0 0 1*

lemma
assumes *gcd r' r* = *gcd a b*
assumes *s'* ∗ *a* + *t'* ∗ *b* = *r'*
assumes *s* ∗ *a* + *t* ∗ *b* = *r*
assumes *euclid-ext-aux s' s t' t r' r* = ((*x*, *y*), *c*)
shows *euclid-ext-aux-eq-gcd*: *c* = *gcd a b*
and *euclid-ext-aux-bezout*: *x* ∗ *a* + *y* ∗ *b* = *gcd a b*
proof −
have *case euclid-ext-aux s' s t' t r' r of* ((*x*, *y*), *c*) ⇒
x ∗ *a* + *y* ∗ *b* = *c* ∧ *c* = *gcd a b* (**is** *?P* (*euclid-ext-aux s' s t' t r' r*))
using *assms*(*1*−*3*)
proof (*induction s' s t' t r' r rule*: *euclid-ext-aux.induct*)
case (*1 s' s t' t r' r*)
show *?case*
proof (*cases r* = *0*)
case *True*
hence *euclid-ext-aux s' s t' t r' r* =
((*s' div unit-factor r'*, *t' div unit-factor r'*), *normalize r'*)
by (*subst euclid-ext-aux.simps*) (*simp add*: *Let-def*)
also have *?P* ...
proof *safe*
have *s' div unit-factor r'* ∗ *a* + *t' div unit-factor r'* ∗ *b* =

```
            (s' * a + t' * b) div unit-factor r'
        by (cases r' = 0) (simp-all add: unit-div-commute)
       also have s' * a + t' * b = r' by fact
       also have ... div unit-factor r' = normalize r' by simp
      finally show s' div unit-factor r' * a + t' div unit-factor r' * b = normalize
r' .
      next
       from 1.prems True show normalize r' = gcd a b
        by simp
      qed
      finally show ?thesis .
    next
      case False
      hence euclid-ext-aux s' s t' t r' r =
            euclid-ext-aux s (s' - r' div r * s) t (t' - r' div r * t) r (r' mod r)
       by (subst euclid-ext-aux.simps) (simp add: Let-def)
      also from 1.prems False have ?P ...
      proof (intro 1.IH)
       have (s' - r' div r * s) * a + (t' - r' div r * t) * b =
          (s' * a + t' * b) - r' div r * (s * a + t * b) by (simp add: algebra-simps)
       also have s' * a + t' * b = r' by fact
       also have s * a + t * b = r by fact
       also have r' - r' div r * r = r' mod r using div-mult-mod-eq [of r' r]
        by (simp add: algebra-simps)
       finally show (s' - r' div r * s) * a + (t' - r' div r * t) * b = r' mod r .
     qed (auto simp: algebra-simps minus-mod-eq-div-mult [symmetric] gcd.commute)
      finally show ?thesis .
    qed
  qed
  with assms(4) show c = gcd a b x * a + y * b = gcd a b
    by simp-all
qed

declare euclid-ext-aux.simps [simp del]

definition bezout-coefficients :: 'a ⇒ 'a ⇒ 'a × 'a
  where [code]: bezout-coefficients a b = fst (euclid-ext a b)

lemma bezout-coefficients-0:
  bezout-coefficients a 0 = (1 div unit-factor a, 0)
  by (simp add: bezout-coefficients-def euclid-ext-aux.simps)

lemma bezout-coefficients-left-0:
  bezout-coefficients 0 a = (0, 1 div unit-factor a)
  by (simp add: bezout-coefficients-def euclid-ext-aux.simps)

lemma bezout-coefficients:
  assumes bezout-coefficients a b = (x, y)
  shows x * a + y * b = gcd a b
```

using *assms* **by** (*simp add*: *bezout-coefficients-def*
euclid-ext-aux-bezout [*of a b a b 1 0 0 1 x y*] *prod-eq-iff*)

lemma *bezout-coefficients-fst-snd*:
fst (*bezout-coefficients a b*) ∗ *a* + *snd* (*bezout-coefficients a b*) ∗ *b* = *gcd a b*
by (*rule bezout-coefficients*) *simp*

lemma *euclid-ext-eq* [*simp*]:
euclid-ext a b = (*bezout-coefficients a b*, *gcd a b*) (**is** *?p* = *?q*)
proof
show *fst ?p* = *fst ?q*
by (*simp add*: *bezout-coefficients-def*)
have *snd* (*euclid-ext-aux 1 0 0 1 a b*) = *gcd a b*
by (*rule euclid-ext-aux-eq-gcd* [*of a b a b 1 0 0 1*])
(*simp-all add*: *prod-eq-iff*)
then show *snd ?p* = *snd ?q*
by *simp*
qed

declare *euclid-ext-eq* [*symmetric*, *code-unfold*]

end

class *normalization-euclidean-semiring-multiplicative* =
normalization-euclidean-semiring + *normalization-semidom-multiplicative*
begin

subclass *factorial-semiring-multiplicative* **..**

end

class *field-gcd* =
field + *unique-euclidean-ring* + *euclidean-ring-gcd* + *normalization-semidom-multiplicative*
begin

subclass *normalization-euclidean-semiring-multiplicative* **..**

subclass *normalization-euclidean-semiring* **..**

subclass *semiring-gcd-mult-normalize* **..**

end

2.4 Typical instances

instance *nat* :: *normalization-euclidean-semiring* **..**

instance *nat* :: *euclidean-semiring-gcd*
proof

```
  interpret semiring-Gcd 1 0 times
    Euclidean-Algorithm.gcd Euclidean-Algorithm.lcm
    Euclidean-Algorithm.Gcd Euclidean-Algorithm.Lcm
    divide plus minus unit-factor normalize
    rewrites dvd.dvd (∗) = Rings.dvd
    by (fact semiring-Gcd) (simp add: dvd.dvd-def dvd-def fun-eq-iff)
  show [simp]: (Euclidean-Algorithm.gcd :: nat ⇒ -) = gcd
  proof (rule ext)+
    fix m n :: nat
    show Euclidean-Algorithm.gcd m n = gcd m n
    proof (induct m n rule: eucl-induct)
      case zero
      then show ?case
        by simp
    next
      case (mod m n)
      then have gcd n (m mod n) = gcd n m
        using gcd-nat.simps [of m n] by (simp add: ac-simps)
      with mod show ?case
        by (simp add: Euclidean-Algorithm.gcd-mod ac-simps)
    qed
  qed
  show [simp]: (Euclidean-Algorithm.Lcm :: nat set ⇒ -) = Lcm
    by (auto intro!: ext Lcm-eqI)
  show (Euclidean-Algorithm.lcm :: nat ⇒ -) = lcm
   by (simp add: fun-eq-iff Euclidean-Algorithm.lcm-def semiring-gcd-class.lcm-gcd)
  show (Euclidean-Algorithm.Gcd :: nat set ⇒ -) = Gcd
   by (simp add: fun-eq-iff Euclidean-Algorithm.Gcd-def semiring-Gcd-class.Gcd-Lcm)
qed

instance nat :: normalization-euclidean-semiring-multiplicative ..

lemma prime-factorization-Suc-0 [simp]: prime-factorization (Suc 0) = {#}
  unfolding One-nat-def [symmetric] using prime-factorization-1 .

instance int :: normalization-euclidean-semiring ..

instance int :: euclidean-ring-gcd
proof
  interpret semiring-Gcd 1 0 times
    Euclidean-Algorithm.gcd Euclidean-Algorithm.lcm
    Euclidean-Algorithm.Gcd Euclidean-Algorithm.Lcm
    divide plus minus unit-factor normalize
    rewrites dvd.dvd (∗) = Rings.dvd
    by (fact semiring-Gcd) (simp add: dvd.dvd-def dvd-def fun-eq-iff)
  show [simp]: (Euclidean-Algorithm.gcd :: int ⇒ -) = gcd
  proof (rule ext)+
    fix k l :: int
    show Euclidean-Algorithm.gcd k l = gcd k l
```

```
    proof (induct k l rule: eucl-induct)
      case zero
      then show ?case
        by simp
    next
      case (mod k l)
      have gcd l (k mod l) = gcd l k
      proof (cases l 0::int rule: linorder-cases)
        case less
        then show ?thesis
          using gcd-non-0-int [of − l − k] by (simp add: ac-simps)
      next
        case equal
        with mod show ?thesis
          by simp
      next
        case greater
        then show ?thesis
          using gcd-non-0-int [of l k] by (simp add: ac-simps)
      qed
      with mod show ?case
        by (simp add: Euclidean-Algorithm.gcd-mod ac-simps)
    qed
  qed
  show [simp]: (Euclidean-Algorithm.Lcm :: int set ⇒ -) = Lcm
    by (auto intro!: ext Lcm-eqI)
  show (Euclidean-Algorithm.lcm :: int ⇒ -) = lcm
   by (simp add: fun-eq-iff Euclidean-Algorithm.lcm-def semiring-gcd-class.lcm-gcd)
  show (Euclidean-Algorithm.Gcd :: int set ⇒ -) = Gcd
   by (simp add: fun-eq-iff Euclidean-Algorithm.Gcd-def semiring-Gcd-class.Gcd-Lcm)
qed

instance int :: normalization-euclidean-semiring-multiplicative ..

lemma (in idom) prime-CHAR-semidom:
  assumes CHAR('a) > 0
  shows   prime CHAR('a)
proof −
  have False if ab: a ≠ 1 b ≠ 1 CHAR('a) = a * b for a b
  proof −
    from assms ab have a > 0 b > 0
      by (auto intro!: Nat.gr0I)
    have of-nat (a * b) = (0 :: 'a)
      using ab by (metis of-nat-CHAR)
    also have of-nat (a * b) = (of-nat a :: 'a) * of-nat b
      by simp
    finally have of-nat a * of-nat b = (0 :: 'a) .
    moreover have of-nat a * of-nat b ≠ (0 :: 'a)
      using ab ‹a > 0› ‹b > 0›
```

by (*intro no-zero-divisors*) (*auto simp*: *of-nat-eq-0-iff-char-dvd*)
ultimately show *False*
by *contradiction*
qed
moreover have *CHAR*('*a*) > *1*
using *assms CHAR-not-1'* **by** *linarith*
ultimately have *prime-elem CHAR*('*a*)
by (*intro irreducible-imp-prime-elem*) (*auto simp*: *Factorial-Ring.irreducible-def*)
thus *?thesis*
by (*auto simp*: *prime-def*)
qed

end

3 Primes

theory *Primes*
imports *Euclidean-Algorithm*
begin

3.1 Primes on *nat* and *int*

lemma *Suc-0-not-prime-nat* [*simp*]: ¬ *prime* (*Suc 0*)
using *not-prime-1* [**where** *?'a* = *nat*] **by** *simp*

lemma *prime-ge-2-nat*:
p ≥ *2* **if** *prime p* **for** *p* :: *nat*
proof −
from *that* **have** *p* ≠ *0* **and** *p* ≠ *1*
by (*auto dest*: *prime-elem-not-zeroI prime-elem-not-unit*)
then show *?thesis*
by *simp*
qed

lemma *prime-ge-2-int*:
p ≥ *2* **if** *prime p* **for** *p* :: *int*
proof −
from *that* **have** *prime-elem p* **and** |*p*| = *p*
by (*auto dest*: *normalize-prime*)
then have *p* ≠ *0* **and** |*p*| ≠ *1* **and** *p* ≥ *0*
by (*auto dest*: *prime-elem-not-zeroI prime-elem-not-unit*)
then show *?thesis*
by *simp*
qed

lemma *prime-ge-0-int*: *prime p* ⟹ *p* ≥ (*0*::*int*)
using *prime-ge-2-int* [*of p*] **by** *simp*

lemma *prime-gt-0-nat*: *prime p* ⟹ *p* > (*0*::*nat*)

using *prime-ge-2-nat* [*of p*] **by** *simp*

lemma *prime-gt-0-int*: *prime p* $\Longrightarrow$ *p* > (*0*::*int*)
using *prime-ge-2-int* [*of p*] **by** *simp*

lemma *prime-ge-1-nat*: *prime p* $\Longrightarrow$ *p* $\geq$ (*1*::*nat*)
using *prime-ge-2-nat* [*of p*] **by** *simp*

lemma *prime-ge-Suc-0-nat*: *prime p* $\Longrightarrow$ *p* $\geq$ *Suc 0*
using *prime-ge-1-nat* [*of p*] **by** *simp*

lemma *prime-ge-1-int*: *prime p* $\Longrightarrow$ *p* $\geq$ (*1*::*int*)
using *prime-ge-2-int* [*of p*] **by** *simp*

lemma *prime-gt-1-nat*: *prime p* $\Longrightarrow$ *p* > (*1*::*nat*)
using *prime-ge-2-nat* [*of p*] **by** *simp*

lemma *prime-gt-Suc-0-nat*: *prime p* $\Longrightarrow$ *p* > *Suc 0*
using *prime-gt-1-nat* [*of p*] **by** *simp*

lemma *prime-gt-1-int*: *prime p* $\Longrightarrow$ *p* > (*1*::*int*)
using *prime-ge-2-int* [*of p*] **by** *simp*

lemma *prime-natI*:
prime p **if** *p* $\geq$ *2* **and** $\bigwedge$*m n*. *p dvd m* * *n* $\Longrightarrow$ *p dvd m* $\vee$ *p dvd n* **for** *p* :: *nat*
using *that* **by** (*auto intro*!: *primeI prime-elemI*)

lemma *prime-intI*:
prime p **if** *p* $\geq$ *2* **and** $\bigwedge$*m n*. *p dvd m* * *n* $\Longrightarrow$ *p dvd m* $\vee$ *p dvd n* **for** *p* :: *int*
using *that* **by** (*auto intro*!: *primeI prime-elemI*)

lemma *prime-elem-nat-iff* [*simp*]:
prime-elem n $\longleftrightarrow$ *prime n* **for** *n* :: *nat*
by (*simp add*: *prime-def*)

lemma *prime-elem-iff-prime-abs* [*simp*]:
prime-elem k $\longleftrightarrow$ *prime* |*k*| **for** *k* :: *int*
by (*auto intro*: *primeI*)

lemma *prime-nat-int-transfer* [*simp*]:
prime (*int n*) $\longleftrightarrow$ *prime n* (**is** *?P* $\longleftrightarrow$ *?Q*)
proof
assume *?P*
then have *n* $\geq$ *2*
by (*auto dest*: *prime-ge-2-int*)
then show *?Q*
proof (*rule prime-natI*)

```
    fix r s
    assume n dvd r * s
    with of-nat-dvd-iff [of n r * s] have int n dvd int r * int s
      by simp
    with ‹?P› have int n dvd int r ∨ int n dvd int s
      using prime-dvd-mult-iff [of int n int r int s]
      by simp
    then show n dvd r ∨ n dvd s
      by simp
  qed
next
  assume ?Q
  then have int n ≥ 2
    by (auto dest: prime-ge-2-nat)
  then show ?P
  proof (rule prime-intI)
    fix r s
    assume int n dvd r * s
    then have n dvd nat |r * s|
      by simp
    then have n dvd nat |r| * nat |s|
      by (simp add: nat-abs-mult-distrib)
    with ‹?Q› have n dvd nat |r| ∨ n dvd nat |s|
      using prime-dvd-mult-iff [of n nat |r| nat |s|]
      by simp
    then show int n dvd r ∨ int n dvd s
      by simp
  qed
qed

lemma prime-nat-iff-prime [simp]:
  prime (nat k) ⟷ prime k
proof (cases k ≥ 0)
  case True
  then show ?thesis
    using prime-nat-int-transfer [of nat k] by simp
next
  case False
  then show ?thesis
    by (auto dest: prime-ge-2-int)
qed

lemma prime-int-nat-transfer:
  prime k ⟷ k ≥ 0 ∧ prime (nat k)
  by (auto dest: prime-ge-2-int)

lemma prime-nat-naiveI:
  prime p if p ≥ 2 and dvd: ⋀n. n dvd p ⟹ n = 1 ∨ n = p for p :: nat
proof (rule primeI, rule prime-elemI)
```

```
  fix m n :: nat
  assume p dvd m * n
  then obtain r s where p = r * s r dvd m s dvd n
    by (blast dest: division-decomp)
  moreover have r = 1 ∨ r = p
    using ‹r dvd m› ‹p = r * s› dvd [of r] by simp
  ultimately show p dvd m ∨ p dvd n
    by auto
qed (use ‹p ≥ 2› in simp-all)

lemma prime-int-naiveI:
  prime p if p ≥ 2 and dvd: ⋀k. k dvd p ⟹ |k| = 1 ∨ |k| = p for p :: int
proof −
  from ‹p ≥ 2› have nat p ≥ 2
    by simp
  then have prime (nat p)
  proof (rule prime-nat-naiveI)
    fix n
    assume n dvd nat p
    with ‹p ≥ 2› have n dvd nat |p|
      by simp
    then have int n dvd p
      by simp
    with dvd [of int n] show n = 1 ∨ n = nat p
      by auto
  qed
  then show ?thesis
    by simp
qed

lemma prime-nat-iff:
  prime (n :: nat) ⟷ (1 < n ∧ (∀ m. m dvd n ⟶ m = 1 ∨ m = n))
proof (safe intro!: prime-gt-1-nat)
  assume prime n
  then have *: prime-elem n
    by simp
  fix m assume m: m dvd n m ≠ n
  from * ‹m dvd n› have n dvd m ∨ is-unit m
    by (intro irreducibleD' prime-elem-imp-irreducible)
  with m show m = 1 by (auto dest: dvd-antisym)
next
  assume n > 1 ∀ m. m dvd n ⟶ m = 1 ∨ m = n
  then show prime n
    using prime-nat-naiveI [of n] by auto
qed

lemma prime-nat-iff':
  prime (p :: nat) ⟷ p > 1 ∧ (∀ n ∈ {2..<p}. ¬ n dvd p)
proof safe
```

```
  assume p > 1 and *: ∀n∈{2..<p}. ¬n dvd p
  show prime p unfolding prime-nat-iff
  proof (intro conjI allI impI)
    fix m assume m dvd p
    with ‹p > 1› have m ≠ 0 by (intro notI) auto
    hence m ≥ 1 by simp
    moreover from ‹m dvd p› and * have m ∉ {2..<p} by blast
    with ‹m dvd p› and ‹p > 1› have m ≤ 1 ∨ m = p by (auto dest: dvd-imp-le)
    ultimately show m = 1 ∨ m = p by simp
  qed fact+
qed (auto simp: prime-nat-iff)

lemma prime-int-iff:
  prime (n::int) ⟷ (1 < n ∧ (∀m. m ≥ 0 ∧ m dvd n ⟶ m = 1 ∨ m = n))
proof (intro iffI conjI allI impI; (elim conjE)?)
  assume *: prime n
  hence irred: irreducible n by (auto intro: prime-elem-imp-irreducible)
  from * have n ≥ 0 n ≠ 0 n ≠ 1
    by (auto simp add: prime-ge-0-int)
  thus n > 1 by presburger
  fix m assume m dvd n ‹m ≥ 0›
  with irred have m dvd 1 ∨ n dvd m by (auto simp: irreducible-altdef)
  with ‹m dvd n› ‹m ≥ 0› ‹n > 1› show m = 1 ∨ m = n
    using associated-iff-dvd[of m n] by auto
next
  assume n: 1 < n ∀m. m ≥ 0 ∧ m dvd n ⟶ m = 1 ∨ m = n
  hence nat n > 1 by simp
  moreover have ∀m. m dvd nat n ⟶ m = 1 ∨ m = nat n
  proof (intro allI impI)
    fix m assume m dvd nat n
    with ‹n > 1› have m dvd nat |n|
      by simp
    then have int m dvd n
      by simp
    with n(2) have int m = 1 ∨ int m = n
      using of-nat-0-le-iff by blast
    thus m = 1 ∨ m = nat n by auto
  qed
  ultimately show prime n
    unfolding prime-int-nat-transfer prime-nat-iff by auto
qed

lemma prime-int-iff':
  prime (p :: int) ⟷ p > 1 ∧ (∀n ∈ {2..<p}. ¬ n dvd p) (is ?P ⟷ ?Q)
proof (cases p ≥ 0)
  case True
  have ?P ⟷ prime (nat p)
    by simp
  also have ... ⟷ p > 1 ∧ (∀n∈{2..<nat p}. ¬ n dvd nat |p|)
```

```
    using True by (simp add: prime-nat-iff')
  also have {2..<nat p} = nat ` {2..<p}
    using True int-eq-iff by fastforce
  finally show ?P ⟷ ?Q by simp
next
  case False
  then show ?thesis
    by (auto simp add: prime-ge-0-int)
qed

lemma prime-nat-not-dvd:
  assumes prime p p > n n ≠ (1::nat)
  shows   ¬n dvd p
proof
  assume n dvd p
  from assms(1) have irreducible p by (simp add: prime-elem-imp-irreducible)
  from irreducibleD'[OF this ‹n dvd p›] ‹n dvd p› ‹p > n› assms show False
    by (cases n = 0) (auto dest!: dvd-imp-le)
qed

lemma prime-int-not-dvd:
  assumes prime p p > n n > (1::int)
  shows   ¬n dvd p
proof
  assume n dvd p
  from assms(1) have irreducible p by (auto intro: prime-elem-imp-irreducible)
  from irreducibleD'[OF this ‹n dvd p›] ‹n dvd p› ‹p > n› assms show False
    by (auto dest!: zdvd-imp-le)
qed

lemma prime-odd-nat: prime p ⟹ p > (2::nat) ⟹ odd p
  by (intro prime-nat-not-dvd) auto

lemma prime-odd-int: prime p ⟹ p > (2::int) ⟹ odd p
  by (intro prime-int-not-dvd) auto

lemma prime-int-altdef:
  prime p = (1 < p ∧ (∀ m::int. m ≥ 0 ⟶ m dvd p ⟶
    m = 1 ∨ m = p))
  unfolding prime-int-iff by blast

lemma not-prime-eq-prod-nat:
  assumes m > 1 ¬ prime (m::nat)
  shows   ∃ n k. n = m * k ∧ 1 < m ∧ m < n ∧ 1 < k ∧ k < n
  using assms irreducible-altdef[of m]
  by (auto simp: prime-elem-iff-irreducible irreducible-altdef)
```

3.2 Make prime naively executable

lemma *prime-int-numeral-eq* [*simp*]:
prime (*numeral m* :: *int*) ⟷ *prime* (*numeral m* :: *nat*)
by (*simp add*: *prime-int-nat-transfer*)

class *check-prime-by-range* = *normalization-semidom* + *discrete-linordered-semidom* +
assumes *prime-iff*: ‹*prime a* ⟷ *1* < *a* ∧ (∀ *d*∈{*2*..*a div 2*}. ¬ *d dvd a*)›
begin

lemma *two-is-prime* [*simp*]:
‹*prime 2*›
by (*simp add*: *prime-iff*)

end

lemma *divisor-less-eq-half-nat*:
‹*m* ≤ *n div 2*› **if** ‹*m dvd n*› ‹*m* < *n*› **for** *m n* :: *nat*
using *that* **by** (*auto simp add*: *less-eq-div-iff-mult-less-eq*)

instance *nat* :: *check-prime-by-range*
apply *standard*
apply (*auto simp add*: *prime-nat-iff*)
apply (*rule ccontr*)
apply (*auto simp add*: *neq-iff*)
apply (*metis One-nat-def Suc-1 Suc-leI atLeastAtMost-iff divisor-less-eq-half-nat*)
done

lemma *two-is-prime-nat* [*simp*]:
‹*prime* (*2*::*nat*)›
by (*fact two-is-prime*)

lemma *divisor-less-eq-half-int*:
‹*k* ≤ *l div 2*› **if** ‹*k dvd l*› ‹*k* < *l*› ‹*l* ≥ *0*› ‹*k* ≥ *0*› **for** *k l* :: *int*
proof −
define *m n* **where** ‹*m* = *nat* |*k*|› ‹*n* = *nat* |*l*|›
with ‹*l* ≥ *0*› ‹*k* ≥ *0*› **have** ‹*k* = *int m*› ‹*l* = *int n*›
by *simp-all*
with *that* **show** *?thesis*
using *divisor-less-eq-half-nat* [*of m n*] **by** *simp*
qed

instance *int* :: *check-prime-by-range*
apply *standard*
apply (*auto simp add*: *prime-int-iff*)
apply (*smt* (*verit*) *int-div-less-self*)
apply (*rule ccontr*)
apply (*auto simp add*: *neq-iff zdvd-not-zless*)
apply (*metis div-by-0 dvd-div-eq-0-iff less-le-not-le one-dvd order-le-less*

zdvd-not-zless)
apply (*metis atLeastAtMost-iff divisor-less-eq-half-int dvd-div-eq-0-iff int-one-le-iff-zero-less nle-le one-add-one pos-imp-zdiv-nonneg-iff zdiv-eq-0-iff zless-imp-add1-zle*)
done

lemma *prime-nat-numeral-eq* [*simp*]: — TODO Sieve Of Erathosthenes might speed this up
prime (*numeral m* :: *nat*) ⟷
(*1*::*nat*) < *numeral m* ∧
(∀ *n*::*nat* ∈ *set* [*2*..<*Suc* (*numeral m div 2*)]. ¬ *n dvd numeral m*)
using *prime-iff* [*of* ‹*numeral m* :: *nat*›]
by (*simp only*: *set-upt atLeastLessThanSuc-atLeastAtMost*)

context *check-prime-by-range*
begin

definition *check-divisors* :: ‹′*a* ⇒ ′*a* ⇒ ′*a* ⇒ *bool*›
where ‹*check-divisors l u a* ⟷ (∀ *d*∈{*l*..*u*}. ¬ *d dvd a*)›

lemma *check-divisors-rec* [*code*]:
‹*check-divisors l u a* ⟷ *u* < *l* ∨ (¬ *l dvd a* ∧ *check-divisors* (*l* + *1*) *u a*)›
apply (*auto simp add*: *check-divisors-def not-less*)
apply (*metis local.add-increasing2 local.atLeastAtMost-iff local.linear local.order-eq-iff local.zero-le-one*)
subgoal for *d*
apply (*cases* ‹*l* + *1* ≤ *d*›)
apply (*auto simp add*: *not-le*)
apply (*metis local.dual-order.antisym local.less-eq-iff-succ-less*)
done
done

lemma *prime-eq-check-divisors* [*code*]:
‹*prime a* ⟷ *a* > *1* ∧ *check-divisors 2* (*a div 2*) *a*›
by (*simp add*: *check-divisors-def prime-iff*)

end

3.3 Largest exponent of a prime factor

lemma *prime-factor-nat*:
n ≠ (*1*::*nat*) ⟹ ∃ *p*. *prime p* ∧ *p dvd n*
using *prime-divisor-exists*[*of n*]
by (*cases n* = *0*) (*auto intro*: *exI*[*of* - *2*::*nat*])

lemma *prime-factor-int*:
fixes *k* :: *int*
assumes |*k*| ≠ *1*
obtains *p* **where** *prime p p dvd k*

proof (*cases k = 0*)
case *True*
then have *prime* (*2*::*int*) **and** *2 dvd k*
by *simp-all*
with *that* **show** *thesis*
by *blast*
next
case *False*
with *assms prime-divisor-exists* [*of k*] **obtain** *p* **where** *prime p p dvd k*
by *auto*
with *that* **show** *thesis*
by *blast*
qed

Possibly duplicates other material, but avoid the complexities of multisets.

lemma *prime-power-cancel-less*:
assumes *prime p* **and** *eq*: $m * (p \hat{} k) = m' * (p \hat{} k')$ **and** *less*: $k < k'$ **and** $\neg$ *p dvd m*
shows *False*
proof −
obtain *l* **where** *l*: $k' = k + l$ **and** $l > 0$
using *less less-imp-add-positive* **by** *auto*
have $m = m * (p \hat{} k)$ *div* $(p \hat{} k)$
using ‹*prime p*› **by** *simp*
also have $\ldots = m' * (p \hat{} k')$ *div* $(p \hat{} k)$
using *eq* **by** *simp*
also have $\ldots = m' * (p \hat{} l) * (p \hat{} k)$ *div* $(p \hat{} k)$
by (*simp add*: *l mult.commute mult.left-commute power-add*)
also have ... $= m' * (p \hat{} l)$
using ‹*prime p*› **by** *simp*
finally have *p dvd m*
using ‹$l > 0$› **by** *simp*
with *assms* **show** *False*
by *simp*
qed

lemma *prime-power-cancel*:
assumes *prime p* **and** *eq*: $m * (p \hat{} k) = m' * (p \hat{} k')$ **and** $\neg$ *p dvd m* $\neg$ *p dvd m'*
shows $k = k'$
using *prime-power-cancel-less* [*OF* ‹*prime p*›] *assms*
by (*metis linorder-neqE-nat*)

lemma *prime-power-cancel2*:
assumes *prime p* $m * (p \hat{} k) = m' * (p \hat{} k')$ $\neg$ *p dvd m* $\neg$ *p dvd m'*
obtains $m = m'$ $k = k'$
using *prime-power-cancel* [*OF assms*] *assms* **by** *auto*

lemma *prime-power-canonical*:

```
  fixes m :: nat
  assumes prime p m > 0
  shows ∃k n. ¬ p dvd n ∧ m = n * p ^ k
using ‹m > 0›
proof (induction m rule: less-induct)
  case (less m)
  show ?case
  proof (cases p dvd m)
    case True
    then obtain m' where m': m = p * m'
      using dvdE by blast
    with ‹prime p› have 0 < m' m' < m
      using less.prems prime-nat-iff by auto
    with m' less show ?thesis
      by (metis power-Suc mult.left-commute)
  next
    case False
    then show ?thesis
      by (metis mult.right-neutral power-0)
  qed
qed
```

3.4 Infinitely many primes

```
lemma next-prime-bound: ∃p::nat. prime p ∧ n < p ∧ p ≤ fact n + 1
proof−
  have f1: fact n + 1 ≠ (1::nat) using fact-ge-1 [of n, where 'a=nat] by arith
  from prime-factor-nat [OF f1]
  obtain p :: nat where prime p and p dvd fact n + 1 by auto
  then have p ≤ fact n + 1 apply (intro dvd-imp-le) apply auto done
  { assume p ≤ n
    from ‹prime p› have p ≥ 1
      by (cases p, simp-all)
    with ‹p <= n› have p dvd fact n
      by (intro dvd-fact)
    with ‹p dvd fact n + 1› have p dvd fact n + 1 − fact n
      by (rule dvd-diff-nat)
    then have p dvd 1 by simp
    then have p <= 1 by auto
    moreover from ‹prime p› have p > 1
      using prime-nat-iff by blast
    ultimately have False by auto}
  then have n < p by presburger
  with ‹prime p› and ‹p <= fact n + 1› show ?thesis by auto
qed

lemma bigger-prime: ∃p. prime p ∧ p > (n::nat)
  using next-prime-bound by auto
```

lemma *primes-infinite*: ¬ (*finite* {(*p*::*nat*). *prime p*})
proof
  assume *finite* {(*p*::*nat*). *prime p*}
  with *Max-ge* **have** (∃ *b*. (∀ *x* ∈ {(*p*::*nat*). *prime p*}. *x* ≤ *b*))
    by *auto*
  then obtain *b* **where** ∀ (*x*::*nat*). *prime x* ⟶ *x* ≤ *b*
    by *auto*
  with *bigger-prime* [*of b*] **show** *False*
    by *auto*
qed

3.5 Powers of Primes

Versions for type nat only

lemma *prime-product*:
  fixes *p*::*nat*
  assumes *prime* (*p* ∗ *q*)
  shows *p* = *1* ∨ *q* = *1*
proof −
  from *assms* **have**
    1 < *p* ∗ *q* **and** *P*: ⋀*m*. *m dvd p* ∗ *q* ⟹ *m* = *1* ∨ *m* = *p* ∗ *q*
    unfolding *prime-nat-iff* **by** *auto*
  from ‹*1* < *p* ∗ *q*› **have** *p* ≠ *0* **by** (*cases p*) *auto*
  then have *Q*: *p* = *p* ∗ *q* ⟷ *q* = *1* **by** *auto*
  have *p dvd p* ∗ *q* **by** *simp*
  then have *p* = *1* ∨ *p* = *p* ∗ *q* **by** (*rule P*)
  then show *?thesis* **by** (*simp add*: *Q*)
qed

lemma *prime-power-mult-nat*:
  fixes *p* :: *nat*
  assumes *p*: *prime p* **and** *xy*: *x* ∗ *y* = *p* ^ *k*
  shows ∃ *i j*. *x* = *p* ^ *i* ∧ *y* = *p* ^ *j*
using *xy*
proof(*induct k arbitrary*: *x y*)
  case *0* **thus** *?case* **apply** *simp* **by** (*rule exI*[**where** *x*=*0*], *simp*)
next
  case (*Suc k x y*)
  from *Suc.prems* **have** *pxy*: *p dvd x*∗*y* **by** *auto*
  from *prime-dvd-multD* [*OF p pxy*] **have** *pxyc*: *p dvd x* ∨ *p dvd y* **.**
  from *p* **have** *p0*: *p* ≠ *0* **by** − (*rule ccontr*, *simp*)
  {assume *px*: *p dvd x*
    then obtain *d* **where** *d*: *x* = *p*∗*d* **unfolding** *dvd-def* **by** *blast*
    from *Suc.prems d* **have** *p*∗*d*∗*y* = *p*^*Suc k* **by** *simp*
    hence *th*: *d*∗*y* = *p*^*k* **using** *p0* **by** *simp*
    from *Suc.hyps*[*OF th*] **obtain** *i j* **where** *ij*: *d* = *p*^*i y* = *p*^*j* **by** *blast*
    with *d* **have** *x* = *p*^*Suc i* **by** *simp*
    with *ij*(*2*) **have** *?case* **by** *blast***}**

```
  moreover
  {assume px: p dvd y
    then obtain d where d: y = p*d unfolding dvd-def by blast
    from Suc.prems d  have p*d*x = p^Suc k by (simp add: mult.commute)
    hence th: d*x = p^k using p0 by simp
    from Suc.hyps[OF th] obtain i j where ij: d = p^i x = p^j by blast
    with d have y = p^Suc i by simp
    with ij(2) have ?case by blast}
  ultimately show ?case  using pxyc by blast
qed

lemma prime-power-exp-nat:
  fixes p::nat
  assumes p: prime p and n: n ≠ 0
    and xn: x^n = p^k shows ∃ i. x = p^i
  using n xn
proof(induct n arbitrary: k)
  case 0 thus ?case by simp
next
  case (Suc n k) hence th: x*x^n = p^k by simp
  {assume n = 0 with Suc have ?case by simp (rule exI[where x=k], simp)}
  moreover
  {assume n: n ≠ 0
    from prime-power-mult-nat[OF p th]
    obtain i j where ij: x = p^i x^n = p^jby blast
    from Suc.hyps[OF n ij(2)] have ?case .}
  ultimately show ?case by blast
qed

lemma divides-primepow-nat:
  fixes p :: nat
  assumes p: prime p
  shows d dvd p ^ k ⟷ (∃ i≤k. d = p ^ i)
  using assms divides-primepow [of p d k] by (auto intro: le-imp-power-dvd)

lemma gcd-prime-int:
  assumes prime (p :: int)
  shows   gcd p k = (if p dvd k then p else 1)
proof −
  have p ≥ 0
    using assms prime-ge-0-int by auto
  show ?thesis
  proof (cases p dvd k)
    case True
    thus ?thesis using assms ‹p ≥ 0› by auto
  next
    case False
    hence coprime p k
      using assms by (simp add: prime-imp-coprime)
```

with *False* **show** *?thesis*
by *auto*
qed
qed

3.6 Chinese Remainder Theorem Variants

lemma *bezout-gcd-nat*:
fixes *a*::*nat* **shows** $\exists x\ y.\ a * x - b * y = gcd\ a\ b \vee b * x - a * y = gcd\ a\ b$
using *bezout-nat*[*of a b*]
by (*metis bezout-nat diff-add-inverse gcd-add-mult gcd.commute*
gcd-nat.right-neutral mult-0)

lemma *gcd-bezout-sum-nat*:
fixes *a*::*nat*
assumes $a * x + b * y = d$
shows *gcd a b dvd d*
proof−
let *?g* = *gcd a b*
have *dv*: *?g dvd a*∗*x ?g dvd b* ∗ *y*
by *simp-all*
from *dvd-add*[*OF dv*] *assms*
show *?thesis* **by** *auto*
qed

A binary form of the Chinese Remainder Theorem.

lemma *chinese-remainder*:
fixes *a*::*nat* **assumes** *ab*: *coprime a b* **and** *a*: $a \neq 0$ **and** *b*: $b \neq 0$
shows $\exists x\ q1\ q2.\ x = u + q1 * a \wedge x = v + q2 * b$
proof−
from *bezout-add-strong-nat*[*OF a*, *of b*] *bezout-add-strong-nat*[*OF b*, *of a*]
obtain *d1 x1 y1 d2 x2 y2* **where** *dxy1*: *d1 dvd a d1 dvd b a* ∗ *x1* = *b* ∗ *y1* + *d1*
and *dxy2*: *d2 dvd b d2 dvd a b* ∗ *x2* = *a* ∗ *y2* + *d2* **by** *blast*
then have *d12*: *d1* = *1 d2* = *1*
using *ab coprime-common-divisor-nat* [*of a b*] **by** *blast*+
let *?x* = *v* ∗ *a* ∗ *x1* + *u* ∗ *b* ∗ *x2*
let *?q1* = *v* ∗ *x1* + *u* ∗ *y2*
let *?q2* = *v* ∗ *y1* + *u* ∗ *x2*
from *dxy2*(*3*)[*simplified d12*] *dxy1*(*3*)[*simplified d12*]
have *?x* = *u* + *?q1* ∗ *a ?x* = *v* + *?q2* ∗ *b*
by *algebra*+
thus *?thesis* **by** *blast*
qed

Primality

lemma *coprime-bezout-strong*:
fixes *a*::*nat* **assumes** *coprime a b* $b \neq 1$
shows $\exists x\ y.\ a * x = b * y + 1$

by (*metis add.commute add.right-neutral assms(1) assms(2) chinese-remainder coprime-1-left coprime-1-right coprime-crossproduct-nat mult.commute mult.right-neutral mult-cancel-left*)

lemma *bezout-prime*:
assumes *p*: *prime p* **and** *pa*: ¬ *p dvd a*
shows ∃ *x y*. *a*∗*x* = *Suc* (*p*∗*y*)
proof −
have *ap*: *coprime a p*
using *coprime-commute p pa prime-imp-coprime* **by** *auto*
moreover from *p* **have** *p* ≠ *1* **by** *auto*
ultimately have ∃ *x y*. *a* ∗ *x* = *p* ∗ *y* + *1*
by (*rule coprime-bezout-strong*)
then show *?thesis* **by** *simp*
qed

3.7 Multiplicity and primality for natural numbers and integers

lemma *prime-factors-gt-0-nat*:
p ∈ *prime-factors x* ⟹ *p* > (*0*::*nat*)
by (*simp add*: *in-prime-factors-imp-prime prime-gt-0-nat*)

lemma *prime-factors-gt-0-int*:
p ∈ *prime-factors x* ⟹ *p* > (*0*::*int*)
by (*simp add*: *in-prime-factors-imp-prime prime-gt-0-int*)

lemma *prime-factors-ge-0-int* [*elim*]:
fixes *n* :: *int*
shows *p* ∈ *prime-factors n* ⟹ *p* ≥ *0*
by (*drule prime-factors-gt-0-int*) *simp*

lemma *prod-mset-prime-factorization-int*:
fixes *n* :: *int*
assumes *n* > *0*
shows *prod-mset* (*prime-factorization n*) = *n*
using *assms* **by** (*simp add*: *prod-mset-prime-factorization*)

lemma *prime-factorization-exists-nat*:
n > *0* ⟹ (∃ *M*. (∀ *p*::*nat* ∈ *set-mset M*. *prime p*) ∧ *n* = (∏ *i* ∈# *M*. *i*))
using *prime-factorization-exists*[*of n*] **by** *auto*

lemma *prod-mset-prime-factorization-nat* [*simp*]:
(*n*::*nat*) > *0* ⟹ *prod-mset* (*prime-factorization n*) = *n*
by (*subst prod-mset-prime-factorization*) *simp-all*

lemma *prime-factorization-nat*:
n > (*0*::*nat*) ⟹ *n* = (∏ *p* ∈ *prime-factors n*. *p* ^ *multiplicity p n*)
by (*simp add*: *prod-prime-factors*)

lemma *prime-factorization-int*:
  $n > (0::int) \Longrightarrow n = (\prod p \in \textit{prime-factors}\ n.\ p\ \hat{}\ \textit{multiplicity}\ p\ n)$
 by (*simp add*: *prod-prime-factors*)

lemma *prime-factorization-unique-nat*:
 fixes $f :: nat \Rightarrow -$
 assumes *S-eq*: $S = \{p.\ 0 < f\ p\}$
  and *finite S*
  and *S*: $\forall p{\in}S.\ \textit{prime}\ p\ n = (\prod p{\in}S.\ p\ \hat{}\ f\ p)$
 shows $S = \textit{prime-factors}\ n \wedge (\forall p.\ \textit{prime}\ p \longrightarrow f\ p = \textit{multiplicity}\ p\ n)$
 using *assms* **by** (*intro prime-factorization-unique''*) *auto*

lemma *prime-factorization-unique-int*:
 fixes $f :: int \Rightarrow -$
 assumes *S-eq*: $S = \{p.\ 0 < f\ p\}$
  and *finite S*
  and *S*: $\forall p{\in}S.\ \textit{prime}\ p\ \textit{abs}\ n = (\prod p{\in}S.\ p\ \hat{}\ f\ p)$
 shows $S = \textit{prime-factors}\ n \wedge (\forall p.\ \textit{prime}\ p \longrightarrow f\ p = \textit{multiplicity}\ p\ n)$
 using *assms* **by** (*intro prime-factorization-unique''*) *auto*

lemma *prime-factors-characterization-nat*:
 $S = \{p.\ 0 < f\ (p::nat)\} \Longrightarrow$
  $\textit{finite}\ S \Longrightarrow \forall p{\in}S.\ \textit{prime}\ p \Longrightarrow n = (\prod p{\in}S.\ p\ \hat{}\ f\ p) \Longrightarrow \textit{prime-factors}\ n = S$
 by (*rule prime-factorization-unique-nat* [*THEN conjunct1*, *symmetric*])

lemma *prime-factors-characterization'-nat*:
 $\textit{finite}\ \{p.\ 0 < f\ (p::nat)\} \Longrightarrow$
  $(\forall p.\ 0 < f\ p \longrightarrow \textit{prime}\ p) \Longrightarrow$
   $\textit{prime-factors}\ (\prod p \mid 0 < f\ p.\ p\ \hat{}\ f\ p) = \{p.\ 0 < f\ p\}$
 by (*rule prime-factors-characterization-nat*) *auto*

lemma *prime-factors-characterization-int*:
 $S = \{p.\ 0 < f\ (p::int)\} \Longrightarrow \textit{finite}\ S \Longrightarrow$
  $\forall p{\in}S.\ \textit{prime}\ p \Longrightarrow \textit{abs}\ n = (\prod p{\in}S.\ p\ \hat{}\ f\ p) \Longrightarrow \textit{prime-factors}\ n = S$
 by (*rule prime-factorization-unique-int* [*THEN conjunct1*, *symmetric*])

lemma *abs-prod*: $\textit{abs}\ (\textit{prod}\ f\ A :: 'a :: \textit{linordered-idom}) = \textit{prod}\ (\lambda x.\ \textit{abs}\ (f\ x))\ A$
 by (*cases finite A*, *induction A rule*: *finite-induct*) (*simp-all add*: *abs-mult*)

lemma *primes-characterization'-int* [*rule-format*]:
 $\textit{finite}\ \{p.\ p \geq 0 \wedge 0 < f\ (p::int)\} \Longrightarrow \forall p.\ 0 < f\ p \longrightarrow \textit{prime}\ p \Longrightarrow$
   $\textit{prime-factors}\ (\prod p \mid p \geq 0 \wedge 0 < f\ p.\ p\ \hat{}\ f\ p) = \{p.\ p \geq 0 \wedge 0 < f\ p\}$
 by (*rule prime-factors-characterization-int*) (*auto simp*: *abs-prod prime-ge-0-int*)

lemma *multiplicity-characterization-nat*:
 $S = \{p.\ 0 < f\ (p::nat)\} \Longrightarrow \textit{finite}\ S \Longrightarrow \forall p{\in}S.\ \textit{prime}\ p \Longrightarrow \textit{prime}\ p \Longrightarrow$
  $n = (\prod p{\in}S.\ p\ \hat{}\ f\ p) \Longrightarrow \textit{multiplicity}\ p\ n = f\ p$

by (*frule prime-factorization-unique-nat* [*of S f n*, *THEN conjunct2*, *rule-format*, *symmetric*]) *auto*

lemma *multiplicity-characterization'-nat*: *finite* {*p*. *0* < *f* (*p*::*nat*)} ⟶
(∀ *p*. *0* < *f p* ⟶ *prime p*) ⟶ *prime p* ⟶
multiplicity p (∏ *p* | *0* < *f p*. *p* ^ *f p*) = *f p*
by (*intro impI*, *rule multiplicity-characterization-nat*) *auto*

lemma *multiplicity-characterization-int*: *S* = {*p*. *0* < *f* (*p*::*int*)} ⟹
finite S ⟹ ∀ *p*∈*S*. *prime p* ⟹ *prime p* ⟹ *n* = (∏ *p*∈*S*. *p* ^ *f p*) ⟹
multiplicity p n = *f p*
by (*frule prime-factorization-unique-int* [*of S f n*, *THEN conjunct2*, *rule-format*, *symmetric*])
(*auto simp*: *abs-prod power-abs prime-ge-0-int intro*!: *prod.cong*)

lemma *multiplicity-characterization'-int* [*rule-format*]:
finite {*p*. *p* ≥ *0* ∧ *0* < *f* (*p*::*int*)} ⟹
(∀ *p*. *0* < *f p* ⟶ *prime p*) ⟹ *prime p* ⟹
multiplicity p (∏ *p* | *p* ≥ *0* ∧ *0* < *f p*. *p* ^ *f p*) = *f p*
by (*rule multiplicity-characterization-int*) (*auto simp*: *prime-ge-0-int*)

lemma *multiplicity-one-nat* [*simp*]: *multiplicity p* (*Suc 0*) = *0*
unfolding *One-nat-def* [*symmetric*] **by** (*rule multiplicity-one*)

lemma *multiplicity-eq-nat*:
fixes *x* **and** *y*::*nat*
assumes *x* > *0 y* > *0* ⋀*p*. *prime p* ⟹ *multiplicity p x* = *multiplicity p y*
shows *x* = *y*
using *multiplicity-eq-imp-eq*[*of x y*] *assms* **by** *simp*

lemma *multiplicity-eq-int*:
fixes *x y* :: *int*
assumes *x* > *0 y* > *0* ⋀*p*. *prime p* ⟹ *multiplicity p x* = *multiplicity p y*
shows *x* = *y*
using *multiplicity-eq-imp-eq*[*of x y*] *assms* **by** *simp*

lemma *multiplicity-prod-prime-powers*:
assumes *finite S* ⋀*x*. *x* ∈ *S* ⟹ *prime x prime p*
shows *multiplicity p* (∏ *p* ∈ *S*. *p* ^ *f p*) = (*if p* ∈ *S then f p else 0*)
proof −
define *g* **where** *g* = (λ*x*. *if x* ∈ *S then f x else 0*)
define *A* **where** *A* = *Abs-multiset g*
have {*x*. *g x* > *0*} ⊆ *S* **by** (*auto simp*: *g-def*)
from *finite-subset*[*OF this assms*(*1*)] **have** [*simp*]: *finite* {*x*. *0* < *g x*}
by *simp*
from *assms* **have** *count-A*: *count A x* = *g x* **for** *x* **unfolding** *A-def*
by *simp*
have *set-mset-A*: *set-mset A* = {*x*∈*S*. *f x* > *0*}
unfolding *set-mset-def count-A* **by** (*auto simp*: *g-def*)

with *assms* **have** *prime*: *prime x* **if** $x \in\# A$ **for** *x* **using** *that* **by** *auto*
from *set-mset-A assms* **have** $(\prod p \in S.\ p\ \widehat{}\ f\ p) = (\prod p \in S.\ p\ \widehat{}\ g\ p)$
by (*intro prod.cong*) (*auto simp*: *g-def*)
also from *set-mset-A assms* **have** $\ldots = (\prod p \in \textit{set-mset}\ A.\ p\ \widehat{}\ g\ p)$
by (*intro prod.mono-neutral-right*) (*auto simp*: *g-def set-mset-A*)
also have $\ldots = \textit{prod-mset}\ A$
by (*auto simp*: *prod-mset-multiplicity count-A set-mset-A intro*!: *prod.cong*)
also from *assms* **have** *multiplicity p* $\ldots$ = *sum-mset* (*image-mset* (*multiplicity p*) *A*)
by (*subst prime-elem-multiplicity-prod-mset-distrib*) (*auto dest*: *prime*)
also from *assms* **have** *image-mset* (*multiplicity p*) *A* = *image-mset* (λx. *if x* = *p then 1 else 0*) *A*
by (*intro image-mset-cong*) (*auto simp*: *prime-multiplicity-other dest*: *prime*)
also have *sum-mset* $\ldots$ = (*if p* $\in$ *S then f p else 0*) **by** (*simp add*: *sum-mset-delta count-A g-def*)
finally show *?thesis* **.**
qed

lemma *prime-factorization-prod-mset*:
assumes $0 \notin\# A$
shows *prime-factorization* (*prod-mset A*) = $\sum_{\#}$(*image-mset prime-factorization A*)
using *assms* **by** (*induct A*) (*auto simp add*: *prime-factorization-mult*)

lemma *prime-factors-prod*:
assumes *finite A* **and** $0 \notin f\ `\ A$
shows *prime-factors* (*prod f A*) = $\bigcup$((*prime-factors* $\circ$ *f*) ` *A*)
using *assms* **by** (*simp add*: *prod-unfold-prod-mset prime-factorization-prod-mset*)

lemma *prime-factors-fact*:
prime-factors (*fact n*) = {*p* $\in$ {*2*..*n*}. *prime p*} (**is** *?M* = *?N*)
proof (*rule set-eqI*)
fix *p*
{ fix *m* :: *nat*
assume *p* $\in$ *prime-factors m*
then have *prime p* **and** *p dvd m* **by** *auto*
moreover assume *m* > *0*
ultimately have *2* $\leq$ *p* **and** *p* $\leq$ *m*
by (*auto intro*: *prime-ge-2-nat dest*: *dvd-imp-le*)
moreover assume *m* $\leq$ *n*
ultimately have *2* $\leq$ *p* **and** *p* $\leq$ *n*
by (*auto intro*: *order-trans*)
} note $*$ = *this*
show *p* $\in$ *?M* $\longleftrightarrow$ *p* $\in$ *?N*
by (*auto simp add*: *fact-prod prime-factors-prod Suc-le-eq dest*!: *prime-prime-factors intro*: $*$)
qed

lemma *prime-dvd-fact-iff*:

assumes *prime p*
shows *p dvd fact n* $\longleftrightarrow$ $p \leq n$
using *assms*
by (*auto simp add*: *prime-factorization-subset-iff-dvd* [*symmetric*]
prime-factorization-prime prime-factors-fact prime-ge-2-nat)

lemma *dvd-choose-prime*:
assumes *kn*: $k < n$ **and** *k*: $k \neq 0$ **and** *n*: $n \neq 0$ **and** *prime-n*: *prime n*
shows *n dvd* (*n choose k*)
proof −
have *n dvd* (*fact n*) **by** (*simp add*: *fact-num-eq-if n*)
moreover have ¬ *n dvd* (*fact k* ∗ *fact* (*n*−*k*))
by (*metis prime-dvd-fact-iff prime-dvd-mult-iff assms neq0-conv diff-less linorder-not-less*)
moreover have (*fact n*::*nat*) = *fact k* ∗ *fact* (*n*−*k*) ∗ (*n choose k*)
using *binomial-fact-lemma kn* **by** *auto*
ultimately show *?thesis* **using** *prime-n*
by (*auto simp add*: *prime-dvd-mult-iff*)
qed

lemma (**in** *ring-1*) *minus-power-prime-CHAR*:
assumes *p* = *CHAR*('*a*) *prime p*
shows (−*x* :: '*a*) ^ *p* = −(*x* ^ *p*)
proof (*cases p* = *2*)
case *False*
have *prime p*
using *assms* **by** *blast*
hence *odd p*
using *prime-imp-coprime assms False coprime-right-2-iff-odd gcd-nat.strict-iff-not*
by *blast*
thus *?thesis*
by *simp*
qed (*use assms* **in** ‹*auto simp*: *uminus-CHAR-2*›)

3.8 Rings and fields with prime characteristic

We introduce some type classes for rings and fields with prime characteristic.

class *semiring-prime-char* = *semiring-1* +
assumes *prime-char-aux*: ∃ *n*. *prime n* ∧ *of-nat n* = (*0* :: '*a*)
begin

lemma *CHAR-pos* [*intro*, *simp*]: *CHAR*('*a*) > *0*
using *local.CHAR-pos-iff local.prime-char-aux prime-gt-0-nat* **by** *blast*

lemma *CHAR-nonzero* [*simp*]: *CHAR*('*a*) ≠ *0*
using *CHAR-pos* **by** *auto*

lemma *CHAR-prime* [*intro*, *simp*]: *prime CHAR*('*a*)
by (*metis* (*mono-tags*, *lifting*) *gcd-nat.order-iff-strict local.of-nat-1 local.of-nat-eq-0-iff-char-dvd*
local.one-neq-zero local.prime-char-aux prime-nat-iff)

end

lemma *semiring-prime-charI* [*intro?*]:
prime CHAR('*a* :: *semiring-1*) ⟹ *OFCLASS*('*a*, *semiring-prime-char-class*)
by *standard auto*

lemma *idom-prime-charI* [*intro?*]:
assumes *CHAR*('*a* :: *idom*) > *0*
shows *OFCLASS*('*a*, *semiring-prime-char-class*)
proof
show *prime CHAR*('*a*)
using *assms prime-CHAR-semidom* **by** *blast*
qed

class *comm-semiring-prime-char* = *comm-semiring-1* + *semiring-prime-char*
class *comm-ring-prime-char* = *comm-ring-1* + *semiring-prime-char*
begin
subclass *comm-semiring-prime-char* **..**
end
class *idom-prime-char* = *idom* + *semiring-prime-char*
begin
subclass *comm-ring-prime-char* **..**
end

class *field-prime-char* = *field* +
assumes *pos-char-exists*: ∃ *n*>*0*. *of-nat n* = (*0* :: '*a*)
begin
subclass *idom-prime-char*
apply *standard*
using *pos-char-exists local.CHAR-pos-iff local.of-nat-CHAR local.prime-CHAR-semidom*
by *blast*
end

lemma *field-prime-charI* [*intro?*]:
n > *0* ⟹ *of-nat n* = (*0* :: '*a* :: *field*) ⟹ *OFCLASS*('*a*, *field-prime-char-class*)
by *standard auto*

lemma *field-prime-charI'* [*intro?*]:
CHAR('*a* :: *field*) > *0* ⟹ *OFCLASS*('*a*, *field-prime-char-class*)
by *standard auto*

3.9 Finite fields

class *finite-field* = *field-prime-char* + *finite*

lemma *finite-fieldI* [*intro?*]:
assumes *finite* (*UNIV* :: '*a* :: *field set*)
shows *OFCLASS*('*a*, *finite-field-class*)

proof *standard*
show $\exists n{>}0.$ *of-nat* $n = (0 :: 'a)$
using *assms prime-CHAR-semidom*[**where** *?'a* = *'a*] *finite-imp-CHAR-pos*[*OF assms*]
by (*intro exI*[*of* - *CHAR*('*a*)]) *auto*
qed *fact+*

On a finite field with n elements, taking the n-th power of an element is the identity. This is an obvious consequence of the fact that the multiplicative group of the field is a finite group of order $n - 1$, so $x \hat{} n = 1$ for any non-zero x.

Note that this result is sharp in the sense that the multiplicative group of a finite field is cyclic, i.e. it contains an element of order $n - 1$. (We don't prove this here.)

lemma *finite-field-power-card-eq-same*:
fixes $x :: 'a :: finite\text{-}field$
shows $x \hat{} card\ (UNIV :: 'a\ set) = x$
proof (*cases* $x = 0$)
case *False*
have $x * (\prod y{\in}UNIV{-}\{0\}.\ x * y) = x * x \hat{} (card\ (UNIV :: 'a\ set) - 1) * \prod (UNIV{-}\{0\})$
by (*simp add*: *prod.distrib mult-ac*)
also have $x * x \hat{} (card\ (UNIV :: 'a\ set) - 1) = x \hat{} Suc\ (card\ (UNIV :: 'a\ set) - 1)$
by (*subst power-Suc*) *auto*
also have $Suc\ (card\ (UNIV :: 'a\ set) - 1) = card\ (UNIV :: 'a\ set)$
using *finite-UNIV-card-ge-0*[**where** *?'a* = *'a*] **by** *simp*
also have $(\prod y{\in}UNIV{-}\{0\}.\ x * y) = (\prod y{\in}UNIV{-}\{0\}.\ y)$
by (*rule prod.reindex-bij-witness*[*of* - $\lambda y.\ y$ / x $\lambda y.\ x * y$]) (*use False* **in** *auto*)
finally show *?thesis*
by *simp*
qed (*use finite-UNIV-card-ge-0*[**where** *?'a* = *'a*] **in** *auto*)

lemma *finite-field-power-card-power-eq-same*:
fixes $x :: 'a :: finite\text{-}field$
assumes $m = card\ (UNIV :: 'a\ set) \hat{} n$
shows $x \hat{} m = x$
unfolding *assms*
by (*induction n*) (*simp-all add*: *finite-field-power-card-eq-same power-mult*)

class *enum-finite-field* = *finite-field* +
fixes *enum-finite-field* :: *nat* ⇒ *'a*
assumes *enum-finite-field*: *enum-finite-field* ' $\{..{<}card\ (UNIV :: 'a\ set)\} = UNIV$
begin

lemma *inj-on-enum-finite-field*: *inj-on enum-finite-field* $\{..{<}card\ (UNIV :: 'a\ set)\}$
using *enum-finite-field* **by** (*simp add*: *eq-card-imp-inj-on*)

end

To get rid of the pending sort hypotheses, we prove that the field with 2 elements is indeed a finite field.

typedef *gf2* = {*0*, *1* :: *nat*}
  by *auto*

setup-lifting *type-definition-gf2*

instantiation *gf2* :: *field*
begin
lift-definition *zero-gf2* :: *gf2* **is** *0* **by** *auto*
lift-definition *one-gf2* :: *gf2* **is** *1* **by** *auto*
lift-definition *uminus-gf2* :: *gf2* ⇒ *gf2* **is** λ*x*. *x* **.**
lift-definition *plus-gf2* :: *gf2* ⇒ *gf2* ⇒ *gf2* **is** λ*x y*. *if x* = *y then 0 else 1* **by** *auto*
lift-definition *minus-gf2* :: *gf2* ⇒ *gf2* ⇒ *gf2* **is** λ*x y*. *if x* = *y then 0 else 1* **by** *auto*
lift-definition *times-gf2* :: *gf2* ⇒ *gf2* ⇒ *gf2* **is** λ*x y*. *x* ∗ *y* **by** *auto*
lift-definition *inverse-gf2* :: *gf2* ⇒ *gf2* **is** λ*x*. *x* **.**
lift-definition *divide-gf2* :: *gf2* ⇒ *gf2* ⇒ *gf2* **is** λ*x y*. *x* ∗ *y* **by** *auto*

instance
  by *standard* (*transfer*; *fastforce*)+

end

instance *gf2* :: *finite-field*
proof
  interpret *type-definition Rep-gf2 Abs-gf2* {*0*, *1* :: *nat*}
    by (*rule type-definition-gf2*)
  show *finite* (*UNIV* :: *gf2 set*)
    by (*metis Abs-image finite.emptyI finite.insertI finite-imageI*)
qed

3.10 The Freshman's Dream in rings of prime characteristic

lemma (**in** *comm-semiring-1*) *freshmans-dream*:
  fixes *x y* :: *'a* **and** *n* :: *nat*
  assumes *prime CHAR*(*'a*)
  assumes *n-def*: *n* = *CHAR*(*'a*)
  shows (*x* + *y*) ^ *n* = *x* ^ *n* + *y* ^ *n*
proof −
  interpret *comm-semiring-prime-char*
    by *standard* (*auto intro*!: *exI*[*of* - *CHAR*(*'a*)] *assms*)
  have *n* > *0*
    unfolding *n-def* **by** *simp*
  have (*x* + *y*) ^ *n* = (∑ *k*≤*n*. *of-nat* (*n choose k*) ∗ *x* ^ *k* ∗ *y* ^ (*n* − *k*))
    by (*rule binomial-ring*)
  also have ... = (∑ *k*∈{*0*,*n*}. *of-nat* (*n choose k*) ∗ *x* ^ *k* ∗ *y* ^ (*n* − *k*))

proof (*intro sum.mono-neutral-right ballI*)
fix k **assume** $k \in \{..n\} - \{0, n\}$
hence k: $k > 0$ $k < n$
by *auto*
have *CHAR*('a) *dvd* (n *choose* k)
unfolding *n-def*
by (*rule dvd-choose-prime*) (*use* k **in** ‹*auto simp*: *n-def*›)
hence *of-nat* (n *choose* k) = (0 :: 'a)
using *of-nat-eq-0-iff-char-dvd* **by** *blast*
thus *of-nat* (n *choose* k) * x ^ k * y ^ (n − k) = 0
by *simp*
qed *auto*
finally show *?thesis*
using ‹$n > 0$› **by** (*simp add*: *add-ac*)
qed

lemma (**in** *comm-semiring-1*) *freshmans-dream'*:
assumes [*simp*]: *prime CHAR*('a) **and** m = *CHAR*('a) ^ n
shows (x + y :: 'a) ^ m = x ^ m + y ^ m
unfolding *assms*(*2*)
proof (*induction n*)
case (*Suc n*)
have (x + y) ^ (*CHAR*('a) ^ n * *CHAR*('a)) = ((x + y) ^ (*CHAR*('a) ^ n)) ^ *CHAR*('a)
by (*rule power-mult*)
thus *?case*
by (*simp add*: *Suc.IH freshmans-dream Groups.mult-ac flip*: *power-mult*)
qed *auto*

lemma (**in** *comm-semiring-1*) *freshmans-dream-sum*:
fixes f :: 'b ⇒ 'a
assumes *prime CHAR*('a) **and** n = *CHAR*('a)
shows *sum* f A ^ n = *sum* (λi. f i ^ n) A
using *assms*
by (*induct A rule*: *infinite-finite-induct*)
(*auto simp add*: *power-0-left freshmans-dream*)

lemma (**in** *comm-semiring-1*) *freshmans-dream-sum'*:
fixes f :: 'b ⇒ 'a
assumes *prime CHAR*('a) m = *CHAR*('a) ^ n
shows *sum* f A ^ m = *sum* (λi. f i ^ m) A
using *assms*
by (*induction A rule*: *infinite-finite-induct*)
(*auto simp*: *freshmans-dream' power-0-left*)

lemmas *prime-imp-coprime-nat* = *prime-imp-coprime*[**where** *?'a* = *nat*]

lemmas *prime-imp-coprime-int* = *prime-imp-coprime*[**where** *?'a* = *int*]
lemmas *prime-dvd-mult-nat* = *prime-dvd-mult-iff*[**where** *?'a* = *nat*]
lemmas *prime-dvd-mult-int* = *prime-dvd-mult-iff*[**where** *?'a* = *int*]
lemmas *prime-dvd-mult-eq-nat* = *prime-dvd-mult-iff*[**where** *?'a* = *nat*]
lemmas *prime-dvd-mult-eq-int* = *prime-dvd-mult-iff*[**where** *?'a* = *int*]
lemmas *prime-dvd-power-nat* = *prime-dvd-power*[**where** *?'a* = *nat*]
lemmas *prime-dvd-power-int* = *prime-dvd-power*[**where** *?'a* = *int*]
lemmas *prime-dvd-power-nat-iff* = *prime-dvd-power-iff*[**where** *?'a* = *nat*]
lemmas *prime-dvd-power-int-iff* = *prime-dvd-power-iff*[**where** *?'a* = *int*]
lemmas *prime-imp-power-coprime-nat* = *prime-imp-power-coprime*[**where** *?'a* = *nat*]
lemmas *prime-imp-power-coprime-int* = *prime-imp-power-coprime*[**where** *?'a* = *int*]
lemmas *primes-coprime-nat* = *primes-coprime*[**where** *?'a* = *nat*]
lemmas *primes-coprime-int* = *primes-coprime*[**where** *?'a* = *nat*]
lemmas *prime-divprod-pow-nat* = *prime-elem-divprod-pow*[**where** *?'a* = *nat*]
lemmas *prime-exp* = *prime-elem-power-iff*[**where** *?'a* = *nat*]

end

4 Polynomials as type over a ring structure

theory *Polynomial*
imports
Complex-Main
HOL−Library.More-List
HOL−Library.Infinite-Set
Primes
begin

context *semidom-modulo*
begin

lemma *not-dvd-imp-mod-neq-0*:
‹*a mod b* ≠ *0*› **if** ‹¬ *b dvd a*›
using *that mod-0-imp-dvd* [*of a b*] **by** *blast*

end

4.1 Auxiliary: operations for lists (later) representing coefficients

definition *cCons* :: *'a*::*zero* ⇒ *'a list* ⇒ *'a list* (**infixr** ‹##› *65*)
where *x ## xs* = (*if xs* = [] ∧ *x* = *0 then* [] *else x # xs*)

lemma *cCons-0-Nil-eq* [*simp*]: *0 ##* [] = []
by (*simp add*: *cCons-def*)

lemma *cCons-Cons-eq* [*simp*]: *x ## y # ys* = *x # y # ys*

by (*simp add*: *cCons-def*)

lemma *cCons-append-Cons-eq* [*simp*]: $x \mathbin{\#\#} xs \mathbin{@} y \mathbin{\#} ys = x \mathbin{\#} xs \mathbin{@} y \mathbin{\#} ys$
by (*simp add*: *cCons-def*)

lemma *cCons-not-0-eq* [*simp*]: $x \neq 0 \Longrightarrow x \mathbin{\#\#} xs = x \mathbin{\#} xs$
by (*simp add*: *cCons-def*)

lemma *strip-while-not-0-Cons-eq* [*simp*]:
strip-while $(\lambda x.\ x = 0)\ (x \mathbin{\#} xs) = x \mathbin{\#\#}$ *strip-while* $(\lambda x.\ x = 0)\ xs$
proof (*cases* $x = 0$)
case *False*
then show *?thesis* **by** *simp*
next
case *True*
show *?thesis*
proof (*induct xs rule*: *rev-induct*)
case *Nil*
with *True* **show** *?case* **by** *simp*
next
case (*snoc y ys*)
then show *?case*
by (*cases* $y = 0$) (*simp-all add*: *append-Cons* [*symmetric*] *del*: *append-Cons*)
qed
qed

lemma *tl-cCons* [*simp*]: *tl* $(x \mathbin{\#\#} xs) = xs$
by (*simp add*: *cCons-def*)

4.2 Definition of type *poly*

typedef (**overloaded**) $'a$ *poly* $= \{f :: nat \Rightarrow 'a{::}zero.\ \forall_\infty n.\ f\ n = 0\}$
morphisms *coeff Abs-poly*
by (*auto intro*!: *ALL-MOST*)

setup-lifting *type-definition-poly*

lemma *poly-eq-iff*: $p = q \longleftrightarrow (\forall n.\ coeff\ p\ n = coeff\ q\ n)$
by (*simp add*: *coeff-inject* [*symmetric*] *fun-eq-iff*)

lemma *poly-eqI*: $(\bigwedge n.\ coeff\ p\ n = coeff\ q\ n) \Longrightarrow p = q$
by (*simp add*: *poly-eq-iff*)

lemma *MOST-coeff-eq-0*: $\forall_\infty n.\ coeff\ p\ n = 0$
using *coeff* [*of p*] **by** *simp*

lemma *coeff-Abs-poly*:
assumes $\bigwedge i.\ i > n \Longrightarrow f\ i = 0$
shows *coeff* (*Abs-poly f*) $= f$

proof (*rule Abs-poly-inverse*, *clarify*)
have *eventually* (λ*i*. *i* > *n*) *cofinite*
by (*auto simp*: *MOST-nat*)
thus *eventually* (λ*i*. *f i* = *0*) *cofinite*
by *eventually-elim* (*use assms* **in** *auto*)
qed

4.3 Degree of a polynomial

definition *degree* :: *'a*::*zero poly* ⇒ *nat*
where *degree p* = (*LEAST n*. ∀ *i*>*n*. *coeff p i* = *0*)

lemma *degree-cong*:
assumes ⋀*i*. *coeff p i* = *0* ⟷ *coeff q i* = *0*
shows *degree p* = *degree q*
proof −
have (λ*n*. ∀ *i*>*n*. *poly.coeff p i* = *0*) = (λ*n*. ∀ *i*>*n*. *poly.coeff q i* = *0*)
using *assms* **by** (*auto simp*: *fun-eq-iff*)
thus *?thesis*
by (*simp only*: *degree-def*)
qed

lemma *coeff-Abs-poly-If-le*:
coeff (*Abs-poly* (λ*i*. *if i* ≤ *n then f i else 0*)) = (λ*i*. *if i* ≤ *n then f i else 0*)
proof (*rule Abs-poly-inverse*, *clarify*)
have *eventually* (λ*i*. *i* > *n*) *cofinite*
by (*auto simp*: *MOST-nat*)
thus *eventually* (λ*i*. (*if i* ≤ *n then f i else 0*) = *0*) *cofinite*
by *eventually-elim auto*
qed

lemma *coeff-eq-0*:
assumes *degree p* < *n*
shows *coeff p n* = *0*
proof −
have ∃ *n*. ∀ *i*>*n*. *coeff p i* = *0*
using *MOST-coeff-eq-0* **by** (*simp add*: *MOST-nat*)
then have ∀ *i*>*degree p*. *coeff p i* = *0*
unfolding *degree-def* **by** (*rule LeastI-ex*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *le-degree*: *coeff p n* ≠ *0* ⟹ *n* ≤ *degree p*
using *coeff-eq-0 linorder-le-less-linear* **by** *blast*

lemma *degree-le*: ∀ *i*>*n*. *coeff p i* = *0* ⟹ *degree p* ≤ *n*
unfolding *degree-def* **by** (*erule Least-le*)

lemma *less-degree-imp*: *n* < *degree p* ⟹ ∃ *i*>*n*. *coeff p i* ≠ *0*

unfolding *degree-def* **by** (*drule not-less-Least*, *simp*)

lemma *poly-eqI2*:
assumes *degree p* = *degree q* **and** $\bigwedge i.\ i \leq \textit{degree}\ p \Longrightarrow \textit{coeff}\ p\ i = \textit{coeff}\ q\ i$
shows $p = q$
by (*metis assms le-degree poly-eqI*)

4.4 The zero polynomial

instantiation *poly* :: (*zero*) *zero*
begin

lift-definition *zero-poly* :: *'a poly*
is λ-. *0*
by (*rule MOST-I*) *simp*

instance ..

end

lemma *coeff-0* [*simp*]: *coeff 0 n* = *0*
by *transfer rule*

lemma *degree-0* [*simp*]: *degree 0* = *0*
by (*rule order-antisym* [*OF degree-le le0*]) *simp*

lemma *leading-coeff-neq-0*:
assumes $p \neq 0$
shows *coeff p* (*degree p*) $\neq$ *0*
proof (*cases degree p*)
case *0*
from ‹$p \neq 0$› **obtain** *n* **where** *coeff p n* $\neq$ *0*
by (*auto simp add*: *poly-eq-iff*)
then have $n \leq$ *degree p*
by (*rule le-degree*)
with ‹*coeff p n* $\neq$ *0*› **and** ‹*degree p* = *0*› **show** *coeff p* (*degree p*) $\neq$ *0*
by *simp*
next
case (*Suc n*)
from ‹*degree p* = *Suc n*› **have** $n <$ *degree p*
by *simp*
then have $\exists i{>}n.$ *coeff p i* $\neq$ *0*
by (*rule less-degree-imp*)
then obtain *i* **where** $n < i$ **and** *coeff p i* $\neq$ *0*
by *blast*
from ‹*degree p* = *Suc n*› **and** ‹$n < i$› **have** *degree p* $\leq i$
by *simp*
also from ‹*coeff p i* $\neq$ *0*› **have** $i \leq$ *degree p*
by (*rule le-degree*)

```
  finally have degree p = i .
  with ‹coeff p i ≠ 0› show coeff p (degree p) ≠ 0 by simp
qed

lemma leading-coeff-0-iff [simp]: coeff p (degree p) = 0 ⟷ p = 0
  by (cases p = 0) (simp-all add: leading-coeff-neq-0)

lemma degree-lessI:
  assumes p ≠ 0 ∨ n > 0 ∀ k≥n. coeff p k = 0
  shows   degree p < n
proof (cases p = 0)
  case False
  show ?thesis
  proof (rule ccontr)
    assume *: ¬(degree p < n)
    define d where d = degree p
    from ‹p ≠ 0› have coeff p d ≠ 0
      by (auto simp: d-def)
    moreover have coeff p d = 0
      using assms(2) * by (auto simp: not-less)
    ultimately show False by contradiction
  qed
qed (use assms in auto)

lemma eq-zero-or-degree-less:
  assumes degree p ≤ n and coeff p n = 0
  shows p = 0 ∨ degree p < n
proof (cases n)
  case 0
  with ‹degree p ≤ n› and ‹coeff p n = 0› have coeff p (degree p) = 0
    by simp
  then have p = 0 by simp
  then show ?thesis ..
next
  case (Suc m)
  from ‹degree p ≤ n› have ∀ i>n. coeff p i = 0
    by (simp add: coeff-eq-0)
  with ‹coeff p n = 0› have ∀ i≥n. coeff p i = 0
    by (simp add: le-less)
  with ‹n = Suc m› have ∀ i>m. coeff p i = 0
    by (simp add: less-eq-Suc-le)
  then have degree p ≤ m
    by (rule degree-le)
  with ‹n = Suc m› have degree p < n
    by (simp add: less-Suc-eq-le)
  then show ?thesis ..
qed

lemma coeff-0-degree-minus-1: coeff rrr dr = 0 ⟹ degree rrr ≤ dr ⟹ degree
```

rrr $\leq$ *dr* $-$ *1*
using *eq-zero-or-degree-less* **by** *fastforce*

4.5 List-style constructor for polynomials

lift-definition *pCons* :: *'a::zero* $\Rightarrow$ *'a poly* $\Rightarrow$ *'a poly*
is λ*a p. case-nat a* (*coeff p*)
by (*rule MOST-SucD*) (*simp add*: *MOST-coeff-eq-0*)

lemmas *coeff-pCons* = *pCons.rep-eq*

lemma *coeff-pCons'*: *poly.coeff* (*pCons c p*) *n* = (*if n* = *0 then c else poly.coeff p* (*n* $-$ *1*))
by *transfer'*(*auto split*: *nat.splits*)

lemma *coeff-pCons-0* [*simp*]: *coeff* (*pCons a p*) *0* = *a*
by *transfer simp*

lemma *coeff-pCons-Suc* [*simp*]: *coeff* (*pCons a p*) (*Suc n*) = *coeff p n*
by (*simp add*: *coeff-pCons*)

lemma *degree-pCons-le*: *degree* (*pCons a p*) $\leq$ *Suc* (*degree p*)
by (*rule degree-le*) (*simp add*: *coeff-eq-0 coeff-pCons split*: *nat.split*)

lemma *degree-pCons-eq*: *p* $\neq$ *0* $\Longrightarrow$ *degree* (*pCons a p*) = *Suc* (*degree p*)
by (*simp add*: *degree-pCons-le le-antisym le-degree*)

lemma *degree-pCons-0*: *degree* (*pCons a 0*) = *0*
proof $-$
have *degree* (*pCons a 0*) $\leq$ *Suc 0*
by (*metis* (*no-types*) *degree-0 degree-pCons-le*)
then show *?thesis*
by (*metis coeff-0 coeff-pCons-Suc degree-0 eq-zero-or-degree-less less-Suc0*)
qed

lemma *degree-pCons-eq-if* [*simp*]: *degree* (*pCons a p*) = (*if p* = *0 then 0 else Suc* (*degree p*))
by (*simp add*: *degree-pCons-0 degree-pCons-eq*)

lemma *pCons-0-0* [*simp*]: *pCons 0 0* = *0*
by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *pCons-eq-iff* [*simp*]: *pCons a p* = *pCons b q* $\longleftrightarrow$ *a* = *b* $\wedge$ *p* = *q*
proof *safe*
assume *pCons a p* = *pCons b q*
then have *coeff* (*pCons a p*) *0* = *coeff* (*pCons b q*) *0*
by *simp*
then show *a* = *b*
by *simp*

next
assume *pCons a p = pCons b q*
then have *coeff (pCons a p) (Suc n) = coeff (pCons b q) (Suc n)* **for** *n*
by *simp*
then show *p = q*
by (*simp add*: *poly-eq-iff*)
qed

lemma *pCons-eq-0-iff* [*simp*]: *pCons a p = 0 ⟷ a = 0 ∧ p = 0*
using *pCons-eq-iff* [*of a p 0 0*] **by** *simp*

lemma *pCons-cases* [*cases type*: *poly*]:
obtains (*pCons*) *a q* **where** *p = pCons a q*
proof
show *p = pCons* (*coeff p 0*) (*Abs-poly* (*λn. coeff p* (*Suc n*)))
by *transfer*
(*simp-all add*: *MOST-inj*[**where** *f*=*Suc* **and** *P*=*λn. p n = 0* **for** *p*] *fun-eq-iff*
Abs-poly-inverse
split: *nat.split*)
qed

lemma *pCons-induct* [*case-names 0 pCons*, *induct type*: *poly*]:
assumes *zero*: *P 0*
assumes *pCons*: *⋀a p. a ≠ 0 ∨ p ≠ 0 ⟹ P p ⟹ P* (*pCons a p*)
shows *P p*
proof (*induct p rule*: *measure-induct-rule* [**where** *f*=*degree*])
case (*less p*)
obtain *a q* **where** *p = pCons a q* **by** (*rule pCons-cases*)
have *P q*
proof (*cases q = 0*)
case *True*
then show *P q* **by** (*simp add*: *zero*)
next
case *False*
then have *degree* (*pCons a q*) = *Suc* (*degree q*)
by (*rule degree-pCons-eq*)
with ‹*p = pCons a q*› **have** *degree q < degree p*
by *simp*
then show *P q*
by (*rule less.hyps*)
qed
have *P* (*pCons a q*)
proof (*cases a ≠ 0 ∨ q ≠ 0*)
case *True*
with ‹*P q*› **show** *?thesis* **by** (*auto intro*: *pCons*)
next
case *False*
with *zero* **show** *?thesis* **by** *simp*
qed

with ‹*p* = *pCons a q*› **show** *?case*
by *simp*
qed

lemma *degree-eq-zeroE*:
fixes *p* :: *'a*::*zero poly*
assumes *degree p* = *0*
obtains *a* **where** *p* = *pCons a 0*
proof −
obtain *a q* **where** *p*: *p* = *pCons a q*
by (*cases p*)
with *assms* **have** *q* = *0*
by (*cases q* = *0*) *simp-all*
with *p* **have** *p* = *pCons a 0*
by *simp*
then show *thesis* **..**
qed

4.6 Quickcheck generator for polynomials

quickcheck-generator *poly constructors*: *0* :: *- poly*, *pCons*

4.7 List-style syntax for polynomials

syntax
-poly :: *args* ⇒ *'a poly* (‹(‹*indent=2 notation=‹mixfix polynomial enumeration›*›[:-:])›)
syntax-consts
-poly ⇌ *pCons*
translations
[:*x*, *xs*:] ⇌ *CONST pCons x* [:*xs*:]
[:*x*:] ⇌ *CONST pCons x 0*

lemma *degree-0-id*:
assumes *degree p* = *0*
shows [: *coeff p 0* :] = *p*
by (*metis assms coeff-pCons-0 degree-eq-zeroE*)

lemma *degree0-coeffs*: *degree p* = *0* ⟹ ∃ *a*. *p* = [: *a* :]
by (*meson degree-eq-zeroE*)

lemma *degree1-coeffs*:
fixes *p* :: *'a*::*zero poly*
assumes *degree p* = *1*
obtains *a b* **where** *p* = [: *b*, *a* :] *a* ≠ *0*
proof −
obtain *b a q* **where** *p* = *pCons b q q* = *pCons a 0*
by (*metis assms degree0-coeffs degree-0 degree-pCons-eq-if lessI less-one pCons-cases*)
then show *thesis*
using *assms that* **by** *force*

qed

lemma *degree2-coeffs*:
fixes *p* :: *'a*::*zero poly*
assumes *degree p* = *2*
obtains *a b c* **where** *p* = [: *c*, *b*, *a* :] *a* ≠ *0*
proof −
obtain *c q* **where** *p* = *pCons c q degree q* = *1*
by (*metis One-nat-def assms degree-0 degree-pCons-eq-if fact-0 fact-2 nat.inject numeral-2-eq-2 pCons-cases*)
then show *thesis*
by (*metis degree1-coeffs that*)
qed

4.8 Representation of polynomials by lists of coefficients

primrec *Poly* :: *'a*::*zero list* ⇒ *'a poly*
where
[*code-post*]: *Poly* [] = *0*
| [*code-post*]: *Poly* (*a* # *as*) = *pCons a* (*Poly as*)

lemma *Poly-replicate-0* [*simp*]: *Poly* (*replicate n 0*) = *0*
by (*induct n*) *simp-all*

lemma *Poly-eq-0*: *Poly as* = *0* ⟷ (∃ *n*. *as* = *replicate n 0*)
by (*induct as*) (*auto simp add*: *Cons-replicate-eq*)

lemma *Poly-append-replicate-zero* [*simp*]: *Poly* (*as* @ *replicate n 0*) = *Poly as*
by (*induct as*) *simp-all*

lemma *Poly-snoc-zero* [*simp*]: *Poly* (*as* @ [*0*]) = *Poly as*
using *Poly-append-replicate-zero* [*of as 1*] **by** *simp*

lemma *Poly-cCons-eq-pCons-Poly* [*simp*]: *Poly* (*a* ## *p*) = *pCons a* (*Poly p*)
by (*simp add*: *cCons-def*)

lemma *Poly-on-rev-starting-with-0* [*simp*]: *hd as* = *0* ⟹ *Poly* (*rev* (*tl as*)) = *Poly* (*rev as*)
by (*cases as*) *simp-all*

lemma *degree-Poly*: *degree* (*Poly xs*) ≤ *length xs*
by (*induct xs*) *simp-all*

lemma *coeff-Poly-eq* [*simp*]: *coeff* (*Poly xs*) = *nth-default 0 xs*
by (*induct xs*) (*simp-all add*: *fun-eq-iff coeff-pCons split*: *nat.splits*)

definition *coeffs* :: *'a poly* ⇒ *'a*::*zero list*
where *coeffs p* = (*if p* = *0 then* [] *else map* (λ*i*. *coeff p i*) [*0* ..< *Suc* (*degree p*)])

lemma *coeffs-eq-Nil* [*simp*]: *coeffs p* = [] ⟷ *p* = *0*
by (*simp add*: *coeffs-def*)

lemma *not-0-coeffs-not-Nil*: *p* ≠ *0* ⟹ *coeffs p* ≠ []
by *simp*

lemma *coeffs-0-eq-Nil* [*simp*]: *coeffs 0* = []
by *simp*

lemma *coeffs-pCons-eq-cCons* [*simp*]: *coeffs* (*pCons a p*) = *a* ## *coeffs p*
proof −
have ∗: ∀ *m*∈*set ms*. *m* > *0* ⟹ *map* (*case-nat x f*) *ms* = *map f* (*map* (λ*n*. *n* − *1*) *ms*)
for *ms* :: *nat list* **and** *f* :: *nat* ⇒ *'a* **and** *x* :: *'a*
by (*induct ms*) (*auto split*: *nat.split*)
show *?thesis*
by (*simp add*: ∗ *coeffs-def upt-conv-Cons coeff-pCons map-decr-upt del*: *upt-Suc*)
qed

lemma *length-coeffs*: *p* ≠ *0* ⟹ *length* (*coeffs p*) = *degree p* + *1*
by (*simp add*: *coeffs-def*)

lemma *coeffs-nth*: *p* ≠ *0* ⟹ *n* ≤ *degree p* ⟹ *coeffs p* ! *n* = *coeff p n*
by (*auto simp*: *coeffs-def simp del*: *upt-Suc*)

lemma *coeff-in-coeffs*: *p* ≠ *0* ⟹ *n* ≤ *degree p* ⟹ *coeff p n* ∈ *set* (*coeffs p*)
using *coeffs-nth* [*of p n*, *symmetric*] **by** (*simp add*: *length-coeffs*)

lemma *not-0-cCons-eq* [*simp*]: *p* ≠ *0* ⟹ *a* ## *coeffs p* = *a* # *coeffs p*
by (*simp add*: *cCons-def*)

lemma *Poly-coeffs* [*simp*, *code abstype*]: *Poly* (*coeffs p*) = *p*
by (*induct p*) *auto*

lemma *coeffs-Poly* [*simp*]: *coeffs* (*Poly as*) = *strip-while* (*HOL.eq 0*) *as*
proof (*induct as*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons a as*)
from *replicate-length-same* [*of as 0*] **have** (∀ *n*. *as* ≠ *replicate n 0*) ⟷ (∃ *a*∈*set as*. *a* ≠ *0*)
by (*auto dest*: *sym* [*of* - *as*])
with *Cons* **show** *?case* **by** *auto*
qed

lemma *no-trailing-coeffs* [*simp*]:
no-trailing (*HOL.eq 0*) (*coeffs p*)
by (*induct p*) *auto*

lemma *strip-while-coeffs* [*simp*]:
strip-while (*HOL.eq 0*) (*coeffs p*) = *coeffs p*
by *simp*

lemma *coeffs-eq-iff*: *p* = *q* ⟷ *coeffs p* = *coeffs q*
(**is** *?P* ⟷ *?Q*)
proof
assume *?P*
then show *?Q* **by** *simp*
next
assume *?Q*
then have *Poly* (*coeffs p*) = *Poly* (*coeffs q*) **by** *simp*
then show *?P* **by** *simp*
qed

lemma *nth-default-coeffs-eq*: *nth-default 0* (*coeffs p*) = *coeff p*
by (*metis Poly-coeffs coeff-Poly-eq*)

lemma *range-coeff*: *range* (*coeff p*) = *insert 0* (*set* (*coeffs p*))
by (*metis nth-default-coeffs-eq range-nth-default*)

lemma [*code*]: *coeff p* = *nth-default 0* (*coeffs p*)
by (*simp add*: *nth-default-coeffs-eq*)

lemma *coeffs-eqI*:
assumes *coeff*: ⋀*n*. *coeff p n* = *nth-default 0 xs n*
assumes *zero*: *no-trailing* (*HOL.eq 0*) *xs*
shows *coeffs p* = *xs*
proof −
from *coeff* **have** *p* = *Poly xs*
by (*simp add*: *poly-eq-iff*)
with *zero* **show** *?thesis* **by** *simp*
qed

lemma *degree-eq-length-coeffs* [*code*]: *degree p* = *length* (*coeffs p*) − *1*
by (*simp add*: *coeffs-def*)

lemma *length-coeffs-degree*: *p* ≠ *0* ⟹ *length* (*coeffs p*) = *Suc* (*degree p*)
by (*induct p*) (*auto simp*: *cCons-def*)

lemma [*code abstract*]: *coeffs 0* = []
by (*fact coeffs-0-eq-Nil*)

lemma [*code abstract*]: *coeffs* (*pCons a p*) = *a* ## *coeffs p*
by (*fact coeffs-pCons-eq-cCons*)

lemma *set-coeffs-subset-singleton-0-iff* [*simp*]:
set (*coeffs p*) ⊆ {*0*} ⟷ *p* = *0*

by (*auto simp add*: *coeffs-def intro*: *classical*)

lemma *set-coeffs-not-only-0* [*simp*]:
set (*coeffs p*) ≠ {*0*}
by (*auto simp add*: *set-eq-subset*)

lemma *forall-coeffs-conv*:
(∀ *n*. *P* (*coeff p n*)) ⟷ (∀ *c* ∈ *set* (*coeffs p*). *P c*) **if** *P 0*
using *that* **by** (*auto simp add*: *coeffs-def*)
(*metis atLeastLessThan-iff coeff-eq-0 not-less-iff-gr-or-eq zero-le*)

instantiation *poly* :: ({*zero*, *equal*}) *equal*
begin

definition [*code*]: *HOL.equal* (*p*::'*a poly*) *q* ⟷ *HOL.equal* (*coeffs p*) (*coeffs q*)

instance
by *standard* (*simp add*: *equal equal-poly-def coeffs-eq-iff*)

end

lemma [*code nbe*]: *HOL.equal* (*p* :: - *poly*) *p* ⟷ *True*
by (*fact equal-refl*)

definition *is-zero* :: '*a*::*zero poly* ⇒ *bool*
where [*code*]: *is-zero p* ⟷ *List.null* (*coeffs p*)

lemma *is-zero-null* [*code-abbrev*]: *is-zero p* ⟷ *p* = *0*
by (*simp add*: *is-zero-def*)

Reconstructing the polynomial from the list

definition *poly-of-list* :: '*a*::*comm-monoid-add list* ⇒ '*a poly*
where [*simp*]: *poly-of-list* = *Poly*

lemma *poly-of-list-impl* [*code abstract*]: *coeffs* (*poly-of-list as*) = *strip-while* (*HOL.eq 0*) *as*
by *simp*

4.9 Fold combinator for polynomials

definition *fold-coeffs* :: ('*a*::*zero* ⇒ '*b* ⇒ '*b*) ⇒ '*a poly* ⇒ '*b* ⇒ '*b*
where *fold-coeffs f p* = *foldr f* (*coeffs p*)

lemma *fold-coeffs-0-eq* [*simp*]: *fold-coeffs f 0* = *id*
by (*simp add*: *fold-coeffs-def*)

lemma *fold-coeffs-pCons-eq* [*simp*]: *f 0* = *id* ⟹ *fold-coeffs f* (*pCons a p*) = *f a* ∘ *fold-coeffs f p*
by (*simp add*: *fold-coeffs-def cCons-def fun-eq-iff*)

lemma *fold-coeffs-pCons-0-0-eq* [*simp*]: *fold-coeffs f* (*pCons 0 0*) = *id*
by (*simp add*: *fold-coeffs-def*)

lemma *fold-coeffs-pCons-coeff-not-0-eq* [*simp*]:
a ≠ *0* ⟹ *fold-coeffs f* (*pCons a p*) = *f a* ∘ *fold-coeffs f p*
by (*simp add*: *fold-coeffs-def*)

lemma *fold-coeffs-pCons-not-0-0-eq* [*simp*]:
p ≠ *0* ⟹ *fold-coeffs f* (*pCons a p*) = *f a* ∘ *fold-coeffs f p*
by (*simp add*: *fold-coeffs-def*)

4.10 Canonical morphism on polynomials – evaluation

definition *poly* :: ‹'*a*::*comm-semiring-0 poly* ⇒ '*a* ⇒ '*a*›
where ‹*poly p a* = *horner-sum id a* (*coeffs p*)›

lemma *poly-eq-fold-coeffs*:
‹*poly p* = *fold-coeffs* (λ*a f x*. *a* + *x* ∗ *f x*) *p* (λ*x*. *0*)›
by (*induction p*) (*auto simp add*: *fun-eq-iff poly-def*)

lemma *poly-0* [*simp*]: *poly 0 x* = *0*
by (*simp add*: *poly-def*)

lemma *poly-pCons* [*simp*]: *poly* (*pCons a p*) *x* = *a* + *x* ∗ *poly p x*
by (*cases p* = *0* ∧ *a* = *0*) (*auto simp add*: *poly-def*)

lemma *poly-altdef*: *poly p x* = (∑ *i*≤*degree p*. *coeff p i* ∗ *x* ^ *i*)
for *x* :: '*a*::{*comm-semiring-0*,*semiring-1*}
proof (*induction p rule*: *pCons-induct*)
case *0*
then show *?case*
by *simp*
next
case (*pCons a p*)
show *?case*
proof (*cases p* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
let *?p'* = *pCons a p*
note *poly-pCons*[*of a p x*]
also note *pCons.IH*
also have *a* + *x* ∗ (∑ *i*≤*degree p*. *coeff p i* ∗ *x* ^ *i*) =
coeff ?p' 0 ∗ *x*^*0* + (∑ *i*≤*degree p*. *coeff ?p'* (*Suc i*) ∗ *x*^*Suc i*)
by (*simp add*: *field-simps sum-distrib-left coeff-pCons*)
also note *sum.atMost-Suc-shift*[*symmetric*]
also note *degree-pCons-eq*[*OF* ‹*p* ≠ *0*›, *of a*, *symmetric*]

finally show *?thesis* **.**
qed
qed

lemma *poly-0-coeff-0*: *poly p 0 = coeff p 0*
by (*cases p*) (*auto simp*: *poly-altdef*)

lemma *poly-zero*:
fixes *p* :: *'a* :: *comm-ring-1 poly*
assumes *x*: *poly p x = 0* **shows** *p = 0* ⟷ *degree p = 0*
proof
assume *degp*: *degree p = 0*
hence *poly p x = coeff p* (*degree p*) **by**(*subst degree-0-id*[*OF degp*,*symmetric*], *simp*)
hence *coeff p* (*degree p*) *= 0* **using** *x* **by** *auto*
thus *p = 0* **by** *auto*
qed *auto*

4.11 Monomials

lift-definition *monom* :: *'a* ⇒ *nat* ⇒ *'a*::*zero poly*
is λ*a m n*. *if m = n then a else 0*
by (*simp add*: *MOST-iff-cofinite*)

lemma *coeff-monom* [*simp*]: *coeff* (*monom a m*) *n* = (*if m = n then a else 0*)
by *transfer rule*

lemma *monom-0*: *monom a 0* = [:*a*:]
by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *monom-Suc*: *monom a* (*Suc n*) = *pCons 0* (*monom a n*)
by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *monom-eq-0* [*simp*]: *monom 0 n = 0*
by (*rule poly-eqI*) *simp*

lemma *monom-eq-0-iff* [*simp*]: *monom a n = 0* ⟷ *a = 0*
by (*simp add*: *poly-eq-iff*)

lemma *monom-eq-iff* [*simp*]: *monom a n = monom b n* ⟷ *a = b*
by (*simp add*: *poly-eq-iff*)

lemma *degree-monom-le*: *degree* (*monom a n*) ≤ *n*
by (*rule degree-le*, *simp*)

lemma *degree-monom-eq*: *a* ≠ *0* ⟹ *degree* (*monom a n*) = *n*
by (*metis coeff-monom leading-coeff-0-iff*)

lemma *coeffs-monom* [*code abstract*]:

coeffs (*monom a n*) = (*if a = 0 then* [] *else replicate n 0* @ [*a*])
by (*induct n*) (*simp-all add*: *monom-0 monom-Suc*)

lemma *fold-coeffs-monom* [*simp*]: *a* ≠ *0* ⟹ *fold-coeffs f* (*monom a n*) = *f 0* ^^ *n* ∘ *f a*
by (*simp add*: *fold-coeffs-def coeffs-monom fun-eq-iff*)

lemma *poly-monom*: *poly* (*monom a n*) *x* = *a* ∗ *x* ^ *n*
for *a x* :: *'a*::*comm-semiring-1*
by (*cases a* = *0*, *simp-all*) (*induct n*, *simp-all add*: *mult.left-commute poly-eq-fold-coeffs*)

lemma *monom-eq-iff'*: *monom c n* = *monom d m* ⟷ *c* = *d* ∧ (*c* = *0* ∨ *n* = *m*)
by (*auto simp*: *poly-eq-iff*)

lemma *monom-eq-const-iff*: *monom c n* = [:*d*:] ⟷ *c* = *d* ∧ (*c* = *0* ∨ *n* = *0*)
using *monom-eq-iff'*[*of c n d 0*] **by** (*simp add*: *monom-0*)

4.12 Leading coefficient

abbreviation *lead-coeff*:: *'a*::*zero poly* ⇒ *'a*
where *lead-coeff p* ≡ *coeff p* (*degree p*)

lemma *lead-coeff-pCons*[*simp*]:
p ≠ *0* ⟹ *lead-coeff* (*pCons a p*) = *lead-coeff p*
p = *0* ⟹ *lead-coeff* (*pCons a p*) = *a*
by *auto*

lemma *lead-coeff-monom* [*simp*]: *lead-coeff* (*monom c n*) = *c*
by (*cases c* = *0*) (*simp-all add*: *degree-monom-eq*)

lemma *last-coeffs-eq-coeff-degree*:
last (*coeffs p*) = *lead-coeff p* **if** *p* ≠ *0*
using *that* **by** (*simp add*: *coeffs-def*)

lemma *lead-coeff-list-def*:
lead-coeff p = (*if coeffs p*=[] *then 0 else last* (*coeffs p*))
by (*simp add*: *last-coeffs-eq-coeff-degree*)

4.13 Addition and subtraction

instantiation *poly* :: (*comm-monoid-add*) *comm-monoid-add*
begin

lift-definition *plus-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly*
is λ*p q n*. *coeff p n* + *coeff q n*
proof −
fix *q p* :: *'a poly*
show ∀ ∞*n*. *coeff p n* + *coeff q n* = *0*
using *MOST-coeff-eq-0*[*of p*] *MOST-coeff-eq-0*[*of q*] **by** *eventually-elim simp*

qed

lemma *coeff-add* [*simp*]: *coeff* (*p* + *q*) *n* = *coeff p n* + *coeff q n*
by (*simp add*: *plus-poly.rep-eq*)

instance
proof
fix *p q r* :: *'a poly*
show (*p* + *q*) + *r* = *p* + (*q* + *r*)
by (*simp add*: *poly-eq-iff add.assoc*)
show *p* + *q* = *q* + *p*
by (*simp add*: *poly-eq-iff add.commute*)
show *0* + *p* = *p*
by (*simp add*: *poly-eq-iff*)
qed

end

instantiation *poly* :: (*cancel-comm-monoid-add*) *cancel-comm-monoid-add*
begin

lift-definition *minus-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly*
is λ*p q n*. *coeff p n* − *coeff q n*
proof −
fix *q p* :: *'a poly*
show ∀ ∞*n*. *coeff p n* − *coeff q n* = *0*
using *MOST-coeff-eq-0*[*of p*] *MOST-coeff-eq-0*[*of q*] **by** *eventually-elim simp*
qed

lemma *coeff-diff* [*simp*]: *coeff* (*p* − *q*) *n* = *coeff p n* − *coeff q n*
by (*simp add*: *minus-poly.rep-eq*)

instance
proof
fix *p q r* :: *'a poly*
show *p* + *q* − *p* = *q*
by (*simp add*: *poly-eq-iff*)
show *p* − *q* − *r* = *p* − (*q* + *r*)
by (*simp add*: *poly-eq-iff diff-diff-eq*)
qed

end

instantiation *poly* :: (*ab-group-add*) *ab-group-add*
begin

lift-definition *uminus-poly* :: *'a poly* ⇒ *'a poly*
is λ*p n*. − *coeff p n*
proof −

fix $p :: 'a\ poly$
show $\forall_\infty n.\ -\ coeff\ p\ n = 0$
using *MOST-coeff-eq-0* **by** *simp*
qed

lemma *coeff-minus* [*simp*]: $coeff\ (-\ p)\ n = -\ coeff\ p\ n$
by (*simp add*: *uminus-poly.rep-eq*)

instance
proof
fix $p\ q :: 'a\ poly$
show $-\ p + p = 0$
by (*simp add*: *poly-eq-iff*)
show $p - q = p + -\ q$
by (*simp add*: *poly-eq-iff*)
qed

end

lemma *add-pCons* [*simp*]: $pCons\ a\ p + pCons\ b\ q = pCons\ (a + b)\ (p + q)$
by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *minus-pCons* [*simp*]: $-\ pCons\ a\ p = pCons\ (-\ a)\ (-\ p)$
by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *diff-pCons* [*simp*]: $pCons\ a\ p - pCons\ b\ q = pCons\ (a - b)\ (p - q)$
by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *degree-add-le-max*: $degree\ (p + q) \leq max\ (degree\ p)\ (degree\ q)$
by (*rule degree-le*) (*auto simp add*: *coeff-eq-0*)

lemma *degree-add-le*: $degree\ p \leq n \Longrightarrow degree\ q \leq n \Longrightarrow degree\ (p + q) \leq n$
by (*auto intro*: *order-trans degree-add-le-max*)

lemma *degree-add-less*: $degree\ p < n \Longrightarrow degree\ q < n \Longrightarrow degree\ (p + q) < n$
by (*auto intro*: *le-less-trans degree-add-le-max*)

lemma *degree-add-eq-right*: **assumes** $degree\ p < degree\ q$ **shows** $degree\ (p + q) = degree\ q$
proof (*cases* $q = 0$)
case *False*
show *?thesis*
proof (*rule order-antisym*)
show $degree\ (p + q) \leq degree\ q$
by (*simp add*: *assms degree-add-le order.strict-implies-order*)
show $degree\ q \leq degree\ (p + q)$
by (*simp add*: *False assms coeff-eq-0 le-degree*)
qed
qed (*use assms* **in** *auto*)

lemma *degree-add-eq-left*: *degree q* < *degree p* ⟹ *degree* (*p* + *q*) = *degree p*
using *degree-add-eq-right* [*of q p*] **by** (*simp add*: *add.commute*)

lemma *degree-minus* [*simp*]: *degree* (− *p*) = *degree p*
by (*simp add*: *degree-def*)

lemma *lead-coeff-add-le*: *degree p* < *degree q* ⟹ *lead-coeff* (*p* + *q*) = *lead-coeff q*
by (*metis coeff-add coeff-eq-0 monoid-add-class.add.left-neutral degree-add-eq-right*)

lemma *lead-coeff-minus*: *lead-coeff* (− *p*) = − *lead-coeff p*
by (*metis coeff-minus degree-minus*)

lemma *degree-diff-le-max*: *degree* (*p* − *q*) ≤ *max* (*degree p*) (*degree q*)
for *p q* :: *'a*::*ab-group-add poly*
using *degree-add-le* [**where** *p*=*p* **and** *q*=−*q*] **by** *simp*

lemma *degree-diff-le*: *degree p* ≤ *n* ⟹ *degree q* ≤ *n* ⟹ *degree* (*p* − *q*) ≤ *n*
for *p q* :: *'a*::*ab-group-add poly*
using *degree-add-le* [*of p n* − *q*] **by** *simp*

lemma *degree-diff-less*: *degree p* < *n* ⟹ *degree q* < *n* ⟹ *degree* (*p* − *q*) < *n*
for *p q* :: *'a*::*ab-group-add poly*
using *degree-add-less* [*of p n* − *q*] **by** *simp*

lemma *add-monom*: *monom a n* + *monom b n* = *monom* (*a* + *b*) *n*
by (*rule poly-eqI*) *simp*

lemma *diff-monom*: *monom a n* − *monom b n* = *monom* (*a* − *b*) *n*
by (*rule poly-eqI*) *simp*

lemma *minus-monom*: − *monom a n* = *monom* (− *a*) *n*
by (*rule poly-eqI*) *simp*

lemma *coeff-sum*: *coeff* (∑ *x*∈*A*. *p x*) *i* = (∑ *x*∈*A*. *coeff* (*p x*) *i*)
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *monom-sum*: *monom* (∑ *x*∈*A*. *a x*) *n* = (∑ *x*∈*A*. *monom* (*a x*) *n*)
by (*rule poly-eqI*) (*simp add*: *coeff-sum*)

fun *plus-coeffs* :: *'a*::*comm-monoid-add list* ⇒ *'a list* ⇒ *'a list*
where
plus-coeffs xs [] = *xs*
| *plus-coeffs* [] *ys* = *ys*
| *plus-coeffs* (*x* # *xs*) (*y* # *ys*) = (*x* + *y*) ## *plus-coeffs xs ys*

lemma *coeffs-plus-eq-plus-coeffs* [*code abstract*]:
coeffs (*p* + *q*) = *plus-coeffs* (*coeffs p*) (*coeffs q*)
proof −

```
  have ∗: nth-default 0 (plus-coeffs xs ys) n = nth-default 0 xs n + nth-default 0
ys n
    for xs ys :: 'a list and n
  proof (induct xs ys arbitrary: n rule: plus-coeffs.induct)
    case (3 x xs y ys n)
    then show ?case
      by (cases n) (auto simp add: cCons-def)
  qed simp-all
  have ∗∗: no-trailing (HOL.eq 0) (plus-coeffs xs ys)
    if no-trailing (HOL.eq 0) xs and no-trailing (HOL.eq 0) ys
    for xs ys :: 'a list
    using that by (induct xs ys rule: plus-coeffs.induct) (simp-all add: cCons-def)
  show ?thesis
    by (rule coeffs-eqI) (auto simp add: ∗ nth-default-coeffs-eq intro: ∗∗)
qed

lemma coeffs-uminus [code abstract]:
  coeffs (− p) = map uminus (coeffs p)
proof −
  have eq-0: HOL.eq 0 ∘ uminus = HOL.eq (0::'a)
    by (simp add: fun-eq-iff)
  show ?thesis
   by (rule coeffs-eqI) (simp-all add: nth-default-map-eq nth-default-coeffs-eq no-trailing-map
eq-0)
qed

lemma [code]: p − q = p + − q
  for p q :: 'a::ab-group-add poly
  by (fact diff-conv-add-uminus)

lemma poly-add [simp]: poly (p + q) x = poly p x + poly q x
proof (induction p arbitrary: q)
  case (pCons a p)
  then show ?case
    by (cases q) (simp add: algebra-simps)
qed auto

lemma poly-minus [simp]: poly (− p) x = − poly p x
  for x :: 'a::comm-ring
  by (induct p) simp-all

lemma poly-diff [simp]: poly (p − q) x = poly p x − poly q x
  for x :: 'a::comm-ring
  using poly-add [of p − q x] by simp

lemma poly-sum: poly (∑ k∈A. p k) x = (∑ k∈A. poly (p k) x)
  by (induct A rule: infinite-finite-induct) simp-all

lemma poly-sum-list: poly (∑ p←ps. p) y = (∑ p←ps. poly p y)
```

by (*induction ps*) *auto*

lemma *poly-sum-mset*: *poly* ($\sum x{\in}\#A.\ p\ x$) y = ($\sum x{\in}\#A.$ *poly* ($p\ x$) y)
by (*induction A*) *auto*

lemma *degree-sum-le*: *finite* $S \Longrightarrow (\bigwedge p.\ p \in S \Longrightarrow$ *degree* ($f\ p$) $\leq n) \Longrightarrow$ *degree* (*sum f S*) $\leq n$
proof (*induct S rule*: *finite-induct*)
case *empty*
then show *?case* **by** *simp*
next
case (*insert p S*)
then have *degree* (*sum f S*) $\leq n$ *degree* ($f\ p$) $\leq n$
by *auto*
then show *?case*
unfolding *sum.insert*[*OF insert*(*1*−*2*)] **by** (*metis degree-add-le*)
qed

lemma *degree-sum-less*:
assumes $\bigwedge x.\ x \in A \Longrightarrow$ *degree* ($f\ x$) $< n$ $n > 0$
shows *degree* (*sum f A*) $< n$
using *assms* **by** (*induction rule*: *infinite-finite-induct*) (*auto intro*!: *degree-add-less*)

lemma *poly-as-sum-of-monoms'*:
assumes *degree* $p \leq n$
shows ($\sum i{\leq}n.$ *monom* (*coeff p i*) i) = p
proof −
have *eq*: $\bigwedge i.$ {..n} ∩ {i} = (*if* $i \leq n$ *then* {i} *else* {})
by *auto*
from *assms* **show** *?thesis*
by (*simp add*: *poly-eq-iff coeff-sum coeff-eq-0 sum.If-cases eq*
if-distrib[**where** $f{=}\lambda x.\ x * a$ **for** a])
qed

lemma *poly-as-sum-of-monoms*: ($\sum i{\leq}$*degree p. monom* (*coeff p i*) i) = p
by (*intro poly-as-sum-of-monoms' order-refl*)

lemma *Poly-snoc*: *Poly* (*xs* @ [x]) = *Poly xs* + *monom x* (*length xs*)
by (*induct xs*) (*simp-all add*: *monom-0 monom-Suc*)

4.14 Multiplication by a constant, polynomial multiplication and the unit polynomial

lift-definition *smult* :: *'a*::*comm-semiring-0* ⇒ *'a poly* ⇒ *'a poly*
is $\lambda a\ p\ n.\ a * coeff\ p\ n$
proof −
fix a :: *'a* **and** p :: *'a poly*
show $\forall_\infty i.\ a * coeff\ p\ i = 0$
using *MOST-coeff-eq-0*[*of p*] **by** *eventually-elim simp*

qed

lemma *coeff-smult* [*simp*]: *coeff* (*smult a p*) *n* = *a* ∗ *coeff p n*
by (*simp add*: *smult.rep-eq*)

lemma *degree-smult-le*: *degree* (*smult a p*) ≤ *degree p*
by (*rule degree-le*) (*simp add*: *coeff-eq-0*)

lemma *smult-smult* [*simp*]: *smult a* (*smult b p*) = *smult* (*a* ∗ *b*) *p*
by (*rule poly-eqI*) (*simp add*: *mult.assoc*)

lemma *smult-0-right* [*simp*]: *smult a 0* = *0*
by (*rule poly-eqI*) *simp*

lemma *smult-0-left* [*simp*]: *smult 0 p* = *0*
by (*rule poly-eqI*) *simp*

lemma *smult-1-left* [*simp*]: *smult* (*1*::′*a*::*comm-semiring-1*) *p* = *p*
by (*rule poly-eqI*) *simp*

lemma *smult-add-right*: *smult a* (*p* + *q*) = *smult a p* + *smult a q*
by (*rule poly-eqI*) (*simp add*: *algebra-simps*)

lemma *smult-add-left*: *smult* (*a* + *b*) *p* = *smult a p* + *smult b p*
by (*rule poly-eqI*) (*simp add*: *algebra-simps*)

lemma *smult-minus-right* [*simp*]: *smult a* (− *p*) = − *smult a p*
for *a* :: ′*a*::*comm-ring*
by (*rule poly-eqI*) *simp*

lemma *smult-minus-left* [*simp*]: *smult* (− *a*) *p* = − *smult a p*
for *a* :: ′*a*::*comm-ring*
by (*rule poly-eqI*) *simp*

lemma *smult-diff-right*: *smult a* (*p* − *q*) = *smult a p* − *smult a q*
for *a* :: ′*a*::*comm-ring*
by (*rule poly-eqI*) (*simp add*: *algebra-simps*)

lemma *smult-diff-left*: *smult* (*a* − *b*) *p* = *smult a p* − *smult b p*
for *a b* :: ′*a*::*comm-ring*
by (*rule poly-eqI*) (*simp add*: *algebra-simps*)

lemmas *smult-distribs* =
smult-add-left smult-add-right
smult-diff-left smult-diff-right

lemma *smult-pCons* [*simp*]: *smult a* (*pCons b p*) = *pCons* (*a* ∗ *b*) (*smult a p*)
by (*rule poly-eqI*) (*simp add*: *coeff-pCons split*: *nat.split*)

lemma *smult-monom*: *smult a* (*monom b n*) = *monom* (*a* ∗ *b*) *n*
by (*induct n*) (*simp-all add*: *monom-0 monom-Suc*)

lemma *smult-Poly*: *smult c* (*Poly xs*) = *Poly* (*map* ((∗) *c*) *xs*)
by (*auto simp*: *poly-eq-iff nth-default-def*)

lemma *degree-smult-eq* [*simp*]: *degree* (*smult a p*) = (*if a* = *0 then 0 else degree p*)
for *a* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*}
by (*cases a* = *0*) (*simp-all add*: *degree-def*)

lemma *smult-eq-0-iff* [*simp*]: *smult a p* = *0* ⟷ *a* = *0* ∨ *p* = *0*
for *a* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*}
by (*simp add*: *poly-eq-iff*)

lemma *coeffs-smult* [*code abstract*]:
coeffs (*smult a p*) = (*if a* = *0 then* [] *else map* (*Groups.times a*) (*coeffs p*))
for *p* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
proof −
have *eq-0*: *HOL.eq 0* ∘ *times a* = *HOL.eq* (*0*::*'a*) **if** *a* ≠ *0*
using *that* **by** (*simp add*: *fun-eq-iff*)
show *?thesis*
by (*rule coeffs-eqI*) (*auto simp add*: *no-trailing-map nth-default-map-eq nth-default-coeffs-eq eq-0*)
qed

lemma *smult-eq-iff*:
fixes *b* :: *'a* :: *field*
assumes *b* ≠ *0*
shows *smult a p* = *smult b q* ⟷ *smult* (*a* / *b*) *p* = *q*
(**is** *?lhs* ⟷ *?rhs*)
proof
assume *?lhs*
also from *assms* **have** *smult* (*inverse b*) ... = *q*
by *simp*
finally show *?rhs*
by (*simp add*: *field-simps*)
next
assume *?rhs*
with *assms* **show** *?lhs* **by** *auto*
qed

lemma *smult-cancel*:
fixes *p*::*'a*::*idom poly*
assumes *c*≠*0* **and** *smult*: *smult c p* = *smult c q*
shows *p*=*q*
proof −
have *smult c* (*p*−*q*) = *0* **using** *smult* **by** (*metis diff-self smult-diff-right*)
thus *?thesis* **using** ‹*c*≠*0*› **by** *auto*
qed

instantiation *poly* :: (*comm-semiring-0*) *comm-semiring-0*
begin

definition $p * q = \mathit{fold\text{-}coeffs}\ (\lambda a\ p.\ \mathit{smult}\ a\ q + \mathit{pCons}\ 0\ p)\ p\ 0$

lemma *mult-poly-0-left*: $(0::'a\ \mathit{poly}) * q = 0$
by (*simp add*: *times-poly-def*)

lemma *mult-pCons-left* [*simp*]: $\mathit{pCons}\ a\ p * q = \mathit{smult}\ a\ q + \mathit{pCons}\ 0\ (p * q)$
by (*cases* $p = 0 \wedge a = 0$) (*auto simp add*: *times-poly-def*)

lemma *mult-poly-0-right*: $p * (0::'a\ \mathit{poly}) = 0$
by (*induct p*) (*simp-all add*: *mult-poly-0-left*)

lemma *mult-pCons-right* [*simp*]: $p * \mathit{pCons}\ a\ q = \mathit{smult}\ a\ p + \mathit{pCons}\ 0\ (p * q)$
by (*induct p*) (*simp-all add*: *mult-poly-0-left algebra-simps*)

lemmas *mult-poly-0* = *mult-poly-0-left mult-poly-0-right*

lemma *mult-smult-left* [*simp*]: $\mathit{smult}\ a\ p * q = \mathit{smult}\ a\ (p * q)$
by (*induct p*) (*simp-all add*: *mult-poly-0 smult-add-right*)

lemma *mult-smult-right* [*simp*]: $p * \mathit{smult}\ a\ q = \mathit{smult}\ a\ (p * q)$
by (*induct q*) (*simp-all add*: *mult-poly-0 smult-add-right*)

lemma *mult-poly-add-left*: $(p + q) * r = p * r + q * r$
for $p\ q\ r :: 'a\ \mathit{poly}$
by (*induct r*) (*simp-all add*: *mult-poly-0 smult-distribs algebra-simps*)

instance
proof
fix $p\ q\ r :: 'a\ \mathit{poly}$
show $0 * p = 0$
by (*rule mult-poly-0-left*)
show $p * 0 = 0$
by (*rule mult-poly-0-right*)
show $(p + q) * r = p * r + q * r$
by (*rule mult-poly-add-left*)
show $(p * q) * r = p * (q * r)$
by (*induct p*) (*simp-all add*: *mult-poly-0 mult-poly-add-left*)
show $p * q = q * p$
by (*induct p*) (*simp-all add*: *mult-poly-0*)
qed

end

lemma *coeff-mult-degree-sum*:
$\mathit{coeff}\ (p * q)\ (\mathit{degree}\ p + \mathit{degree}\ q) = \mathit{coeff}\ p\ (\mathit{degree}\ p) * \mathit{coeff}\ q\ (\mathit{degree}\ q)$

by (*induct p*) (*simp-all add*: *coeff-eq-0*)

instance *poly* :: ({*comm-semiring-0*,*semiring-no-zero-divisors*}) *semiring-no-zero-divisors*
proof
fix *p q* :: *'a poly*
assume *p* ≠ *0* **and** *q* ≠ *0*
have *coeff* (*p* ∗ *q*) (*degree p* + *degree q*) = *coeff p* (*degree p*) ∗ *coeff q* (*degree q*)
by (*rule coeff-mult-degree-sum*)
also from ‹*p* ≠ *0*› ‹*q* ≠ *0*› **have** *coeff p* (*degree p*) ∗ *coeff q* (*degree q*) ≠ *0*
by *simp*
finally have ∃ *n*. *coeff* (*p* ∗ *q*) *n* ≠ *0* **..**
then show *p* ∗ *q* ≠ *0*
by (*simp add*: *poly-eq-iff*)
qed

instance *poly* :: (*comm-semiring-0-cancel*) *comm-semiring-0-cancel* **..**

lemma *coeff-mult*: *coeff* (*p* ∗ *q*) *n* = (∑ *i*≤*n*. *coeff p i* ∗ *coeff q* (*n*−*i*))
proof (*induct p arbitrary*: *n*)
case *0*
show *?case* **by** *simp*
next
case (*pCons a p n*)
then show *?case*
by (*cases n*) (*simp-all add*: *sum.atMost-Suc-shift del*: *sum.atMost-Suc*)
qed

lemma *coeff-mult-0*: *coeff* (*p* ∗ *q*) *0* = *coeff p 0* ∗ *coeff q 0*
by (*simp add*: *coeff-mult*)

lemma *degree-mult-le*: *degree* (*p* ∗ *q*) ≤ *degree p* + *degree q*
proof (*rule degree-le*)
show ∀ *i*>*degree p* + *degree q*. *coeff* (*p* ∗ *q*) *i* = *0*
by (*induct p*) (*simp-all add*: *coeff-eq-0 coeff-pCons split*: *nat.split*)
qed

lemma *mult-monom*: *monom a m* ∗ *monom b n* = *monom* (*a* ∗ *b*) (*m* + *n*)
by (*induct m*) (*simp add*: *monom-0 smult-monom*, *simp add*: *monom-Suc*)

instantiation *poly* :: (*comm-semiring-1*) *comm-semiring-1*
begin

lift-definition *one-poly* :: *'a poly*
is λ*n*. *of-bool* (*n* = *0*)
by (*rule MOST-SucD*) *simp*

lemma *coeff-1* [*simp*]:
coeff 1 n = *of-bool* (*n* = *0*)
by (*simp add*: *one-poly.rep-eq*)

lemma *one-pCons*:
1 = [:*1*:]
by (*simp add*: *poly-eq-iff coeff-pCons split*: *nat.splits*)

lemma *pCons-one*:
[:*1*:] = *1*
by (*simp add*: *one-pCons*)

instance
by *standard* (*simp-all add*: *one-pCons*)

end

lemma *poly-1* [*simp*]:
poly 1 x = *1*
by (*simp add*: *one-pCons*)

lemma *one-poly-eq-simps* [*simp*]:
1 = [:*1*:] ⟷ *True*
[:*1*:] = *1* ⟷ *True*
by (*simp-all add*: *one-pCons*)

lemma *degree-1* [*simp*]:
degree 1 = *0*
by (*simp add*: *one-pCons*)

lemma *coeffs-1-eq* [*simp*, *code abstract*]:
coeffs 1 = [*1*]
by (*simp add*: *one-pCons*)

lemma *smult-one* [*simp*]:
smult c 1 = [:*c*:]
by (*simp add*: *one-pCons*)

lemma *smult-sum*: *smult* (∑ *i* ∈ *S*. *f i*) *p* = (∑ *i* ∈ *S*. *smult* (*f i*) *p*)
by (*induct S rule*: *infinite-finite-induct*, *auto simp*: *smult-add-left*)

lemma *smult-power*: (*smult a p*) ^ *n* = *smult* (*a* ^ *n*) (*p* ^ *n*)
by (*induct n*, *auto simp*: *field-simps*)

lemma *monom-eq-1* [*simp*]:
monom 1 0 = *1*
by (*simp add*: *monom-0 one-pCons*)

lemma *monom-eq-1-iff*:
monom c n = *1* ⟷ *c* = *1* ∧ *n* = *0*
using *monom-eq-const-iff* [*of c n 1*] **by** *auto*

lemma *monom-altdef*:
  monom c n = smult c ([:0, 1:] ^ n)
  by (*induct n*) (*simp-all add*: *monom-0 monom-Suc*)

lemma *degree-sum-list-le*: (⋀ *p* . *p* ∈ *set ps* ⟹ *degree p* ≤ *n*)
  ⟹ *degree* (*sum-list ps*) ≤ *n*
proof (*induct ps*)
  case (*Cons p ps*)
  hence *degree* (*sum-list ps*) ≤ *n degree p* ≤ *n* **by** *auto*
  thus *?case* **unfolding** *sum-list.Cons* **by** (*metis degree-add-le*)
qed *simp*

lemma *degree-prod-list-le*: *degree* (*prod-list ps*) ≤ *sum-list* (*map degree ps*)
proof (*induct ps*)
  case (*Cons p ps*)
  show *?case* **unfolding** *prod-list.Cons*
    by (*rule order.trans*[*OF degree-mult-le*], *insert Cons*, *auto*)
qed *simp*

instance *poly* :: ({*comm-semiring-1*,*semiring-1-no-zero-divisors*}) *semiring-1-no-zero-divisors*
..
instance *poly* :: (*comm-ring*) *comm-ring* **..**
instance *poly* :: (*comm-ring-1*) *comm-ring-1* **..**
instance *poly* :: (*comm-ring-1*) *comm-semiring-1-cancel* **..**

lemma *prod-smult*: (∏ *x*∈*A*. *smult* (*c x*) (*p x*)) = *smult* (*prod c A*) (*prod p A*)
  by (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *mult-ac*)

lemma *degree-power-le*: *degree* (*p* ^ *n*) ≤ *degree p* ∗ *n*
  by (*induct n*) (*auto intro*: *order-trans degree-mult-le*)

lemma *coeff-0-power*: *coeff* (*p* ^ *n*) *0* = *coeff p 0* ^ *n*
  by (*induct n*) (*simp-all add*: *coeff-mult*)

lemma *poly-smult* [*simp*]: *poly* (*smult a p*) *x* = *a* ∗ *poly p x*
  by (*induct p*) (*simp-all add*: *algebra-simps*)

lemma *poly-mult* [*simp*]: *poly* (*p* ∗ *q*) *x* = *poly p x* ∗ *poly q x*
  by (*induct p*) (*simp-all add*: *algebra-simps*)

lemma *poly-power* [*simp*]: *poly* (*p* ^ *n*) *x* = *poly p x* ^ *n*
  for *p* :: *'a*::*comm-semiring-1 poly*
  by (*induct n*) *simp-all*

lemma *poly-prod*: *poly* (∏ *k*∈*A*. *p k*) *x* = (∏ *k*∈*A*. *poly* (*p k*) *x*)
  by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *poly-prod-list*: *poly* (∏ *p*←*ps*. *p*) *y* = (∏ *p*←*ps*. *poly p y*)
  by (*induction ps*) *auto*

lemma *poly-prod-mset*: *poly* $(\prod x{\in}\#A.\ p\ x)\ y = (\prod x{\in}\#A.\ poly\ (p\ x)\ y)$
by (*induction A*) *auto*

lemma *poly-const-pow*: $[:\ c\ :]\ \hat{}\ n = [:\ c\ \hat{}\ n\ :]$
by (*induction n*) (*auto simp*: *algebra-simps*)

lemma *monom-power*: *monom c n* $\hat{}\ k = monom\ (c\ \hat{}\ k)\ (n * k)$
by (*induction k*) (*auto simp*: *mult-monom*)

lemma *degree-prod-sum-le*: *finite S* $\Longrightarrow$ *degree* (*prod f S*) $\leq$ *sum* (*degree* $\circ$ *f*) *S*
proof (*induct S rule*: *finite-induct*)
case *empty*
then show *?case* **by** *simp*
next
case (*insert a S*)
show *?case*
unfolding *prod.insert*[*OF insert*(*1*−*2*)] *sum.insert*[*OF insert*(*1*−*2*)]
by (*rule le-trans*[*OF degree-mult-le*]) (*use insert* **in** *auto*)
qed

lemma *coeff-0-prod-list*: *coeff* (*prod-list xs*) *0* = *prod-list* (*map* (λ*p*. *coeff p 0*) *xs*)
by (*induct xs*) (*simp-all add*: *coeff-mult*)

lemma *coeff-monom-mult*: *coeff* (*monom c n* $*$ *p*) *k* = (*if k* $<$ *n then 0 else c* $*$ *coeff p* (*k* − *n*))
proof −
have *coeff* (*monom c n* $*$ *p*) *k* = ($\sum$ *i*$\leq$*k*. (*if n* = *i then c else 0*) $*$ *coeff p* (*k* − *i*))
by (*simp add*: *coeff-mult*)
also have ... = ($\sum$ *i*$\leq$*k*. (*if n* = *i then c* $*$ *coeff p* (*k* − *i*) *else 0*))
by (*intro sum.cong*) *simp-all*
also have ... = (*if k* $<$ *n then 0 else c* $*$ *coeff p* (*k* − *n*))
by *simp*
finally show *?thesis* **.**
qed

lemma *coeff-monom-Suc*: *coeff* (*monom a* (*Suc d*) $*$ *p*) (*Suc i*) = *coeff* (*monom a d* $*$ *p*) *i*
by (*simp add*: *monom-Suc*)

lemma *monom-1-dvd-iff'*: *monom 1 n dvd p* $\longleftrightarrow$ ($\forall$ *k*$<$*n*. *coeff p k* = *0*)
proof
assume *monom 1 n dvd p*
then obtain *r* **where** *p* = *monom 1 n* $*$ *r*
by (*rule dvdE*)
then show $\forall$ *k*$<$*n*. *coeff p k* = *0*
by (*simp add*: *coeff-mult*)
next

assume *zero*: $(\forall k<n.\ coeff\ p\ k = 0)$
define r **where** $r = Abs\text{-}poly\ (\lambda k.\ coeff\ p\ (k + n))$
have $\forall_\infty k.\ coeff\ p\ (k + n) = 0$
by (*subst cofinite-eq-sequentially*, *subst eventually-sequentially-seg*,
subst cofinite-eq-sequentially [*symmetric*]) *transfer*
then have *coeff-r* [*simp*]: $coeff\ r\ k = coeff\ p\ (k + n)$ **for** k
unfolding *r-def* **by** (*subst poly.Abs-poly-inverse*) *simp-all*
have $p = monom\ 1\ n * r$
by (*rule poly-eqI*, *subst coeff-monom-mult*) (*simp-all add*: *zero*)
then show *monom 1 n dvd p* **by** *simp*
qed

lemma *coeff-sum-monom*:
assumes n: $n \le d$
shows $coeff\ (\sum i{\le}d.\ monom\ (f\ i)\ i)\ n = f\ n$ (**is** *?l* = -)
proof −
have *?l* = $(\sum i{\le}d.\ coeff\ (monom\ (f\ i)\ i)\ n)$ (**is** - = *sum ?cmf* -)
using *coeff-sum*.
also have $\{..d\} = insert\ n\ (\{..d\}-\{n\})$ **using** n **by** *auto*
hence *sum ?cmf* $\{..d\}$ = *sum ?cmf* ... **by** *auto*
also have ... = *sum ?cmf* $(\{..d\}-\{n\})$ + *?cmf n* **by** (*subst sum.insert*,*auto*)
also have *sum ?cmf* $(\{..d\}-\{n\}) = 0$ **by** (*subst sum.neutral*, *auto*)
finally show *?thesis* **by** *simp*
qed

4.15 Mapping polynomials

definition *map-poly* :: $('a :: zero \Rightarrow 'b :: zero) \Rightarrow 'a\ poly \Rightarrow 'b\ poly$
where *map-poly f p* = *Poly* (*map f* (*coeffs p*))

lemma *map-poly-0* [*simp*]: *map-poly f 0* = *0*
by (*simp add*: *map-poly-def*)

lemma *map-poly-1*: *map-poly f 1* = [:*f 1*:]
by (*simp add*: *map-poly-def*)

lemma *map-poly-1'* [*simp*]: *f 1* = *1* $\Longrightarrow$ *map-poly f 1* = *1*
by (*simp add*: *map-poly-def one-pCons*)

lemma *coeff-map-poly*:
assumes *f 0* = *0*
shows *coeff* (*map-poly f p*) *n* = *f* (*coeff p n*)
by (*auto simp*: *assms map-poly-def nth-default-def coeffs-def not-less Suc-le-eq coeff-eq-0*
simp del: *upt-Suc*)

lemma *lead-coeff-map-poly-nz*:
assumes *f* (*lead-coeff p*) ≠ *0 f 0* = *0*
shows *lead-coeff* (*map-poly f p*) = *f* (*lead-coeff p*)

by (*metis* (*no-types*, *lifting*) *antisym assms coeff-0 coeff-map-poly le-degree leading-coeff-0-iff*)

lemma *coeffs-map-poly* [*code abstract*]:
coeffs (*map-poly f p*) = *strip-while* ((=) *0*) (*map f* (*coeffs p*))
by (*simp add*: *map-poly-def*)

lemma *coeffs-map-poly'*:
assumes $\bigwedge x.\ x \neq 0 \Longrightarrow f\ x \neq 0$
shows *coeffs* (*map-poly f p*) = *map f* (*coeffs p*)
using *assms*
by (*auto simp add*: *coeffs-map-poly strip-while-idem-iff*
last-coeffs-eq-coeff-degree no-trailing-unfold last-map)

lemma *set-coeffs-map-poly*:
$(\bigwedge x.\ f\ x = 0 \longleftrightarrow x = 0) \Longrightarrow$ *set* (*coeffs* (*map-poly f p*)) = *f* ‘ *set* (*coeffs p*)
by (*simp add*: *coeffs-map-poly'*)

lemma *degree-map-poly*:
assumes $\bigwedge x.\ x \neq 0 \Longrightarrow f\ x \neq 0$
shows *degree* (*map-poly f p*) = *degree p*
by (*simp add*: *degree-eq-length-coeffs coeffs-map-poly' assms*)

lemma *map-poly-eq-0-iff*:
assumes *f 0* = *0* $\bigwedge x.\ x \in$ *set* (*coeffs p*) $\Longrightarrow x \neq 0 \Longrightarrow f\ x \neq 0$
shows *map-poly f p* = *0* $\longleftrightarrow$ *p* = *0*
proof −
have (*coeff* (*map-poly f p*) *n* = *0*) = (*coeff p n* = *0*) **for** *n*
proof −
have *coeff* (*map-poly f p*) *n* = *f* (*coeff p n*)
by (*simp add*: *coeff-map-poly assms*)
also have ... = *0* $\longleftrightarrow$ *coeff p n* = *0*
proof (*cases n* < *length* (*coeffs p*))
case *True*
then have *coeff p n* ∈ *set* (*coeffs p*)
by (*auto simp*: *coeffs-def simp del*: *upt-Suc*)
with *assms* **show** *f* (*coeff p n*) = *0* $\longleftrightarrow$ *coeff p n* = *0*
by *auto*
next
case *False*
then show *?thesis*
by (*auto simp*: *assms length-coeffs nth-default-coeffs-eq* [*symmetric*] *nth-default-def*)
qed
finally show *?thesis* **.**
qed
then show *?thesis* **by** (*auto simp*: *poly-eq-iff*)
qed

lemma *map-poly-smult*:

assumes $f\ 0 = 0 \bigwedge c\ x.\ f\ (c * x) = f\ c * f\ x$
shows *map-poly f* (*smult c p*) = *smult* (*f c*) (*map-poly f p*)
by (*intro poly-eqI*) (*simp-all add*: *assms coeff-map-poly*)

lemma *map-poly-pCons*:
assumes *f 0* = *0*
shows *map-poly f* (*pCons c p*) = *pCons* (*f c*) (*map-poly f p*)
by (*intro poly-eqI*) (*simp-all add*: *assms coeff-map-poly coeff-pCons split*: *nat.splits*)

lemma *map-poly-map-poly*:
assumes *f 0* = *0 g 0* = *0*
shows *map-poly f* (*map-poly g p*) = *map-poly* (*f* ∘ *g*) *p*
by (*intro poly-eqI*) (*simp add*: *coeff-map-poly assms*)

lemma *map-poly-id* [*simp*]: *map-poly id p* = *p*
by (*simp add*: *map-poly-def*)

lemma *map-poly-id′* [*simp*]: *map-poly* (λ*x*. *x*) *p* = *p*
by (*simp add*: *map-poly-def*)

lemma *map-poly-cong*:
assumes (⋀*x*. *x* ∈ *set* (*coeffs p*) ⟹ *f x* = *g x*)
shows *map-poly f p* = *map-poly g p*
proof −
from *assms* **have** *map f* (*coeffs p*) = *map g* (*coeffs p*)
by (*intro map-cong*) *simp-all*
then show *?thesis*
by (*simp only*: *coeffs-eq-iff coeffs-map-poly*)
qed

lemma *map-poly-monom*: *f 0* = *0* ⟹ *map-poly f* (*monom c n*) = *monom* (*f c*) *n*
by (*intro poly-eqI*) (*simp-all add*: *coeff-map-poly*)

lemma *map-poly-idI*:
assumes ⋀*x*. *x* ∈ *set* (*coeffs p*) ⟹ *f x* = *x*
shows *map-poly f p* = *p*
using *map-poly-cong*[*OF assms*, *of - id*] **by** *simp*

lemma *map-poly-idI′*:
assumes ⋀*x*. *x* ∈ *set* (*coeffs p*) ⟹ *f x* = *x*
shows *p* = *map-poly f p*
using *map-poly-cong*[*OF assms*, *of - id*] **by** *simp*

lemma *smult-conv-map-poly*: *smult c p* = *map-poly* (λ*x*. *c* * *x*) *p*
by (*intro poly-eqI*) (*simp-all add*: *coeff-map-poly*)

lemma *poly-cnj*: *cnj* (*poly p z*) = *poly* (*map-poly cnj p*) (*cnj z*)
by (*simp add*: *poly-altdef degree-map-poly coeff-map-poly*)

lemma *poly-cnj-real*:
assumes $\bigwedge n.$ *poly.coeff p n* $\in \mathbb{R}$
shows *cnj* (*poly p z*) = *poly p* (*cnj z*)
proof −
from *assms* **have** *map-poly cnj p* = *p*
by (*intro poly-eqI*) (*auto simp*: *coeff-map-poly Reals-cnj-iff*)
with *poly-cnj*[*of p z*] **show** *?thesis* **by** *simp*
qed

lemma *real-poly-cnj-root-iff*:
assumes $\bigwedge n.$ *poly.coeff p n* $\in \mathbb{R}$
shows *poly p* (*cnj z*) = *0* $\longleftrightarrow$ *poly p z* = *0*
proof −
have *poly p* (*cnj z*) = *cnj* (*poly p z*)
by (*simp add*: *poly-cnj-real assms*)
also have ... = *0* $\longleftrightarrow$ *poly p z* = *0* **by** *simp*
finally show *?thesis* **.**
qed

lemma *sum-to-poly*: $(\sum x{\in}A.\ [:f\ x:]) = [:\sum x{\in}A.\ f\ x:]$
by (*induction A rule*: *infinite-finite-induct*) *auto*

lemma *diff-to-poly*: [:*c*:] − [:*d*:] = [:*c* − *d*:]
by (*simp add*: *poly-eq-iff mult-ac*)

lemma *mult-to-poly*: [:*c*:] ∗ [:*d*:] = [:*c* ∗ *d*:]
by (*simp add*: *poly-eq-iff mult-ac*)

lemma *prod-to-poly*: $(\prod x{\in}A.\ [:f\ x:]) = [:\prod x{\in}A.\ f\ x:]$
by (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *mult-to-poly mult-ac*)

lemma *poly-map-poly-cnj* [*simp*]: *poly* (*map-poly cnj p*) *x* = *cnj* (*poly p* (*cnj x*))
using *complex-cnj-cnj poly-cnj* **by** *force*

lemma *map-poly-degree-eq*:
assumes *f* (*lead-coeff p*) ≠ *0*
shows *degree* (*map-poly f p*) = *degree p*
using *assms*
unfolding *map-poly-def degree-eq-length-coeffs coeffs-Poly lead-coeff-list-def*
by (*metis* (*full-types*) *last-conv-nth-default length-map no-trailing-unfold nth-default-coeffs-eq*

nth-default-map-eq strip-while-idem)

lemma *map-poly-degree-less*:
assumes *f* (*lead-coeff p*) =*0 degree p*≠*0*
shows *degree* (*map-poly f p*) < *degree p*
proof −
have *length* (*coeffs p*) >*1*
using ‹*degree p*≠*0*› **by** (*simp add*: *degree-eq-length-coeffs*)

then obtain *xs x* **where** *xs-def*:*coeffs p*=*xs*@[*x*] *length xs*>*0*
by (*metis One-nat-def add-0 append-Nil length-greater-0-conv list.size*(*4*) *nat-neq-iff not-less-zero rev-exhaust*)
have *f x*=*0* **using** *assms*(*1*) **by** (*simp add*: *lead-coeff-list-def xs-def*(*1*))
have *degree* (*map-poly f p*) = *length* (*strip-while* ((=) *0*) (*map f* (*xs*@[*x*]))) − *1*
unfolding *map-poly-def degree-eq-length-coeffs coeffs-Poly*
by (*subst xs-def*,*auto*)
also have ... = *length* (*strip-while* ((=) *0*) (*map f xs*)) − *1*
using ‹*f x*=*0*› **by** *simp*
also have ... ≤ *length xs* −*1*
using *length-strip-while-le* **by** (*metis diff-le-mono length-map*)
also have ... < *length* (*xs*@[*x*]) − *1*
using *xs-def*(*2*) **by** *auto*
also have ... = *degree p*
unfolding *degree-eq-length-coeffs xs-def* **by** *simp*
finally show *?thesis* **.**
qed

lemma *map-poly-degree-leq*:
shows *degree* (*map-poly f p*) ≤ *degree p*
unfolding *map-poly-def degree-eq-length-coeffs*
by (*metis coeffs-Poly diff-le-mono length-map length-strip-while-le*)

4.16 Conversions

lemma *of-nat-poly*: *of-nat n* = [:*of-nat n*:]
by (*induct n*) (*simp-all add*: *one-pCons*)

lemma *of-nat-monom*: *of-nat n* = *monom* (*of-nat n*) *0*
by (*simp add*: *of-nat-poly monom-0*)

lemma *degree-of-nat* [*simp*]: *degree* (*of-nat n*) = *0*
by (*simp add*: *of-nat-poly*)

lemma *lead-coeff-of-nat* [*simp*]: *lead-coeff* (*of-nat n*) = *of-nat n*
by (*simp add*: *of-nat-poly*)

lemma *of-int-poly*: *of-int k* = [:*of-int k*:]
by (*simp only*: *of-int-of-nat of-nat-poly*) *simp*

lemma *of-int-monom*: *of-int k* = *monom* (*of-int k*) *0*
by (*simp add*: *of-int-poly monom-0*)

lemma *degree-of-int* [*simp*]: *degree* (*of-int k*) = *0*
by (*simp add*: *of-int-poly*)

lemma *lead-coeff-of-int* [*simp*]: *lead-coeff* (*of-int k*) = *of-int k*
by (*simp add*: *of-int-poly*)

lemma *poly-of-nat* [*simp*]: *poly* (*of-nat n*) *x* = *of-nat n*
by (*simp add*: *of-nat-poly*)

lemma *poly-of-int* [*simp*]: *poly* (*of-int n*) *x* = *of-int n*
by (*simp add*: *of-int-poly*)

lemma *poly-numeral* [*simp*]: *poly* (*numeral n*) *x* = *numeral n*
by (*metis of-nat-numeral poly-of-nat*)

lemma *numeral-poly*: *numeral n* = [:*numeral n*:]
proof −
have *numeral n* = *of-nat* (*numeral n*)
by *simp*
also have ... = [:*of-nat* (*numeral n*):]
by (*simp add*: *of-nat-poly*)
finally show *?thesis*
by *simp*
qed

lemma *numeral-monom*:
numeral n = *monom* (*numeral n*) *0*
by (*simp add*: *numeral-poly monom-0*)

lemma *degree-numeral* [*simp*]:
degree (*numeral n*) = *0*
by (*simp add*: *numeral-poly*)

lemma *lead-coeff-numeral* [*simp*]:
lead-coeff (*numeral n*) = *numeral n*
by (*simp add*: *numeral-poly*)

lemma *coeff-linear-poly-power*:
fixes *c* :: *'a* :: *semiring-1*
assumes *i* ≤ *n*
shows *coeff* ([:*a*, *b*:] ^ *n*) *i* = *of-nat* (*n choose i*) ∗ *b* ^ *i* ∗ *a* ^ (*n* − *i*)
proof −
have [:*a*, *b*:] = *monom b 1* + [:*a*:]
by (*simp add*: *monom-altdef*)
also have *coeff* (... ^ *n*) *i* = (∑ *k*≤*n*. *a*^(*n*−*k*) ∗ *of-nat* (*n choose k*) ∗ (*if k* = *i then b* ^ *k else 0*))
by (*subst binomial-ring*) (*simp add*: *coeff-sum of-nat-poly monom-power poly-const-pow mult-ac*)
also have ... = (∑ *k*∈{*i*}. *a* ^ (*n* − *i*) ∗ *b* ^ *i* ∗ *of-nat* (*n choose k*))
using *assms* **by** (*intro sum.mono-neutral-cong-right*) (*auto simp*: *mult-ac*)
finally show ∗: *?thesis* **by** (*simp add*: *mult-ac*)
qed

4.17 Lemmas about divisibility

lemma *dvd-smult*:
  assumes *p dvd q*
  shows *p dvd smult a q*
proof −
  from *assms* **obtain** *k* **where** *q* = *p* ∗ *k* **..**
  then have *smult a q* = *p* ∗ *smult a k* **by** *simp*
  then show *p dvd smult a q* **..**
qed

lemma *dvd-smult-cancel*: *p dvd smult a q* ⟹ *a* ≠ *0* ⟹ *p dvd q*
  for *a* :: *'a*::*field*
  by (*drule dvd-smult* [**where** *a*=*inverse a*]) *simp*

lemma *dvd-smult-iff*: *a* ≠ *0* ⟹ *p dvd smult a q* ⟷ *p dvd q*
  for *a* :: *'a*::*field*
  by (*safe elim*!: *dvd-smult dvd-smult-cancel*)

lemma *smult-dvd-cancel*:
  assumes *smult a p dvd q*
  shows *p dvd q*
proof −
  from *assms* **obtain** *k* **where** *q* = *smult a p* ∗ *k* **..**
  then have *q* = *p* ∗ *smult a k* **by** *simp*
  then show *p dvd q* **..**
qed

lemma *smult-dvd*: *p dvd q* ⟹ *a* ≠ *0* ⟹ *smult a p dvd q*
  for *a* :: *'a*::*field*
  by (*rule smult-dvd-cancel* [**where** *a*=*inverse a*]) *simp*

lemma *smult-dvd-iff*: *smult a p dvd q* ⟷ (*if a* = *0 then q* = *0 else p dvd q*)
  for *a* :: *'a*::*field*
  by (*auto elim*: *smult-dvd smult-dvd-cancel*)

lemma *is-unit-smult-iff*: *smult c p dvd 1* ⟷ *c dvd 1* ∧ *p dvd 1*
proof −
  have *smult c p* = [:*c*:] ∗ *p* **by** *simp*
  also have ... *dvd 1* ⟷ *c dvd 1* ∧ *p dvd 1*
  proof *safe*
    assume ∗: [:*c*:] ∗ *p dvd 1*
    then show *p dvd 1*
      by (*rule dvd-mult-right*)
    from ∗ **obtain** *q* **where** *q*: *1* = [:*c*:] ∗ *p* ∗ *q*
      by (*rule dvdE*)
    have *c dvd c* ∗ (*coeff p 0* ∗ *coeff q 0*)
      by *simp*
    also have ... = *coeff* ([:*c*:] ∗ *p* ∗ *q*) *0*
      by (*simp add*: *mult.assoc coeff-mult*)

also note *q* [*symmetric*]
finally have *c dvd coeff 1 0* **.**
then show *c dvd 1* **by** *simp*
next
assume *c dvd 1 p dvd 1*
from *this*(*1*) **obtain** *d* **where** *1 = c * d*
by (*rule dvdE*)
then have *1 =* [:*c*:] * [:*d*:]
by (*simp add*: *one-pCons ac-simps*)
then have [:*c*:] *dvd 1*
by (*rule dvdI*)
from *mult-dvd-mono*[*OF this* ‹*p dvd 1*›] **show** [:*c*:] * *p dvd 1*
by *simp*
qed
finally show *?thesis* **.**
qed

4.18 Polynomials form an integral domain

instance *poly* :: (*idom*) *idom* **..**

instance *poly* :: ({*ring-char-0*, *comm-ring-1*}) *ring-char-0*
by *standard* (*auto simp add*: *of-nat-poly intro*: *injI*)

lemma *semiring-char-poly* [*simp*]: *CHAR*('*a* :: *comm-semiring-1 poly*) = *CHAR*('*a*)
by (*rule CHAR-eqI*) (*auto simp*: *of-nat-poly of-nat-eq-0-iff-char-dvd*)

instance *poly* :: ({*semiring-prime-char*,*comm-semiring-1*}) *semiring-prime-char*
by (*rule semiring-prime-charI*) *auto*
instance *poly* :: ({*comm-semiring-prime-char*,*comm-semiring-1*}) *comm-semiring-prime-char*
by *standard*
instance *poly* :: ({*comm-ring-prime-char*,*comm-semiring-1*}) *comm-ring-prime-char*
by *standard*
instance *poly* :: ({*idom-prime-char*,*comm-semiring-1*}) *idom-prime-char*
by *standard*

lemma *degree-mult-eq*: *p* ≠ *0* ⟹ *q* ≠ *0* ⟹ *degree* (*p * q*) = *degree p* + *degree q*
for *p q* :: '*a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
by (*rule order-antisym* [*OF degree-mult-le le-degree*]) (*simp add*: *coeff-mult-degree-sum*)

lemma *degree-prod-sum-eq*:
(⋀*x*. *x* ∈ *A* ⟹ *f x* ≠ *0*) ⟹
degree (*prod f A* :: '*a* :: *idom poly*) = (∑ *x*∈*A*. *degree* (*f x*))
by (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *degree-mult-eq*)

lemma *dvd-imp-degree*:
‹*degree x* ≤ *degree y*› **if** ‹*x dvd y*› ‹*x* ≠ *0*› ‹*y* ≠ *0*›
for *x y* :: ‹'*a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*›
proof −

from ‹*x dvd y*› **obtain** *z* **where** ‹*y* = *x* ∗ *z*› **..**
with ‹*x* ≠ *0*› ‹*y* ≠ *0*› **show** *?thesis*
by (*simp add*: *degree-mult-eq*)
qed

lemma *degree-prod-eq-sum-degree*:
fixes *A* :: *'a set*
and *f* :: *'a* ⇒ *'b*::*idom poly*
assumes *f0*: ∀ *i*∈*A*. *f i* ≠ *0*
shows *degree* (∏ *i*∈*A*. (*f i*)) = (∑ *i*∈*A*. *degree* (*f i*))
using *assms*
by (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *degree-mult-eq*)

lemma *degree-mult-eq-0*:
degree (*p* ∗ *q*) = *0* ⟷ *p* = *0* ∨ *q* = *0* ∨ (*p* ≠ *0* ∧ *q* ≠ *0* ∧ *degree p* = *0* ∧ *degree q* = *0*)
for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
by (*auto simp*: *degree-mult-eq*)

lemma *degree-power-eq*: *p* ≠ *0* ⟹ *degree* ((*p* :: *'a* :: *idom poly*) ^ *n*) = *n* ∗ *degree p*
by (*induction n*) (*simp-all add*: *degree-mult-eq*)

lemma *degree-mult-right-le*:
fixes *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
assumes *q* ≠ *0*
shows *degree p* ≤ *degree* (*p* ∗ *q*)
using *assms* **by** (*cases p* = *0*) (*simp-all add*: *degree-mult-eq*)

lemma *coeff-degree-mult*: *coeff* (*p* ∗ *q*) (*degree* (*p* ∗ *q*)) = *coeff q* (*degree q*) ∗ *coeff p* (*degree p*)
for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
by (*cases p* = *0* ∨ *q* = *0*) (*auto simp*: *degree-mult-eq coeff-mult-degree-sum mult-ac*)

lemma *dvd-imp-degree-le*: *p dvd q* ⟹ *q* ≠ *0* ⟹ *degree p* ≤ *degree q*
for *p q* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
by (*erule dvdE*, *hypsubst*, *subst degree-mult-eq*) *auto*

lemma *divides-degree*:
fixes *p q* :: *'a* ::{*comm-semiring-1*,*semiring-no-zero-divisors*} *poly*
assumes *p dvd q*
shows *degree p* ≤ *degree q* ∨ *q* = *0*
by (*metis dvd-imp-degree-le assms*)

lemma *const-poly-dvd-iff*:
fixes *c* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*}
shows [:*c*:] *dvd p* ⟷ (∀ *n*. *c dvd coeff p n*)
proof (*cases c* = *0* ∨ *p* = *0*)

```
  case True
  then show ?thesis
    by (auto intro!: poly-eqI)
next
  case False
  show ?thesis
  proof
    assume [:c:] dvd p
    then show ∀ n. c dvd coeff p n
      by (auto simp: coeffs-def)
  next
    assume *: ∀ n. c dvd coeff p n
    define mydiv where mydiv x y = (SOME z. x = y * z) for x y :: 'a
    have mydiv: x = y * mydiv x y if y dvd x for x y
      using that unfolding mydiv-def dvd-def by (rule someI-ex)
    define q where q = Poly (map (λa. mydiv a c) (coeffs p))
    from False * have p = q * [:c:]
      by (intro poly-eqI)
        (auto simp: q-def nth-default-def not-less length-coeffs-degree coeffs-nth
          intro!: coeff-eq-0 mydiv)
    then show [:c:] dvd p
      by (simp only: dvd-triv-right)
  qed
qed

lemma const-poly-dvd-const-poly-iff [simp]: [:a:] dvd [:b:] ⟷ a dvd b
  for a b :: 'a::{comm-semiring-1,semiring-no-zero-divisors}
  by (subst const-poly-dvd-iff) (auto simp: coeff-pCons split: nat.splits)

lemma lead-coeff-mult: lead-coeff (p * q) = lead-coeff p * lead-coeff q
  for p q :: 'a::{comm-semiring-0, semiring-no-zero-divisors} poly
  by (cases p = 0 ∨ q = 0) (auto simp: coeff-mult-degree-sum degree-mult-eq)

lemma lead-coeff-prod: lead-coeff (prod f A) = (∏ x∈A. lead-coeff (f x))
  for f :: 'a ⇒ 'b::{comm-semiring-1, semiring-no-zero-divisors} poly
  by (induction A rule: infinite-finite-induct) (auto simp: lead-coeff-mult)

lemma lead-coeff-smult: lead-coeff (smult c p) = c * lead-coeff p
  for p :: 'a::{comm-semiring-0,semiring-no-zero-divisors} poly
proof −
  have smult c p = [:c:] * p by simp
  also have lead-coeff ... = c * lead-coeff p
    by (subst lead-coeff-mult) simp-all
  finally show ?thesis .
qed

lemma lead-coeff-1 [simp]: lead-coeff 1 = 1
  by simp
```

lemma *lead-coeff-power*: *lead-coeff* $(p \mathbin{\hat{}} n) = \mathit{lead\text{-}coeff}\ p \mathbin{\hat{}} n$
for $p :: 'a::\{\mathit{comm\text{-}semiring\text{-}1}, \mathit{semiring\text{-}no\text{-}zero\text{-}divisors}\}\ \mathit{poly}$
by (*induct n*) (*simp-all add*: *lead-coeff-mult*)

4.19 Polynomials form an ordered integral domain

definition *pos-poly* :: $'a::\mathit{linordered\text{-}semidom}\ \mathit{poly} \Rightarrow \mathit{bool}$
where $\mathit{pos\text{-}poly}\ p \longleftrightarrow 0 < \mathit{coeff}\ p\ (\mathit{degree}\ p)$

lemma *pos-poly-pCons*: $\mathit{pos\text{-}poly}\ (\mathit{pCons}\ a\ p) \longleftrightarrow \mathit{pos\text{-}poly}\ p \vee (p = 0 \wedge 0 < a)$
by (*simp add*: *pos-poly-def*)

lemma *not-pos-poly-0* [*simp*]: $\neg\ \mathit{pos\text{-}poly}\ 0$
by (*simp add*: *pos-poly-def*)

lemma *pos-poly-add*: $\mathit{pos\text{-}poly}\ p \Longrightarrow \mathit{pos\text{-}poly}\ q \Longrightarrow \mathit{pos\text{-}poly}\ (p + q)$
proof (*induction p arbitrary*: *q*)
case (*pCons a p*)
then show *?case*
by (*cases q*; *force simp add*: *pos-poly-pCons add-pos-pos*)
qed *auto*

lemma *pos-poly-mult*: $\mathit{pos\text{-}poly}\ p \Longrightarrow \mathit{pos\text{-}poly}\ q \Longrightarrow \mathit{pos\text{-}poly}\ (p * q)$
by (*simp add*: *pos-poly-def coeff-degree-mult*)

lemma *pos-poly-total*: $p = 0 \vee \mathit{pos\text{-}poly}\ p \vee \mathit{pos\text{-}poly}\ (-\ p)$
for $p :: 'a::\mathit{linordered\text{-}idom}\ \mathit{poly}$
by (*induct p*) (*auto simp*: *pos-poly-pCons*)

lemma *pos-poly-coeffs* [*code*]: $\mathit{pos\text{-}poly}\ p \longleftrightarrow (\mathit{let}\ \mathit{as} = \mathit{coeffs}\ p\ \mathit{in}\ \mathit{as} \neq [] \wedge \mathit{last}\ \mathit{as} > 0)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?rhs*
then show *?lhs*
by (*auto simp add*: *pos-poly-def last-coeffs-eq-coeff-degree*)
next
assume *?lhs*
then have $*$: $0 < \mathit{coeff}\ p\ (\mathit{degree}\ p)$
by (*simp add*: *pos-poly-def*)
then have $p \neq 0$
by *auto*
with $*$ **show** *?rhs*
by (*simp add*: *last-coeffs-eq-coeff-degree*)
qed

instantiation *poly* :: (*linordered-idom*) *linordered-idom*
begin

definition $x < y \longleftrightarrow$ *pos-poly* $(y - x)$

definition $x \leq y \longleftrightarrow x = y \vee$ *pos-poly* $(y - x)$

definition $|x::'a\ poly| = (if\ x < 0\ then - x\ else\ x)$

definition $sgn\ (x::'a\ poly) = (if\ x = 0\ then\ 0\ else\ if\ 0 < x\ then\ 1\ else - 1)$

instance
proof
  fix $x\ y\ z :: 'a\ poly$
  show $x < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
    unfolding *less-eq-poly-def less-poly-def*
    using *pos-poly-add* **by** *force*
  then show $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
    using *less-eq-poly-def less-poly-def* **by** *force*
  show $x \leq x$
    by (*simp add*: *less-eq-poly-def*)
  show $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
    using *less-eq-poly-def pos-poly-add* **by** *fastforce*
  show $x \leq y \Longrightarrow z + x \leq z + y$
    by (*simp add*: *less-eq-poly-def*)
  show $x \leq y \vee y \leq x$
    unfolding *less-eq-poly-def*
    using *pos-poly-total* [*of* $x - y$]
    by *auto*
  show $x < y \Longrightarrow 0 < z \Longrightarrow z * x < z * y$
    by (*simp add*: *less-poly-def right-diff-distrib* [*symmetric*] *pos-poly-mult*)
  show $|x| = (if\ x < 0\ then - x\ else\ x)$
    by (*rule abs-poly-def*)
  show $sgn\ x = (if\ x = 0\ then\ 0\ else\ if\ 0 < x\ then\ 1\ else - 1)$
    by (*rule sgn-poly-def*)
qed

end

TODO: Simplification rules for comparisons

4.20 Synthetic division and polynomial roots

4.20.1 Synthetic division

Synthetic division is simply division by the linear polynomial $x - c$.

definition *synthetic-divmod* :: $'a$::*comm-semiring-0 poly* $\Rightarrow$ $'a \Rightarrow$ $'a$ *poly* $\times$ $'a$
  where *synthetic-divmod p c* = *fold-coeffs* $(\lambda a\ (q,\ r).\ (pCons\ r\ q,\ a + c * r))\ p$ $(0,\ 0)$

definition *synthetic-div* :: $'a$::*comm-semiring-0 poly* $\Rightarrow$ $'a \Rightarrow$ $'a$ *poly*
  where *synthetic-div p c* = *fst* (*synthetic-divmod p c*)

lemma *synthetic-divmod-0* [*simp*]: *synthetic-divmod 0 c* = (*0*, *0*)
by (*simp add*: *synthetic-divmod-def*)

lemma *synthetic-divmod-pCons* [*simp*]:
synthetic-divmod (*pCons a p*) *c* = (λ(*q*, *r*). (*pCons r q*, *a* + *c* ∗ *r*)) (*synthetic-divmod p c*)
by (*cases p* = *0* ∧ *a* = *0*) (*auto simp add*: *synthetic-divmod-def*)

lemma *synthetic-div-0* [*simp*]: *synthetic-div 0 c* = *0*
by (*simp add*: *synthetic-div-def*)

lemma *synthetic-div-unique-lemma*: *smult c p* = *pCons a p* ⟹ *p* = *0*
by (*induct p arbitrary*: *a*) *simp-all*

lemma *snd-synthetic-divmod*: *snd* (*synthetic-divmod p c*) = *poly p c*
by (*induct p*) (*simp-all add*: *split-def*)

lemma *synthetic-div-pCons* [*simp*]:
synthetic-div (*pCons a p*) *c* = *pCons* (*poly p c*) (*synthetic-div p c*)
by (*simp add*: *synthetic-div-def split-def snd-synthetic-divmod*)

lemma *synthetic-div-eq-0-iff*: *synthetic-div p c* = *0* ⟷ *degree p* = *0*
proof (*induct p*)
case *0*
then show *?case* **by** *simp*
next
case (*pCons a p*)
then show *?case* **by** (*cases p*) *simp*
qed

lemma *degree-synthetic-div*: *degree* (*synthetic-div p c*) = *degree p* − *1*
by (*induct p*) (*simp-all add*: *synthetic-div-eq-0-iff*)

lemma *synthetic-div-correct*:
p + *smult c* (*synthetic-div p c*) = *pCons* (*poly p c*) (*synthetic-div p c*)
by (*induct p*) *simp-all*

lemma *synthetic-div-unique*: *p* + *smult c q* = *pCons r q* ⟹ *r* = *poly p c* ∧ *q* = *synthetic-div p c*
proof (*induction p arbitrary*: *q r*)
case *0*
then show *?case*
using *synthetic-div-unique-lemma* **by** *fastforce*
next
case (*pCons a p*)
then show *?case*
by (*cases q*; *force*)
qed

lemma *synthetic-div-correct'*: $[:-c,\ 1:] * \textit{synthetic-div}\ p\ c + [:\textit{poly}\ p\ c:] = p$
for $c :: 'a$::*comm-ring-1*
using *synthetic-div-correct* [*of p c*] **by** (*simp add*: *algebra-simps*)

4.20.2 Polynomial roots

lemma *poly-eq-0-iff-dvd*: $\textit{poly}\ p\ c = 0 \longleftrightarrow [:-\ c,\ 1:]\ \textit{dvd}\ p$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
for $c :: 'a$::*comm-ring-1*
proof
assume *?lhs*
with *synthetic-div-correct'* [*of c p*] **have** $p = [:-c,\ 1:] * \textit{synthetic-div}\ p\ c$ **by** *simp*
then show *?rhs* **..**
next
assume *?rhs*
then obtain *k* **where** $p = [:-c,\ 1:] * k$ **by** (*rule dvdE*)
then show *?lhs* **by** *simp*
qed

lemma *dvd-iff-poly-eq-0*: $[:c,\ 1:]\ \textit{dvd}\ p \longleftrightarrow \textit{poly}\ p\ (-\ c) = 0$
for $c :: 'a$::*comm-ring-1*
by (*simp add*: *poly-eq-0-iff-dvd*)

lemma *poly-roots-finite*: $p \neq 0 \Longrightarrow \textit{finite}\ \{x.\ \textit{poly}\ p\ x = 0\}$
for $p :: 'a$::{*comm-ring-1*,*ring-no-zero-divisors*} *poly*
proof (*induct n* $\equiv$ *degree p arbitrary*: *p*)
case *0*
then obtain *a* **where** $a \neq 0$ **and** $p = [:a:]$
by (*cases p*) (*simp split*: *if-splits*)
then show *finite* $\{x.\ \textit{poly}\ p\ x = 0\}$
by *simp*
next
case (*Suc n*)
show *finite* $\{x.\ \textit{poly}\ p\ x = 0\}$
proof (*cases* $\exists x.\ \textit{poly}\ p\ x = 0$)
case *False*
then show *finite* $\{x.\ \textit{poly}\ p\ x = 0\}$ **by** *simp*
next
case *True*
then obtain *a* **where** *poly p a* $= 0$ **..**
then have $[:-a,\ 1:]\ \textit{dvd}\ p$
by (*simp only*: *poly-eq-0-iff-dvd*)
then obtain *k* **where** *k*: $p = [:-a,\ 1:] * k$ **..**
with ‹$p \neq 0$› **have** $k \neq 0$
by *auto*
with *k* **have** *degree p* $=$ *Suc* (*degree k*)
by (*simp add*: *degree-mult-eq del*: *mult-pCons-left*)
with ‹*Suc n* $=$ *degree p*› **have** *n* $=$ *degree k*

```
    by simp
  from this ‹k ≠ 0› have finite {x. poly k x = 0}
    by (rule Suc.hyps)
  then have finite (insert a {x. poly k x = 0})
    by simp
  then show finite {x. poly p x = 0}
    by (simp add: k Collect-disj-eq del: mult-pCons-left)
 qed
qed

lemma poly-eq-poly-eq-iff: poly p = poly q ⟷ p = q
 (is ?lhs ⟷ ?rhs)
 for p q :: 'a::{comm-ring-1,ring-no-zero-divisors,ring-char-0} poly
proof
 assume ?rhs
 then show ?lhs by simp
next
 assume ?lhs
 have poly p = poly 0 ⟷ p = 0 for p :: 'a poly
 proof (cases p = 0)
  case False
  then show ?thesis
   by (auto simp add: infinite-UNIV-char-0 dest: poly-roots-finite)
 qed auto
 from ‹?lhs› and this [of p − q] show ?rhs
  by auto
qed
```

A nice extension rule for polynomials.

```
lemma poly-ext:
 fixes p q :: 'a :: {ring-char-0, idom} poly
 assumes ⋀x. poly p x = poly q x shows p = q
 unfolding poly-eq-poly-eq-iff[symmetric]
 using assms by (rule ext)
```

Copied from non-negative variants.

```
lemma coeff-linear-power-neg[simp]:
 fixes a :: 'a::comm-ring-1
 shows coeff ([:a, −1:] ^ n) n = (−1) ^ n
proof (induct n)
 case 0
 then show ?case by simp
next
 case (Suc n)
 then have degree ([:a, − 1:] ^ n) < Suc n
  by (auto intro: le-less-trans degree-power-le)
 with Suc show ?case
  by (simp add: coeff-eq-0)
qed
```

lemma *degree-linear-power-neg*[*simp*]:
fixes *a* :: *'a*::{*idom*,*comm-ring-1*}
shows *degree* ([:*a*, −*1*:] ^ *n*) = *n*
by (*simp add*: *degree-power-eq*)

lemma *poly-all-0-iff-0*: (∀ *x*. *poly p x* = *0*) ⟷ *p* = *0*
for *p* :: *'a*::{*ring-char-0*,*comm-ring-1*,*ring-no-zero-divisors*} *poly*
by (*auto simp add*: *poly-eq-poly-eq-iff* [*symmetric*])

lemma *card-poly-roots-bound*:
fixes *p* :: *'a*::{*comm-ring-1*,*ring-no-zero-divisors*} *poly*
assumes *p* ≠ *0*
shows *card* {*x*. *poly p x* = *0*} ≤ *degree p*
using *assms*
proof (*induction degree p arbitrary*: *p rule*: *less-induct*)
case (*less p*)
show *?case*
proof (*cases* ∃ *x*. *poly p x* = *0*)
case *False*
hence {*x*. *poly p x* = *0*} = {} **by** *blast*
thus *?thesis* **by** *simp*
next
case *True*
then obtain *x* **where** *x*: *poly p x* = *0* **by** *blast*
hence [:−*x*, *1*:] *dvd p* **by** (*subst* (*asm*) *poly-eq-0-iff-dvd*)
then obtain *q* **where** *q*: *p* = [:−*x*, *1*:] ∗ *q* **by** (*auto simp*: *dvd-def*)
with ‹*p* ≠ *0*› **have** [*simp*]: *q* ≠ *0* **by** *auto*
have *deg*: *degree p* = *Suc* (*degree q*)
by (*subst q*, *subst degree-mult-eq*) *auto*
have *card* {*x*. *poly p x* = *0*} ≤ *card* (*insert x* {*x*. *poly q x* = *0*})
by (*intro card-mono*) (*auto intro*: *poly-roots-finite simp*: *q*)
also have ... ≤ *Suc* (*card* {*x*. *poly q x* = *0*})
by (*rule card-insert-le-m1*) *auto*
also from *deg* **have** *card* {*x*. *poly q x* = *0*} ≤ *degree q*
using ‹*p* ≠ *0*› **and** *q* **by** (*intro less*) *auto*
also have *Suc* ... = *degree p* **by** (*simp add*: *deg*)
finally show *?thesis* **by** − *simp-all*
qed
qed

lemma *poly-eqI-degree*:
fixes *p q* :: *'a* :: {*comm-ring-1*, *ring-no-zero-divisors*} *poly*
assumes ⋀*x*. *x* ∈ *A* ⟹ *poly p x* = *poly q x*
assumes *card A* > *degree p card A* > *degree q*
shows *p* = *q*
proof (*rule ccontr*)
assume *neq*: *p* ≠ *q*
have *degree* (*p* − *q*) ≤ *max* (*degree p*) (*degree q*)

```
    by (rule degree-diff-le-max)
  also from assms have ... < card A by linarith
  also have ... ≤ card {x. poly (p − q) x = 0}
    using neq and assms by (intro card-mono poly-roots-finite) auto
  finally have degree (p − q) < card {x. poly (p − q) x = 0} .
  moreover have degree (p − q) ≥ card {x. poly (p − q) x = 0}
    using neq by (intro card-poly-roots-bound) auto
  ultimately show False by linarith
qed

lemma poly-eqI-degree-lead-coeff:
  fixes p q :: 'a :: {comm-ring-1, ring-no-zero-divisors} poly
  assumes poly.coeff p n = poly.coeff q n card A ≥ n degree p ≤ n degree q ≤ n
  assumes ⋀z. z ∈ A ⟹ poly p z = poly q z
  shows   p = q
proof (rule ccontr)
  assume p ≠ q

  have n > 0
  proof (rule ccontr)
    assume ¬(n > 0)
    thus False
      using assms ‹p ≠ q› by (auto elim!: degree-eq-zeroE)
  qed

  have n ≤ card A
    by fact
  also have card A ≤ card {x. poly (p − q) x = 0}
    by (intro card-mono poly-roots-finite) (use ‹p ≠ q› assms in auto)
  also have card {x. poly (p − q) x = 0} ≤ degree (p − q)
    by (rule card-poly-roots-bound) (use ‹p ≠ q› in auto)
  also have degree (p − q) < n
  proof (intro degree-lessI allI impI)
    fix k assume k ≥ n
    show poly.coeff (p − q) k = 0
    proof (cases k = n)
      case False
      hence poly.coeff p k = 0 poly.coeff q k = 0
        using assms ‹k ≥ n› by (auto simp: coeff-eq-0)
      thus ?thesis
        by simp
    qed (use assms in auto)
  qed (use ‹n > 0› in auto)
  finally show False
    by simp
qed
```

4.20.3 Order of polynomial roots

definition *order* :: *'a::idom ⇒ 'a poly ⇒ nat*
where *order a p = (LEAST n. ¬ [:−a, 1:] ^ Suc n dvd p)*

lemma *coeff-linear-power*: *coeff ([:a, 1:] ^ n) n = 1*
for *a* :: *'a::comm-semiring-1*
proof (*induct n*)
case (*Suc n*)
have *degree ([:a, 1:] ^ n) ≤ 1 * n*
by (*metis One-nat-def degree-pCons-eq-if degree-power-le one-neq-zero one-pCons*)
then have *coeff ([:a, 1:] ^ n) (Suc n) = 0*
by (*simp add*: *coeff-eq-0*)
then show *?case*
using *Suc.hyps* **by** *fastforce*
qed *auto*

lemma *degree-linear-power*: *degree ([:a, 1:] ^ n) = n*
for *a* :: *'a::comm-semiring-1*
proof (*rule order-antisym*)
show *degree ([:a, 1:] ^ n) ≤ n*
by (*metis One-nat-def degree-pCons-eq-if degree-power-le mult.left-neutral one-neq-zero one-pCons*)
qed (*simp add*: *coeff-linear-power le-degree*)

lemma *order-1*: *[:−a, 1:] ^ order a p dvd p*
proof (*cases p = 0*)
case *False*
show *?thesis*
proof (*cases order a p*)
case (*Suc n*)
then show *?thesis*
by (*metis lessI not-less-Least order-def*)
qed *auto*
qed *auto*

lemma *order-2*:
assumes *p ≠ 0*
shows *¬ [:−a, 1:] ^ Suc (order a p) dvd p*
proof −
have *False* **if** *[:− a, 1:] ^ Suc (degree p) dvd p*
using *dvd-imp-degree-le* [*OF that*]
by (*metis Suc-n-not-le-n assms degree-linear-power*)
then show *?thesis*
unfolding *order-def*
by (*metis* (*no-types*, *lifting*) *LeastI*)
qed

lemma *order*: *p ≠ 0 ⟹ [:−a, 1:] ^ order a p dvd p ∧ ¬ [:−a, 1:] ^ Suc (order a p) dvd p*

by (*rule conjI* [*OF order-1 order-2*])

lemma *order-degree*:
assumes *p*: $p \neq 0$
shows $order\ a\ p \leq degree\ p$
proof −
have $order\ a\ p = degree\ ([:-a,\ 1:] \ ^\frown\ order\ a\ p)$
by (*simp only*: *degree-linear-power*)
also from *order-1 p* **have** $\ldots \leq degree\ p$
by (*rule dvd-imp-degree-le*)
finally show *?thesis* **.**
qed

lemma *order-root*: $poly\ p\ a = 0 \longleftrightarrow p = 0 \vee order\ a\ p \neq 0$ (**is** *?lhs* = *?rhs*)
proof
show *?lhs* $\Longrightarrow$ *?rhs*
by (*metis One-nat-def order-2 poly-eq-0-iff-dvd power-one-right*)
show *?rhs* $\Longrightarrow$ *?lhs*
by (*meson dvd-power dvd-trans neq0-conv order-1 poly-0 poly-eq-0-iff-dvd*)
qed

lemma *order-0I*: $poly\ p\ a \neq 0 \Longrightarrow order\ a\ p = 0$
by (*subst* (*asm*) *order-root*) *auto*

lemma *order-unique-lemma*:
fixes $p :: 'a::idom\ poly$
assumes $[:-a,\ 1:] \ ^\frown\ n\ dvd\ p \neg [:-a,\ 1:] \ ^\frown\ Suc\ n\ dvd\ p$
shows $order\ a\ p = n$
unfolding *Polynomial.order-def*
by (*metis* (*mono-tags*, *lifting*) *Least-equality assms not-less-eq-eq power-le-dvd*)

lemma *order-mult*:
assumes $p * q \neq 0$ **shows** $order\ a\ (p * q) = order\ a\ p + order\ a\ q$
proof −
define *i* **where** $i \equiv order\ a\ p$
define *j* **where** $j \equiv order\ a\ q$
define *t* **where** $t \equiv [:-a,\ 1:]$
have *t-dvd-iff*: $\bigwedge u.\ t\ dvd\ u \longleftrightarrow poly\ u\ a = 0$
by (*simp add*: *t-def dvd-iff-poly-eq-0*)
have *dvd*: $t \ ^\frown\ i\ dvd\ p\ t \ ^\frown\ j\ dvd\ q$ **and** $\neg\ t \ ^\frown\ Suc\ i\ dvd\ p \neg\ t \ ^\frown\ Suc\ j\ dvd\ q$
using *assms i-def j-def order-1 order-2 t-def* **by** *auto*
then have $\neg\ t \ ^\frown\ Suc(i + j)\ dvd\ p * q$
by (*elim dvdE*) (*simp add*: *power-add t-dvd-iff*)
moreover have $t \ ^\frown\ (i + j)\ dvd\ p * q$
using *dvd* **by** (*simp add*: *mult-dvd-mono power-add*)
ultimately show $order\ a\ (p * q) = i + j$
using *order-unique-lemma t-def* **by** *blast*
qed

lemma *order-smult*:
  assumes $c \neq 0$
  shows *order x* (*smult c p*) = *order x p*
proof (*cases p* = *0*)
  case *True*
  then show *?thesis*
    by *simp*
next
  case *False*
  have *smult c p* = [:*c*:] ∗ *p* **by** *simp*
  also from *assms False* **have** *order x* ... = *order x* [:*c*:] + *order x p*
    by (*subst order-mult*) *simp-all*
  also have *order x* [:*c*:] = *0*
    by (*rule order-0I*) (*use assms* **in** *auto*)
  finally show *?thesis*
    by *simp*
qed

lemma *order-gt-0-iff*: $p \neq 0 \Longrightarrow$ *order x p* > *0* $\longleftrightarrow$ *poly p x* = *0*
  by (*subst order-root*) *auto*

lemma *order-eq-0-iff*: $p \neq 0 \Longrightarrow$ *order x p* = *0* $\longleftrightarrow$ *poly p x* $\neq$ *0*
  by (*subst order-root*) *auto*

Next three lemmas contributed by Wenda Li

lemma *order-1-eq-0* [*simp*]:*order x 1* = *0*
  by (*metis order-root poly-1 zero-neq-one*)

lemma *order-uminus*[*simp*]: *order x* (−*p*) = *order x p*
  by (*metis neg-equal-0-iff-equal order-smult smult-1-left smult-minus-left*)

lemma *order-power-n-n*: *order a* ([:−*a*,*1*:]^*n*)=*n*
proof (*induct n*)
  case *0*
  then show *?case*
    by (*metis order-root poly-1 power-0 zero-neq-one*)
next
  case (*Suc n*)
  have *order a* ([:− *a*, *1*:] ^ *Suc n*) = *order a* ([:− *a*, *1*:] ^ *n*) + *order a* [:−*a*,*1*:]
   by (*metis* (*no-types*, *opaque-lifting*) *One-nat-def add-Suc-right monoid-add-class.add.right-neutral*
    one-neq-zero order-mult pCons-eq-0-iff power-add power-eq-0-iff power-one-right)
  moreover have *order a* [:−*a*,*1*:] = *1*
    unfolding *order-def*
  proof (*rule Least-equality*, *rule notI*)
    assume [:− *a*, *1*:] ^ *Suc 1 dvd* [:− *a*, *1*:]
    then have *degree* ([:− *a*, *1*:] ^ *Suc 1*) ≤ *degree* ([:− *a*, *1*:])
      by (*rule dvd-imp-degree-le*) *auto*
    then show *False*

```
    by auto
  next
    fix y
    assume *: ¬ [:− a, 1:] ^ Suc y dvd [:− a, 1:]
    show 1 ≤ y
    proof (rule ccontr)
      assume ¬ 1 ≤ y
      then have y = 0 by auto
      then have [:− a, 1:] ^ Suc y dvd [:− a, 1:] by auto
      with * show False by auto
    qed
  qed
  ultimately show ?case
    using Suc by auto
qed

lemma order-0-monom [simp]: c ≠ 0 ⟹ order 0 (monom c n) = n
  using order-power-n-n[of 0 n] by (simp add: monom-altdef order-smult)

lemma dvd-imp-order-le: q ≠ 0 ⟹ p dvd q ⟹ Polynomial.order a p ≤ Polynomial.order a q
  by (auto simp: order-mult)
```

Now justify the standard squarefree decomposition, i.e. $f\ /\ gcd\ f\ f'$.

```
lemma order-divides: [:−a, 1:] ^ n dvd p ⟷ p = 0 ∨ n ≤ order a p
  by (meson dvd-0-right not-less-eq-eq order-1 order-2 power-le-dvd)

lemma order-decomp:
  assumes p ≠ 0
  shows ∃ q. p = [:− a, 1:] ^ order a p * q ∧ ¬ [:− a, 1:] dvd q
proof −
  from assms have *: [:− a, 1:] ^ order a p dvd p
    and **: ¬ [:− a, 1:] ^ Suc (order a p) dvd p
    by (auto dest: order)
  from * obtain q where q: p = [:− a, 1:] ^ order a p * q ..
  with ** have ¬ [:− a, 1:] ^ Suc (order a p) dvd [:− a, 1:] ^ order a p * q
    by simp
  then have ¬ [:− a, 1:] ^ order a p * [:− a, 1:] dvd [:− a, 1:] ^ order a p * q
    by simp
  with idom-class.dvd-mult-cancel-left [of [:− a, 1:] ^ order a p [:− a, 1:] q]
  have ¬ [:− a, 1:] dvd q by auto
  with q show ?thesis by blast
qed

lemma monom-1-dvd-iff: p ≠ 0 ⟹ monom 1 n dvd p ⟷ n ≤ order 0 p
  using order-divides[of 0 n p] by (simp add: monom-altdef)

lemma poly-root-order-induct [case-names 0 no-roots root]:
  fixes p :: 'a :: idom poly
```

```
  assumes P 0 ⋀p. (⋀x. poly p x ≠ 0) ⟹ P p
        ⋀p x n. n > 0 ⟹ poly p x ≠ 0 ⟹ P p ⟹ P ([:−x, 1:] ^ n * p)
  shows   P p
proof (induction degree p arbitrary: p rule: less-induct)
  case (less p)
  consider p = 0 | p ≠ 0 ∃x. poly p x = 0 | ⋀x. poly p x ≠ 0 by blast
  thus ?case
  proof cases
    case 3
    with assms(2)[of p] show ?thesis by simp
  next
    case 2
    then obtain x where x: poly p x = 0 by auto
    have [:−x, 1:] ^ order x p dvd p by (intro order-1)
    then obtain q where q: p = [:−x, 1:] ^ order x p * q by (auto simp: dvd-def)
    with 2 have [simp]: q ≠ 0 by auto
    have order-pos: order x p > 0
      using ‹p ≠ 0› and x by (auto simp: order-root)
    have order x p = order x p + order x q
      by (subst q, subst order-mult) (auto simp: order-power-n-n)
    hence [simp]: order x q = 0 by simp
    have deg: degree p = order x p + degree q
      by (subst q, subst degree-mult-eq) (auto simp: degree-power-eq)
    with order-pos have degree q < degree p by simp
    hence P q by (rule less)
    with order-pos have P ([:−x, 1:] ^ order x p * q)
      by (intro assms(3)) (auto simp: order-root)
    with q show ?thesis by simp
  qed (simp-all add: assms(1))
qed


context
  includes multiset.lifting
begin

lift-definition proots :: ('a :: idom) poly ⇒ 'a multiset is
  λ(p :: 'a poly) (x :: 'a). if p = 0 then 0 else order x p
proof −
  fix p :: 'a poly
  show finite {x. 0 < (if p = 0 then 0 else order x p)}
    by (cases p = 0)
       (auto simp: order-gt-0-iff intro: finite-subset[OF - poly-roots-finite[of p]])
qed

lemma proots-0 [simp]: proots (0 :: 'a :: idom poly) = {#}
  by transfer' auto

lemma proots-1 [simp]: proots (1 :: 'a :: idom poly) = {#}
```

by *transfer′ auto*

lemma *proots-const* [*simp*]: *proots* [: *x* :] = *0*
by *transfer′* (*auto split*: *if-splits simp*: *fun-eq-iff order-eq-0-iff*)

lemma *proots-numeral* [*simp*]: *proots* (*numeral n*) = *0*
by (*simp add*: *numeral-poly*)

lemma *count-proots* [*simp*]:
p ≠ *0* ⟹ *count* (*proots p*) *a* = *order a p*
by *transfer′ auto*

lemma *set-count-proots* [*simp*]:
p ≠ *0* ⟹ *set-mset* (*proots p*) = {*x*. *poly p x* = *0*}
by (*auto simp*: *set-mset-def order-gt-0-iff*)

lemma *proots-uminus* [*simp*]: *proots* (−*p*) = *proots p*
by (*cases p* = *0*; *rule multiset-eqI*) *auto*

lemma *proots-smult* [*simp*]: *c* ≠ *0* ⟹ *proots* (*smult c p*) = *proots p*
by (*cases p* = *0*; *rule multiset-eqI*) (*auto simp*: *order-smult*)

lemma *proots-mult*:
assumes *p* ≠ *0 q* ≠ *0*
shows *proots* (*p* ∗ *q*) = *proots p* + *proots q*
using *assms* **by** (*intro multiset-eqI*) (*auto simp*: *order-mult*)

lemma *proots-prod*:
assumes ⋀*x*. *x* ∈ *A* ⟹ *f x* ≠ *0*
shows *proots* (∏ *x*∈*A*. *f x*) = (∑ *x*∈*A*. *proots* (*f x*))
using *assms* **by** (*induction A rule*: *infinite-finite-induct*) (*auto simp*: *proots-mult*)

lemma *proots-prod-mset*:
assumes *0* ∉# *A*
shows *proots* (∏ *p*∈#*A*. *p*) = (∑ *p*∈#*A*. *proots p*)
using *assms* **by** (*induction A*) (*auto simp*: *proots-mult*)

lemma *proots-prod-list*:
assumes *0* ∉ *set ps*
shows *proots* (∏ *p*←*ps*. *p*) = (∑ *p*←*ps*. *proots p*)
using *assms* **by** (*induction ps*) (*auto simp*: *proots-mult prod-list-zero-iff*)

lemma *proots-power*: *proots* (*p* ^ *n*) = *repeat-mset n* (*proots p*)
proof (*cases p* = *0*)
case *False*
thus *?thesis*
by (*induction n*) (*auto simp*: *proots-mult*)
qed (*auto simp*: *power-0-left*)

lemma *proots-linear-factor* [*simp*]: *proots* [:*x*, *1*:] = {#−*x*#}
proof −
  have *order* (−*x*) [:*x*, *1*:] > *0*
    by (*subst order-gt-0-iff*) *auto*
  moreover have *order* (−*x*) [:*x*, *1*:] ≤ *degree* [:*x*, *1*:]
    by (*rule order-degree*) *auto*
  moreover have *order y* [:*x*, *1*:] = *0* **if** *y* ≠ −*x* **for** *y*
    by (*rule order-0I*) (*use this* **in** ‹*auto simp*: *add-eq-0-iff*›)
  ultimately show *?thesis*
    by (*intro multiset-eqI*) *auto*
qed

lemma *size-proots-le*: *size* (*proots p*) ≤ *degree p*
proof (*induction p rule*: *poly-root-order-induct*)
  case (*no-roots p*)
  hence *proots p* = *0*
    by (*simp add*: *multiset-eqI order-root*)
  thus *?case* **by** *simp*
next
  case (*root p x n*)
  have [*simp*]: *p* ≠ *0*
    using *root.hyps* **by** *auto*
  from *root.IH* **show** *?case*
    by (*auto simp*: *proots-mult proots-power degree-mult-eq degree-power-eq*)
qed *auto*

end

4.21 Additional induction rules on polynomials

An induction rule for induction over the roots of a polynomial with a certain property. (e.g. all positive roots)

lemma *poly-root-induct* [*case-names 0 no-roots root*]:
  fixes *p* :: *'a* :: *idom poly*
  assumes *Q 0*
    and ⋀*p*. (⋀*a*. *P a* ⟹ *poly p a* ≠ *0*) ⟹ *Q p*
    and ⋀*a p*. *P a* ⟹ *Q p* ⟹ *Q* ([:*a*, −*1*:] ∗ *p*)
  shows *Q p*
proof (*induction degree p arbitrary*: *p rule*: *less-induct*)
  case (*less p*)
  show *?case*
  proof (*cases p* = *0*)
    case *True*
    with *assms*(*1*) **show** *?thesis* **by** *simp*
  next
    case *False*
    show *?thesis*
    proof (*cases* ∃ *a*. *P a* ∧ *poly p a* = *0*)
      case *False*

```
    then show ?thesis by (intro assms(2)) blast
  next
    case True
    then obtain a where a: P a poly p a = 0
      by blast
    then have −[:−a, 1:] dvd p
      by (subst minus-dvd-iff) (simp add: poly-eq-0-iff-dvd)
    then obtain q where q: p = [:a, −1:] ∗ q by (elim dvdE) simp
    with False have q ≠ 0 by auto
    have degree p = Suc (degree q)
      by (subst q, subst degree-mult-eq) (simp-all add: ‹q ≠ 0›)
    then have Q q by (intro less) simp
    with a(1) have Q ([:a, −1:] ∗ q)
      by (rule assms(3))
    with q show ?thesis by simp
  qed
 qed
qed

lemma dropWhile-replicate-append:
 dropWhile ((=) a) (replicate n a @ ys) = dropWhile ((=) a) ys
 by (induct n) simp-all

lemma Poly-append-replicate-0: Poly (xs @ replicate n 0) = Poly xs
 by (subst coeffs-eq-iff) (simp-all add: strip-while-def dropWhile-replicate-append)
```

An induction rule for simultaneous induction over two polynomials, prepending one coefficient in each step.

```
lemma poly-induct2 [case-names 0 pCons]:
 assumes P 0 0 ⋀a p b q. P p q ⟹ P (pCons a p) (pCons b q)
 shows P p q
proof −
 define n where n = max (length (coeffs p)) (length (coeffs q))
 define xs where xs = coeffs p @ (replicate (n − length (coeffs p)) 0)
 define ys where ys = coeffs q @ (replicate (n − length (coeffs q)) 0)
 have length xs = length ys
   by (simp add: xs-def ys-def n-def)
 then have P (Poly xs) (Poly ys)
   by (induct rule: list-induct2) (simp-all add: assms)
 also have Poly xs = p
   by (simp add: xs-def Poly-append-replicate-0)
 also have Poly ys = q
   by (simp add: ys-def Poly-append-replicate-0)
 finally show ?thesis .
qed
```

4.22 Composition of polynomials

```
definition pcompose :: 'a::comm-semiring-0 poly ⇒ 'a poly ⇒ 'a poly
```

where *pcompose p q = fold-coeffs* ($\lambda a\ c.\ [:a:] + q * c$) *p 0*

notation *pcompose* (**infixl** ‹$\circ_p$› *71*)

lemma *pcompose-0* [*simp*]: *pcompose 0 q = 0*
by (*simp add*: *pcompose-def*)

lemma *pcompose-pCons*: *pcompose* (*pCons a p*) *q* = [:*a*:] + *q* * *pcompose p q*
by (*cases p = 0* ∧ *a = 0*) (*auto simp add*: *pcompose-def*)

lemma *pcompose-altdef*: *pcompose p q = poly* (*map-poly* ($\lambda x.$ [:*x*:]) *p*) *q*
by (*induction p*) (*simp-all add*: *map-poly-pCons pcompose-pCons*)

lemma *coeff-pcompose-0* [*simp*]:
coeff (*pcompose p q*) *0 = poly p* (*coeff q 0*)
by (*induction p*) (*simp-all add*: *coeff-mult-0 pcompose-pCons*)

lemma *pcompose-1*: *pcompose 1 p = 1*
for *p* :: *'a*::*comm-semiring-1 poly*
by (*auto simp*: *one-pCons pcompose-pCons*)

lemma *poly-pcompose*: *poly* (*pcompose p q*) *x = poly p* (*poly q x*)
by (*induct p*) (*simp-all add*: *pcompose-pCons*)

lemma *degree-pcompose-le*: *degree* (*pcompose p q*) ≤ *degree p* * *degree q*
proof (*induction p*)
case (*pCons a p*)
then show *?case*
proof (*clarsimp simp add*: *pcompose-pCons*)
assume *degree* (*p* ∘$_p$ *q*) ≤ *degree p* * *degree q p* ≠ *0*
then have *degree* (*q* * *p* ∘$_p$ *q*) ≤ *degree q* + *degree p* * *degree q*
by (*meson add-le-cancel-left degree-mult-le dual-order.trans pCons.IH*)
then show *degree* ([:*a*:] + *q* * *p* ∘$_p$ *q*) ≤ *degree q* + *degree p* * *degree q*
by (*simp add*: *degree-add-le*)
qed
qed *auto*

lemma *pcompose-add*: *pcompose* (*p* + *q*) *r = pcompose p r* + *pcompose q r*
for *p q r* :: *'a*::{*comm-semiring-0*, *ab-semigroup-add*} *poly*
proof (*induction p q rule*: *poly-induct2*)
case *0*
then show *?case* **by** *simp*
next
case (*pCons a p b q*)
have *pcompose* (*pCons a p* + *pCons b q*) *r* = [:*a* + *b*:] + *r* * *pcompose p r* + *r* * *pcompose q r*
by (*simp-all add*: *pcompose-pCons pCons.IH algebra-simps*)
also have [:*a* + *b*:] = [:*a*:] + [:*b*:] **by** *simp*
also have ... + *r* * *pcompose p r* + *r* * *pcompose q r* = *pcompose* (*pCons a p*)

$r + pcompose\ (pCons\ b\ q)\ r$
by (*simp only*: *pcompose-pCons add-ac*)
finally show *?case* **.**
qed

lemma *pcompose-uminus*: $pcompose\ (-p)\ r = -pcompose\ p\ r$
for $p\ r :: 'a$::*comm-ring poly*
by (*induct p*) (*simp-all add*: *pcompose-pCons*)

lemma *pcompose-diff*: $pcompose\ (p - q)\ r = pcompose\ p\ r - pcompose\ q\ r$
for $p\ q\ r :: 'a$::*comm-ring poly*
using *pcompose-add*[*of p* $-q$] **by** (*simp add*: *pcompose-uminus*)

lemma *pcompose-smult*: $pcompose\ (smult\ a\ p)\ r = smult\ a\ (pcompose\ p\ r)$
for $p\ r :: 'a$::*comm-semiring-0 poly*
by (*induct p*) (*simp-all add*: *pcompose-pCons pcompose-add smult-add-right*)

lemma *pcompose-mult*: $pcompose\ (p * q)\ r = pcompose\ p\ r * pcompose\ q\ r$
for $p\ q\ r :: 'a$::*comm-semiring-0 poly*
by (*induct p arbitrary*: *q*) (*simp-all add*: *pcompose-add pcompose-smult pcompose-pCons algebra-simps*)

lemma *pcompose-assoc*: $pcompose\ p\ (pcompose\ q\ r) = pcompose\ (pcompose\ p\ q)\ r$
for $p\ q\ r :: 'a$::*comm-semiring-0 poly*
by (*induct p arbitrary*: *q*) (*simp-all add*: *pcompose-pCons pcompose-add pcompose-mult*)

lemma *pcompose-idR*[*simp*]: $pcompose\ p\ [:\ 0,\ 1\ :] = p$
for $p :: 'a$::*comm-semiring-1 poly*
by (*induct p*) (*simp-all add*: *pcompose-pCons*)

lemma *pcompose-sum*: $pcompose\ (sum\ f\ A)\ p = sum\ (\lambda i.\ pcompose\ (f\ i)\ p)\ A$
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *pcompose-1 pcompose-add*)

lemma *pcompose-prod*: $pcompose\ (prod\ f\ A)\ p = prod\ (\lambda i.\ pcompose\ (f\ i)\ p)\ A$
by (*induct A rule*: *infinite-finite-induct*) (*simp-all add*: *pcompose-1 pcompose-mult*)

lemma *pcompose-const* [*simp*]: $pcompose\ [:a:]\ q = [:a:]$
by (*subst pcompose-pCons*) *simp*

lemma *pcompose-0'*: $pcompose\ p\ 0 = [:coeff\ p\ 0:]$
by (*induct p*) (*auto simp add*: *pcompose-pCons*)

lemma *pcompose-coeff-0*:
$coeff\ (pcompose\ p\ q)\ 0 = poly\ p\ (coeff\ q\ 0)$
by (*metis poly-0-coeff-0 poly-pcompose*)

lemma *pcompose-pCons-0*: $pcompose\ p\ [:a:] = [:poly\ p\ a:]$
by (*metis* (*no-types*, *lifting*) *coeff-pCons-0 pcompose-0' pcompose-assoc poly-0-coeff-0*

poly-pcompose)

lemma *degree-pcompose*: *degree* (*pcompose p q*) = *degree p* ∗ *degree q*
for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
proof (*induct p*)
case *0*
then show *?case* **by** *auto*
next
case (*pCons a p*)
consider *degree* (*q* ∗ *pcompose p q*) = *0* | *degree* (*q* ∗ *pcompose p q*) > *0*
by *blast*
then show *?case*
proof *cases*
case *prems*: *1*
show *?thesis*
proof (*cases p* = *0*)
case *True*
then show *?thesis* **by** *auto*
next
case *False*
from *prems* **have** *degree q* = *0* ∨ *pcompose p q* = *0*
by (*auto simp add*: *degree-mult-eq-0*)
moreover have *False* **if** *pcompose p q* = *0 degree q* ≠ *0*
proof −
from *pCons.hyps*(*2*) *that* **have** *degree p* = *0*
by *auto*
then obtain *a1* **where** *p* = [:*a1*:]
by (*metis degree-pCons-eq-if old.nat.distinct*(*2*) *pCons-cases*)
with ‹*pcompose p q* = *0*› ‹*p* ≠ *0*› **show** *False*
by *auto*
qed
ultimately have *degree* (*pCons a p*) ∗ *degree q* = *0*
by *auto*
moreover have *degree* (*pcompose* (*pCons a p*) *q*) = *0*
proof −
from *prems* **have** *0* = *max* (*degree* [:*a*:]) (*degree* (*q* ∗ *pcompose p q*))
by *simp*
also have ... ≥ *degree* ([:*a*:] + *q* ∗ *pcompose p q*)
by (*rule degree-add-le-max*)
finally show *?thesis*
by (*auto simp add*: *pcompose-pCons*)
qed
ultimately show *?thesis* **by** *simp*
qed
next
case *prems*: *2*
then have *p* ≠ *0 q* ≠ *0 pcompose p q* ≠ *0*
by *auto*
from *prems degree-add-eq-right* [*of* [:*a*:]]

```
    have degree (pcompose (pCons a p) q) = degree (q * pcompose p q)
      by (auto simp: pcompose-pCons)
    with pCons.hyps(2) degree-mult-eq[OF ‹q≠0› ‹pcompose p q≠0›] show ?thesis
      by auto
  qed
qed

lemma pcompose-eq-0:
  fixes p q :: 'a::{comm-semiring-0,semiring-no-zero-divisors} poly
  assumes pcompose p q = 0 degree q > 0
  shows p = 0
proof −
  from assms degree-pcompose [of p q] have degree p = 0
    by auto
  then obtain a where p = [:a:]
    by (metis degree-pCons-eq-if gr0-conv-Suc neq0-conv pCons-cases)
  with assms(1) have a = 0
    by auto
  with ‹p = [:a:]› show ?thesis
    by simp
qed

lemma pcompose-eq-0-iff:
  fixes p q :: 'a::{comm-semiring-0,semiring-no-zero-divisors} poly
  assumes degree q > 0
  shows pcompose p q = 0 ⟷ p = 0
  using pcompose-eq-0[OF - assms] by auto

lemma coeff-pcompose-linear:
  coeff (pcompose p [:0, a :: 'a :: comm-semiring-1:]) i = a ^ i * coeff p i
  by (induction p arbitrary: i) (auto simp: pcompose-pCons coeff-pCons mult-ac
split: nat.splits)

lemma lead-coeff-comp:
  fixes p q :: 'a::{comm-semiring-1,semiring-no-zero-divisors} poly
  assumes degree q > 0
  shows lead-coeff (pcompose p q) = lead-coeff p * lead-coeff q ^ (degree p)
proof (induct p)
  case 0
  then show ?case by auto
next
  case (pCons a p)
  consider degree (q * pcompose p q) = 0 | degree (q * pcompose p q) > 0
    by blast
  then show ?case
  proof cases
    case prems: 1
    then have pcompose p q = 0
      by (metis assms degree-0 degree-mult-eq-0 neq0-conv)
```

```
    with pcompose-eq-0[OF - ‹degree q > 0›] have p = 0
      by simp
    then show ?thesis
      by auto
  next
    case prems: 2
    then have degree [:a:] < degree (q * pcompose p q)
      by simp
    then have lead-coeff ([:a:] + q * p ∘p q) = lead-coeff (q * p ∘p q)
      by (rule lead-coeff-add-le)
    then have lead-coeff (pcompose (pCons a p) q) = lead-coeff (q * pcompose p
q)
      by (simp add: pcompose-pCons)
    also have ... = lead-coeff q * (lead-coeff p * lead-coeff q ^ degree p)
      using pCons.hyps(2) lead-coeff-mult[of q pcompose p q] by simp
    also have ... = lead-coeff p * lead-coeff q ^ (degree p + 1)
      by (auto simp: mult-ac)
    finally show ?thesis by auto
  qed
qed

lemma coeff-pcompose-monom-linear [simp]:
  fixes p :: 'a :: comm-ring-1 poly
  shows coeff (pcompose p (monom c (Suc 0))) k = c ^ k * coeff p k
  by (induction p arbitrary: k)
     (auto simp: coeff-pCons coeff-monom-mult pcompose-pCons split: nat.splits)

lemma of-nat-mult-conv-smult: of-nat n * P = smult (of-nat n) P
  by (simp add: monom-0 of-nat-monom)

lemma numeral-mult-conv-smult: numeral n * P = smult (numeral n) P
  by (simp add: numeral-poly)

lemma sum-order-le-degree:
  assumes p ≠ 0
  shows   (∑ x | poly p x = 0. order x p) ≤ degree p
  using assms
proof (induction degree p arbitrary: p rule: less-induct)
  case (less p)
  show ?case
  proof (cases ∃ x. poly p x = 0)
    case False
    thus ?thesis
      by auto
  next
    case True
    then obtain x where x: poly p x = 0
      by auto
    have [:−x, 1:] ^ order x p dvd p
```

by (*simp add*: *order-1*)
then obtain *q* **where** *q*: *p* = [:−*x*, *1*:] ^ *order x p* ∗ *q*
by (*elim dvdE*)
have [*simp*]: *q* ≠ *0*
using *q less.prems* **by** *auto*
have *order x p* = *order x p* + *order x q*
by (*subst q*, *subst order-mult*) (*auto simp*: *order-power-n-n*)
hence *order x q* = *0*
by *auto*
hence [*simp*]: *poly q x* ≠ *0*
by (*simp add*: *order-root*)
have *deg-p*: *degree p* = *degree q* + *order x p*
by (*subst q*, *subst degree-mult-eq*) (*auto simp*: *degree-power-eq*)
moreover have *order x p* > *0*
using *x less.prems* **by** (*simp add*: *order-root*)
ultimately have *degree q* < *degree p*
by *linarith*
hence (∑ *x* | *poly q x* = *0*. *order x q*) ≤ *degree q*
by (*intro less.hyps*) *auto*
hence *order x p* + (∑ *x* | *poly q x* = *0*. *order x q*) ≤ *degree p*
by (*simp add*: *deg-p*)
also have {*y*. *poly q y* = *0*} = {*y*. *poly p y* = *0*} − {*x*}
by (*subst q*) *auto*
also have (∑ *y* ∈ {*y*. *poly p y* = *0*} − {*x*}. *order y q*) =
(∑ *y* ∈ {*y*. *poly p y* = *0*} − {*x*}. *order y p*)
by (*intro sum.cong refl*, *subst q*)
(*auto simp*: *order-mult order-power-n-n intro*!: *order-0I*)
also have *order x p* + ... = (∑ *y* ∈ *insert x* ({*y*. *poly p y* = *0*} − {*x*}). *order y p*)
using ‹*p* ≠ *0*› **by** (*subst sum.insert*) (*auto simp*: *poly-roots-finite*)
also have *insert x* ({*y*. *poly p y* = *0*} − {*x*}) = {*y*. *poly p y* = *0*}
using ‹*poly p x* = *0*› **by** *auto*
finally show *?thesis* **.**
qed
qed

4.23 Closure properties of coefficients

context
fixes *R* :: *'a* :: *comm-semiring-1 set*
assumes *R-0*: *0* ∈ *R*
assumes *R-plus*: ⋀*x y*. *x* ∈ *R* ⟹ *y* ∈ *R* ⟹ *x* + *y* ∈ *R*
assumes *R-mult*: ⋀*x y*. *x* ∈ *R* ⟹ *y* ∈ *R* ⟹ *x* ∗ *y* ∈ *R*
begin

lemma *coeff-mult-semiring-closed*:
assumes ⋀*i*. *coeff p i* ∈ *R* ⋀*i*. *coeff q i* ∈ *R*
shows *coeff* (*p* ∗ *q*) *i* ∈ *R*
proof −

have *R-sum*: *sum f A* ∈ *R* **if** ⋀*x*. *x* ∈ *A* ⟹ *f x* ∈ *R* **for** *A* **and** *f* :: *nat* ⇒ *'a*
using *that* **by** (*induction A rule*: *infinite-finite-induct*) (*auto intro*: *R-0 R-plus*)
show *?thesis*
unfolding *coeff-mult* **by** (*auto intro*!: *R-sum R-mult assms*)
qed

lemma *coeff-pcompose-semiring-closed*:
assumes ⋀*i*. *coeff p i* ∈ *R* ⋀*i*. *coeff q i* ∈ *R*
shows *coeff* (*pcompose p q*) *i* ∈ *R*
using *assms*(*1*)
proof (*induction p arbitrary*: *i*)
case (*pCons a p i*)
have [*simp*]: *a* ∈ *R*
using *pCons.prems*[*of 0*] **by** *auto*
have *coeff p i* ∈ *R* **for** *i*
using *pCons.prems*[*of Suc i*] **by** *auto*
hence *coeff* (*p* ∘$_p$ *q*) *i* ∈ *R* **for** *i*
using *pCons.prems* **by** (*intro pCons.IH*)
thus *?case*
by (*auto simp*: *pcompose-pCons coeff-pCons split*: *nat.splits*
intro!: *assms R-plus coeff-mult-semiring-closed*)
qed *auto*

end

4.24 Shifting polynomials

definition *poly-shift* :: *nat* ⇒ *'a*::*zero poly* ⇒ *'a poly*
where *poly-shift n p* = *Abs-poly* (λ*i*. *coeff p* (*i* + *n*))

lemma *nth-default-drop*: *nth-default x* (*drop n xs*) *m* = *nth-default x xs* (*m* + *n*)
by (*auto simp add*: *nth-default-def add-ac*)

lemma *nth-default-take*: *nth-default x* (*take n xs*) *m* = (*if m* < *n then nth-default x xs m else x*)
by (*auto simp add*: *nth-default-def add-ac*)

lemma *coeff-poly-shift*: *coeff* (*poly-shift n p*) *i* = *coeff p* (*i* + *n*)
proof −
from *MOST-coeff-eq-0*[*of p*] **obtain** *m* **where** ∀ *k*>*m*. *coeff p k* = *0*
by (*auto simp*: *MOST-nat*)
then have ∀ *k*>*m*. *coeff p* (*k* + *n*) = *0*
by *auto*
then have ∀ ∞*k*. *coeff p* (*k* + *n*) = *0*
by (*auto simp*: *MOST-nat*)
then show *?thesis*
by (*simp add*: *poly-shift-def poly.Abs-poly-inverse*)
qed

lemma *poly-shift-id* [*simp*]: *poly-shift 0* = (λ*x*. *x*)
by (*simp add*: *poly-eq-iff fun-eq-iff coeff-poly-shift*)

lemma *poly-shift-0* [*simp*]: *poly-shift n 0* = *0*
by (*simp add*: *poly-eq-iff coeff-poly-shift*)

lemma *poly-shift-1*: *poly-shift n 1* = (*if n* = *0 then 1 else 0*)
by (*simp add*: *poly-eq-iff coeff-poly-shift*)

lemma *poly-shift-monom*: *poly-shift n* (*monom c m*) = (*if m* ≥ *n then monom c* (*m* − *n*) *else 0*)
by (*auto simp add*: *poly-eq-iff coeff-poly-shift*)

lemma *coeffs-shift-poly* [*code abstract*]:
coeffs (*poly-shift n p*) = *drop n* (*coeffs p*)
proof (*cases p* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then show *?thesis*
by (*intro coeffs-eqI*)
(*simp-all add*: *coeff-poly-shift nth-default-drop nth-default-coeffs-eq*)
qed

4.25 Truncating polynomials

definition *poly-cutoff*
where *poly-cutoff n p* = *Abs-poly* (λ*k*. *if k* < *n then coeff p k else 0*)

lemma *coeff-poly-cutoff*: *coeff* (*poly-cutoff n p*) *k* = (*if k* < *n then coeff p k else 0*)
unfolding *poly-cutoff-def*
by (*subst poly.Abs-poly-inverse*) (*auto simp*: *MOST-nat intro*: *exI*[*of* - *n*])

lemma *poly-cutoff-0* [*simp*]: *poly-cutoff n 0* = *0*
by (*simp add*: *poly-eq-iff coeff-poly-cutoff*)

lemma *poly-cutoff-1* [*simp*]: *poly-cutoff n 1* = (*if n* = *0 then 0 else 1*)
by (*simp add*: *poly-eq-iff coeff-poly-cutoff*)

lemma *coeffs-poly-cutoff* [*code abstract*]:
coeffs (*poly-cutoff n p*) = *strip-while* ((=) *0*) (*take n* (*coeffs p*))
proof (*cases strip-while* ((=) *0*) (*take n* (*coeffs p*)) = [])
case *True*
then have *coeff* (*poly-cutoff n p*) *k* = *0* **for** *k*
unfolding *coeff-poly-cutoff*
by (*auto simp*: *nth-default-coeffs-eq* [*symmetric*] *nth-default-def set-conv-nth*)
then have *poly-cutoff n p* = *0*

```
    by (simp add: poly-eq-iff)
  then show ?thesis
    by (subst True) simp-all
next
  case False
  have no-trailing ((=) 0) (strip-while ((=) 0) (take n (coeffs p)))
    by simp
  with False have last (strip-while ((=) 0) (take n (coeffs p))) ≠ 0
    unfolding no-trailing-unfold by auto
  then show ?thesis
    by (intro coeffs-eqI)
      (simp-all add: coeff-poly-cutoff nth-default-take nth-default-coeffs-eq)
qed
```

4.26 Reflecting polynomials

```
definition reflect-poly :: 'a::zero poly ⇒ 'a poly
  where reflect-poly p = Poly (rev (coeffs p))

lemma coeffs-reflect-poly [code abstract]:
  coeffs (reflect-poly p) = rev (dropWhile ((=) 0) (coeffs p))
  by (simp add: reflect-poly-def)

lemma reflect-poly-0 [simp]: reflect-poly 0 = 0
  by (simp add: reflect-poly-def)

lemma reflect-poly-1 [simp]: reflect-poly 1 = 1
  by (simp add: reflect-poly-def one-pCons)

lemma coeff-reflect-poly:
  coeff (reflect-poly p) n = (if n > degree p then 0 else coeff p (degree p − n))
  by (cases p = 0)
    (auto simp add: reflect-poly-def nth-default-def
      rev-nth degree-eq-length-coeffs coeffs-nth not-less
      dest: le-imp-less-Suc)

lemma coeff-0-reflect-poly-0-iff [simp]: coeff (reflect-poly p) 0 = 0 ⟷ p = 0
  by (simp add: coeff-reflect-poly)

lemma reflect-poly-at-0-eq-0-iff [simp]: poly (reflect-poly p) 0 = 0 ⟷ p = 0
  by (simp add: coeff-reflect-poly poly-0-coeff-0)

lemma reflect-poly-pCons':
  p ≠ 0 ⟹ reflect-poly (pCons c p) = reflect-poly p + monom c (Suc (degree p))
  by (intro poly-eqI)
    (auto simp: coeff-reflect-poly coeff-pCons not-less Suc-diff-le split: nat.split)

lemma reflect-poly-const [simp]: reflect-poly [:a:] = [:a:]
  by (cases a = 0) (simp-all add: reflect-poly-def)
```

lemma *poly-reflect-poly-nz*:
  $x \neq 0 \Longrightarrow$ *poly* (*reflect-poly p*) *x* = *x* ^ *degree p* ∗ *poly p* (*inverse x*)
  for *x* :: *'a*::*field*
 by (*induct rule*: *pCons-induct*) (*simp-all add*: *field-simps reflect-poly-pCons' poly-monom*)

lemma *coeff-0-reflect-poly* [*simp*]: *coeff* (*reflect-poly p*) *0* = *lead-coeff p*
  by (*simp add*: *coeff-reflect-poly*)

lemma *poly-reflect-poly-0* [*simp*]: *poly* (*reflect-poly p*) *0* = *lead-coeff p*
  by (*simp add*: *poly-0-coeff-0*)

lemma *reflect-poly-reflect-poly* [*simp*]: *coeff p 0* ≠ *0* ⟹ *reflect-poly* (*reflect-poly p*) = *p*
  by (*cases p rule*: *pCons-cases*) (*simp add*: *reflect-poly-def*)

lemma *degree-reflect-poly-le*: *degree* (*reflect-poly p*) ≤ *degree p*
 by (*simp add*: *degree-eq-length-coeffs coeffs-reflect-poly length-dropWhile-le diff-le-mono*)

lemma *reflect-poly-pCons*: *a* ≠ *0* ⟹ *reflect-poly* (*pCons a p*) = *Poly* (*rev* (*a* # *coeffs p*))
  by (*subst coeffs-eq-iff*) (*simp add*: *coeffs-reflect-poly*)

lemma *degree-reflect-poly-eq* [*simp*]: *coeff p 0* ≠ *0* ⟹ *degree* (*reflect-poly p*) = *degree p*
 by (*cases p rule*: *pCons-cases*) (*simp add*: *reflect-poly-pCons degree-eq-length-coeffs*)

lemma *reflect-poly-eq-0-iff* [*simp*]: *reflect-poly p* = *0* ⟷ *p* = *0*
  using *coeff-0-reflect-poly-0-iff* **by** *fastforce*

lemma *reflect-poly-mult*: *reflect-poly* (*p* ∗ *q*) = *reflect-poly p* ∗ *reflect-poly q*
  for *p q* :: *'a*::{*comm-semiring-0*,*semiring-no-zero-divisors*} *poly*
proof (*cases p* = *0* ∨ *q* = *0*)
  case *False*
  then have [*simp*]: *p* ≠ *0 q* ≠ *0* **by** *auto*
  show *?thesis*
  proof (*rule poly-eqI*)
   show *coeff* (*reflect-poly* (*p* ∗ *q*)) *i* = *coeff* (*reflect-poly p* ∗ *reflect-poly q*) *i* **for** *i*
   proof (*cases i* ≤ *degree* (*p* ∗ *q*))
     case *True*
     define *A* **where** *A* = {..*i*} ∩ {*i* − *degree q*..*degree p*}
     define *B* **where** *B* = {..*degree p*} ∩ {*degree p* − *i*..*degree* (*p*∗*q*) − *i*}
     let *?f* = λ*j*. *degree p* − *j*

     from *True* **have** *coeff* (*reflect-poly* (*p* ∗ *q*)) *i* = *coeff* (*p* ∗ *q*) (*degree* (*p* ∗ *q*) − *i*)
       by (*simp add*: *coeff-reflect-poly*)
     also have ... = (∑ *j*≤*degree* (*p* ∗ *q*) − *i*. *coeff p j* ∗ *coeff q* (*degree* (*p* ∗ *q*)

```
− i − j))
      by (simp add: coeff-mult)
    also have ... = (∑j∈B. coeff p j * coeff q (degree (p * q) − i − j))
      by (intro sum.mono-neutral-right) (auto simp: B-def degree-mult-eq not-le coeff-eq-0)
    also from True have ... = (∑j∈A. coeff p (degree p − j) * coeff q (degree q − (i − j)))
      by (intro sum.reindex-bij-witness[of - ?f ?f])
        (auto simp: A-def B-def degree-mult-eq add-ac)
    also have ... =
      (∑j≤i.
        if j ∈ {i − degree q..degree p}
        then coeff p (degree p − j) * coeff q (degree q − (i − j))
        else 0)
      by (subst sum.inter-restrict [symmetric]) (simp-all add: A-def)
    also have ... = coeff (reflect-poly p * reflect-poly q) i
      by (fastforce simp: coeff-mult coeff-reflect-poly intro!: sum.cong)
    finally show ?thesis .
   qed (auto simp: coeff-mult coeff-reflect-poly coeff-eq-0 degree-mult-eq intro!: sum.neutral)
 qed
qed auto

lemma reflect-poly-smult: reflect-poly (smult c p) = smult c (reflect-poly p)
 for p :: 'a::{comm-semiring-0,semiring-no-zero-divisors} poly
 using reflect-poly-mult[of [:c:] p] by simp

lemma reflect-poly-power: reflect-poly (p ^ n) = reflect-poly p ^ n
 for p :: 'a::{comm-semiring-1,semiring-no-zero-divisors} poly
 by (induct n) (simp-all add: reflect-poly-mult)

lemma reflect-poly-prod: reflect-poly (prod f A) = prod (λx. reflect-poly (f x)) A
 for f :: - ⇒ -::{comm-semiring-0,semiring-no-zero-divisors} poly
 by (induct A rule: infinite-finite-induct) (simp-all add: reflect-poly-mult)

lemma reflect-poly-prod-list: reflect-poly (prod-list xs) = prod-list (map reflect-poly xs)
 for xs :: -::{comm-semiring-0,semiring-no-zero-divisors} poly list
 by (induct xs) (simp-all add: reflect-poly-mult)

lemma reflect-poly-Poly-nz:
 no-trailing (HOL.eq 0) xs ⟹ reflect-poly (Poly xs) = Poly (rev xs)
 by (simp add: reflect-poly-def)

lemmas reflect-poly-simps =
 reflect-poly-0 reflect-poly-1 reflect-poly-const reflect-poly-smult reflect-poly-mult
 reflect-poly-power reflect-poly-prod reflect-poly-prod-list
```

4.27 Derivatives

function *pderiv* :: ('*a* :: {*comm-semiring-1*,*semiring-no-zero-divisors*}) *poly* ⇒ '*a poly*
  where *pderiv* (*pCons a p*) = (*if p* = *0 then 0 else p* + *pCons 0* (*pderiv p*))
  by (*auto intro*: *pCons-cases*)

termination *pderiv*
  by (*relation measure degree*) *simp-all*

declare *pderiv.simps*[*simp del*]

lemma *pderiv-0* [*simp*]: *pderiv 0* = *0*
  using *pderiv.simps* [*of 0 0*] **by** *simp*

lemma *pderiv-pCons*: *pderiv* (*pCons a p*) = *p* + *pCons 0* (*pderiv p*)
  by (*simp add*: *pderiv.simps*)

lemma *pderiv-1* [*simp*]: *pderiv 1* = *0*
  by (*simp add*: *one-pCons pderiv-pCons*)

lemma *pderiv-of-nat* [*simp*]: *pderiv* (*of-nat n*) = *0*
  and *pderiv-numeral* [*simp*]: *pderiv* (*numeral m*) = *0*
  by (*simp-all add*: *of-nat-poly numeral-poly pderiv-pCons*)

lemma *coeff-pderiv*: *coeff* (*pderiv p*) *n* = *of-nat* (*Suc n*) ∗ *coeff p* (*Suc n*)
  by (*induct p arbitrary*: *n*)
    (*auto simp add*: *pderiv-pCons coeff-pCons algebra-simps split*: *nat.split*)

fun *pderiv-coeffs-code* :: '*a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} ⇒ '*a list* ⇒ '*a list*
  where
    pderiv-coeffs-code f (*x* # *xs*) = *cCons* (*f* ∗ *x*) (*pderiv-coeffs-code* (*f*+*1*) *xs*)
  | *pderiv-coeffs-code f* [] = []

definition *pderiv-coeffs* :: '*a*::{*comm-semiring-1*,*semiring-no-zero-divisors*} *list* ⇒ '*a list*
  where *pderiv-coeffs xs* = *pderiv-coeffs-code 1* (*tl xs*)

lemma *pderiv-coeffs-code*:
  nth-default 0 (*pderiv-coeffs-code f xs*) *n* = (*f* + *of-nat n*) ∗ *nth-default 0 xs n*
proof (*induct xs arbitrary*: *f n*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case (*Cons x xs*)
  show *?case*
  proof (*cases n*)
    case *0*

```
    then show ?thesis
     by (cases pderiv-coeffs-code (f + 1) xs = [] ∧ f * x = 0) (auto simp: cCons-def)
  next
    case n: (Suc m)
    show ?thesis
    proof (cases pderiv-coeffs-code (f + 1) xs = [] ∧ f * x = 0)
      case False
      then have nth-default 0 (pderiv-coeffs-code f (x # xs)) n =
          nth-default 0 (pderiv-coeffs-code (f + 1) xs) m
        by (auto simp: cCons-def n)
      also have ... = (f + of-nat n) * nth-default 0 xs m
        by (simp add: Cons n add-ac)
      finally show ?thesis
        by (simp add: n)
    next
      case True
      have empty: pderiv-coeffs-code g xs = [] ⟹ g + of-nat m = 0 ∨ nth-default
0 xs m = 0 for g
      proof (induct xs arbitrary: g m)
        case Nil
        then show ?case by simp
      next
        case (Cons x xs)
         from Cons(2) have empty: pderiv-coeffs-code (g + 1) xs = [] and g: g =
0 ∨ x = 0
          by (auto simp: cCons-def split: if-splits)
        note IH = Cons(1)[OF empty]
        from IH[of m] IH[of m − 1] g show ?case
          by (cases m) (auto simp: field-simps)
      qed
      from True have nth-default 0 (pderiv-coeffs-code f (x # xs)) n = 0
        by (auto simp: cCons-def n)
      moreover from True have (f + of-nat n) * nth-default 0 (x # xs) n = 0
        by (simp add: n) (use empty[of f+1] in ‹auto simp: field-simps›)
      ultimately show ?thesis by simp
    qed
  qed
qed

lemma coeffs-pderiv-code [code abstract]: coeffs (pderiv p) = pderiv-coeffs (coeffs
p)
  unfolding pderiv-coeffs-def
proof (rule coeffs-eqI, unfold pderiv-coeffs-code coeff-pderiv, goal-cases)
  case (1 n)
  have id: coeff p (Suc n) = nth-default 0 (map (λi. coeff p (Suc i)) [0..<degree
p]) n
    by (cases n < degree p) (auto simp: nth-default-def coeff-eq-0)
  show ?case
    unfolding coeffs-def map-upt-Suc by (auto simp: id)
```

next
case *2*
obtain *n* :: *'a* **and** *xs* **where** *defs*: *tl* (*coeffs p*) = *xs 1* = *n*
by *simp*
from *2* **show** *?case*
unfolding *defs* **by** (*induct xs arbitrary*: *n*) (*auto simp*: *cCons-def*)
qed

lemma *pderiv-eq-0-iff*: *pderiv p* = *0* ⟷ *degree p* = *0*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
proof (*cases degree p*)
case *0*
then show *?thesis*
by (*metis degree-eq-zeroE pderiv.simps*)
next
case (*Suc n*)
then show *?thesis*
using *coeff-0 coeff-pderiv degree-0 leading-coeff-0-iff mult-eq-0-iff nat.distinct*(*1*) *of-nat-eq-0-iff*
by (*metis coeff-0 coeff-pderiv degree-0 leading-coeff-0-iff mult-eq-0-iff nat.distinct*(*1*) *of-nat-eq-0-iff*)
qed

lemma *degree-pderiv*: *degree* (*pderiv p*) = *degree p* − *1*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
proof −
have *degree p* − *1* ≤ *degree* (*pderiv p*)
proof (*cases degree p*)
case (*Suc n*)
then show *?thesis*
by (*metis coeff-pderiv degree-0 diff-Suc-1 le-degree leading-coeff-0-iff mult-eq-0-iff nat.distinct*(*1*) *of-nat-eq-0-iff*)
qed *auto*
moreover have ∀ *i*>*degree p* − *1*. *coeff* (*pderiv p*) *i* = *0*
by (*simp add*: *coeff-eq-0 coeff-pderiv*)
ultimately show *?thesis*
using *order-antisym* [*OF degree-le*] **by** *blast*
qed

lemma *not-dvd-pderiv*:
fixes *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
assumes *degree p* ≠ *0*
shows ¬ *p dvd pderiv p*
proof
assume *dvd*: *p dvd pderiv p*
then obtain *q* **where** *p*: *pderiv p* = *p* ∗ *q*
unfolding *dvd-def* **by** *auto*
from *dvd* **have** *le*: *degree p* ≤ *degree* (*pderiv p*)
by (*simp add*: *assms dvd-imp-degree-le pderiv-eq-0-iff*)

from *assms* **and** *this* [*unfolded degree-pderiv*]
show *False* **by** *auto*
qed

lemma *dvd-pderiv-iff* [*simp*]: *p dvd pderiv p* $\longleftrightarrow$ *degree p = 0*
for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
using *not-dvd-pderiv*[*of p*] **by** (*auto simp*: *pderiv-eq-0-iff* [*symmetric*])

lemma *pderiv-singleton* [*simp*]: *pderiv* [:*a*:] = *0*
by (*simp add*: *pderiv-pCons*)

lemma *pderiv-add*: *pderiv* (*p* + *q*) = *pderiv p* + *pderiv q*
by (*rule poly-eqI*) (*simp add*: *coeff-pderiv algebra-simps*)

lemma *pderiv-minus*: *pderiv* (− *p* :: *'a* :: *idom poly*) = − *pderiv p*
by (*rule poly-eqI*) (*simp add*: *coeff-pderiv algebra-simps*)

lemma *pderiv-diff*: *pderiv* ((*p* :: - :: *idom poly*) − *q*) = *pderiv p* − *pderiv q*
by (*rule poly-eqI*) (*simp add*: *coeff-pderiv algebra-simps*)

lemma *pderiv-smult*: *pderiv* (*smult a p*) = *smult a* (*pderiv p*)
by (*rule poly-eqI*) (*simp add*: *coeff-pderiv algebra-simps*)

lemma *pderiv-mult*: *pderiv* (*p* ∗ *q*) = *p* ∗ *pderiv q* + *q* ∗ *pderiv p*
by (*induct p*) (*auto simp*: *pderiv-add pderiv-smult pderiv-pCons algebra-simps*)

lemma *pderiv-power-Suc*: *pderiv* (*p* ^ *Suc n*) = *smult* (*of-nat* (*Suc n*)) (*p* ^ *n*) ∗ *pderiv p*
proof (*induction n*)
case (*Suc n*)
then show *?case*
by (*simp add*: *pderiv-mult smult-add-left algebra-simps*)
qed *auto*

lemma *pderiv-power*:
pderiv (*p* ^ *n*) = *smult* (*of-nat n*) (*p* ^ (*n* − *1*) ∗ *pderiv p*)
by (*cases n*) (*simp-all add*: *pderiv-power-Suc del*: *power-Suc*)

lemma *pderiv-monom*:
pderiv (*monom c n*) = *monom* (*of-nat n* ∗ *c*) (*n* − *1*)
by (*cases n*)
(*simp-all add*: *monom-altdef pderiv-power-Suc pderiv-smult pderiv-pCons mult-ac del*: *power-Suc*)

lemma *pderiv-pcompose*: *pderiv* (*pcompose p q*) = *pcompose* (*pderiv p*) *q* ∗ *pderiv q*
by (*induction p rule*: *pCons-induct*)
(*auto simp*: *pcompose-pCons pderiv-add pderiv-mult pderiv-pCons pcompose-add algebra-simps*)

lemma *pderiv-prod*: *pderiv* (*prod f* (*as*)) = (∑ *a*∈*as*. *prod f* (*as* − {*a*}) ∗ *pderiv* (*f a*))
proof (*induct as rule*: *infinite-finite-induct*)
case (*insert a as*)
then have *id*: *prod f* (*insert a as*) = *f a* ∗ *prod f as*
⋀*g*. *sum g* (*insert a as*) = *g a* + *sum g as*
insert a as − {*a*} = *as*
by *auto*
have *prod f* (*insert a as* − {*b*}) = *f a* ∗ *prod f* (*as* − {*b*}) **if** *b* ∈ *as* **for** *b*
proof −
from ‹*a* ∉ *as*› *that* **have** ∗: *insert a as* − {*b*} = *insert a* (*as* − {*b*})
by *auto*
show *?thesis*
unfolding ∗ **by** (*subst prod.insert*) (*use insert* **in** *auto*)
qed
then show *?case*
unfolding *id pderiv-mult insert*(*3*) *sum-distrib-left*
by (*auto simp add*: *ac-simps intro*!: *sum.cong*)
qed *auto*

lemma *coeff-higher-pderiv*:
coeff ((*pderiv* ^^ *m*) *f*) *n* = *pochhammer* (*of-nat* (*Suc n*)) *m* ∗ *coeff f* (*n* + *m*)
by (*induction m arbitrary*: *n*) (*simp-all add*: *coeff-pderiv pochhammer-rec algebra-simps*)

lemma *higher-pderiv-0* [*simp*]: (*pderiv* ^^ *n*) *0* = *0*
by (*induction n*) *simp-all*

lemma *higher-pderiv-add*: (*pderiv* ^^ *n*) (*p* + *q*) = (*pderiv* ^^ *n*) *p* + (*pderiv* ^^ *n*) *q*
by (*induction n arbitrary*: *p q*) (*simp-all del*: *funpow.simps add*: *funpow-Suc-right pderiv-add*)

lemma *higher-pderiv-smult*: (*pderiv* ^^ *n*) (*smult c p*) = *smult c* ((*pderiv* ^^ *n*) *p*)
by (*induction n arbitrary*: *p*) (*simp-all del*: *funpow.simps add*: *funpow-Suc-right pderiv-smult*)

lemma *higher-pderiv-monom*:
m ≤ *n* + *1* ⟹ (*pderiv* ^^ *m*) (*monom c n*) = *monom* (*pochhammer* (*int n* − *int m* + *1*) *m* ∗ *c*) (*n* − *m*)
proof (*induction m arbitrary*: *c n*)
case (*Suc m*)
thus *?case*
by (*cases n*)
(*simp-all del*: *funpow.simps add*: *funpow-Suc-right pderiv-monom pochhammer-rec*′ *Suc.IH*)
qed *simp-all*

lemma *higher-pderiv-monom-eq-zero*:
  $m > n + 1 \Longrightarrow (pderiv \mathbin{\frown\frown} m)\ (monom\ c\ n) = 0$
proof (*induction m arbitrary*: *c n*)
  case (*Suc m*)
  thus *?case*
    by (*cases n*)
      (*simp-all del*: *funpow.simps add*: *funpow-Suc-right pderiv-monom pochhammer-rec' Suc.IH*)
qed *simp-all*

lemma *higher-pderiv-sum*: $(pderiv \mathbin{\frown\frown} n)\ (sum\ f\ A) = (\sum x{\in}A.\ (pderiv \mathbin{\frown\frown} n)\ (f\ x))$
  by (*induction A rule*: *infinite-finite-induct*) (*simp-all add*: *higher-pderiv-add*)

lemma *higher-pderiv-sum-mset*: $(pderiv \mathbin{\frown\frown} n)\ (sum\text{-}mset\ A) = (\sum p{\in}\#A.\ (pderiv \mathbin{\frown\frown} n)\ p)$
  by (*induction A*) (*simp-all add*: *higher-pderiv-add*)

lemma *higher-pderiv-sum-list*: $(pderiv \mathbin{\frown\frown} n)\ (sum\text{-}list\ ps) = (\sum p{\leftarrow}ps.\ (pderiv \mathbin{\frown\frown} n)\ p)$
  by (*induction ps*) (*simp-all add*: *higher-pderiv-add*)

lemma *degree-higher-pderiv*: $Polynomial.degree\ ((pderiv \mathbin{\frown\frown} n)\ p) = Polynomial.degree\ p - n$
  for *p* :: *'a*::{*comm-semiring-1*,*semiring-no-zero-divisors*,*semiring-char-0*} *poly*
  by (*induction n*) (*auto simp*: *degree-pderiv*)

lemma *DERIV-pow2*: $DERIV\ (\lambda x.\ x \mathbin{\hat{}} Suc\ n)\ x :> real\ (Suc\ n) * (x \mathbin{\hat{}} n)$
  by (*rule DERIV-cong*, *rule DERIV-pow*) *simp*
declare *DERIV-pow2* [*simp*] *DERIV-pow* [*simp*]

lemma *DERIV-add-const*: $DERIV\ f\ x :> D \Longrightarrow DERIV\ (\lambda x.\ a + f\ x :: 'a{::}real\text{-}normed\text{-}field)\ x :> D$
  by (*rule DERIV-cong*, *rule DERIV-add*) *auto*

lemma *poly-DERIV* [*simp*]: $DERIV\ (\lambda x.\ poly\ p\ x)\ x :> poly\ (pderiv\ p)\ x$
  by (*induct p*) (*auto intro*!: *derivative-eq-intros simp add*: *pderiv-pCons*)

lemma *poly-isCont*[*simp*]:
  fixes *x*::*'a*::*real-normed-field*
  shows *isCont* $(\lambda x.\ poly\ p\ x)\ x$
by (*rule poly-DERIV* [*THEN DERIV-isCont*])

lemma *tendsto-poly* [*tendsto-intros*]: $(f \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ poly\ p\ (f\ x)) \longrightarrow poly\ p\ a)\ F$
  for *f* :: *- ⇒ 'a*::*real-normed-field*
  by (*rule isCont-tendsto-compose* [*OF poly-isCont*])

lemma *continuous-within-poly*: *continuous* (*at z within s*) (*poly p*)
for *z* :: *'a*::{*real-normed-field*}
by (*simp add*: *continuous-within tendsto-poly*)

lemma *continuous-poly* [*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *poly p* (*f x*))
for *f* :: - ⇒ *'a*::*real-normed-field*
unfolding *continuous-def* **by** (*rule tendsto-poly*)

lemma *continuous-on-poly* [*continuous-intros*]:
fixes *p* :: *'a* :: {*real-normed-field*} *poly*
assumes *continuous-on A f*
shows *continuous-on A* (λ*x*. *poly p* (*f x*))
by (*metis DERIV-continuous-on assms continuous-on-compose2 poly-DERIV subset-UNIV*)

Consequences of the derivative theorem above.

lemma *poly-differentiable*[*simp*]: (λ*x*. *poly p x*) *differentiable* (*at x*)
for *x* :: *real*
by (*simp add*: *real-differentiable-def*) (*blast intro*: *poly-DERIV*)

lemma *poly-IVT-pos*: *a* < *b* ⟹ *poly p a* < *0* ⟹ *0* < *poly p b* ⟹ ∃ *x*. *a* < *x* ∧ *x* < *b* ∧ *poly p x* = *0*
for *a b* :: *real*
using *IVT* [*of poly p a 0 b*] **by** (*auto simp add*: *order-le-less*)

lemma *poly-IVT-neg*: *a* < *b* ⟹ *0* < *poly p a* ⟹ *poly p b* < *0* ⟹ ∃ *x*. *a* < *x* ∧ *x* < *b* ∧ *poly p x* = *0*
for *a b* :: *real*
using *poly-IVT-pos* [**where** *p* = − *p*] **by** *simp*

lemma *poly-IVT*: *a* < *b* ⟹ *poly p a* ∗ *poly p b* < *0* ⟹ ∃ *x*>*a*. *x* < *b* ∧ *poly p x* = *0*
for *p* :: *real poly*
by (*metis less-not-sym mult-less-0-iff poly-IVT-neg poly-IVT-pos*)

lemma *poly-MVT*: *a* < *b* ⟹ ∃ *x*. *a* < *x* ∧ *x* < *b* ∧ *poly p b* − *poly p a* = (*b* − *a*) ∗ *poly* (*pderiv p*) *x*
for *a b* :: *real*
by (*simp add*: *MVT2*)

lemma *poly-MVT'*:
fixes *a b* :: *real*
assumes {*min a b*..*max a b*} ⊆ *A*
shows ∃ *x*∈*A*. *poly p b* − *poly p a* = (*b* − *a*) ∗ *poly* (*pderiv p*) *x*
proof (*cases a b rule*: *linorder-cases*)
case *less*
from *poly-MVT*[*OF less*, *of p*] **obtain** *x*
where *a* < *x x* < *b poly p b* − *poly p a* = (*b* − *a*) ∗ *poly* (*pderiv p*) *x*

```
    by auto
  then show ?thesis by (intro bexI[of - x]) (auto intro!: subsetD[OF assms])
next
  case greater
  from poly-MVT[OF greater, of p] obtain x
    where b < x x < a poly p a − poly p b = (a − b) ∗ poly (pderiv p) x by auto
  then show ?thesis by (intro bexI[of - x]) (auto simp: algebra-simps intro!: sub-
setD[OF assms])
qed (use assms in auto)

lemma poly-pinfty-gt-lc:
  fixes p :: real poly
  assumes lead-coeff p > 0
  shows ∃ n. ∀ x ≥ n. poly p x ≥ lead-coeff p
  using assms
proof (induct p)
  case 0
  then show ?case by auto
next
  case (pCons a p)
  from this(1) consider a ≠ 0 p = 0 | p ≠ 0 by auto
  then show ?case
  proof cases
    case 1
    then show ?thesis by auto
  next
    case 2
    with pCons obtain n1 where gte-lcoeff: ∀ x≥n1. lead-coeff p ≤ poly p x
      by auto
    from pCons(3) ‹p ≠ 0› have gt-0: lead-coeff p > 0 by auto
    define n where n = max n1 (1 + |a| / lead-coeff p)
    have lead-coeff (pCons a p) ≤ poly (pCons a p) x if n ≤ x for x
    proof −
      from gte-lcoeff that have lead-coeff p ≤ poly p x
        by (auto simp: n-def)
      with gt-0 have |a| / lead-coeff p ≥ |a| / poly p x and poly p x > 0
        by (auto intro: frac-le)
      with ‹n ≤ x›[unfolded n-def] have x ≥ 1 + |a| / poly p x
        by auto
      with ‹lead-coeff p ≤ poly p x› ‹poly p x > 0› ‹p ≠ 0›
      show lead-coeff (pCons a p) ≤ poly (pCons a p) x
        by (auto simp: field-simps)
    qed
    then show ?thesis by blast
  qed
qed

lemma dvd-monic:
  fixes p q:: 'a :: idom poly
```

assumes *monic*:*lead-coeff p=1* **and** *p dvd* (*smult c q*) **and** *c≠0*
shows *p dvd q* **using** *assms*
proof (*cases q=0 ∨ degree p=0*)
case *True*
thus *?thesis* **using** *assms*
by (*auto elim*!: *degree-eq-zeroE simp add*: *const-poly-dvd-iff*)
next
case *False*
hence *q≠0* **and** *degree p≠0* **by** *auto*
obtain *k* **where** *k*:*smult c q = p∗k* **using** *assms dvd-def* **by** *metis*
hence *k≠0* **by** (*metis False assms*(*3*) *mult-zero-right smult-eq-0-iff*)
hence *deg-eq*:*degree q=degree p + degree k*
by (*metis False assms*(*3*) *degree-0 degree-mult-eq degree-smult-eq k*)
have *c-dvd*:∀ *n*≤*degree k*. *c dvd coeff k* (*degree k − n*)
proof (*rule*,*rule*)
fix *n* **assume** *n* ≤ *degree k*
thus *c dvd coeff k* (*degree k − n*)
proof (*induct n rule*:*nat-less-induct*)
case (*1 n*)
define *T* **where** *T*≡(λ*i*. *coeff p i ∗ coeff k* (*degree p*+*degree k − n − i*))
have *c ∗ coeff q* (*degree q − n*) = (∑ *i*≤*degree q − n*. *coeff p i ∗ coeff k* (*degree q − n − i*))
using *coeff-mult*[*of p k degree q − n*] *k coeff-smult*[*of c q degree q −n*] **by** *auto*
also have ...=(∑ *i*≤*degree p*+*degree k − n*. *T i*)
using *deg-eq* **unfolding** *T-def* **by** *auto*
also have ...=(∑ *i*∈{*0*..<*degree p*}. *T i*) + *sum T* {(*degree p*)}+
sum T {*degree p + 1*..*degree p + degree k − n*}
proof −
define *C* **where** *C*≡{{*0*..<*degree p*}, {*degree p*},{*degree p*+*1*..*degree p*+*degree k*−*n*}}
have ∀ *A*∈*C*. *finite A* **unfolding** *C-def* **by** *auto*
moreover have ∀ *A*∈*C*. ∀ *B*∈*C*. *A* ≠ *B* ⟶ *A* ∩ *B* = {}
unfolding *C-def* **by** *auto*
ultimately have *sum T* (⋃ *C*) = *sum* (*sum T*) *C*
using *sum.Union-disjoint* **by** *auto*
moreover have ⋃ *C*={..*degree p + degree k − n*}
using ‹*n* ≤ *degree k*› **unfolding** *C-def* **by** *auto*
moreover have *sum* (*sum T*) *C*= *sum T* {*0*..<*degree p*} + *sum T* {(*degree p*)} +
sum T {*degree p + 1*..*degree p + degree k − n*}
proof −
have {*0*..<*degree p*}≠{*degree p*}
by (*metis atLeast0LessThan insertI1 lessThan-iff less-imp-not-eq*)
moreover have {*degree p*}≠{*degree p + 1*..*degree p + degree k − n*}
by (*metis add.commute add-diff-cancel-right*′ *atLeastAtMost-singleton-iff*
diff-self-eq-0 eq-imp-le not-one-le-zero)
moreover have {*0*..<*degree p*}≠{*degree p + 1*..*degree p + degree k − n*}
using ‹*degree k*≥*n*› ‹*degree p≠0*› **by** *fastforce*

```
      ultimately show ?thesis unfolding C-def by auto
    qed
    ultimately show ?thesis by auto
  qed
  also have ...=(∑ i∈{0..<degree p}. T i) +  coeff k (degree k − n)
  proof −
    have ∀ x∈{degree p + 1..degree p + degree k − n}. T x=0
      using coeff-eq-0[of p] unfolding T-def by simp
    hence sum T {degree p + 1..degree p + degree k − n}=0 by auto
    moreover have T (degree p)=coeff k (degree k − n)
      using monic by (simp add: T-def)
    ultimately show ?thesis by auto
  qed
  finally have c-coeff: c ∗ coeff q (degree q − n) = sum T {0..<degree p}
        + coeff k (degree k − n) .
  moreover have n≠0⟹c dvd sum T {0..<degree p}
  proof (rule dvd-sum)
    fix i assume i:i ∈ {0..<degree p} and n≠0
    hence (n+i−degree p)≤degree k using ‹n ≤ degree k› by auto
    moreover have n + i − degree p <n using i ‹n≠0› by auto
    ultimately have c dvd coeff k (degree k − (n+i−degree p))
      using 1(1) by auto
    hence c dvd coeff k (degree p + degree k − n − i)
      by (metis add-diff-cancel-left' deg-eq diff-diff-left dvd-0-right le-degree
        le-diff-conv add.commute ordered-cancel-comm-monoid-diff-class.diff-diff-right)
    thus c dvd T i unfolding T-def by auto
  qed
  moreover have n=0 ⟹?case
  proof −
    assume n=0
    hence ∀ i∈{0..<degree p}. coeff k (degree p + degree k − n − i) =0
      using coeff-eq-0[of k] by simp
    hence c ∗ coeff q (degree q − n) = coeff k (degree k − n)
      using c-coeff unfolding T-def by auto
    thus ?thesis by (metis dvdI)
  qed
  ultimately show ?case by (metis dvd-add-right-iff dvd-triv-left)
 qed
qed
hence ∀ n. c dvd coeff k n
 by (metis diff-diff-cancel dvd-0-right le-add2 le-add-diff-inverse le-degree)
then obtain f where f:∀ n. c ∗ f n=coeff k n unfolding dvd-def by metis
have  ∀ ∞ n. f n = 0
 by (metis (mono-tags, lifting) MOST-coeff-eq-0 MOST-mono assms(3) f mult-eq-0-iff)
hence smult c (Abs-poly f)=k
 using f smult.abs-eq[of c Abs-poly f] Abs-poly-inverse[of f] coeff-inverse[of k]
 by simp
hence q=p∗ Abs-poly f using k ‹c≠0› smult-cancel by auto
thus ?thesis unfolding dvd-def by auto
```

qed

lemma *lemma-order-pderiv1*:
pderiv ([:− *a*, *1*:] ^ *Suc n* ∗ *q*)
= [:− *a*, *1*:] ^ *Suc n* ∗ *pderiv q* + *smult* (*of-nat* (*Suc n*)) (*q* ∗ [:− *a*, *1*:] ^ *n*)
unfolding *pderiv-mult pderiv-power-Suc*
by (*simp del*: *power-Suc of-nat-Suc add*: *pderiv-pCons*)

lemma *order-pderiv*:
fixes *p*::*'a*::{*idom*,*semiring-char-0*} *poly*
assumes *p*≠*0 poly p x* = *0*
shows *order x p* = *Suc* (*order x* (*pderiv p*)) **using** *assms*
proof −
define *xx op* **where** *xx*=[:− *x*, *1*:] **and** *op* = *order x p*
have *op* ≠ *0* **unfolding** *op-def* **using** *assms order-root* **by** *blast*
obtain *pp* **where** *pp*:*p* = *xx* ^ *op* ∗ *pp* ¬ *xx dvd pp*
using *order-decomp*[*OF* ‹*p*≠*0*›,*of x*,*folded xx-def op-def*] **by** *auto*
have *p-der*:*pderiv p* = *smult* (*of-nat op*) (*xx*^(*op* −*1*)) ∗ *pp* + *xx*^*op*∗*pderiv pp*
unfolding *pp*(*1*) **by** (*auto simp*:*pderiv-mult pderiv-power xx-def algebra-simps pderiv-pCons*)
have *xx*^(*op* −*1*) *dvd* (*pderiv p*)
unfolding *p-der*
by (*metis* ‹*op* ≠ *0*› *dvd-add-left-iff dvd-mult2 dvd-refl dvd-smult dvd-triv-right power-eq-if*)
moreover have ¬ *xx*^*op dvd* (*pderiv p*)
proof
assume *xx* ^ *op dvd pderiv p*
then have *xx* ^ *op dvd smult* (*of-nat op*) (*xx*^(*op* −*1*) ∗ *pp*)
unfolding *p-der* **by** (*simp add*: *dvd-add-left-iff*)
then have *xx* ^ *op dvd* (*xx*^(*op* −*1*)) ∗ *pp*
apply (*elim dvd-monic*[*rotated*])
using ‹*op*≠*0*› **by** (*auto simp*:*lead-coeff-power xx-def*)
then have *xx* ^ (*op*−*1*) ∗ *xx dvd* (*xx*^(*op* −*1*))
using ‹¬ *xx dvd pp*› **by** (*simp add*: ‹*op* ≠ *0*› *mult.commute power-eq-if*)
then have *xx dvd 1*
using *assms*(*1*) *pp*(*1*) **by** *auto*
then show *False* **unfolding** *xx-def* **by** (*meson assms*(*1*) *dvd-trans one-dvd order-decomp*)
qed
ultimately have *op* − *1* = *order x* (*pderiv p*)
using *order-unique-lemma*[*of x op*−*1 pderiv p*,*folded xx-def*] ‹*op*≠*0*›
by *auto*
then show *?thesis* **using** ‹*op*≠*0*› **unfolding** *op-def* **by** *auto*
qed

lemma *lemma-order-pderiv*:
fixes *p* :: *'a* :: *field-char-0 poly*
assumes *n*: *0* < *n*
and *pd*: *pderiv p* ≠ *0*

and *pe*: $p = [:- a, 1:] \mathbin{\hat{}} n * q$
and *nd*: $\neg [:- a, 1:] \ dvd \ q$
shows $n = Suc \ (order \ a \ (pderiv \ p))$
by (*metis add.right-neutral gr0-conv-Suc n nat.case nd order-mult order-pderiv order-power-n-n order-root pd pderiv-0 pe poly-eq-0-iff-dvd*)

lemma *poly-squarefree-decomp-order*:
fixes $p :: 'a::field\text{-}char\text{-}0 \ poly$
assumes $pderiv \ p \neq 0$
and *p*: $p = q * d$
and *p'*: $pderiv \ p = e * d$
and *d*: $d = r * p + s * pderiv \ p$
shows $order \ a \ q = (if \ order \ a \ p = 0 \ then \ 0 \ else \ 1)$
proof (*rule classical*)
assume *1*: $\neg$ *?thesis*
from ‹$pderiv \ p \neq 0$› **have** $p \neq 0$ **by** *auto*
with *p* **have** $order \ a \ p = order \ a \ q + order \ a \ d$
by (*simp add*: *order-mult*)
with *1* **have** $order \ a \ p \neq 0$
by (*auto split*: *if-splits*)
from ‹$pderiv \ p \neq 0$› ‹$pderiv \ p = e * d$› **have** *oapp*: $order \ a \ (pderiv \ p) = order \ a \ e + order \ a \ d$
by (*simp add*: *order-mult*)
from ‹$pderiv \ p \neq 0$› ‹$order \ a \ p \neq 0$› **have** *oap*: $order \ a \ p = Suc \ (order \ a \ (pderiv \ p))$
using ‹$p \neq 0$› *order-pderiv order-root* **by** *blast*
from ‹$p \neq 0$› ‹$p = q * d$› **have** $d \neq 0$
by *simp*
have $[:- a, 1:] \mathbin{\hat{}} order \ a \ (pderiv \ p) \ dvd \ r * p$
by (*metis dvd-trans dvd-triv-right oap order-1 power-Suc*)
then have $([:-a, 1:] \mathbin{\hat{}} (order \ a \ (pderiv \ p))) \ dvd \ d$
by (*simp add*: *d order-1*)
with ‹$d \neq 0$› **have** $order \ a \ (pderiv \ p) \leq order \ a \ d$
by (*simp add*: *order-divides*)
show *?thesis*
using ‹$order \ a \ p = order \ a \ q + order \ a \ d$›
and *oapp oap*
and ‹$order \ a \ (pderiv \ p) \leq order \ a \ d$›
by *auto*
qed

lemma *poly-squarefree-decomp-order2*:
$pderiv \ p \neq 0 \Longrightarrow p = q * d \Longrightarrow pderiv \ p = e * d \Longrightarrow$
$d = r * p + s * pderiv \ p \Longrightarrow \forall a. \ order \ a \ q = (if \ order \ a \ p = 0 \ then \ 0 \ else \ 1)$
for $p :: 'a::field\text{-}char\text{-}0 \ poly$
by (*blast intro*: *poly-squarefree-decomp-order*)

lemma *order-pderiv2*:
$pderiv \ p \neq 0 \Longrightarrow order \ a \ p \neq 0 \Longrightarrow order \ a \ (pderiv \ p) = n \longleftrightarrow order \ a \ p = Suc$

```
n
  for p :: 'a::field-char-0 poly
  by (metis nat.inject order-pderiv order-root pderiv-0)

definition rsquarefree :: 'a::idom poly ⇒ bool
  where rsquarefree p ⟷ p ≠ 0 ∧ (∀ a. order a p = 0 ∨ order a p = 1)

lemma pderiv-iszero: pderiv p = 0 ⟹ ∃ h. p = [:h:]
  for p :: 'a::{semidom,semiring-char-0} poly
  by (cases p) (auto simp: pderiv-eq-0-iff split: if-splits)

lemma rsquarefree-roots: rsquarefree p ⟷ (∀ a. ¬ (poly p a = 0 ∧ poly (pderiv
p) a = 0))
  for p :: 'a::field-char-0 poly
proof (cases p = 0)
  case False
  show ?thesis
  proof (cases pderiv p = 0)
    case True
    with ‹p ≠ 0› pderiv-iszero show ?thesis
      by (force simp add: order-0I rsquarefree-def)
  next
    case False
    with ‹p ≠ 0› order-pderiv2 show ?thesis
      by (force simp add: rsquarefree-def order-root)
  qed
qed (simp add: rsquarefree-def)

lemma rsquarefree-root-order:
  assumes rsquarefree p poly p z = 0 p ≠ 0
  shows   order z p = 1
proof −
  from assms have order z p ∈ {0, 1} by (auto simp: rsquarefree-def)
  moreover from assms have order z p > 0 by (auto simp: order-root)
  ultimately show order z p = 1 by auto
qed

lemma poly-squarefree-decomp:
  fixes p :: 'a::field-char-0 poly
  assumes pderiv p ≠ 0
    and p = q * d
    and pderiv p = e * d
    and d = r * p + s * pderiv p
  shows rsquarefree q ∧ (∀ a. poly q a = 0 ⟷ poly p a = 0)
proof −
  from ‹pderiv p ≠ 0› have p ≠ 0 by auto
  with ‹p = q * d› have q ≠ 0 by simp
  from assms have ∀ a. order a q = (if order a p = 0 then 0 else 1)
    by (rule poly-squarefree-decomp-order2)
```

with ‹$p \neq 0$› ‹$q \neq 0$› **show** *?thesis*
by (*simp add*: *rsquarefree-def order-root*)
qed

lemma *has-field-derivative-poly* [*derivative-intros*]:
assumes (*f has-field-derivative f'*) (*at x within A*)
shows ((λ*x*. *poly p* (*f x*)) *has-field-derivative*
(*f'* ∗ *poly* (*pderiv p*) (*f x*))) (*at x within A*)
using *DERIV-chain*[*OF poly-DERIV assms*, *of p*] **by** (*simp add*: *o-def mult-ac*)

4.28 Algebraic numbers

lemma *intpolyE*:
assumes ⋀*i*. *poly.coeff p i* ∈ ℤ
obtains *q* **where** *p* = *map-poly of-int q*
proof −
have ∀ *i*∈{..*Polynomial.degree p*}. ∃ *x*. *poly.coeff p i* = *of-int x*
using *assms* **by** (*auto simp*: *Ints-def*)
from *bchoice*[*OF this*] **obtain** *f*
where *f*: ⋀*i*. *i* ≤ *Polynomial.degree p* ⟹ *poly.coeff p i* = *of-int* (*f i*) **by** *blast*
define *q* **where** *q* = *Poly* (*map f* [*0*..<*Suc* (*Polynomial.degree p*)])
have *p* = *map-poly of-int q*
by (*intro poly-eqI*)
(*auto simp*: *coeff-map-poly q-def nth-default-def f coeff-eq-0 simp del*: *upt-Suc*)
with *that* **show** *?thesis* **by** *blast*
qed

lemma *ratpolyE*:
assumes ⋀*i*. *poly.coeff p i* ∈ ℚ
obtains *q* **where** *p* = *map-poly of-rat q*
proof −
have ∀ *i*∈{..*Polynomial.degree p*}. ∃ *x*. *poly.coeff p i* = *of-rat x*
using *assms* **by** (*auto simp*: *Rats-def*)
from *bchoice*[*OF this*] **obtain** *f*
where *f*: ⋀*i*. *i* ≤ *Polynomial.degree p* ⟹ *poly.coeff p i* = *of-rat* (*f i*) **by** *blast*
define *q* **where** *q* = *Poly* (*map f* [*0*..<*Suc* (*Polynomial.degree p*)])
have *p* = *map-poly of-rat q*
by (*intro poly-eqI*)
(*auto simp*: *coeff-map-poly q-def nth-default-def f coeff-eq-0 simp del*: *upt-Suc*)
with *that* **show** *?thesis* **by** *blast*
qed

Algebraic numbers can be defined in two equivalent ways: all real numbers that are roots of rational polynomials or of integer polynomials. The Algebraic-Numbers AFP entry uses the rational definition, but we need the integer definition.

The equivalence is obvious since any rational polynomial can be multiplied with the LCM of its coefficients, yielding an integer polynomial with the same roots.

definition *algebraic* :: *'a* :: *field-char-0* ⇒ *bool*
  where *algebraic x* ⟷ (∃ *p*. (∀ *i*. *coeff p i* ∈ ℤ) ∧ *p* ≠ *0* ∧ *poly p x* = *0*)

lemma *algebraicI*: (⋀*i*. *coeff p i* ∈ ℤ) ⟹ *p* ≠ *0* ⟹ *poly p x* = *0* ⟹ *algebraic x*
  unfolding *algebraic-def* **by** *blast*

lemma *algebraicE*:
  assumes *algebraic x*
  obtains *p* **where** ⋀*i*. *coeff p i* ∈ ℤ *p* ≠ *0 poly p x* = *0*
  using *assms* **unfolding** *algebraic-def* **by** *blast*

lemma *algebraic-altdef*: *algebraic x* ⟷ (∃ *p*. (∀ *i*. *coeff p i* ∈ ℚ) ∧ *p* ≠ *0* ∧ *poly p x* = *0*)
  for *p* :: *'a*::*field-char-0 poly*
proof *safe*
  fix *p*
  assume *rat*: ∀ *i*. *coeff p i* ∈ ℚ **and** *root*: *poly p x* = *0* **and** *nz*: *p* ≠ *0*
  define *cs* **where** *cs* = *coeffs p*
  from *rat* **have** ∀ *c*∈*range* (*coeff p*). ∃ *c'*. *c* = *of-rat c'*
    unfolding *Rats-def* **by** *blast*
  then obtain *f* **where** *f*: *coeff p i* = *of-rat* (*f* (*coeff p i*)) **for** *i*
    by (*subst* (*asm*) *bchoice-iff*) *blast*
  define *cs'* **where** *cs'* = *map* (*quotient-of* ∘ *f*) (*coeffs p*)
  define *d* **where** *d* = *Lcm* (*set* (*map snd cs'*))
  define *p'* **where** *p'* = *smult* (*of-int d*) *p*

  have *coeff p' n* ∈ ℤ **for** *n*
  proof (*cases n* ≤ *degree p*)
    case *True*
    define *c* **where** *c* = *coeff p n*
    define *a* **where** *a* = *fst* (*quotient-of* (*f* (*coeff p n*)))
    define *b* **where** *b* = *snd* (*quotient-of* (*f* (*coeff p n*)))
    have *b-pos*: *b* > *0*
      unfolding *b-def* **using** *quotient-of-denom-pos'* **by** *simp*
    have *coeff p' n* = *of-int d* ∗ *coeff p n*
      by (*simp add*: *p'-def*)
    also have *coeff p n* = *of-rat* (*of-int a* / *of-int b*)
      unfolding *a-def b-def*
        by (*subst quotient-of-div* [*of f* (*coeff p n*), *symmetric*]) (*simp-all add*: *f* [*symmetric*])
    also have *of-int d* ∗ ... = *of-rat* (*of-int* (*a*∗*d*) / *of-int b*)
      by (*simp add*: *of-rat-mult of-rat-divide*)
    also from *nz True* **have** *b* ∈ *snd* ‘ *set cs'*
      by (*force simp*: *cs'-def o-def b-def coeffs-def simp del*: *upt-Suc*)
    then have *b dvd* (*a* ∗ *d*)
      by (*simp add*: *d-def*)
    then have *of-int* (*a* ∗ *d*) / *of-int b* ∈ (ℤ :: *rat set*)
      by (*rule of-int-divide-in-Ints*)

```
    then have of-rat (of-int (a * d)) / of-int b) ∈ ℤ by (elim Ints-cases) auto
    finally show ?thesis .
  next
    case False
    then show ?thesis
      by (auto simp: p'-def not-le coeff-eq-0)
  qed
  moreover have set (map snd cs') ⊆ {0<..}
    unfolding cs'-def using quotient-of-denom-pos' by (auto simp: coeffs-def simp
del: upt-Suc)
  then have d ≠ 0
    unfolding d-def by (induct cs') simp-all
  with nz have p' ≠ 0 by (simp add: p'-def)
  moreover from root have poly p' x = 0
    by (simp add: p'-def)
  ultimately show algebraic x
    unfolding algebraic-def by blast
next
  assume algebraic x
  then obtain p where p: coeff p i ∈ ℤ poly p x = 0 p ≠ 0 for i
    by (force simp: algebraic-def)
  moreover have coeff p i ∈ ℤ ⟹ coeff p i ∈ ℚ for i
    by (elim Ints-cases) simp
  ultimately show ∃ p. (∀ i. coeff p i ∈ ℚ) ∧ p ≠ 0 ∧ poly p x = 0 by auto
qed

lemma algebraicI': (⋀i. coeff p i ∈ ℚ) ⟹ p ≠ 0 ⟹ poly p x = 0 ⟹ algebraic
x
  unfolding algebraic-altdef by blast

lemma algebraicE':
  assumes algebraic (x :: 'a :: field-char-0)
  obtains p where p ≠ 0 poly (map-poly of-int p) x = 0
proof −
  from assms obtain q where q: ⋀i. coeff q i ∈ ℤ q ≠ 0 poly q x = 0
    by (erule algebraicE)
  moreover from this(1) obtain q' where q': q = map-poly of-int q' by (erule
intpolyE)
  moreover have q' ≠ 0
    using q' q by auto
  ultimately show ?thesis by (intro that[of q']) simp-all
qed

lemma algebraicE'-nonzero:
  assumes algebraic (x :: 'a :: field-char-0) x ≠ 0
  obtains p where p ≠ 0 coeff p 0 ≠ 0 poly (map-poly of-int p) x = 0
proof −
  from assms(1) obtain p where p: p ≠ 0 poly (map-poly of-int p) x = 0
    by (erule algebraicE')
```

```
  define n :: nat where n = order 0 p
  have monom 1 n dvd p by (simp add: monom-1-dvd-iff p n-def)
  then obtain q where q: p = monom 1 n * q by (erule dvdE)
  have [simp]: map-poly of-int (monom 1 n * q) = monom (1 :: 'a) n * map-poly
of-int q
    by (induction n) (auto simp: monom-0 monom-Suc map-poly-pCons)
  from p have q ≠ 0 poly (map-poly of-int q) x = 0 by (auto simp: q poly-monom
assms(2))
  moreover from this have order 0 p = n + order 0 q by (simp add: q order-mult)
  hence order 0 q = 0 by (simp add: n-def)
  with ‹q ≠ 0› have poly q 0 ≠ 0 by (simp add: order-root)
  ultimately show ?thesis using that[of q] by (auto simp: poly-0-coeff-0)
qed

lemma rat-imp-algebraic: x ∈ ℚ ⟹ algebraic x
proof (rule algebraicI')
  show poly [:-x, 1:] x = 0
    by simp
qed (auto simp: coeff-pCons split: nat.splits)

lemma algebraic-0 [simp, intro]: algebraic 0
  and algebraic-1 [simp, intro]: algebraic 1
  and algebraic-numeral [simp, intro]: algebraic (numeral n)
  and algebraic-of-nat [simp, intro]: algebraic (of-nat k)
  and algebraic-of-int [simp, intro]: algebraic (of-int m)
  by (simp-all add: rat-imp-algebraic)

lemma algebraic-ii [simp, intro]: algebraic i
proof (rule algebraicI)
  show poly [:1, 0, 1:] i = 0
    by simp
qed (auto simp: coeff-pCons split: nat.splits)

lemma algebraic-minus [intro]:
  assumes algebraic x
  shows   algebraic (-x)
proof -
  from assms obtain p where p: ∀ i. coeff p i ∈ ℤ poly p x = 0 p ≠ 0
    by (elim algebraicE) auto
  define s where s = (if even (degree p) then 1 else -1 :: 'a)

  define q where q = Polynomial.smult s (pcompose p [:0, -1:])
  have poly q (-x) = 0
    using p by (auto simp: q-def poly-pcompose s-def)
  moreover have q ≠ 0
    using p by (auto simp: q-def s-def pcompose-eq-0-iff)
  find-theorems pcompose - - = 0
  moreover have coeff q i ∈ ℤ for i
  proof -
```

```
    have coeff (pcompose p [:0, −1:]) i ∈ ℤ
      using p by (intro coeff-pcompose-semiring-closed) (auto simp: coeff-pCons
split: nat.splits)
    thus ?thesis by (simp add: q-def s-def)
  qed
  ultimately show ?thesis
    by (auto simp: algebraic-def)
qed

lemma algebraic-minus-iff [simp]:
  algebraic (−x) ⟷ algebraic (x :: 'a :: field-char-0)
  using algebraic-minus[of x] algebraic-minus[of −x] by auto

lemma algebraic-inverse [intro]:
  assumes algebraic x
  shows   algebraic (inverse x)
proof (cases x = 0)
  case [simp]: False
  from assms obtain p where p: ∀ i. coeff p i ∈ ℤ poly p x = 0 p ≠ 0
    by (elim algebraicE) auto
  show ?thesis
  proof (rule algebraicI)
    show poly (reflect-poly p) (inverse x) = 0
      using assms p by (simp add: poly-reflect-poly-nz)
  qed (use assms p in ‹auto simp: coeff-reflect-poly›)
qed auto

lemma algebraic-root:
  assumes algebraic y
      and poly p x = y and ∀ i. coeff p i ∈ ℤ and lead-coeff p = 1 and degree p
> 0
  shows   algebraic x
proof −
  from assms obtain q where q: poly q y = 0 ∀ i. coeff q i ∈ ℤ q ≠ 0
    by (elim algebraicE) auto
  show ?thesis
  proof (rule algebraicI)
    from assms q show pcompose q p ≠ 0
      by (auto simp: pcompose-eq-0-iff)
    from assms q show coeff (pcompose q p) i ∈ ℤ for i
      by (intro allI coeff-pcompose-semiring-closed) auto
    show poly (pcompose q p) x = 0
      using assms q by (simp add: poly-pcompose)
  qed
qed

lemma algebraic-abs-real [simp]:
  algebraic |x :: real| ⟷ algebraic x
  by (auto simp: abs-if)
```

lemma *algebraic-nth-root-real* [*intro*]:
assumes *algebraic x*
shows *algebraic* (*root n x*)
proof (*cases n = 0*)
case *False*
show *?thesis*
proof (*rule algebraic-root*)
show *poly* (*monom 1 n*) (*root n x*) = (*if even n then* |*x*| *else x*)
using *sgn-power-root*[*of n x*] *False*
by (*auto simp add*: *poly-monom sgn-if split*: *if-splits*)
qed (*use False assms* **in** ‹*auto simp*: *degree-monom-eq*›)
qed *auto*

lemma *algebraic-sqrt* [*intro*]: *algebraic x* ⟹ *algebraic* (*sqrt x*)
by (*auto simp*: *sqrt-def*)

lemma *algebraic-csqrt* [*intro*]: *algebraic x* ⟹ *algebraic* (*csqrt x*)
by (*rule algebraic-root*[**where** *p* = *monom 1 2*])
(*auto simp*: *poly-monom degree-monom-eq*)

lemma *algebraic-cnj* [*intro*]:
assumes *algebraic x*
shows *algebraic* (*cnj x*)
proof −
from *assms* **obtain** *p* **where** *p*: *poly p x* = *0* ∀ *i*. *coeff p i* ∈ ℤ *p* ≠ *0*
by (*elim algebraicE*) *auto*
show *?thesis*
proof (*rule algebraicI*)
show *poly* (*map-poly cnj p*) (*cnj x*) = *0*
using *p* **by** *simp*
show *map-poly cnj p* ≠ *0*
using *p* **by** (*auto simp*: *map-poly-eq-0-iff*)
show *coeff* (*map-poly cnj p*) *i* ∈ ℤ **for** *i*
using *p* **by** (*auto simp*: *coeff-map-poly*)
qed
qed

lemma *algebraic-cnj-iff* [*simp*]: *algebraic* (*cnj x*) ⟷ *algebraic x*
using *algebraic-cnj*[*of x*] *algebraic-cnj*[*of cnj x*] **by** *auto*

lemma *algebraic-of-real* [*intro*]:
assumes *algebraic x*
shows *algebraic* (*of-real x*)
proof −
from *assms* **obtain** *p* **where** *p*: *p* ≠ *0 poly* (*map-poly of-int p*) *x* = *0* **by** (*erule algebraicE'*)
have *1*: *map-poly of-int p* ≠ (*0* :: *'a poly*)
using *p* **by** (*metis coeff-0 coeff-map-poly leading-coeff-0-iff of-int-eq-0-iff*)

have *poly* (*map-poly of-int p*) (*of-real x* :: *'a*) = *of-real* (*poly* (*map-poly of-int p*) *x*)
by (*simp add*: *poly-altdef degree-map-poly coeff-map-poly*)
also note *p*(*2*)
finally have *2*: *poly* (*map-poly of-int p*) (*of-real x* :: *'a*) = *0*
by *simp*

from *1 2* **show** *algebraic* (*of-real x* :: *'a*)
by (*intro algebraicI*[*of map-poly of-int p*]) (*auto simp*: *coeff-map-poly*)
qed

lemma *algebraic-of-real-iff* [*simp*]:
algebraic (*of-real x* :: *'a* :: {*real-algebra-1*,*field-char-0*}) ⟷ *algebraic x*
proof
assume *algebraic* (*of-real x* :: *'a*)
then obtain *p* **where** *p*: *p* ≠ *0 poly* (*map-poly of-int p*) (*of-real x* :: *'a*) = *0*
by (*erule algebraicE'*)
have *1*: (*map-poly of-int p* :: *real poly*) ≠ *0*
using *p* **by** (*metis coeff-0 coeff-map-poly leading-coeff-0-iff of-int-0 of-int-eq-iff*)

note *p*(*2*)
also have *poly* (*map-poly of-int p*) (*of-real x* :: *'a*) = *of-real* (*poly* (*map-poly of-int p*) *x*)
by (*simp add*: *poly-altdef degree-map-poly coeff-map-poly*)
also have ... = *0* ⟷ *poly* (*map-poly of-int p*) *x* = *0*
using *of-real-eq-0-iff* **by** *blast*
finally have *2*: *poly* (*map-poly real-of-int p*) *x* = *0* **.**

from *1* **and** *2* **show** *algebraic x*
by (*intro algebraicI*[*of map-poly of-int p*]) (*auto simp*: *coeff-map-poly*)
qed *auto*

4.29 Algebraic integers

inductive *algebraic-int* :: *'a* :: *field* ⇒ *bool* **where**
⟦*lead-coeff p* = *1*; ∀ *i*. *coeff p i* ∈ ℤ; *poly p x* = *0*⟧ ⟹ *algebraic-int x*

lemma *algebraic-int-altdef-ipoly*:
fixes *x* :: *'a* :: *field-char-0*
shows *algebraic-int x* ⟷ (∃ *p*. *poly* (*map-poly of-int p*) *x* = *0* ∧ *lead-coeff p* = *1*)
proof
assume *algebraic-int x*
then obtain *p* **where** *p*: *lead-coeff p* = *1* ∀ *i*. *coeff p i* ∈ ℤ *poly p x* = *0*
by (*auto elim*: *algebraic-int.cases*)
define *the-int* **where** *the-int* = (λ*x*::*'a*. *THE r*. *x* = *of-int r*)
define *p'* **where** *p'* = *map-poly the-int p*
have *of-int-the-int*: *of-int* (*the-int x*) = *x* **if** *x* ∈ ℤ **for** *x*

```
  unfolding the-int-def by (rule sym, rule theI') (insert that, auto simp: Ints-def)
 have the-int-0-iff: the-int x = 0 ⟷ x = 0 if x ∈ ℤ for x
  using of-int-the-int[OF that] by auto
 have [simp]: the-int 0 = 0
  by (subst the-int-0-iff) auto
 have map-poly of-int p' = map-poly (of-int ∘ the-int) p
  by (simp add: p'-def map-poly-map-poly)
 also from p of-int-the-int have ... = p
  by (subst poly-eq-iff) (auto simp: coeff-map-poly)
 finally have p-p': map-poly of-int p' = p .

 show (∃ p. poly (map-poly of-int p) x = 0 ∧ lead-coeff p = 1)
 proof (intro exI conjI notI)
  from p show poly (map-poly of-int p') x = 0 by (simp add: p-p')
 next
  show lead-coeff p' = 1
   using p by (simp flip: p-p' add: degree-map-poly coeff-map-poly)
 qed
next
 assume ∃ p. poly (map-poly of-int p) x = 0 ∧ lead-coeff p = 1
 then obtain p where p: poly (map-poly of-int p) x = 0 lead-coeff p = 1
  by auto
 define p' where p' = (map-poly of-int p :: 'a poly)
 from p have lead-coeff p' = 1 poly p' x = 0 ∀ i. coeff p' i ∈ ℤ
  by (auto simp: p'-def coeff-map-poly degree-map-poly)
 thus algebraic-int x
  by (intro algebraic-int.intros)
qed

theorem rational-algebraic-int-is-int:
 assumes algebraic-int x and x ∈ ℚ
 shows   x ∈ ℤ
proof −
 from assms(2) obtain a b where ab: b > 0 Rings.coprime a b and x-eq: x =
of-int a / of-int b
  by (auto elim: Rats-cases')
 from ‹b > 0› have [simp]: b ≠ 0
  by auto
 from assms(1) obtain p
  where p: lead-coeff p = 1 ∀ i. coeff p i ∈ ℤ poly p x = 0
  by (auto simp: algebraic-int.simps)

 define q :: 'a poly where q = [:−of-int a, of-int b:]
 have poly q x = 0 q ≠ 0 ∀ i. coeff q i ∈ ℤ
  by (auto simp: x-eq q-def coeff-pCons split: nat.splits)
 define n where n = degree p
 have n > 0
  using p by (intro Nat.gr0I) (auto simp: n-def elim!: degree-eq-zeroE)
 have (∑ i<n. coeff p i * of-int (a ^ i * b ^ (n − i − 1))) ∈ ℤ
```

```
    using p by auto
  then obtain R where R: of-int R = (∑ i<n. coeff p i * of-int (a ^ i * b ^ (n
- i - 1)))
    by (auto simp: Ints-def)
  have [simp]: coeff p n = 1
    using p by (auto simp: n-def)

  have 0 = poly p x * of-int b ^ n
    using p by simp
  also have ... = (∑ i≤n. coeff p i * x ^ i * of-int b ^ n)
    by (simp add: poly-altdef n-def sum-distrib-right)
  also have ... = (∑ i≤n. coeff p i * of-int (a ^ i * b ^ (n - i)))
    by (intro sum.cong) (auto simp: x-eq field-simps simp flip: power-add)
  also have {..n} = insert n {..<n}
    using ‹n > 0› by auto
  also have (∑ i∈.... coeff p i * of-int (a ^ i * b ^ (n - i))) =
              coeff p n * of-int (a ^ n) + (∑ i<n. coeff p i * of-int (a ^ i * b ^ (n
- i)))
    by (subst sum.insert) auto
  also have (∑ i<n. coeff p i * of-int (a ^ i * b ^ (n - i))) =
            (∑ i<n. coeff p i * of-int (a ^ i * b * b ^ (n - i - 1)))
    by (intro sum.cong) (auto simp flip: power-add power-Suc simp: Suc-diff-Suc)
  also have ... = of-int (b * R)
    by (simp add: R sum-distrib-left sum-distrib-right mult-ac)
  finally have of-int (a ^ n) = (-of-int (b * R) :: 'a)
    by (auto simp: add-eq-0-iff)
  hence a ^ n = -b * R
    by (simp flip: of-int-mult of-int-power of-int-minus)
  hence b dvd a ^ n
    by simp
  with ‹Rings.coprime a b› have b dvd 1
    by (meson coprime-power-left-iff dvd-refl not-coprimeI)
  with x-eq and ‹b > 0› show ?thesis
    by auto
qed

lemma algebraic-int-imp-algebraic [dest]: algebraic-int x ⟹ algebraic x
  by (auto simp: algebraic-int.simps algebraic-def)

lemma int-imp-algebraic-int:
  assumes x ∈ ℤ
  shows   algebraic-int x
proof
  show ∀ i. coeff [:-x, 1:] i ∈ ℤ
    using assms by (auto simp: coeff-pCons split: nat.splits)
qed auto

lemma algebraic-int-0 [simp, intro]: algebraic-int 0
  and algebraic-int-1 [simp, intro]: algebraic-int 1
```

and *algebraic-int-numeral* [*simp*, *intro*]: *algebraic-int* (*numeral n*)
and *algebraic-int-of-nat* [*simp*, *intro*]: *algebraic-int* (*of-nat k*)
and *algebraic-int-of-int* [*simp*, *intro*]: *algebraic-int* (*of-int m*)
by (*simp-all add*: *int-imp-algebraic-int*)

lemma *algebraic-int-ii* [*simp*, *intro*]: *algebraic-int* i
proof
show *poly* [:*1*, *0*, *1*:] i = *0*
by *simp*
qed (*auto simp*: *coeff-pCons split*: *nat.splits*)

lemma *algebraic-int-minus* [*intro*]:
assumes *algebraic-int x*
shows *algebraic-int* (−*x*)
proof −
from *assms* **obtain** *p* **where** *p*: *lead-coeff p* = *1* ∀ *i*. *coeff p i* ∈ ℤ *poly p x* = *0*
by (*auto simp*: *algebraic-int.simps*)
define *s* **where** *s* = (*if even* (*degree p*) *then 1 else* −*1* :: *'a*)

define *q* **where** *q* = *Polynomial.smult s* (*pcompose p* [:*0*, −*1*:])
have *lead-coeff q* = *s* ∗ *lead-coeff* (*pcompose p* [:*0*, −*1*:])
by (*simp add*: *q-def*)
also have *lead-coeff* (*pcompose p* [:*0*, −*1*:]) = *lead-coeff p* ∗ (− *1*) ^ *degree p*
by (*subst lead-coeff-comp*) *auto*
finally have *poly q* (−*x*) = *0* **and** *lead-coeff q* = *1*
using *p* **by** (*auto simp*: *q-def poly-pcompose s-def*)
moreover have *coeff q i* ∈ ℤ **for** *i*
proof −
have *coeff* (*pcompose p* [:*0*, −*1*:]) *i* ∈ ℤ
using *p* **by** (*intro coeff-pcompose-semiring-closed*) (*auto simp*: *coeff-pCons split*: *nat.splits*)
thus *?thesis* **by** (*simp add*: *q-def s-def*)
qed
ultimately show *?thesis*
by (*auto simp*: *algebraic-int.simps*)
qed

lemma *algebraic-int-minus-iff* [*simp*]:
algebraic-int (−*x*) ⟷ *algebraic-int* (*x* :: *'a* :: *field-char-0*)
using *algebraic-int-minus*[*of x*] *algebraic-int-minus*[*of* −*x*] **by** *auto*

lemma *algebraic-int-inverse* [*intro*]:
assumes *poly p x* = *0* **and** ∀ *i*. *coeff p i* ∈ ℤ **and** *coeff p 0* = *1*
shows *algebraic-int* (*inverse x*)
proof
from *assms* **have** [*simp*]: *x* ≠ *0*
by (*auto simp*: *poly-0-coeff-0*)
show *poly* (*reflect-poly p*) (*inverse x*) = *0*
using *assms* **by** (*simp add*: *poly-reflect-poly-nz*)

qed (*use assms* **in** ‹*auto simp*: *coeff-reflect-poly*›)

lemma *algebraic-int-root*:
assumes *algebraic-int y*
and *poly p x* = *y* **and** ∀ *i*. *coeff p i* ∈ ℤ **and** *lead-coeff p* = *1* **and** *degree p* > *0*
shows *algebraic-int x*
proof −
from *assms* **obtain** *q* **where** *q*: *poly q y* = *0* ∀ *i*. *coeff q i* ∈ ℤ *lead-coeff q* = *1*
by (*auto simp*: *algebraic-int.simps*)
show *?thesis*
proof
from *assms q* **show** *lead-coeff* (*pcompose q p*) = *1*
by (*subst lead-coeff-comp*) *auto*
from *assms q* **show** ∀ *i*. *coeff* (*pcompose q p*) *i* ∈ ℤ
by (*intro allI coeff-pcompose-semiring-closed*) *auto*
show *poly* (*pcompose q p*) *x* = *0*
using *assms q* **by** (*simp add*: *poly-pcompose*)
qed
qed

lemma *algebraic-int-abs-real* [*simp*]:
algebraic-int |*x* :: *real*| ⟷ *algebraic-int x*
by (*auto simp*: *abs-if*)

lemma *algebraic-int-nth-root-real* [*intro*]:
assumes *algebraic-int x*
shows *algebraic-int* (*root n x*)
proof (*cases n* = *0*)
case *False*
show *?thesis*
proof (*rule algebraic-int-root*)
show *poly* (*monom 1 n*) (*root n x*) = (*if even n then* |*x*| *else x*)
using *sgn-power-root*[*of n x*] *False*
by (*auto simp add*: *poly-monom sgn-if split*: *if-splits*)
qed (*use False assms* **in** ‹*auto simp*: *degree-monom-eq*›)
qed *auto*

lemma *algebraic-int-sqrt* [*intro*]: *algebraic-int x* ⟹ *algebraic-int* (*sqrt x*)
by (*auto simp*: *sqrt-def*)

lemma *algebraic-int-csqrt* [*intro*]: *algebraic-int x* ⟹ *algebraic-int* (*csqrt x*)
by (*rule algebraic-int-root*[**where** *p* = *monom 1 2*])
(*auto simp*: *poly-monom degree-monom-eq*)

lemma *algebraic-int-cnj* [*intro*]:
assumes *algebraic-int x*
shows *algebraic-int* (*cnj x*)
proof −

```
  from assms obtain p where p: lead-coeff p = 1 ∀ i. coeff p i ∈ ℤ poly p x = 0
    by (auto simp: algebraic-int.simps)
  show ?thesis
  proof
    show poly (map-poly cnj p) (cnj x) = 0
      using p by simp
    show lead-coeff (map-poly cnj p) = 1
      using p by (simp add: coeff-map-poly degree-map-poly)
    show ∀ i. coeff (map-poly cnj p) i ∈ ℤ
      using p by (auto simp: coeff-map-poly)
  qed
qed

lemma algebraic-int-cnj-iff [simp]: algebraic-int (cnj x) ⟷ algebraic-int x
  using algebraic-int-cnj[of x] algebraic-int-cnj[of cnj x] by auto

lemma algebraic-int-of-real [intro]:
  assumes algebraic-int x
  shows   algebraic-int (of-real x)
proof −
  from assms obtain p where p: poly p x = 0 ∀ i. coeff p i ∈ ℤ lead-coeff p = 1
    by (auto simp: algebraic-int.simps)
  show algebraic-int (of-real x :: 'a)
  proof
    have poly (map-poly of-real p) (of-real x) = (of-real (poly p x) :: 'a)
      by (induction p) (auto simp: map-poly-pCons)
    thus poly (map-poly of-real p) (of-real x) = (0 :: 'a)
      using p by simp
  qed (use p in ‹auto simp: coeff-map-poly degree-map-poly›)
qed

lemma algebraic-int-of-real-iff [simp]:
  algebraic-int (of-real x :: 'a :: {field-char-0, real-algebra-1}) ⟷ algebraic-int x
proof
  assume algebraic-int (of-real x :: 'a)
  then obtain p
    where p: poly (map-poly of-int p) (of-real x :: 'a) = 0 lead-coeff p = 1
    by (auto simp: algebraic-int-altdef-ipoly)
  show algebraic-int x
    unfolding algebraic-int-altdef-ipoly
  proof (intro exI[of - p] conjI)
    have of-real (poly (map-poly real-of-int p) x) = poly (map-poly of-int p) (of-real
x :: 'a)
      by (induction p) (auto simp: map-poly-pCons)
    also note p(1)
    finally show poly (map-poly real-of-int p) x = 0 by simp
  qed (use p in auto)
qed auto
```

4.30 Division of polynomials

4.30.1 Division in general

instantiation *poly* :: (*idom-divide*) *idom-divide*
begin

fun *divide-poly-main* :: *'a* ⇒ *'a poly* ⇒ *'a poly* ⇒ *'a poly* ⇒ *nat* ⇒ *nat* ⇒ *'a poly*
where
divide-poly-main lc q r d dr (*Suc n*) =
(*let cr* = *coeff r dr*; *a* = *cr div lc*; *mon* = *monom a n in*
if False ∨ *a* ∗ *lc* = *cr then* — *False* ∨ is only because of problem in function-package
divide-poly-main
lc
(*q* + *mon*)
(*r* − *mon* ∗ *d*)
d (*dr* − *1*) *n else 0*)
| *divide-poly-main lc q r d dr 0* = *q*

definition *divide-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly*
where *divide-poly f g* =
(*if g* = *0 then 0*
else
divide-poly-main (*coeff g* (*degree g*)) *0 f g* (*degree f*)
(*1* + *length* (*coeffs f*) − *length* (*coeffs g*)))

lemma *divide-poly-main*:
assumes *d*: *d* ≠ *0 lc* = *coeff d* (*degree d*)
and *degree* (*d* ∗ *r*) ≤ *dr divide-poly-main lc q* (*d* ∗ *r*) *d dr n* = *q'*
and *n* = *1* + *dr* − *degree d* ∨ *dr* = *0* ∧ *n* = *0* ∧ *d* ∗ *r* = *0*
shows *q'* = *q* + *r*
using *assms*(*3*−)
proof (*induct n arbitrary*: *q r dr*)
case (*Suc n*)
let *?rr* = *d* ∗ *r*
let *?a* = *coeff ?rr dr*
let *?qq* = *?a div lc*
define *b* **where** [*simp*]: *b* = *monom ?qq n*
let *?rrr* = *d* ∗ (*r* − *b*)
let *?qqq* = *q* + *b*
note *res* = *Suc*(*3*)
from *Suc*(*4*) **have** *dr*: *dr* = *n* + *degree d* **by** *auto*
from *d* **have** *lc*: *lc* ≠ *0* **by** *auto*
have *coeff* (*b* ∗ *d*) *dr* = *coeff b n* ∗ *coeff d* (*degree d*)
proof (*cases ?qq* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*

```
    then have n: n = degree b
      by (simp add: degree-monom-eq)
    show ?thesis
      unfolding n dr by (simp add: coeff-mult-degree-sum)
  qed
  also have ... = lc * coeff b n
    by (simp add: d)
  finally have c2: coeff (b * d) dr = lc * coeff b n .
  have rrr: ?rrr = ?rr - b * d
    by (simp add: field-simps)
  have c1: coeff (d * r) dr = lc * coeff r n
  proof (cases degree r = n)
    case True
    with Suc(2) show ?thesis
      unfolding dr using coeff-mult-degree-sum[of d r] d by (auto simp: ac-simps)
  next
    case False
    from dr Suc(2) have degree r ≤ n
      by auto
        (metis add.commute add-le-cancel-left d(1) degree-0 degree-mult-eq
          diff-is-0-eq diff-zero le-cases)
    with False have r-n: degree r < n
      by auto
    then have right: lc * coeff r n = 0
      by (simp add: coeff-eq-0)
    have coeff (d * r) dr = coeff (d * r) (degree d + n)
      by (simp add: dr ac-simps)
    also from r-n have ... = 0
      by (metis False Suc.prems(1) add.commute add-left-imp-eq coeff-degree-mult
coeff-eq-0
        coeff-mult-degree-sum degree-mult-le dr le-eq-less-or-eq)
    finally show ?thesis
      by (simp only: right)
  qed
  have c0: coeff ?rrr dr = 0
    and id: lc * (coeff (d * r) dr div lc) = coeff (d * r) dr
    unfolding rrr coeff-diff c2
    unfolding b-def coeff-monom coeff-smult c1 using lc by auto
  from res[unfolded divide-poly-main.simps[of lc q] Let-def] id
  have res: divide-poly-main lc ?qqq ?rrr d (dr - 1) n = q'
    by (simp del: divide-poly-main.simps add: field-simps)
  note IH = Suc(1)[OF - res]
  from Suc(4) have dr: dr = n + degree d by auto
  from Suc(2) have deg-rr: degree ?rr ≤ dr by auto
  have deg-bd: degree (b * d) ≤ dr
    unfolding dr b-def by (rule order.trans[OF degree-mult-le]) (auto simp: de-
gree-monom-le)
  have degree ?rrr ≤ dr
    unfolding rrr by (rule degree-diff-le[OF deg-rr deg-bd])
```

```
  with c0 have deg-rrr: degree ?rrr ≤ (dr − 1)
    by (rule coeff-0-degree-minus-1)
  have n = 1 + (dr − 1) − degree d ∨ dr − 1 = 0 ∧ n = 0 ∧ ?rrr = 0
  proof (cases dr)
    case 0
    with Suc(4) have 0: dr = 0 n = 0 degree d = 0
      by auto
    with deg-rrr have degree ?rrr = 0
      by simp
    from degree-eq-zeroE[OF this] obtain a where rrr: ?rrr = [:a:]
      by metis
    show ?thesis
      unfolding 0 using c0 unfolding rrr 0 by simp
  next
    case -: Suc
    with Suc(4) show ?thesis by auto
  qed
  from IH[OF deg-rrr this] show ?case
    by simp
next
  case 0
  show ?case
  proof (cases r = 0)
    case True
    with 0 show ?thesis by auto
  next
    case False
    from d False have degree (d * r) = degree d + degree r
      by (subst degree-mult-eq) auto
    with 0 d show ?thesis by auto
  qed
qed

lemma divide-poly-main-0: divide-poly-main 0 0 r d dr n = 0
proof (induct n arbitrary: r d dr)
  case 0
  then show ?case by simp
next
  case Suc
  show ?case
    unfolding divide-poly-main.simps[of - - r] Let-def
    by (simp add: Suc del: divide-poly-main.simps)
qed

lemma divide-poly:
  assumes g: g ≠ 0
  shows (f * g) div g = (f :: 'a poly)
proof −
  have len: length (coeffs f) = Suc (degree f) if f ≠ 0 for f :: 'a poly
```

```
    using that unfolding degree-eq-length-coeffs by auto
  have divide-poly-main (coeff g (degree g)) 0 (g * f) g (degree (g * f))
    (1 + length (coeffs (g * f)) - length (coeffs  g)) = (f * g) div g
    by (simp add: divide-poly-def Let-def ac-simps)
  note main = divide-poly-main[OF g refl le-refl this]
  have (f * g) div g = 0 + f
  proof (rule main, goal-cases)
    case 1
    show ?case
    proof (cases f = 0)
      case True
      with g show ?thesis
        by (auto simp: degree-eq-length-coeffs)
    next
      case False
      with g have fg: g * f ≠ 0 by auto
      show ?thesis
        unfolding len[OF fg] len[OF g] by auto
    qed
  qed
  then show ?thesis by simp
qed

lemma divide-poly-0: f div 0 = 0
  for f :: 'a poly
  by (simp add: divide-poly-def Let-def divide-poly-main-0)

instance
  by standard (auto simp: divide-poly divide-poly-0)

end

instance poly :: (idom-divide) algebraic-semidom ..

lemma div-const-poly-conv-map-poly:
  assumes [:c:] dvd p
  shows p div [:c:] = map-poly (λx. x div c) p
proof (cases c = 0)
  case True
  then show ?thesis
    by (auto intro!: poly-eqI simp: coeff-map-poly)
next
  case False
  from assms obtain q where p: p = [:c:] * q by (rule dvdE)
  moreover {
    have smult c q = [:c:] * q
      by simp
    also have ... div [:c:] = q
      by (rule nonzero-mult-div-cancel-left) (use False in auto)
```

```
    finally have smult c q div [:c:] = q .
  }
  ultimately show ?thesis by (intro poly-eqI) (auto simp: coeff-map-poly False)
qed

lemma is-unit-monom-0:
  fixes a :: 'a::field
  assumes a ≠ 0
  shows is-unit (monom a 0)
proof
  from assms show 1 = monom a 0 * monom (inverse a) 0
    by (simp add: mult-monom)
qed

lemma is-unit-triv: a ≠ 0 ⟹ is-unit [:a:]
  for a :: 'a::field
  by (simp add: is-unit-monom-0 monom-0 [symmetric])

lemma is-unit-iff-degree:
  fixes p :: 'a::field poly
  assumes p ≠ 0
  shows is-unit p ⟷ degree p = 0
    (is ?lhs ⟷ ?rhs)
proof
  assume ?rhs
  then obtain a where p = [:a:]
    by (rule degree-eq-zeroE)
  with assms show ?lhs
    by (simp add: is-unit-triv)
next
  assume ?lhs
  then obtain q where q ≠ 0 p * q = 1 ..
  then have degree (p * q) = degree 1
    by simp
  with ‹p ≠ 0› ‹q ≠ 0› have degree p + degree q = 0
    by (simp add: degree-mult-eq)
  then show ?rhs by simp
qed

lemma is-unit-pCons-iff: is-unit (pCons a p) ⟷ p = 0 ∧ a ≠ 0
  for p :: 'a::field poly
  by (cases p = 0) (auto simp: is-unit-triv is-unit-iff-degree)

lemma is-unit-monom-trivial: is-unit p ⟹ monom (coeff p (degree p)) 0 = p
  for p :: 'a::field poly
  by (cases p) (simp-all add: monom-0 is-unit-pCons-iff)

lemma is-unit-const-poly-iff: [:c:] dvd 1 ⟷ c dvd 1
  for c :: 'a::{comm-semiring-1,semiring-no-zero-divisors}
```

by (*auto simp*: *one-pCons*)

lemma *is-unit-polyE*:
fixes $p :: 'a :: \{comm\text{-}semiring\text{-}1, semiring\text{-}no\text{-}zero\text{-}divisors\}$ *poly*
assumes *p dvd 1*
obtains *c* **where** $p = [:c:]$ *c dvd 1*
proof −
from *assms* **obtain** *q* **where** $1 = p * q$
by (*rule dvdE*)
then have $p \neq 0$ **and** $q \neq 0$
by *auto*
from ‹$1 = p * q$› **have** *degree 1* = *degree* $(p * q)$
by *simp*
also from ‹$p \neq 0$› **and** ‹$q \neq 0$› **have** $\ldots$ = *degree p* + *degree q*
by (*simp add*: *degree-mult-eq*)
finally have *degree p* = *0* **by** *simp*
with *degree-eq-zeroE* **obtain** *c* **where** *c*: $p = [:c:]$ **.**
with ‹*p dvd 1*› **have** *c dvd 1*
by (*simp add*: *is-unit-const-poly-iff*)
with *c* **show** *thesis* **..**
qed

lemma *is-unit-polyE′*:
fixes $p :: 'a::field$ *poly*
assumes *is-unit p*
obtains *a* **where** *p* = *monom a 0* **and** $a \neq 0$
proof −
obtain *a q* **where** *p* = *pCons a q*
by (*cases p*)
with *assms* **have** $p = [:a:]$ **and** $a \neq 0$
by (*simp-all add*: *is-unit-pCons-iff*)
with *that* **show** *thesis* **by** (*simp add*: *monom-0*)
qed

lemma *is-unit-poly-iff*: *p dvd 1* $\longleftrightarrow$ $(\exists\, c.\ p = [:c:] \wedge c\ dvd\ 1)$
for $p :: 'a::\{comm\text{-}semiring\text{-}1, semiring\text{-}no\text{-}zero\text{-}divisors\}$ *poly*
by (*auto elim*: *is-unit-polyE simp add*: *is-unit-const-poly-iff*)

lemma *coprime-poly-0*:
poly p x $\neq$ *0* $\vee$ *poly q x* $\neq$ *0* **if** *coprime p q*
for $x :: 'a :: field$
proof (*rule ccontr*)
assume $\neg$ (*poly p x* $\neq$ *0* $\vee$ *poly q x* $\neq$ *0*)
then have $[:-x,\ 1:]$ *dvd p* $[:-x,\ 1:]$ *dvd q*
by (*simp-all add*: *poly-eq-0-iff-dvd*)
with *that* **have** *is-unit* $[:-x,\ 1:]$
by (*rule coprime-common-divisor*)
then show *False*
by (*auto simp add*: *is-unit-pCons-iff*)

qed

lemma *root-imp-reducible-poly*:
fixes $x :: 'a :: field$
assumes *poly p x = 0* **and** *degree p > 1*
shows ¬*irreducible p*
proof −
from *assms* **have** $p \neq 0$
by *auto*
define *q* **where** $q = [:-x, 1:]$
have *q dvd p*
using *assms* **by** (*simp add*: *poly-eq-0-iff-dvd q-def*)
then obtain *r* **where** *p-eq*: $p = q * r$
by (*elim dvdE*)
have [*simp*]: $q \neq 0$ $r \neq 0$
using ‹$p \neq 0$› **by** (*auto simp*: *p-eq*)
have *degree p = Suc* (*degree r*)
unfolding *p-eq* **by** (*subst degree-mult-eq*) (*auto simp*: *q-def*)
with *assms*(*2*) **have** *degree r > 0*
by *auto*
hence ¬*r dvd 1*
by (*auto simp*: *is-unit-poly-iff*)
moreover have ¬*q dvd 1*
by (*auto simp*: *is-unit-poly-iff q-def*)
ultimately show *?thesis* **using** *p-eq*
by (*auto simp*: *irreducible-def*)
qed

lemma *reducible-polyI*:
fixes $p :: 'a :: field\ poly$
assumes $p = q * r$ *degree q > 0 degree r > 0*
shows ¬*irreducible p*
using *assms* **unfolding** *irreducible-def*
by (*metis* (*no-types*, *opaque-lifting*) *is-unitE is-unit-iff-degree not-gr0*)

4.30.2 Pseudo-Division

This part is by René Thiemann and Akihisa Yamada.

fun *pseudo-divmod-main* ::
$'a :: comm\text{-}ring\text{-}1 \Rightarrow 'a\ poly \Rightarrow 'a\ poly \Rightarrow 'a\ poly \Rightarrow nat \Rightarrow nat \Rightarrow 'a\ poly \times 'a\ poly$
where
pseudo-divmod-main lc q r d dr (*Suc n*) =
(*let*
rr = *smult lc r*;
qq = *coeff r dr*;
rrr = *rr* − *monom qq n* ∗ *d*;
qqq = *smult lc q* + *monom qq n*
in pseudo-divmod-main lc qqq rrr d (*dr* − *1*) *n*)

```
| pseudo-divmod-main lc q r d dr 0 = (q,r)

definition pseudo-divmod :: 'a :: comm-ring-1 poly ⇒ 'a poly ⇒ 'a poly × 'a poly
  where pseudo-divmod p q ≡
    if q = 0 then (0, p)
    else
      pseudo-divmod-main (coeff q (degree q)) 0 p q (degree p)
        (1 + length (coeffs p) − length (coeffs q))

lemma pseudo-divmod-main:
  assumes d: d ≠ 0 lc = coeff d (degree d)
    and degree r ≤ dr pseudo-divmod-main lc q r d dr n = (q',r')
    and n = 1 + dr − degree d ∨ dr = 0 ∧ n = 0 ∧ r = 0
  shows (r' = 0 ∨ degree r' < degree d) ∧ smult (lc^n) (d * q + r) = d * q' + r'
  using assms(3−)
proof (induct n arbitrary: q r dr)
  case 0
  then show ?case by auto
next
  case (Suc n)
  let ?rr = smult lc r
  let ?qq = coeff r dr
  define b where [simp]: b = monom ?qq n
  let ?rrr = ?rr − b * d
  let ?qqq = smult lc q + b
  note res = Suc(3)
  from res[unfolded pseudo-divmod-main.simps[of lc q] Let-def]
  have res: pseudo-divmod-main lc ?qqq ?rrr d (dr − 1) n = (q',r')
    by (simp del: pseudo-divmod-main.simps)
  from Suc(4) have dr: dr = n + degree d by auto
  have coeff (b * d) dr = coeff b n * coeff d (degree d)
  proof (cases ?qq = 0)
    case True
    then show ?thesis by auto
  next
    case False
    then have n: n = degree b
      by (simp add: degree-monom-eq)
    show ?thesis
      unfolding n dr by (simp add: coeff-mult-degree-sum)
  qed
  also have ... = lc * coeff b n by (simp add: d)
  finally have coeff (b * d) dr = lc * coeff b n .
  moreover have coeff ?rr dr = lc * coeff r dr
    by simp
  ultimately have c0: coeff ?rrr dr = 0
    by auto
  from Suc(4) have dr: dr = n + degree d by auto
  have deg-rr: degree ?rr ≤ dr
```

```
    using Suc(2) degree-smult-le dual-order.trans by blast
  have deg-bd: degree (b * d) ≤ dr
   unfolding dr by (rule order.trans[OF degree-mult-le]) (auto simp: degree-monom-le)
  have degree ?rrr ≤ dr
    using degree-diff-le[OF deg-rr deg-bd] by auto
  with c0 have deg-rrr: degree ?rrr ≤ (dr - 1)
    by (rule coeff-0-degree-minus-1)
  have n = 1 + (dr - 1) - degree d ∨ dr - 1 = 0 ∧ n = 0 ∧ ?rrr = 0
  proof (cases dr)
    case 0
    with Suc(4) have 0: dr = 0 n = 0 degree d = 0 by auto
    with deg-rrr have degree ?rrr = 0 by simp
    then have ∃ a. ?rrr = [:a:]
      by (metis degree-pCons-eq-if old.nat.distinct(2) pCons-cases)
    from this obtain a where rrr: ?rrr = [:a:]
      by auto
    show ?thesis
      unfolding 0 using c0 unfolding rrr 0 by simp
  next
    case -: Suc
    with Suc(4) show ?thesis by auto
  qed
  note IH = Suc(1)[OF deg-rrr res this]
  show ?case
  proof (intro conjI)
    from IH show r' = 0 ∨ degree r' < degree d
      by blast
    show smult (lc ^ Suc n) (d * q + r) = d * q' + r'
      unfolding IH[THEN conjunct2,symmetric]
      by (simp add: field-simps smult-add-right)
  qed
qed

lemma pseudo-divmod:
  assumes g: g ≠ 0
    and *: pseudo-divmod f g = (q,r)
  shows smult (coeff g (degree g) ^ (Suc (degree f) - degree g)) f = g * q + r  (is
?A)
    and r = 0 ∨ degree r < degree g  (is ?B)
proof -
  from *[unfolded pseudo-divmod-def Let-def]
  have pseudo-divmod-main (coeff g (degree g)) 0 f g (degree f)
      (1 + length (coeffs f) - length (coeffs g)) = (q, r)
    by (auto simp: g)
  note main = pseudo-divmod-main[OF - - - this, OF g refl le-refl]
  from g have 1 + length (coeffs f) - length (coeffs g) = 1 + degree f - degree
g ∨
    degree f = 0 ∧ 1 + length (coeffs f) - length (coeffs g) = 0 ∧ f = 0
    by (cases f = 0; cases coeffs g) (auto simp: degree-eq-length-coeffs)
```

note *main'* = *main*[*OF this*]
then show *r* = *0* ∨ *degree r* < *degree g* **by** *auto*
show *smult* (*coeff g* (*degree g*) ^ (*Suc* (*degree f*) − *degree g*)) *f* = *g* ∗ *q* + *r*
by (*subst main'*[*THEN conjunct2*, *symmetric*], *simp add*: *degree-eq-length-coeffs*,
cases f = *0*; *cases coeffs g*, *use g* **in** *auto*)
qed

definition *pseudo-mod-main lc r d dr n* = *snd* (*pseudo-divmod-main lc 0 r d dr n*)

lemma *snd-pseudo-divmod-main*:
snd (*pseudo-divmod-main lc q r d dr n*) = *snd* (*pseudo-divmod-main lc q' r d dr n*)
by (*induct n arbitrary*: *q q' lc r d dr*) (*simp-all add*: *Let-def*)

definition *pseudo-mod* :: *'a*::{*comm-ring-1*,*semiring-1-no-zero-divisors*} *poly* ⇒ *'a poly* ⇒ *'a poly*
where *pseudo-mod f g* = *snd* (*pseudo-divmod f g*)

lemma *pseudo-mod*:
fixes *f g* :: *'a*::{*comm-ring-1*,*semiring-1-no-zero-divisors*} *poly*
defines *r* ≡ *pseudo-mod f g*
assumes *g*: *g* ≠ *0*
shows ∃ *a q*. *a* ≠ *0* ∧ *smult a f* = *g* ∗ *q* + *r r* = *0* ∨ *degree r* < *degree g*
proof −
let *?cg* = *coeff g* (*degree g*)
let *?cge* = *?cg* ^ (*Suc* (*degree f*) − *degree g*)
define *a* **where** *a* = *?cge*
from *r-def*[*unfolded pseudo-mod-def*] **obtain** *q* **where** *pdm*: *pseudo-divmod f g* = (*q*, *r*)
by (*cases pseudo-divmod f g*) *auto*
from *pseudo-divmod*[*OF g pdm*] **have** *id*: *smult a f* = *g* ∗ *q* + *r* **and** *r* = *0* ∨ *degree r* < *degree g*
by (*auto simp*: *a-def*)
show *r* = *0* ∨ *degree r* < *degree g* **by** *fact*
from *g* **have** *a* ≠ *0*
by (*auto simp*: *a-def*)
with *id* **show** ∃ *a q*. *a* ≠ *0* ∧ *smult a f* = *g* ∗ *q* + *r*
by *auto*
qed

lemma *fst-pseudo-divmod-main-as-divide-poly-main*:
assumes *d*: *d* ≠ *0*
defines *lc*: *lc* ≡ *coeff d* (*degree d*)
shows *fst* (*pseudo-divmod-main lc q r d dr n*) =
divide-poly-main lc (*smult* (*lc*^*n*) *q*) (*smult* (*lc*^*n*) *r*) *d dr n*
proof (*induct n arbitrary*: *q r dr*)
case *0*
then show *?case* **by** *simp*

next
case (*Suc n*)
note *lc0* = *leading-coeff-neq-0*[*OF d*, *folded lc*]
then have *pseudo-divmod-main lc q r d dr* (*Suc n*) =
pseudo-divmod-main lc (*smult lc q* + *monom* (*coeff r dr*) *n*)
(*smult lc r* − *monom* (*coeff r dr*) *n* ∗ *d*) *d* (*dr* − *1*) *n*
by (*simp add*: *Let-def ac-simps*)
also have *fst* ... = *divide-poly-main lc*
(*smult* (*lc*^*n*) (*smult lc q* + *monom* (*coeff r dr*) *n*))
(*smult* (*lc*^*n*) (*smult lc r* − *monom* (*coeff r dr*) *n* ∗ *d*))
d (*dr* − *1*) *n*
by (*simp only*: *Suc*[*unfolded divide-poly-main.simps Let-def*])
also have ... = *divide-poly-main lc* (*smult* (*lc* ^ *Suc n*) *q*) (*smult* (*lc* ^ *Suc n*) *r*) *d dr* (*Suc n*)
unfolding *smult-monom smult-distribs mult-smult-left*[*symmetric*]
using *lc0* **by** (*simp add*: *Let-def ac-simps*)
finally show *?case* **.**
qed

4.30.3 Division in polynomials over fields

lemma *pseudo-divmod-field*:
fixes *g* :: *'a*::*field poly*
assumes *g*: *g* ≠ *0*
and ∗: *pseudo-divmod f g* = (*q*,*r*)
defines *c* ≡ *coeff g* (*degree g*) ^(*Suc* (*degree f*) − *degree g*)
shows *f* = *g* ∗ *smult* (*1*/*c*) *q* + *smult* (*1*/*c*) *r*
proof −
from *leading-coeff-neq-0*[*OF g*] **have** *c0*: *c* ≠ *0*
by (*auto simp*: *c-def*)
from *pseudo-divmod*(*1*)[*OF g* ∗, *folded c-def*] **have** *smult c f* = *g* ∗ *q* + *r*
by *auto*
also have *smult* (*1* / *c*) ... = *g* ∗ *smult* (*1* / *c*) *q* + *smult* (*1* / *c*) *r*
by (*simp add*: *smult-add-right*)
finally show *?thesis*
using *c0* **by** *auto*
qed

lemma *divide-poly-main-field*:
fixes *d* :: *'a*::*field poly*
assumes *d*: *d* ≠ *0*
defines *lc*: *lc* ≡ *coeff d* (*degree d*)
shows *divide-poly-main lc q r d dr n* =
fst (*pseudo-divmod-main lc* (*smult* ((*1* / *lc*)^*n*) *q*) (*smult* ((*1* / *lc*)^*n*) *r*) *d dr* *n*)
unfolding *lc* **by** (*subst fst-pseudo-divmod-main-as-divide-poly-main*) (*auto simp*: *d power-one-over*)

lemma *divide-poly-field*:

```
  fixes f g :: 'a::field poly
  defines f' ≡ smult ((1 / coeff g (degree g)) ^ (Suc (degree f) - degree g)) f
  shows f div g = fst (pseudo-divmod f' g)
proof (cases g = 0)
  case True
  show ?thesis
    unfolding divide-poly-def pseudo-divmod-def Let-def f'-def True
    by (simp add: divide-poly-main-0)
next
  case False
  from leading-coeff-neq-0[OF False] have degree f' = degree f
    by (auto simp: f'-def)
  then show ?thesis
    using length-coeffs-degree[of f'] length-coeffs-degree[of f]
    unfolding divide-poly-def pseudo-divmod-def Let-def
      divide-poly-main-field[OF False]
      length-coeffs-degree[OF False]
      f'-def
    by force
qed

instantiation poly :: ({semidom-divide-unit-factor,idom-divide}) normalization-semidom
begin

definition unit-factor-poly :: 'a poly ⇒ 'a poly
  where unit-factor-poly p = [:unit-factor (lead-coeff p):]

definition normalize-poly :: 'a poly ⇒ 'a poly
  where normalize p = p div [:unit-factor (lead-coeff p):]

instance
proof
  fix p :: 'a poly
  show unit-factor p * normalize p = p
  proof (cases p = 0)
    case True
    then show ?thesis
      by (simp add: unit-factor-poly-def normalize-poly-def)
  next
    case False
    then have lead-coeff p ≠ 0
      by simp
    then have *: unit-factor (lead-coeff p) ≠ 0
      using unit-factor-is-unit [of lead-coeff p] by auto
    then have unit-factor (lead-coeff p) dvd 1
      by (auto intro: unit-factor-is-unit)
    then have **: unit-factor (lead-coeff p) dvd c for c
      by (rule dvd-trans) simp
    have ***: unit-factor (lead-coeff p) * (c div unit-factor (lead-coeff p)) = c for
```

```
c
    proof −
      from ∗∗ obtain b where c = unit-factor (lead-coeff p) ∗ b ..
      with False ∗ show ?thesis by simp
    qed
    have p div [:unit-factor (lead-coeff p):] =
      map-poly (λc. c div unit-factor (lead-coeff p)) p
      by (simp add: const-poly-dvd-iff div-const-poly-conv-map-poly ∗∗)
    then show ?thesis
      by (simp add: normalize-poly-def unit-factor-poly-def
        smult-conv-map-poly map-poly-map-poly o-def ∗∗∗)
  qed
next
  fix p :: 'a poly
  assume is-unit p
  then obtain c where p: p = [:c:] c dvd 1
    by (auto simp: is-unit-poly-iff)
  then show unit-factor p = p
    by (simp add: unit-factor-poly-def monom-0 is-unit-unit-factor)
next
  fix p :: 'a poly
  assume p ≠ 0
  then show is-unit (unit-factor p)
    by (simp add: unit-factor-poly-def monom-0 is-unit-poly-iff unit-factor-is-unit)
next
  fix a b :: 'a poly assume is-unit a
  thus unit-factor (a ∗ b) = a ∗ unit-factor b
   by (auto simp: unit-factor-poly-def lead-coeff-mult unit-factor-mult elim!: is-unit-polyE)
qed (simp-all add: normalize-poly-def unit-factor-poly-def monom-0 lead-coeff-mult
unit-factor-mult)

end

instance poly :: ({semidom-divide-unit-factor,idom-divide,normalization-semidom-multiplicative})
  normalization-semidom-multiplicative
  by intro-classes (auto simp: unit-factor-poly-def lead-coeff-mult unit-factor-mult)

lemma normalize-poly-eq-map-poly: normalize p = map-poly (λx. x div unit-factor
(lead-coeff p)) p
proof −
  have [:unit-factor (lead-coeff p):] dvd p
    by (metis unit-factor-poly-def unit-factor-self)
  then show ?thesis
    by (simp add: normalize-poly-def div-const-poly-conv-map-poly)
qed

lemma coeff-normalize [simp]:
  coeff (normalize p) n = coeff p n div unit-factor (lead-coeff p)
  by (simp add: normalize-poly-eq-map-poly coeff-map-poly)
```

class *field-unit-factor* = *field* + *unit-factor* +
  assumes *unit-factor-field* [*simp*]: *unit-factor* = *id*
begin

subclass *semidom-divide-unit-factor*
proof
  fix *a*
  assume *a* ≠ *0*
  then have *1* = *a* ∗ *inverse a* **by** *simp*
  then have *a dvd 1* **..**
  then show *unit-factor a dvd 1* **by** *simp*
qed *simp-all*

end

lemma *unit-factor-pCons*:
  unit-factor (*pCons a p*) = (*if p* = *0 then* [:*unit-factor a*:] *else unit-factor p*)
  by (*simp add*: *unit-factor-poly-def*)

lemma *normalize-monom* [*simp*]: *normalize* (*monom a n*) = *monom* (*normalize a*) *n*
  by (*cases a* = *0*) (*simp-all add*: *map-poly-monom normalize-poly-eq-map-poly degree-monom-eq*)

lemma *unit-factor-monom* [*simp*]: *unit-factor* (*monom a n*) = [:*unit-factor a*:]
  by (*cases a* = *0*) (*simp-all add*: *unit-factor-poly-def degree-monom-eq*)

lemma *normalize-const-poly*: *normalize* [:*c*:] = [:*normalize c*:]
  by (*simp add*: *normalize-poly-eq-map-poly map-poly-pCons*)

lemma *normalize-smult*:
  fixes *c* :: *'a* :: {*normalization-semidom-multiplicative*, *idom-divide*}
  shows *normalize* (*smult c p*) = *smult* (*normalize c*) (*normalize p*)
proof −
  have *smult c p* = [:*c*:] ∗ *p* **by** *simp*
  also have *normalize ...* = *smult* (*normalize c*) (*normalize p*)
    by (*subst normalize-mult*) (*simp add*: *normalize-const-poly*)
  finally show *?thesis* **.**
qed

instantiation *poly* :: (*field*) *idom-modulo*
begin

definition *modulo-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly*
  where *mod-poly-def*: *f mod g* =
    (*if g* = *0 then f else pseudo-mod* (*smult* ((*1* / *lead-coeff g*) ^(*Suc* (*degree f*) − *degree g*)) *f*) *g*)

```
instance
proof
  fix x y :: 'a poly
  show x div y * y + x mod y = x
  proof (cases y = 0)
    case True
    then show ?thesis
      by (simp add: divide-poly-0 mod-poly-def)
  next
    case False
    then have pseudo-divmod (smult ((1 / lead-coeff y) ^(Suc (degree x) - degree
y)) x) y =
        (x div y, x mod y)
      by (simp add: divide-poly-field mod-poly-def pseudo-mod-def)
    with False pseudo-divmod [OF False this] show ?thesis
      by (simp add: power-mult-distrib [symmetric] ac-simps)
  qed
qed

end

lemma pseudo-divmod-eq-div-mod:
  ‹pseudo-divmod f g = (f div g, f mod g)› if ‹lead-coeff g = 1›
  using that by (auto simp add: divide-poly-field mod-poly-def pseudo-mod-def)

lemma degree-mod-less-degree:
  ‹degree (x mod y) < degree y› if ‹y ≠ 0› ‹¬ y dvd x›
proof -
  from pseudo-mod(2) [of y] ‹y ≠ 0›
  have *: ‹pseudo-mod f y ≠ 0 ⟹ degree (pseudo-mod f y) < degree y› for f
    by blast
  from ‹¬ y dvd x› have ‹x mod y ≠ 0›
    by blast
  with ‹y ≠ 0› show ?thesis
    by (auto simp add: mod-poly-def intro: *)
qed

instantiation poly :: (field) unique-euclidean-ring
begin

definition euclidean-size-poly :: 'a poly ⇒ nat
  where euclidean-size-poly p = (if p = 0 then 0 else 2 ^ degree p)

definition division-segment-poly :: 'a poly ⇒ 'a poly
  where [simp]: division-segment-poly p = 1

instance proof
  show ‹(q * p + r) div p = q› if ‹p ≠ 0›
    and ‹euclidean-size r < euclidean-size p› for q p r :: ‹'a poly›
```

```
  proof (cases ‹r = 0›)
    case True
    with that show ?thesis
      by simp
  next
    case False
    with ‹p ≠ 0› ‹euclidean-size r < euclidean-size p›
    have ‹degree r < degree p›
      by (simp add: euclidean-size-poly-def)
    with ‹r ≠ 0› have ‹¬ p dvd r›
      by (auto dest: dvd-imp-degree)
    have ‹(q * p + r) div p = q ∧ (q * p + r) mod p = r›
    proof (rule ccontr)
      assume ‹¬ ?thesis›
      moreover have *: ‹((q * p + r) div p - q) * p = r - (q * p + r) mod p›
        by (simp add: algebra-simps)
      ultimately have ‹(q * p + r) div p ≠ q› and ‹(q * p + r) mod p ≠ r›
        using ‹p ≠ 0› by auto
      from ‹¬ p dvd r› have ‹¬ p dvd (q * p + r)›
        by simp
      with ‹p ≠ 0› have ‹degree ((q * p + r) mod p) < degree p›
        by (rule degree-mod-less-degree)
      with ‹degree r < degree p› ‹(q * p + r) mod p ≠ r›
      have ‹degree (r - (q * p + r) mod p) < degree p›
        by (auto intro: degree-diff-less)
      also have ‹degree p ≤ degree ((q * p + r) div p - q) + degree p›
        by simp
      also from ‹(q * p + r) div p ≠ q› ‹p ≠ 0›
      have ‹... = degree (((q * p + r) div p - q) * p)›
        by (simp add: degree-mult-eq)
      also from * have ‹... = degree (r - (q * p + r) mod p)›
        by simp
      finally have ‹degree (r - (q * p + r) mod p) < degree (r - (q * p + r) mod
p)› .
      then show False
        by simp
    qed
    then show ‹(q * p + r) div p = q› ..
  qed
qed (auto simp: euclidean-size-poly-def degree-mult-eq power-add intro: degree-mod-less-degree)

end

lemma euclidean-relation-polyI [case-names by0 divides euclidean-relation]:
  ‹(x div y, x mod y) = (q, r)›
    if by0: ‹y = 0 ⟹ q = 0 ∧ r = x›
    and divides: ‹y ≠ 0 ⟹ y dvd x ⟹ r = 0 ∧ x = q * y›
    and euclidean-relation: ‹y ≠ 0 ⟹ ¬ y dvd x ⟹ degree r < degree y ∧ x = q
* y + r›
```

```
  by (rule euclidean-relationI)
    (use that in ‹simp-all add: euclidean-size-poly-def›)

lemma div-poly-eq-0-iff:
  ‹x div y = 0 ⟷ x = 0 ∨ y = 0 ∨ degree x < degree y› for x y :: ‹'a::field poly›
 by (simp add: unique-euclidean-semiring-class.div-eq-0-iff euclidean-size-poly-def)

lemma div-poly-less:
  ‹x div y = 0› if ‹degree x < degree y› for x y :: ‹'a::field poly›
  using that by (simp add: div-poly-eq-0-iff)

lemma mod-poly-less:
  ‹x mod y = x› if ‹degree x < degree y›
  using that by (simp add: mod-eq-self-iff-div-eq-0 div-poly-eq-0-iff)

lemma degree-div-less:
  ‹degree (x div y) < degree x›
    if ‹degree x > 0› ‹degree y > 0›
    for x y :: ‹'a::field poly›
proof (cases ‹x div y = 0›)
  case True
  with ‹degree x > 0› show ?thesis
    by simp
next
  case False
  from that have ‹x ≠ 0› ‹y ≠ 0›
    and *: ‹degree (x div y * y + x mod y) > 0›
    by auto
  show ?thesis
  proof (cases ‹y dvd x›)
    case True
    then obtain z where ‹x = y * z› ..
    then have ‹degree (x div y) < degree (x div y * y)›
      using ‹y ≠ 0› ‹x ≠ 0› ‹degree y > 0› by (simp add: degree-mult-eq)
    with ‹y dvd x› show ?thesis
      by simp
  next
    case False
    with ‹y ≠ 0› have ‹degree (x mod y) < degree y›
      by (rule degree-mod-less-degree)
    with ‹y ≠ 0› ‹x div y ≠ 0› have ‹degree (x mod y) < degree (x div y * y)›
      by (simp add: degree-mult-eq)
    then have ‹degree (x div y * y + x mod y) = degree (x div y * y)›
      by (rule degree-add-eq-left)
    with ‹y ≠ 0› ‹x div y ≠ 0› ‹degree y > 0› show ?thesis
      by (simp add: degree-mult-eq)
  qed
qed
```

lemma *degree-mod-less'*: $b \neq 0 \Longrightarrow a \bmod b \neq 0 \Longrightarrow \mathit{degree}\ (a \bmod b) < \mathit{degree}\ b$
by (*rule degree-mod-less-degree*) *auto*

lemma *degree-mod-less*: $y \neq 0 \Longrightarrow x \bmod y = 0 \vee \mathit{degree}\ (x \bmod y) < \mathit{degree}\ y$
using *degree-mod-less'* **by** *blast*

lemma *div-smult-left*: ‹*smult a x div y = smult a* (*x div y*)› (**is** *?Q*)
and *mod-smult-left*: ‹*smult a x mod y = smult a* (*x mod y*)› (**is** *?R*)
for *x y* :: ‹*'a*::*field poly*›
proof −
have ‹(*smult a x div y*, *smult a x mod y*) = (*smult a* (*x div y*), *smult a* (*x mod y*))›
proof (*cases* ‹*a = 0*›)
case *True*
then show *?thesis*
by *simp*
next
case *False*
show *?thesis*
by (*rule euclidean-relation-polyI*)
(*use False* **in** ‹*simp-all add*: *dvd-smult-iff degree-mod-less-degree flip*: *smult-add-right*›)
qed
then show *?Q* **and** *?R*
by *simp-all*
qed

lemma *poly-div-minus-left* [*simp*]: (− *x*) *div y* = − (*x div y*)
for *x y* :: *'a*::*field poly*
using *div-smult-left* [*of* − *1*::*'a*] **by** *simp*

lemma *poly-mod-minus-left* [*simp*]: (− *x*) *mod y* = − (*x mod y*)
for *x y* :: *'a*::*field poly*
using *mod-smult-left* [*of* − *1*::*'a*] **by** *simp*

lemma *poly-div-add-left*: ‹(*x* + *y*) *div z* = *x div z* + *y div z*› (**is** *?Q*)
and *poly-mod-add-left*: ‹(*x* + *y*) *mod z* = *x mod z* + *y mod z*› (**is** *?R*)
for *x y z* :: ‹*'a*::*field poly*›
proof −
have ‹((*x* + *y*) *div z*, (*x* + *y*) *mod z*) = (*x div z* + *y div z*, *x mod z* + *y mod z*)›
proof (*induction rule*: *euclidean-relation-polyI*)
case *by0*
then show *?case* **by** *simp*
next
case *divides*
then obtain *w* **where** ‹*x* + *y* = *z* ∗ *w*›
by *blast*
then have *y*: ‹*y* = *z* ∗ *w* − *x*›
by (*simp add*: *algebra-simps*)
from ‹*z* ≠ *0*› **show** *?case*

```
    using mod-mult-self4 [of z w ‹- x›] div-mult-self4 [of z w ‹- x›]
    by (simp add: algebra-simps y)
  next
    case euclidean-relation
    then have ‹degree (x mod z + y mod z) < degree z›
      using degree-mod-less-degree [of z x] degree-mod-less-degree [of z y]
        dvd-add-right-iff [of z x y] dvd-add-left-iff [of z y x]
      by (cases ‹z dvd x ∨ z dvd y›) (auto intro: degree-add-less)
    moreover have ‹x + y = (x div z + y div z) * z + (x mod z + y mod z)›
      by (simp add: algebra-simps)
    ultimately show ?case
      by simp
  qed
  then show ?Q and ?R
    by simp-all
qed

lemma poly-div-diff-left: (x - y) div z = x div z - y div z
  for x y z :: 'a::field poly
  by (simp only: diff-conv-add-uminus poly-div-add-left poly-div-minus-left)

lemma poly-mod-diff-left: (x - y) mod z = x mod z - y mod z
  for x y z :: 'a::field poly
  by (simp only: diff-conv-add-uminus poly-mod-add-left poly-mod-minus-left)

lemma div-smult-right: ‹x div smult a y = smult (inverse a) (x div y)› (is ?Q)
  and mod-smult-right: ‹x mod smult a y = (if a = 0 then x else x mod y)› (is ?R)
proof -
  have ‹(x div smult a y, x mod smult a y) = (smult (inverse a) (x div y), (if a =
0 then x else x mod y))›
  proof (induction rule: euclidean-relation-polyI)
    case by0
    then show ?case by auto
  next
    case divides
    moreover define w where ‹w = x div y›
    ultimately have ‹x = y * w›
      by (simp add: smult-dvd-iff)
    with divides show ?case
      by simp
  next
    case euclidean-relation
    then show ?case
      by (simp add: smult-dvd-iff degree-mod-less-degree)
  qed
  then show ?Q and ?R
    by simp-all
qed
```

lemma *mod-mult-unit-eq*:
  ‹*x mod* (*z* ∗ *y*) = *x mod y*›
  if ‹*is-unit z*›
  for *x y z* :: ‹'*a*::*field poly*›
proof (*cases* ‹*y* = *0*›)
  case *True*
  then show *?thesis*
    by *simp*
next
  case *False*
  moreover have ‹*z* ≠ *0*›
    using *that* **by** *auto*
  moreover define *a* **where** ‹*a* = *lead-coeff z*›
  ultimately have ‹*z* = [:*a*:]› ‹*a* ≠ *0*›
    using *that monom-0* [*of a*] **by** (*simp-all add*: *is-unit-monom-trivial*)
  then show *?thesis*
    by (*simp add*: *mod-smult-right*)
qed

lemma *poly-div-minus-right* [*simp*]: *x div* (− *y*) = − (*x div y*)
  for *x y* :: '*a*::*field poly*
  using *div-smult-right* [*of* - − *1*::'*a*] **by** (*simp add*: *nonzero-inverse-minus-eq*)

lemma *poly-mod-minus-right* [*simp*]: *x mod* (− *y*) = *x mod y*
  for *x y* :: '*a*::*field poly*
  using *mod-smult-right* [*of* - − *1*::'*a*] **by** *simp*

lemma *poly-div-mult-right*: ‹*x div* (*y* ∗ *z*) = (*x div y*) *div z*› (**is** *?Q*)
  and *poly-mod-mult-right*: ‹*x mod* (*y* ∗ *z*) = *y* ∗ (*x div y mod z*) + *x mod y*› (**is** *?R*)
  for *x y z* :: ‹'*a*::*field poly*›
proof −
  have ‹(*x div* (*y* ∗ *z*), *x mod* (*y* ∗ *z*)) = ((*x div y*) *div z*, *y* ∗ (*x div y mod z*) + *x mod y*)›
  proof (*induction rule*: *euclidean-relation-polyI*)
    case *by0*
    then show *?case* **by** *auto*
  next
    case *divides*
    then show *?case* **by** *auto*
  next
    case *euclidean-relation*
    then have ‹*y* ≠ *0*› ‹*z* ≠ *0*›
      by *simp-all*
    with ‹¬ *y* ∗ *z dvd x*› **have** ‹*degree* (*y* ∗ (*x div y mod z*) + *x mod y*) < *degree* (*y* ∗ *z*)›
      using *degree-mod-less-degree* [*of y x*] *degree-mod-less-degree* [*of z* ‹*x div y*›]
        degree-add-eq-left [*of* ‹*x mod y*› ‹*y* ∗ (*x div y mod z*)›]
        by (*cases* ‹*z dvd x div y*›; *cases* ‹*y dvd x*›)

```
      (auto simp add: degree-mult-eq not-dvd-imp-mod-neq-0 dvd-div-iff-mult)
    moreover have ‹x = x div y div z * (y * z) + (y * (x div y mod z) + x mod
y)›
      by (simp add: field-simps flip: distrib-left)
    ultimately show ?case
      by simp
  qed
  then show ?Q and ?R
    by simp-all
qed

lemma dvd-pCons-imp-dvd-pCons-mod:
  ‹y dvd pCons a (x mod y)› if ‹y dvd pCons a x›
proof −
  have ‹pCons a x = pCons a (x div y * y + x mod y)›
    by simp
  also have ‹... = pCons 0 (x div y * y) + pCons a (x mod y)›
    by simp
  also have ‹pCons 0 (x div y * y) = (x div y * monom 1 (Suc 0)) * y›
    by (simp add: monom-Suc)
  finally show ‹y dvd pCons a (x mod y)›
    using ‹y dvd pCons a x› by simp
qed

lemma degree-less-if-less-eqI:
  ‹degree x < degree y› if ‹degree x ≤ degree y› ‹coeff x (degree y) = 0› ‹x ≠ 0›
proof (cases ‹degree x = degree y›)
  case True
  with ‹coeff x (degree y) = 0› have ‹lead-coeff x = 0›
    by simp
  then have ‹x = 0›
    by simp
  with ‹x ≠ 0› show ?thesis
    by simp
next
  case False
  with ‹degree x ≤ degree y› show ?thesis
    by simp
qed

lemma div-pCons-eq:
    ‹pCons a p div q = (if q = 0 then 0 else pCons (coeff (pCons a (p mod q))
(degree q) / lead-coeff q) (p div q))› (is ?Q)
  and mod-pCons-eq:
    ‹pCons a p mod q = (if q = 0 then pCons a p else pCons a (p mod q) − smult
(coeff (pCons a (p mod q)) (degree q) / lead-coeff q) q)› (is ?R)
    for x y :: ‹'a::field poly›
proof −
  have ‹?Q› and ‹?R› if ‹q = 0›
```

```
  using that by simp-all
moreover have ‹?Q› and ‹?R› if ‹q ≠ 0›
proof −
  define b where ‹b = coeff (pCons a (p mod q)) (degree q) / lead-coeff q›
  have ‹(pCons a p div q, pCons a p mod q) =
    (pCons b (p div q), (pCons a (p mod q) − smult b q))› (is ‹- = (?q, ?r)›)
  proof (induction rule: euclidean-relation-polyI)
    case by0
    with ‹q ≠ 0› show ?case by simp
  next
    case divides
    show ?case
    proof (cases ‹pCons a (p mod q) = 0›)
      case True
      then show ?thesis
        by (auto simp add: b-def)
    next
      case False
      have ‹q dvd pCons a (p mod q)›
        using ‹q dvd pCons a p› by (rule dvd-pCons-imp-dvd-pCons-mod)
      then obtain s where ∗: ‹pCons a (p mod q) = q ∗ s› ..
      with False have ‹s ≠ 0›
        by auto
      from ‹q ≠ 0› have ‹degree (pCons a (p mod q)) ≤ degree q›
        by (auto simp add: Suc-le-eq intro: degree-mod-less-degree)
      moreover from ‹s ≠ 0› have ‹degree q ≤ degree (pCons a (p mod q))›
        by (simp add: degree-mult-right-le ∗)
      ultimately have ‹degree (pCons a (p mod q)) = degree q›
        by (rule order.antisym)
      with ‹s ≠ 0› ‹q ≠ 0› have ‹degree s = 0›
        by (simp add: ∗ degree-mult-eq)
      then obtain c where ‹s = [:c:]›
        by (rule degree-eq-zeroE)
      also have ‹c = b›
        using ‹q ≠ 0› by (simp add: b-def ∗ ‹s = [:c:]›)
      finally have ‹smult b q = pCons a (p mod q)›
        by (simp add: ∗)
      then show ?thesis
        by simp
    qed
  next
    case euclidean-relation
    then have ‹degree q > 0›
      using is-unit-iff-degree by blast
    from ‹q ≠ 0› have ‹degree (pCons a (p mod q)) ≤ degree q›
      by (auto simp add: Suc-le-eq intro: degree-mod-less-degree)
    moreover have ‹degree (smult b q) ≤ degree q›
      by (rule degree-smult-le)
    ultimately have ‹degree (pCons a (p mod q) − smult b q) ≤ degree q›
```

```
      by (rule degree-diff-le)
    moreover have ‹coeff (pCons a (p mod q) − smult b q) (degree q) = 0›
      using ‹degree q > 0› by (auto simp add: b-def)
    ultimately have ‹degree (pCons a (p mod q) − smult b q) < degree q›
      using ‹degree q > 0›
      by (cases ‹pCons a (p mod q) = smult b q›)
        (auto intro: degree-less-if-less-eqI)
    then show ?case
      by simp
  qed
  with ‹q ≠ 0› show ?Q and ?R
    by (simp-all add: b-def)
 qed
 ultimately show ?Q and ?R
  by simp-all
qed

lemma div-mod-fold-coeffs:
 (p div q, p mod q) =
  (if q = 0 then (0, p)
   else
    fold-coeffs
      (λa (s, r).
        let b = coeff (pCons a r) (degree q) / coeff q (degree q)
        in (pCons b s, pCons a r − smult b q)) p (0, 0))
 by (rule sym, induct p) (auto simp: div-pCons-eq mod-pCons-eq Let-def)

lemma mod-pCons:
 fixes a :: 'a::field
  and x y :: 'a::field poly
 assumes y: y ≠ 0
 defines b ≡ coeff (pCons a (x mod y)) (degree y) / coeff y (degree y)
 shows (pCons a x) mod y = pCons a (x mod y) − smult b y
 unfolding b-def
 by (simp add: mod-pCons-eq)
```

4.30.4 List-based versions for fast implementation

```
fun minus-poly-rev-list :: 'a :: group-add list ⇒ 'a list ⇒ 'a list
 where
   minus-poly-rev-list (x # xs) (y # ys) = (x − y) # (minus-poly-rev-list xs ys)
 | minus-poly-rev-list xs [] = xs
 | minus-poly-rev-list [] (y # ys) = []

fun pseudo-divmod-main-list ::
 'a::comm-ring-1 ⇒ 'a list ⇒ 'a list ⇒ 'a list ⇒ nat ⇒ 'a list × 'a list
 where
   pseudo-divmod-main-list lc q r d (Suc n) =
    (let
```

```
      rr = map ((*) lc) r;
      a = hd r;
      qqq = cCons a (map ((*) lc) q);
      rrr = tl (if a = 0 then rr else minus-poly-rev-list rr (map ((*) a) d))
     in pseudo-divmod-main-list lc qqq rrr d n)
| pseudo-divmod-main-list lc q r d 0 = (q, r)

fun pseudo-mod-main-list :: 'a::comm-ring-1 ⇒ 'a list ⇒ 'a list ⇒ nat ⇒ 'a list
  where
    pseudo-mod-main-list lc r d (Suc n) =
      (let
        rr = map ((*) lc) r;
        a = hd r;
        rrr = tl (if a = 0 then rr else minus-poly-rev-list rr (map ((*) a) d))
       in pseudo-mod-main-list lc rrr d n)
| pseudo-mod-main-list lc r d 0 = r


fun divmod-poly-one-main-list ::
    'a::comm-ring-1 list ⇒ 'a list ⇒ 'a list ⇒ nat ⇒ 'a list × 'a list
  where
    divmod-poly-one-main-list q r d (Suc n) =
      (let
        a = hd r;
        qqq = cCons a q;
        rr = tl (if a = 0 then r else minus-poly-rev-list r (map ((*) a) d))
       in divmod-poly-one-main-list qqq rr d n)
| divmod-poly-one-main-list q r d 0 = (q, r)

fun mod-poly-one-main-list :: 'a::comm-ring-1 list ⇒ 'a list ⇒ nat ⇒ 'a list
  where
    mod-poly-one-main-list r d (Suc n) =
      (let
        a = hd r;
        rr = tl (if a = 0 then r else minus-poly-rev-list r (map ((*) a) d))
       in mod-poly-one-main-list rr d n)
| mod-poly-one-main-list r d 0 = r

definition pseudo-divmod-list :: 'a::comm-ring-1 list ⇒ 'a list ⇒ 'a list × 'a list
  where pseudo-divmod-list p q =
    (if q = [] then ([], p)
     else
      (let rq = rev q;
          (qu,re) = pseudo-divmod-main-list (hd rq) [] (rev p) rq (1 + length p −
length q)
       in (qu, rev re)))

definition pseudo-mod-list :: 'a::comm-ring-1 list ⇒ 'a list ⇒ 'a list
  where pseudo-mod-list p q =
```

```
    (if q = [] then p
     else
      (let
        rq = rev q;
        re = pseudo-mod-main-list (hd rq) (rev p) rq (1 + length p − length q)
       in rev re))

lemma minus-zero-does-nothing: minus-poly-rev-list x (map ((*) 0) y) = x
  for x :: 'a::ring list
  by (induct x y rule: minus-poly-rev-list.induct) auto

lemma length-minus-poly-rev-list [simp]: length (minus-poly-rev-list xs ys) = length
xs
  by (induct xs ys rule: minus-poly-rev-list.induct) auto

lemma if-0-minus-poly-rev-list:
  (if a = 0 then x else minus-poly-rev-list x (map ((*) a) y)) =
    minus-poly-rev-list x (map ((*) a) y)
  for a :: 'a::ring
  by(cases a = 0) (simp-all add: minus-zero-does-nothing)

lemma Poly-append: Poly (a @ b) = Poly a + monom 1 (length a) * Poly b
  for a :: 'a::comm-semiring-1 list
  by (induct a) (auto simp: monom-0 monom-Suc)

lemma minus-poly-rev-list: length p ≥ length q ⟹
  Poly (rev (minus-poly-rev-list (rev p) (rev q))) =
    Poly p − monom 1 (length p − length q) * Poly q
  for p q :: 'a :: comm-ring-1 list
proof (induct rev p rev q arbitrary: p q rule: minus-poly-rev-list.induct)
  case (1 x xs y ys)
  then have length (rev q) ≤ length (rev p)
    by simp
  from this[folded 1(2,3)] have ys-xs: length ys ≤ length xs
    by simp
  then have *: Poly (rev (minus-poly-rev-list xs ys)) =
      Poly (rev xs) − monom 1 (length xs − length ys) * Poly (rev ys)
    by (subst 1.hyps(1)[of rev xs rev ys, unfolded rev-rev-ident length-rev]) auto
  have Poly p − monom 1 (length p − length q) * Poly q =
    Poly (rev (rev p)) − monom 1 (length (rev (rev p)) − length (rev (rev q))) *
Poly (rev (rev q))
    by simp
  also have ... =
      Poly (rev (x # xs)) − monom 1 (length (x # xs) − length (y # ys)) * Poly
(rev (y # ys))
    unfolding 1(2,3) by simp
  also from ys-xs have ... =
    Poly (rev xs) + monom x (length xs) −
      (monom 1 (length xs − length ys) * Poly (rev ys) + monom y (length xs))
```

by (*simp add*: *Poly-append distrib-left mult-monom smult-monom*)
also have ... = *Poly* (*rev* (*minus-poly-rev-list xs ys*)) + *monom* (*x* − *y*) (*length xs*)
unfolding ∗ *diff-monom*[*symmetric*] **by** *simp*
finally show *?case*
by (*simp add*: *1*(*2*,*3*)[*symmetric*] *smult-monom Poly-append*)
qed *auto*

lemma *smult-monom-mult*: *smult a* (*monom b n* ∗ *f*) = *monom* (*a* ∗ *b*) *n* ∗ *f*
using *smult-monom* [*of a - n*] **by** (*metis mult-smult-left*)

lemma *head-minus-poly-rev-list*:
length d ≤ *length r* ⟹ *d* ≠ [] ⟹
hd (*minus-poly-rev-list* (*map* ((∗) (*last d*)) *r*) (*map* ((∗) (*hd r*)) (*rev d*))) = *0*
for *d r* :: *'a*::*comm-ring list*
proof (*induct r*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons a rs*)
then show *?case* **by** (*cases rev d*) (*simp-all add*: *ac-simps*)
qed

lemma *Poly-map*: *Poly* (*map* ((∗) *a*) *p*) = *smult a* (*Poly p*)
proof (*induct p*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons x xs*)
then show *?case* **by** (*cases Poly xs* = *0*) *auto*
qed

lemma *last-coeff-is-hd*: *xs* ≠ [] ⟹ *coeff* (*Poly xs*) (*length xs* − *1*) = *hd* (*rev xs*)
by (*simp-all add*: *hd-conv-nth rev-nth nth-default-nth nth-append*)

lemma *pseudo-divmod-main-list-invar*:
assumes *leading-nonzero*: *last d* ≠ *0*
and *lc*: *last d* = *lc*
and *d* ≠ []
and *pseudo-divmod-main-list lc q* (*rev r*) (*rev d*) *n* = (*q'*, *rev r'*)
and *n* = *1* + *length r* − *length d*
shows *pseudo-divmod-main lc* (*monom 1 n* ∗ *Poly q*) (*Poly r*) (*Poly d*) (*length r* − *1*) *n* =
(*Poly q'*, *Poly r'*)
using *assms*(*4* −)
proof (*induct n arbitrary*: *r q*)
case (*Suc n*)
from *Suc.prems* **have** ∗: ¬ *Suc* (*length r*) ≤ *length d*
by *simp*

with ‹$d \neq []$› **have** $r \neq []$
using *Suc-leI length-greater-0-conv list.size(3)* **by** *fastforce*
let *?a* = (*hd* (*rev r*))
let *?rr* = *map* ((∗) *lc*) (*rev r*)
let *?rrr* = *rev* (*tl* (*minus-poly-rev-list ?rr* (*map* ((∗) *?a*) (*rev d*))))
let *?qq* = *cCons ?a* (*map* ((∗) *lc*) *q*)
from ∗ *Suc(3)* **have** *n*: *n* = (*1* + *length r* − *length d* − *1*)
by *simp*
from ∗ **have** *rr-val*:(*length ?rrr*) = (*length r* − *1*)
by *auto*
with ‹$r \neq []$› ∗ **have** *rr-smaller*: (*1* + *length r* − *length d* − *1*) = (*1* + *length ?rrr* − *length d*)
by *auto*
from ∗ **have** *id*: *Suc* (*length r*) − *length d* = *Suc* (*length r* − *length d*)
by *auto*
from *Suc.prems* ∗
have *pseudo-divmod-main-list lc ?qq* (*rev ?rrr*) (*rev d*) (*1* + *length r* − *length d* − *1*) = (*q′*, *rev r′*)
by (*simp add*: *Let-def if-0-minus-poly-rev-list id*)
with *n* **have** *v*: *pseudo-divmod-main-list lc ?qq* (*rev ?rrr*) (*rev d*) *n* = (*q′*, *rev r′*)
by *auto*
from ∗ **have** *sucrr*:*Suc* (*length r*) − *length d* = *Suc* (*length r* − *length d*)
using *Suc-diff-le not-less-eq-eq* **by** *blast*
from *Suc(3)* ‹$r \neq []$› **have** *n-ok* : *n* = *1* + (*length ?rrr*) − *length d*
by *simp*
have *cong*: ⋀*x1 x2 x3 x4 y1 y2 y3 y4*. *x1* = *y1* ⟹ *x2* = *y2* ⟹ *x3* = *y3* ⟹ *x4* = *y4* ⟹
pseudo-divmod-main lc x1 x2 x3 x4 n = *pseudo-divmod-main lc y1 y2 y3 y4 n*
by *simp*
have *hd-rev*: *coeff* (*Poly r*) (*length r* − *Suc 0*) = *hd* (*rev r*)
using *last-coeff-is-hd*[*OF* ‹$r \neq []$›] **by** *simp*
show *?case*
unfolding *Suc.hyps(1)*[*OF v n-ok*, *symmetric*] *pseudo-divmod-main.simps Let-def*
proof (*rule cong*[*OF - - refl*], *goal-cases*)
case *1*
show *?case*
by (*simp add*: *monom-Suc hd-rev*[*symmetric*] *smult-monom Poly-map*)
next
case *2*
show *?case*
proof (*subst Poly-on-rev-starting-with-0*, *goal-cases*)
show *hd* (*minus-poly-rev-list* (*map* ((∗) *lc*) (*rev r*)) (*map* ((∗) (*hd* (*rev r*))) (*rev d*))) = *0*
by (*fold lc*, *subst head-minus-poly-rev-list*, *insert* ∗ ‹$d \neq []$›, *auto*)
from ∗ **have** *length d* ≤ *length r*
by *simp*
then show *smult lc* (*Poly r*) − *monom* (*coeff* (*Poly r*) (*length r* − *1*)) *n* ∗ *Poly d* =
Poly (*rev* (*minus-poly-rev-list* (*map* ((∗) *lc*) (*rev r*)) (*map* ((∗) (*hd* (*rev*

```
r))) (rev d))))
        by (fold rev-map) (auto simp add: n smult-monom-mult Poly-map hd-rev [symmetric]
          minus-poly-rev-list)
    qed
  qed simp
qed simp

lemma pseudo-divmod-impl [code]:
  pseudo-divmod f g = map-prod poly-of-list poly-of-list (pseudo-divmod-list (coeffs f) (coeffs g))
    for f g :: 'a::comm-ring-1 poly
proof (cases g = 0)
  case False
  then have last (coeffs g) ≠ 0
    and last (coeffs g) = lead-coeff g
    and coeffs g ≠ []
    by (simp-all add: last-coeffs-eq-coeff-degree)
  moreover obtain q r where qr: pseudo-divmod-main-list
    (last (coeffs g)) (rev [])
    (rev (coeffs f)) (rev (coeffs g))
    (1 + length (coeffs f) −
    length (coeffs g)) = (q, rev (rev r))
    by force
  ultimately have (Poly q, Poly (rev r)) = pseudo-divmod-main (lead-coeff g) 0 f g
    (length (coeffs f) − Suc 0) (Suc (length (coeffs f)) − length (coeffs g))
    by (subst pseudo-divmod-main-list-invar [symmetric]) auto
  moreover have pseudo-divmod-main-list
    (hd (rev (coeffs g))) []
    (rev (coeffs f)) (rev (coeffs g))
    (1 + length (coeffs f) −
    length (coeffs g)) = (q, r)
    by (metis hd-rev qr rev.simps(1) rev-swap)
  ultimately show ?thesis
   by (simp add: degree-eq-length-coeffs pseudo-divmod-def pseudo-divmod-list-def)
next
  case True
  then show ?thesis
    by (auto simp add: pseudo-divmod-def pseudo-divmod-list-def)
qed

lemma pseudo-mod-main-list:
  snd (pseudo-divmod-main-list l q xs ys n) = pseudo-mod-main-list l xs ys n
  by (induct n arbitrary: l q xs ys) (auto simp: Let-def)

lemma pseudo-mod-impl[code]: pseudo-mod f g = poly-of-list (pseudo-mod-list (coeffs f) (coeffs g))
proof −
```

have *snd-case*: $\bigwedge f\ g\ p.\ snd\ ((\lambda(x,y).\ (f\ x,\ g\ y))\ p) = g\ (snd\ p)$
by *auto*
show *?thesis*
unfolding *pseudo-mod-def pseudo-divmod-impl pseudo-divmod-list-def pseudo-mod-list-def Let-def*
by (*simp add*: *snd-case pseudo-mod-main-list*)
qed

4.30.5 Improved Code-Equations for Polynomial (Pseudo) Division

lemma *pdivmod-via-pseudo-divmod*:
‹(*f div g*, *f mod g*) =
(*if g = 0 then* (*0*, *f*)
else
let
ilc = *inverse* (*lead-coeff g*);
h = *smult ilc g*;
(*q*,*r*) = *pseudo-divmod f h*
in (*smult ilc q*, *r*))›
(**is** ‹*?l* = *?r*›)
proof (*cases* ‹*g* = *0*›)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
define *ilc* **where** ‹*ilc* = *inverse* (*lead-coeff g*)›
define *h* **where** ‹*h* = *smult ilc g*›
from *False* **have** ‹*lead-coeff h* = *1*›
and ‹*ilc* ≠ *0*›
by (*auto simp*: *h-def ilc-def*)
define *q r* **where** ‹*q* = *f div h*› **and** ‹*r* = *f mod h*›
with ‹*lead-coeff h* = *1*› **have** *p*: ‹*pseudo-divmod f h* = (*q*, *r*)›
by (*simp add*: *pseudo-divmod-eq-div-mod*)
from ‹*ilc* ≠ *0*› **have** ‹(*f div g*, *f mod g*) = (*smult ilc q*, *r*)›
by (*auto simp*: *h-def div-smult-right mod-smult-right q-def r-def*)
also have ‹(*smult ilc q*, *r*) = *?r*›
using ‹*g* ≠ *0*› **by** (*auto simp*: *Let-def p simp flip*: *h-def ilc-def*)
finally show *?thesis* **.**
qed

lemma *pdivmod-via-pseudo-divmod-list*:
(*f div g*, *f mod g*) =
(*let cg* = *coeffs g in*
if cg = [] *then* (*0*, *f*)
else
let
cf = *coeffs f*;
ilc = *inverse* (*last cg*);

```
        ch = map ((*) ilc) cg;
        (q, r) = pseudo-divmod-main-list 1 [] (rev cf) (rev ch) (1 + length cf −
length cg)
      in (poly-of-list (map ((*) ilc) q), poly-of-list (rev r)))
proof −
  note d = pdivmod-via-pseudo-divmod pseudo-divmod-impl pseudo-divmod-list-def
  show ?thesis
  proof (cases g = 0)
    case True
    with d show ?thesis by auto
  next
    case False
    define ilc where ilc = inverse (coeff g (degree g))
    from False have ilc: ilc ≠ 0
      by (auto simp: ilc-def)
    with False have id: g = 0 ⟷ False coeffs g = [] ⟷ False
      last (coeffs g) = coeff g (degree g)
      coeffs (smult ilc g) = [] ⟷ False
      by (auto simp: last-coeffs-eq-coeff-degree)
    have id2: hd (rev (coeffs (smult ilc g))) = 1
      by (subst hd-rev, insert id ilc, auto simp: coeffs-smult, subst last-map, auto
simp: id ilc-def)
    have id3: length (coeffs (smult ilc g)) = length (coeffs g)
      rev (coeffs (smult ilc g)) = rev (map ((*) ilc) (coeffs g))
      unfolding coeffs-smult using ilc by auto
    obtain q r where pair:
      pseudo-divmod-main-list 1 [] (rev (coeffs f)) (rev (map ((*) ilc) (coeffs g)))
        (1 + length (coeffs f) − length (coeffs g)) = (q, r)
      by force
    show ?thesis
      unfolding d Let-def id if-False ilc-def[symmetric] map-prod-def[symmetric]
id2
      unfolding id3 pair map-prod-def split
      by (auto simp: Poly-map)
  qed
qed

lemma pseudo-divmod-main-list-1: pseudo-divmod-main-list 1 = divmod-poly-one-main-list
proof (intro ext, goal-cases)
  case (1 q r d n)
  have ∗: map ((*) 1) xs = xs for xs :: 'a list
    by (induct xs) auto
  show ?case
    by (induct n arbitrary: q r d) (auto simp: ∗ Let-def)
qed

fun divide-poly-main-list :: 'a::idom-divide ⇒ 'a list ⇒ 'a list ⇒ 'a list ⇒ nat ⇒
'a list
  where
```

```
    divide-poly-main-list lc q r d (Suc n) =
      (let
        cr = hd r
        in if cr = 0 then divide-poly-main-list lc (cCons cr q) (tl r) d n else let
        a = cr div lc;
        qq = cCons a q;
        rr = minus-poly-rev-list r (map ((*) a) d)
       in if hd rr = 0 then divide-poly-main-list lc qq (tl rr) d n else [])
  | divide-poly-main-list lc q r d 0 = q

lemma divide-poly-main-list-simp [simp]:
  divide-poly-main-list lc q r d (Suc n) =
    (let
      cr = hd r;
      a = cr div lc;
      qq = cCons a q;
      rr = minus-poly-rev-list r (map ((*) a) d)
     in if hd rr = 0 then divide-poly-main-list lc qq (tl rr) d n else [])
  by (simp add: Let-def minus-zero-does-nothing)

declare divide-poly-main-list.simps(1)[simp del]

definition divide-poly-list :: 'a::idom-divide poly ⇒ 'a poly ⇒ 'a poly
  where divide-poly-list f g =
    (let cg = coeffs g in
      if cg = [] then g
      else
        let
          cf = coeffs f;
          cgr = rev cg
         in poly-of-list (divide-poly-main-list (hd cgr) [] (rev cf) cgr (1 + length cf
− length cg)))

lemmas pdivmod-via-divmod-list = pdivmod-via-pseudo-divmod-list[unfolded pseudo-divmod-main-list-1]

lemma mod-poly-one-main-list: snd (divmod-poly-one-main-list q r d n) = mod-poly-one-main-list
r d n
  by (induct n arbitrary: q r d) (auto simp: Let-def)

lemma mod-poly-code [code]:
  f mod g =
    (let cg = coeffs g in
      if cg = [] then f
      else
        let
          cf = coeffs f;
          ilc = inverse (last cg);
          ch = map ((*) ilc) cg;
          r = mod-poly-one-main-list (rev cf) (rev ch) (1 + length cf − length cg)
```

```
      in poly-of-list (rev r))
  (is - = ?rhs)
proof -
  have snd (f div g, f mod g) = ?rhs
    unfolding pdivmod-via-divmod-list Let-def mod-poly-one-main-list [symmetric,
of - - - Nil]
    by (auto split: prod.splits)
  then show ?thesis by simp
qed

definition div-field-poly-impl :: 'a :: field poly ⇒ 'a poly ⇒ 'a poly
  where div-field-poly-impl f g =
    (let cg = coeffs g in
      if cg = [] then 0
      else
        let
          cf = coeffs f;
          ilc = inverse (last cg);
          ch = map ((*) ilc) cg;
           q = fst (divmod-poly-one-main-list [] (rev cf) (rev ch) (1 + length cf −
length cg))
        in poly-of-list ((map ((*) ilc) q)))
```

We do not declare the following lemma as code equation, since then polynomial division on non-fields will no longer be executable. However, a code-unfold is possible, since *div-field-poly-impl* is a bit more efficient than the generic polynomial division.

```
lemma div-field-poly-impl[code-unfold]: (div) = div-field-poly-impl
proof (intro ext)
  fix f g :: 'a poly
  have fst (f div g, f mod g) = div-field-poly-impl f g
    unfolding div-field-poly-impl-def pdivmod-via-divmod-list Let-def
    by (auto split: prod.splits)
  then show f div g =  div-field-poly-impl f g
    by simp
qed

lemma divide-poly-main-list:
  assumes lc0: lc ≠ 0
    and lc: last d = lc
    and d: d ≠ []
    and n = (1 + length r − length d)
  shows Poly (divide-poly-main-list lc q (rev r) (rev d) n) =
    divide-poly-main lc (monom 1 n ∗ Poly q) (Poly r) (Poly d) (length r − 1) n
  using assms(4−)
proof (induct n arbitrary: r q)
  case (Suc n)
  from Suc.prems have ifCond: ¬ Suc (length r) ≤ length d
    by simp
```

```
  with d have r: r ≠ []
    using Suc-leI length-greater-0-conv list.size(3) by fastforce
  then obtain rr lcr where r: r = rr @ [lcr]
    by (cases r rule: rev-cases) auto
  from d lc obtain dd where d: d = dd @ [lc]
    by (cases d rule: rev-cases) auto
  from Suc(2) ifCond have n: n = 1 + length rr − length d
    by (auto simp: r)
  from ifCond have len: length dd ≤ length rr
    by (simp add: r d)
  show ?case
  proof (cases lcr div lc * lc = lcr)
    case False
    with r d show ?thesis
      unfolding Suc(2)[symmetric]
      by (auto simp add: Let-def nth-default-append)
  next
    case True
    with r d have id:
      ?thesis ⟷
        Poly (divide-poly-main-list lc (cCons (lcr div lc) q)
          (rev (rev (minus-poly-rev-list (rev rr) (rev (map ((*) (lcr div lc)) dd)))))
(rev d) n) =
          divide-poly-main lc
            (monom 1 (Suc n) * Poly q + monom (lcr div lc) n)
            (Poly r − monom (lcr div lc) n * Poly d)
            (Poly d) (length rr − 1) n
      by (cases r rule: rev-cases; cases d rule: rev-cases)
        (auto simp add: Let-def rev-map nth-default-append)
    have cong: ⋀x1 x2 x3 x4 y1 y2 y3 y4. x1 = y1 ⟹ x2 = y2 ⟹ x3 = y3 ⟹
x4 = y4 ⟹
        divide-poly-main lc x1 x2 x3 x4 n = divide-poly-main lc y1 y2 y3 y4 n
      by simp
    show ?thesis
      unfolding id
    proof (subst Suc(1), simp add: n,
        subst minus-poly-rev-list, force simp: len, rule cong[OF - - refl], goal-cases)
      case 2
      have monom lcr (length rr) = monom (lcr div lc) (length rr − length dd) *
monom lc (length dd)
        by (simp add: mult-monom len True)
      then show ?case unfolding r d Poly-append n ring-distribs
        by (auto simp: Poly-map smult-monom smult-monom-mult)
    qed (auto simp: len monom-Suc smult-monom)
  qed
qed simp

lemma divide-poly-list[code]: f div g = divide-poly-list f g
proof −
```

```
  note d = divide-poly-def divide-poly-list-def
  show ?thesis
  proof (cases g = 0)
    case True
    show ?thesis by (auto simp: d True)
  next
    case False
    then obtain cg lcg where cg: coeffs g = cg @ [lcg]
      by (cases coeffs g rule: rev-cases) auto
    with False have id: (g = 0) = False (cg @ [lcg] = []) = False
      by auto
    from cg False have lcg: coeff g (degree g) = lcg
      using last-coeffs-eq-coeff-degree last-snoc by force
    with False have lcg ≠ 0 by auto
    from cg Poly-coeffs [of g] have ltp: Poly (cg @ [lcg]) = g
      by auto
    show ?thesis
      unfolding d cg Let-def id if-False poly-of-list-def
      by (subst divide-poly-main-list, insert False cg ‹lcg ≠ 0›)
        (auto simp: lcg ltp, simp add: degree-eq-length-coeffs)
  qed
qed

lemma poly-mod:
  poly (p mod q) x = poly p x if poly q x = 0
proof −
  from that have poly (p mod q) x = poly (p div q * q) x + poly (p mod q) x
    by simp
  also have ... = poly p x
    by (simp only: poly-add [symmetric]) simp
  finally show ?thesis .
qed
```

4.31 Primality and irreducibility in polynomial rings

```
lemma prod-mset-const-poly: (∏ x∈#A. [:f x:]) = [:prod-mset (image-mset f A):]
  by (induct A) (simp-all add: ac-simps)

lemma irreducible-const-poly-iff:
  fixes c :: 'a :: {comm-semiring-1,semiring-no-zero-divisors}
  shows irreducible [:c:] ⟷ irreducible c
proof
  assume A: irreducible c
  show irreducible [:c:]
  proof (rule irreducibleI)
    fix a b assume ab: [:c:] = a * b
    hence degree [:c:] = degree (a * b) by (simp only: )
    also from A ab have a ≠ 0 b ≠ 0 by auto
    hence degree (a * b) = degree a + degree b by (simp add: degree-mult-eq)
```

```
    finally have degree a = 0 degree b = 0 by auto
   then obtain a' b' where ab': a = [:a':] b = [:b':] by (auto elim!: degree-eq-zeroE)
    from ab have coeff [:c:] 0 = coeff (a * b) 0 by (simp only: )
    hence c = a' * b' by (simp add: ab' mult-ac)
    from A and this have a' dvd 1 ∨ b' dvd 1 by (rule irreducibleD)
    with ab' show a dvd 1 ∨ b dvd 1
      by (auto simp add: is-unit-const-poly-iff)
  qed (insert A, auto simp: irreducible-def is-unit-poly-iff)
next
  assume A: irreducible [:c:]
  then have c ≠ 0 and ¬ c dvd 1
    by (auto simp add: irreducible-def is-unit-const-poly-iff)
  then show irreducible c
  proof (rule irreducibleI)
    fix a b assume ab: c = a * b
    hence [:c:] = [:a:] * [:b:] by (simp add: mult-ac)
    from A and this have [:a:] dvd 1 ∨ [:b:] dvd 1 by (rule irreducibleD)
    then show a dvd 1 ∨ b dvd 1
      by (auto simp add: is-unit-const-poly-iff)
  qed
qed

lemma lift-prime-elem-poly:
  assumes prime-elem (c :: 'a :: semidom)
  shows   prime-elem [:c:]
proof (rule prime-elemI)
  fix a b assume *: [:c:] dvd a * b
  from * have dvd: c dvd coeff (a * b) n for n
    by (subst (asm) const-poly-dvd-iff) blast
  {
    define m where m = (GREATEST m. ¬c dvd coeff b m)
    assume ¬[:c:] dvd b
    hence A: ∃ i. ¬c dvd coeff b i by (subst (asm) const-poly-dvd-iff) blast
    have B: ⋀i. ¬c dvd coeff b i ⟹ i ≤ degree b
      by (auto intro: le-degree)
    have coeff-m: ¬c dvd coeff b m unfolding m-def by (rule GreatestI-ex-nat[OF
A B])
    have i ≤ m if ¬c dvd coeff b i for i
      unfolding m-def by (metis (mono-tags, lifting) B Greatest-le-nat that)
    hence dvd-b: c dvd coeff b i if i > m for i using that by force

    have c dvd coeff a i for i
    proof (induction i rule: nat-descend-induct[of degree a])
      case (base i)
      thus ?case by (simp add: coeff-eq-0)
    next
      case (descend i)
      let ?A = {..i+m} - {i}
      have c dvd coeff (a * b) (i + m) by (rule dvd)
```

```
    also have coeff (a * b) (i + m) = (∑k≤i + m. coeff a k * coeff b (i + m
− k))
      by (simp add: coeff-mult)
    also have {..i+m} = insert i ?A by auto
    also have (∑k∈.... coeff a k * coeff b (i + m − k)) =
                coeff a i * coeff b m + (∑k∈?A. coeff a k * coeff b (i + m − k))
      (is - = - + ?S)
      by (subst sum.insert) simp-all
    finally have eq: c dvd coeff a i * coeff b m + ?S .
    moreover have c dvd ?S
    proof (rule dvd-sum)
      fix k assume k: k ∈ {..i+m} − {i}
      show c dvd coeff a k * coeff b (i + m − k)
      proof (cases k < i)
        case False
        with k have c dvd coeff a k by (intro descend.IH) simp
        thus ?thesis by simp
      next
        case True
        hence c dvd coeff b (i + m − k) by (intro dvd-b) simp
        thus ?thesis by simp
      qed
    qed
    ultimately have c dvd coeff a i * coeff b m
      by (simp add: dvd-add-left-iff)
    with assms coeff-m show c dvd coeff a i
      by (simp add: prime-elem-dvd-mult-iff)
  qed
  hence [:c:] dvd a by (subst const-poly-dvd-iff) blast
 }
 then show [:c:] dvd a ∨ [:c:] dvd b by blast
next
 from assms show [:c:] ≠ 0 and ¬ [:c:] dvd 1
  by (simp-all add: prime-elem-def is-unit-const-poly-iff)
qed

lemma prime-elem-const-poly-iff:
 fixes c :: 'a :: semidom
 shows   prime-elem [:c:] ⟷ prime-elem c
proof
 assume A: prime-elem [:c:]
 show prime-elem c
 proof (rule prime-elemI)
  fix a b assume c dvd a * b
  hence [:c:] dvd [:a:] * [:b:] by (simp add: mult-ac)
 from A and this have [:c:] dvd [:a:] ∨ [:c:] dvd [:b:] by (rule prime-elem-dvd-multD)
  thus c dvd a ∨ c dvd b by simp
 qed (insert A, auto simp: prime-elem-def is-unit-poly-iff)
qed (auto intro: lift-prime-elem-poly)
```

4.32 Content and primitive part of a polynomial

definition *content* :: *'a::semiring-gcd poly* ⇒ *'a*
where *content p* = *gcd-list* (*coeffs p*)

lemma *content-eq-fold-coeffs* [*code*]: *content p* = *fold-coeffs gcd p 0*
by (*simp add*: *content-def Gcd-fin.set-eq-fold fold-coeffs-def foldr-fold fun-eq-iff ac-simps*)

lemma *content-0* [*simp*]: *content 0* = *0*
by (*simp add*: *content-def*)

lemma *content-1* [*simp*]: *content 1* = *1*
by (*simp add*: *content-def*)

lemma *content-const* [*simp*]: *content* [:*c*:] = *normalize c*
by (*simp add*: *content-def cCons-def*)

lemma *const-poly-dvd-iff-dvd-content*: [:*c*:] *dvd p* ⟷ *c dvd content p*
for *c* :: *'a::semiring-gcd*
proof (*cases p* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
have [:*c*:] *dvd p* ⟷ (∀ *n*. *c dvd coeff p n*)
by (*rule const-poly-dvd-iff*)
also have ... ⟷ (∀ *a*∈*set* (*coeffs p*). *c dvd a*)
proof *safe*
fix *n* :: *nat*
assume ∀ *a*∈*set* (*coeffs p*). *c dvd a*
then show *c dvd coeff p n*
by (*cases n* ≤ *degree p*) (*auto simp*: *coeff-eq-0 coeffs-def split*: *if-splits*)
qed (*auto simp*: *coeffs-def simp del*: *upt-Suc split*: *if-splits*)
also have ... ⟷ *c dvd content p*
by (*simp add*: *content-def dvd-Gcd-fin-iff dvd-mult-unit-iff*)
finally show *?thesis* **.**
qed

lemma *content-dvd* [*simp*]: [:*content p*:] *dvd p*
by (*subst const-poly-dvd-iff-dvd-content*) *simp-all*

lemma *content-dvd-coeff* [*simp*]: *content p dvd coeff p n*
proof (*cases p* = *0*)
case *True*
then show *?thesis*
by *simp*
next
case *False*
then show *?thesis*

by (*cases* $n \leq$ *degree* p)
(*auto simp add*: *content-def not-le coeff-eq-0 coeff-in-coeffs intro*: *Gcd-fin-dvd*)
qed

lemma *content-dvd-coeffs*: $c \in$ *set* (*coeffs* p) $\Longrightarrow$ *content* p *dvd* c
by (*simp add*: *content-def Gcd-fin-dvd*)

lemma *normalize-content* [*simp*]: *normalize* (*content* p) = *content* p
by (*simp add*: *content-def*)

lemma *is-unit-content-iff* [*simp*]: *is-unit* (*content* p) $\longleftrightarrow$ *content* $p = 1$
proof
assume *is-unit* (*content* p)
then have *normalize* (*content* p) = *1* **by** (*simp add*: *is-unit-normalize del*: *normalize-content*)
then show *content* $p = 1$ **by** *simp*
qed *auto*

lemma *content-smult* [*simp*]:
fixes $c :: 'a ::$ {*normalization-semidom-multiplicative*, *semiring-gcd*}
shows *content* (*smult* c p) = *normalize* $c *$ *content* p
by (*simp add*: *content-def coeffs-smult Gcd-fin-mult normalize-mult*)

lemma *content-eq-zero-iff* [*simp*]: *content* $p = 0 \longleftrightarrow p = 0$
by (*auto simp*: *content-def simp*: *poly-eq-iff coeffs-def*)

definition *primitive-part* :: $'a ::$ *semiring-gcd poly* $\Rightarrow$ $'a$ *poly*
where *primitive-part* p = *map-poly* ($\lambda x.$ x *div* *content* p) p

lemma *primitive-part-0* [*simp*]: *primitive-part* 0 = 0
by (*simp add*: *primitive-part-def*)

lemma *content-times-primitive-part* [*simp*]: *smult* (*content* p) (*primitive-part* p) = p
for $p :: 'a ::$ *semiring-gcd poly*
proof (*cases* $p = 0$)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
then show *?thesis*
unfolding *primitive-part-def*
by (*auto simp*: *smult-conv-map-poly map-poly-map-poly o-def content-dvd-coeffs intro*: *map-poly-idI*)
qed

lemma *primitive-part-eq-0-iff* [*simp*]: *primitive-part* $p = 0 \longleftrightarrow p = 0$
proof (*cases* $p = 0$)
case *True*

then show *?thesis* **by** *simp*
next
case *False*
then have *primitive-part p* = *map-poly* (λ*x*. *x div content p*) *p*
by (*simp add*: *primitive-part-def*)
also from *False* **have** ... = *0* ⟷ *p* = *0*
by (*intro map-poly-eq-0-iff*) (*auto simp*: *dvd-div-eq-0-iff content-dvd-coeffs*)
finally show *?thesis*
using *False* **by** *simp*
qed

lemma *content-primitive-part* [*simp*]:
fixes *p* :: *'a* :: {*normalization-semidom-multiplicative*, *semiring-gcd*} *poly*
assumes *p* ≠ *0*
shows *content* (*primitive-part p*) = *1*
proof −
have *p* = *smult* (*content p*) (*primitive-part p*)
by *simp*
also have *content* ... = *content* (*primitive-part p*) ∗ *content p*
by (*simp del*: *content-times-primitive-part add*: *ac-simps*)
finally have *1* ∗ *content p* = *content* (*primitive-part p*) ∗ *content p*
by *simp*
then have *1* ∗ *content p div content p* = *content* (*primitive-part p*) ∗ *content p div content p*
by *simp*
with *assms* **show** *?thesis*
by *simp*
qed

lemma *content-decompose*:
obtains *p'* :: *'a* :: {*normalization-semidom-multiplicative*, *semiring-gcd*} *poly*
where *p* = *smult* (*content p*) *p' content p'* = *1*
proof (*cases p* = *0*)
case *True*
then have *p* = *smult* (*content p*) *1 content 1* = *1*
by *simp-all*
then show *?thesis* **..**
next
case *False*
then have *p* = *smult* (*content p*) (*primitive-part p*) *content* (*primitive-part p*) = *1*
by *simp-all*
then show *?thesis* **..**
qed

lemma *content-dvd-contentI* [*intro*]: *p dvd q* ⟹ *content p dvd content q*
using *const-poly-dvd-iff-dvd-content content-dvd dvd-trans* **by** *blast*

lemma *primitive-part-const-poly* [*simp*]: *primitive-part* [:*x*:] = [:*unit-factor x*:]

by (*simp add*: *primitive-part-def map-poly-pCons*)

lemma *primitive-part-prim*: *content p* = *1* ⟹ *primitive-part p* = *p*
by (*auto simp*: *primitive-part-def*)

lemma *degree-primitive-part* [*simp*]: *degree* (*primitive-part p*) = *degree p*
proof (*cases p* = *0*)
case *True*
then show *?thesis* **by** *simp*
next
case *False*
have *p* = *smult* (*content p*) (*primitive-part p*)
by *simp*
also from *False* **have** *degree* ... = *degree* (*primitive-part p*)
by (*subst degree-smult-eq*) *simp-all*
finally show *?thesis* **..**
qed

lemma *smult-content-normalize-primitive-part* [*simp*]:
fixes *p* :: *'a* :: {*normalization-semidom-multiplicative*, *semiring-gcd*, *idom-divide*}
poly
shows *smult* (*content p*) (*normalize* (*primitive-part p*)) = *normalize p*
proof −
have *smult* (*content p*) (*normalize* (*primitive-part p*)) =
normalize ([:*content p*:] ∗ *primitive-part p*)
by (*subst normalize-mult*) (*simp-all add*: *normalize-const-poly*)
also have [:*content p*:] ∗ *primitive-part p* = *p* **by** *simp*
finally show *?thesis* **.**
qed

context
begin

private

lemma *content-1-mult*:
fixes *f g* :: *'a* :: {*semiring-gcd*, *factorial-semiring*} *poly*
assumes *content f* = *1 content g* = *1*
shows *content* (*f* ∗ *g*) = *1*
proof (*cases f* ∗ *g* = *0*)
case *False*
from *assms* **have** *f* ≠ *0 g* ≠ *0* **by** *auto*

hence *f* ∗ *g* ≠ *0* **by** *auto*
{
assume ¬*is-unit* (*content* (*f* ∗ *g*))
with *False* **have** ∃ *p*. *p dvd content* (*f* ∗ *g*) ∧ *prime p*
by (*intro prime-divisor-exists*) *simp-all*
then obtain *p* **where** *p dvd content* (*f* ∗ *g*) *prime p* **by** *blast*

```
    from ‹p dvd content (f * g)› have [:p:] dvd f * g
      by (simp add: const-poly-dvd-iff-dvd-content)
   moreover from ‹prime p› have prime-elem [:p:] by (simp add: lift-prime-elem-poly)
    ultimately have [:p:] dvd f ∨ [:p:] dvd g
      by (simp add: prime-elem-dvd-mult-iff)
    with assms have is-unit p by (simp add: const-poly-dvd-iff-dvd-content)
    with ‹prime p› have False by simp
  }
  hence is-unit (content (f * g)) by blast
   hence normalize (content (f * g)) = 1 by (simp add: is-unit-normalize del:
normalize-content)
  thus ?thesis by simp
qed (insert assms, auto)

lemma content-mult:
 fixes p q :: 'a :: {factorial-semiring, semiring-gcd, normalization-semidom-multiplicative}
poly
  shows content (p * q) = content p * content q
proof (cases p * q = 0)
  case False
  then have p ≠ 0 and q ≠ 0
    by simp-all
  then have *: content (primitive-part p * primitive-part q) = 1
    by (auto intro: content-1-mult)
 have p * q = smult (content p) (primitive-part p) * smult (content q) (primitive-part
q)
    by simp
  also have ... = smult (content p * content q) (primitive-part p * primitive-part
q)
    by (metis mult.commute mult-smult-right smult-smult)
  with * show ?thesis
  by (simp add: normalize-mult)
next
  case True
  then show ?thesis
    by auto
qed

end

lemma primitive-part-mult:
  fixes p q :: 'a :: {factorial-semiring, semiring-Gcd, ring-gcd, idom-divide,
                      normalization-semidom-multiplicative} poly
  shows primitive-part (p * q) = primitive-part p * primitive-part q
proof -
  have primitive-part (p * q) = p * q div [:content (p * q):]
    by (simp add: primitive-part-def div-const-poly-conv-map-poly)
  also have ... = (p div [:content p:]) * (q div [:content q:])
    by (subst div-mult-div-if-dvd) (simp-all add: content-mult mult-ac)
```

```
  also have ... = primitive-part p * primitive-part q
    by (simp add: primitive-part-def div-const-poly-conv-map-poly)
  finally show ?thesis .
qed

lemma primitive-part-smult:
  fixes p :: 'a :: {factorial-semiring, semiring-Gcd, ring-gcd, idom-divide,
                    normalization-semidom-multiplicative} poly
  shows primitive-part (smult a p) = smult (unit-factor a) (primitive-part p)
proof -
  have smult a p = [:a:] * p by simp
  also have primitive-part ... = smult (unit-factor a) (primitive-part p)
    by (subst primitive-part-mult) simp-all
  finally show ?thesis .
qed

lemma primitive-part-dvd-primitive-partI [intro]:
  fixes p q :: 'a :: {factorial-semiring, semiring-Gcd, ring-gcd, idom-divide,
                      normalization-semidom-multiplicative} poly
  shows p dvd q ⟹ primitive-part p dvd primitive-part q
  by (auto elim!: dvdE simp: primitive-part-mult)

lemma content-prod-mset:
  fixes A :: 'a :: {factorial-semiring, semiring-Gcd, normalization-semidom-multiplicative}
      poly multiset
  shows content (prod-mset A) = prod-mset (image-mset content A)
  by (induction A) (simp-all add: content-mult mult-ac)

lemma content-prod-eq-1-iff:
  fixes p q :: 'a :: {factorial-semiring, semiring-Gcd, normalization-semidom-multiplicative}
poly
  shows content (p * q) = 1 ⟷ content p = 1 ∧ content q = 1
proof safe
  assume A: content (p * q) = 1
  {
    fix p q :: 'a poly assume content p * content q = 1
    hence 1 = content p * content q by simp
    hence content p dvd 1 by (rule dvdI)
    hence content p = 1 by simp
  } note B = this
  from A B[of p q] B [of q p] show content p = 1 content q = 1
    by (simp-all add: content-mult mult-ac)
qed (auto simp: content-mult)
```

4.33 A typeclass for algebraically closed fields

Since the required sort constraints are not available inside the class, we have to resort to a somewhat awkward way of writing the definition of algebraically closed fields:

class *alg-closed-field* = *field* +
  assumes *alg-closed*: $n > 0 \Longrightarrow f\ n \neq 0 \Longrightarrow \exists x.\ (\sum k{\leq}n.\ f\ k * x\ \hat{}\ k) = 0$

We can then however easily show the equivalence to the proper definition:

lemma *alg-closed-imp-poly-has-root*:
  assumes *degree* $(p :: 'a :: \textit{alg-closed-field poly}) > 0$
  shows $\exists x.\ \textit{poly}\ p\ x = 0$
proof −
  have $\exists x.\ (\sum k{\leq}\textit{degree}\ p.\ \textit{coeff}\ p\ k * x\ \hat{}\ k) = 0$
    using *assms* **by** (*intro alg-closed*) *auto*
  thus *?thesis*
    by (*simp add*: *poly-altdef*)
qed

lemma *alg-closedI* [*Pure.intro*]:
  assumes $\bigwedge p :: 'a\ \textit{poly}.\ \textit{degree}\ p > 0 \Longrightarrow \textit{lead-coeff}\ p = 1 \Longrightarrow \exists x.\ \textit{poly}\ p\ x = 0$
  shows *OFCLASS*('a :: *field*, *alg-closed-field-class*)
proof
  fix $n :: \textit{nat}$ **and** $f :: \textit{nat} \Rightarrow 'a$
  assume *n*: $n > 0\ f\ n \neq 0$
  define p **where** $p = \textit{Abs-poly}\ (\lambda k.\ \textit{if}\ k \leq n\ \textit{then}\ f\ k\ \textit{else}\ 0)$
  have *coeff-p*: $\textit{coeff}\ p\ k = (\textit{if}\ k \leq n\ \textit{then}\ f\ k\ \textit{else}\ 0)$ **for** k
  proof −
    have *eventually* $(\lambda k.\ k > n)$ *cofinite*
      by (*auto simp*: *MOST-nat*)
    hence *eventually* $(\lambda k.\ (\textit{if}\ k \leq n\ \textit{then}\ f\ k\ \textit{else}\ 0) = 0)$ *cofinite*
      by *eventually-elim auto*
    thus *?thesis*
      unfolding *p-def* **by** (*subst Abs-poly-inverse*) *auto*
  qed

  from n **have** $\textit{degree}\ p \geq n$
    by (*intro le-degree*) (*auto simp*: *coeff-p*)
  moreover have $\textit{degree}\ p \leq n$
    by (*intro degree-le*) (*auto simp*: *coeff-p*)
  ultimately have *deg-p*: $\textit{degree}\ p = n$
    by *linarith*
  from *deg-p* **and** n **have** [*simp*]: $p \neq 0$
    by *auto*

  define p' **where** $p' = \textit{smult}\ (\textit{inverse}\ (\textit{lead-coeff}\ p))\ p$
  have *deg-p'*: $\textit{degree}\ p' = \textit{degree}\ p$
    by (*auto simp*: *p'-def*)
  have *lead-coeff-p'* [*simp*]: $\textit{lead-coeff}\ p' = 1$
    by (*auto simp*: *p'-def*)

  from *deg-p* **and** *deg-p'* **and** n **have** $\textit{degree}\ p' > 0$
    by *simp*
  from *assms*[*OF this*] **obtain** x **where** $\textit{poly}\ p'\ x = 0$

```
    by auto
  hence poly p x = 0
    by (simp add: p'-def)
  also have poly p x = (∑ k≤n. f k * x ^ k)
    unfolding poly-altdef by (intro sum.cong) (auto simp: deg-p coeff-p)
  finally show ∃ x. (∑ k≤n. f k * x ^ k) = 0 ..
qed

lemma (in alg-closed-field) nth-root-exists:
  assumes n > 0
  shows   ∃ y. y ^ n = (x :: 'a)
proof −
  define f where f = (λi. if i = 0 then −x else if i = n then 1 else 0)
  have ∃ x. (∑ k≤n. f k * x ^ k) = 0
    by (rule alg-closed) (use assms in ‹auto simp: f-def›)
  also have (λx. ∑ k≤n. f k * x ^ k) = (λx. ∑ k∈{0,n}. f k * x ^ k)
    by (intro ext sum.mono-neutral-right) (auto simp: f-def)
  finally show ∃ y. y ^ n = x
    using assms by (simp add: f-def)
qed
```

We can now prove by induction that every polynomial of degree n splits into a product of n linear factors:

```
lemma alg-closed-imp-factorization:
  fixes p :: 'a :: alg-closed-field poly
  assumes p ≠ 0
  shows ∃ A. size A = degree p ∧ p = smult (lead-coeff p) (∏ x∈#A. [:−x, 1:])
  using assms
proof (induction degree p arbitrary: p rule: less-induct)
  case (less p)
  show ?case
  proof (cases degree p = 0)
    case True
    thus ?thesis
      by (intro exI[of - {#}]) (auto elim!: degree-eq-zeroE)
  next
    case False
    then obtain x where x: poly p x = 0
      using alg-closed-imp-poly-has-root by blast
    hence [:−x, 1:] dvd p
      using poly-eq-0-iff-dvd by blast
    then obtain q where p-eq: p = [:−x, 1:] * q
      by (elim dvdE)
    have q ≠ 0
      using less.prems p-eq by auto
    moreover from this have deg: degree p = Suc (degree q)
      unfolding p-eq by (subst degree-mult-eq) auto
    ultimately obtain A where A: size A = degree q q = smult (lead-coeff q)
(∏ x∈#A. [:−x, 1:])
```

```
    using less.hyps[of q] by auto
  have smult (lead-coeff p) (∏ y∈#add-mset x A. [:− y, 1:]) =
      [:− x, 1:] ∗ smult (lead-coeff q) (∏ y∈#A. [:− y, 1:])
    unfolding p-eq lead-coeff-mult by simp
  also note A(2) [symmetric]
  also note p-eq [symmetric]
  finally show ?thesis using A(1)
    by (intro exI[of - add-mset x A]) (auto simp: deg)
 qed
qed
```

As an alternative characterisation of algebraic closure, one can also say that any polynomial of degree at least 2 splits into non-constant factors:

```
lemma alg-closed-imp-reducible:
 assumes degree (p :: 'a :: alg-closed-field poly) > 1
 shows   ¬irreducible p
proof −
 have degree p > 0
   using assms by auto
 then obtain z where z: poly p z = 0
   using alg-closed-imp-poly-has-root[of p] by blast
 then have dvd: [:−z, 1:] dvd p
   by (subst dvd-iff-poly-eq-0) auto
 then obtain q where q: p = [:−z, 1:] ∗ q
   by (erule dvdE)
 have [simp]: q ≠ 0
   using assms q by auto

 show ?thesis
 proof (rule reducible-polyI)
   show p = [:−z, 1:] ∗ q
     by fact
 next
   have degree p = degree ([:−z, 1:] ∗ q)
     by (simp only: q)
   also have ... = degree q + 1
     by (subst degree-mult-eq) auto
   finally show degree q > 0
     using assms by linarith
 qed auto
qed
```

When proving algebraic closure through reducibility, we can assume w.l.o.g. that the polynomial is monic and has a non-zero constant coefficient:

```
lemma alg-closedI-reducible:
 assumes ⋀p :: 'a poly. degree p > 1 ⟹ lead-coeff p = 1 ⟹ coeff p 0 ≠ 0 ⟹
           ¬irreducible p
 shows   OFCLASS('a :: field, alg-closed-field-class)
proof
```

fix $p :: 'a\ poly$ **assume** p: $degree\ p > 0\ lead\text{-}coeff\ p = 1$
show $\exists x.\ poly\ p\ x = 0$
proof $(cases\ coeff\ p\ 0 = 0)$
  case *True*
  hence $poly\ p\ 0 = 0$
    by $(simp\ add: poly\text{-}0\text{-}coeff\text{-}0)$
  thus *?thesis* **by** *blast*
next
  case *False*
  from p **and** *this* **show** *?thesis*
  proof $(induction\ degree\ p\ arbitrary: p\ rule: less\text{-}induct)$
    case $(less\ p)$
    show *?case*
    proof $(cases\ degree\ p = 1)$
      case *True*
      then obtain $a\ b$ **where** p: $p = [:a, b:]$
        by $(cases\ p)$ $(auto\ split: if\text{-}splits\ elim!: degree\text{-}eq\text{-}zeroE)$
      from *True* **have** $[simp]$: $b \neq 0$
        by $(auto\ simp: p)$
      have $poly\ p\ (-a/b) = 0$
        by $(auto\ simp: p)$
      thus *?thesis* **by** *blast*
    next
      case *False*
      hence $degree\ p > 1$
        using *less.prems* **by** *auto*
      from $assms[OF\ \langle degree\ p > 1\rangle\ \langle lead\text{-}coeff\ p = 1\rangle\ \langle coeff\ p\ 0 \neq 0\rangle]$
      have $\neg irreducible\ p$ **by** *auto*
      then obtain $r\ s$ **where** rs: $degree\ r > 0\ degree\ s > 0\ p = r * s$
        using *less.prems* **unfolding** *irreducible-def*
        by $(metis\ is\text{-}unit\text{-}iff\text{-}degree\ mult\text{-}not\text{-}zero\ zero\text{-}less\text{-}iff\text{-}neq\text{-}zero)$
      hence $coeff\ r\ 0 \neq 0$
        using $\langle coeff\ p\ 0 \neq 0\rangle$ **by** $(auto\ simp: coeff\text{-}mult\text{-}0)$

      define r' **where** $r' = smult\ (inverse\ (lead\text{-}coeff\ r))\ r$
      have $[simp]$: $degree\ r' = degree\ r$
        by $(simp\ add: r'\text{-}def)$
      have lc: $lead\text{-}coeff\ r' = 1$
        using rs **by** $(auto\ simp: r'\text{-}def)$
      have nz: $coeff\ r'\ 0 \neq 0$
        using $\langle coeff\ r\ 0 \neq 0\rangle$ **by** $(auto\ simp: r'\text{-}def)$

      have $degree\ r < degree\ r + degree\ s$
        using rs **by** *linarith*
      also have $\ldots = degree\ (r * s)$
        using $rs(3)$ *less.prems* **by** $(subst\ degree\text{-}mult\text{-}eq)$ *auto*
      also have $r * s = p$
        using $rs(3)$ **by** *simp*
      finally have $\exists x.\ poly\ r'\ x = 0$

```
      by (intro less) (use lc rs nz in auto)
    thus ?thesis
      using rs(3) by (auto simp: r'-def)
  qed
 qed
qed
```

Using a clever Tschirnhausen transformation mentioned e.g. in the article by Nowak [1], we can also assume w.l.o.g. that the coefficient a_{n-1} is zero.

```
lemma alg-closedI-reducible-coeff-deg-minus-one-eq-0:
 assumes ⋀p :: 'a poly. degree p > 1 ⟹ lead-coeff p = 1 ⟹ coeff p (degree p
- 1) = 0 ⟹
          coeff p 0 ≠ 0 ⟹ ¬irreducible p
 shows   OFCLASS('a :: field-char-0, alg-closed-field-class)
proof (rule alg-closedI-reducible, goal-cases)
 case (1 p)
 define n where [simp]: n = degree p
 define a where a = coeff p (n - 1)
 define r where r = [: -a / of-nat n, 1 :]
 define s where s = [: a / of-nat n, 1 :]
 define q where q = pcompose p r

 have n > 0
  using 1 by simp
 have r-altdef: r = monom 1 1 + [:-a / of-nat n:]
  by (simp add: r-def monom-altdef)
 have deg-q: degree q = n
  by (simp add: q-def r-def degree-pcompose)
 have lc-q: lead-coeff q = 1
  unfolding q-def using 1 by (subst lead-coeff-comp) (simp-all add: r-def)
 have q ≠ 0
  using 1 deg-q by auto

 have coeff q (n - 1) =
       (∑i≤n. ∑k≤i. coeff p i * (of-nat (i choose k) *
         ((-a / of-nat n) ^ (i - k)) * (if k = n - 1 then 1 else 0))))
  unfolding q-def pcompose-altdef poly-altdef r-altdef
 by (simp-all add: degree-map-poly coeff-map-poly coeff-sum binomial-ring sum-distrib-left
poly-const-pow
          sum-distrib-right mult-ac monom-power coeff-monom-mult of-nat-poly
cong: if-cong)
 also have ... = (∑i≤n. ∑k∈(if i ≥ n - 1 then {n-1} else {}).
                  coeff p i * (of-nat (i choose k) * (-a / of-nat n) ^ (i - k)))
   by (rule sum.cong [OF refl], rule sum.mono-neutral-cong-right) (auto split:
if-splits)
 also have ... = (∑i∈{n-1,n}. ∑k∈(if i ≥ n - 1 then {n-1} else {}).
                  coeff p i * (of-nat (i choose k) * (-a / of-nat n) ^ (i - k)))
  by (rule sum.mono-neutral-right) auto
```

```
  also have ... = a − of-nat (n choose (n − 1)) ∗ a / of-nat n
    using 1 by (simp add: a-def)
  also have n choose (n − 1) = n
    using ‹n > 0› by (subst binomial-symmetric) auto
  also have a − of-nat n ∗ a / of-nat n = 0
    using ‹n > 0› by simp
  finally have coeff q (n − 1) = 0 .

  show ?case
  proof (cases coeff q 0 = 0)
    case True
    hence poly p (− (a / of-nat (degree p))) = 0
      by (auto simp: q-def r-def)
    thus ?thesis
      by (rule root-imp-reducible-poly) (use 1 in auto)
  next
    case False
    hence ¬irreducible q
      using assms[of q] and lc-q and 1 and ‹coeff q (n − 1) = 0›
      by (auto simp: deg-q)
    then obtain u v where uv: degree u > 0 degree v > 0 q = u ∗ v
      using ‹q ≠ 0› 1 deg-q unfolding irreducible-def
     by (metis degree-mult-eq-0 is-unit-iff-degree n-def neq0-conv not-one-less-zero)

    have p = pcompose q s
      by (simp add: q-def r-def s-def pcompose-pCons flip: pcompose-assoc)
    also have q = u ∗ v
      by fact
    finally have p = pcompose u s ∗ pcompose v s
      by (simp add: pcompose-mult)
    moreover have degree (pcompose u s) > 0 degree (pcompose v s) > 0
      using uv by (simp-all add: s-def degree-pcompose)
    ultimately show ¬irreducible p
      using 1 by (intro reducible-polyI)
  qed
qed
```

As a consequence of the full factorisation lemma proven above, we can also show that any polynomial with at least two different roots splits into two non-constant coprime factors:

```
lemma alg-closed-imp-poly-splits-coprime:
  assumes degree (p :: 'a :: {alg-closed-field} poly) > 1
  assumes poly p x = 0 poly p y = 0 x ≠ y
  obtains r s where degree r > 0 degree s > 0 coprime r s p = r ∗ s
proof −
  define n where n = order x p
  have n > 0
    using assms by (metis degree-0 gr0I n-def not-one-less-zero order-root)
  have [:−x, 1:] ^ n dvd p
```

```
    unfolding n-def by (simp add: order-1)
  then obtain q where p-eq: p = [:−x, 1:] ^ n * q
    by (elim dvdE)
  from assms have [simp]: q ≠ 0
    by (auto simp: p-eq)
  have order x p = n + Polynomial.order x q
    unfolding p-eq by (subst order-mult) (auto simp: order-power-n-n)
  hence Polynomial.order x q = 0
    by (simp add: n-def)
  hence poly q x ≠ 0
    by (simp add: order-root)

  show ?thesis
  proof (rule that)
    show coprime ([:−x, 1:] ^ n) q
    proof (rule coprimeI)
      fix d
      assume d: d dvd [:−x, 1:] ^ n d dvd q
      have degree d = 0
      proof (rule ccontr)
        assume ¬(degree d = 0)
        then obtain z where z: poly d z = 0
          using alg-closed-imp-poly-has-root by blast
        moreover from this and d(1) have poly ([:−x, 1:] ^ n) z = 0
          using dvd-trans poly-eq-0-iff-dvd by blast
        ultimately have poly d x = 0
          by auto
        with d(2) have poly q x = 0
          using dvd-trans poly-eq-0-iff-dvd by blast
        with ‹poly q x ≠ 0› show False by contradiction
      qed
      thus is-unit d using d
        by (metis ‹q ≠ 0› dvd-0-left is-unit-iff-degree)
    qed
  next
    have poly q y = 0
      using ‹poly p y = 0› ‹x ≠ y› by (auto simp: p-eq)
    with ‹q ≠ 0› show degree q > 0
      using order-degree order-gt-0-iff order-less-le-trans by blast
  qed (use ‹n > 0› in ‹simp-all add: p-eq degree-power-eq›)
qed
```

4.34 Polynomials and limits

```
lemma filterlim-poly-at-infinity:
  fixes p::'a::real-normed-field poly
  assumes degree p>0
  shows filterlim (poly p) at-infinity at-infinity
using assms
```

```
proof (induct p)
  case 0
  then show ?case by auto
next
  case (pCons a p)
  have ?case when degree p=0
  proof −
    obtain c where c-def:p=[:c:] using ‹degree p = 0› degree-eq-zeroE by blast
    then have c≠0 using ‹0 < degree (pCons a p)› by auto
    then show ?thesis unfolding c-def
      apply (auto intro!:tendsto-add-filterlim-at-infinity)
      apply (subst mult.commute)
      by (auto intro!:tendsto-mult-filterlim-at-infinity filterlim-ident)
  qed
  moreover have ?case when degree p≠0
  proof −
    have filterlim (poly p) at-infinity at-infinity
      using that by (auto intro:pCons)
    then show ?thesis
     by (auto intro!:tendsto-add-filterlim-at-infinity filterlim-at-infinity-times filter-
lim-ident)
  qed
  ultimately show ?case by auto
qed

lemma poly-divide-tendsto-aux:
  fixes p::'a::real-normed-field poly
  shows ((λx. poly p x/x^(degree p)) ⟶ lead-coeff p) at-infinity
proof (induct p)
  case 0
  then show ?case by (auto intro:tendsto-eq-intros)
next
  case (pCons a p)
  have ?case when p=0
    using that by auto
  moreover have ?case when p≠0
  proof −
    define g where g=(λx. a/(x*x^degree p))
    define f where f=(λx. poly p x/x^degree p)
    have ∀F x in at-infinity. poly (pCons a p) x / x ^ degree (pCons a p) = g x +
f x
    proof (rule eventually-at-infinityI[of 1])
      fix x::'a assume norm x≥1
      then have x≠0 by auto
      then show poly (pCons a p) x / x ^ degree (pCons a p) = g x + f x
        using that unfolding g-def f-def by (auto simp add:field-simps)
    qed
    moreover have ((λx. g x+f x) ⟶ lead-coeff (pCons a p)) at-infinity
    proof −
```

```
    have (g ⟶ 0) at-infinity
      unfolding g-def using filterlim-poly-at-infinity[of monom 1 (Suc (degree
p))]
    apply (auto intro!:tendsto-intros tendsto-divide-0 simp add: degree-monom-eq)
     apply (subst filterlim-cong[where g=poly (monom 1 (Suc (degree p)))])
     by (auto simp add:poly-monom)
    moreover have (f ⟶ lead-coeff (pCons a p)) at-infinity
     using pCons ‹p≠0› unfolding f-def by auto
    ultimately show ?thesis by (auto intro:tendsto-eq-intros)
   qed
   ultimately show ?thesis by (auto dest:tendsto-cong)
  qed
  ultimately show ?case by auto
qed

lemma filterlim-power-at-infinity:
  assumes n≠0
  shows filterlim (λx::'a::real-normed-field. x^n) at-infinity at-infinity
  using filterlim-poly-at-infinity[of monom 1 n] assms
  by (simp add: filterlim-ident filterlim-power-at-infinity)

lemma poly-divide-tendsto-0-at-infinity:
  fixes p::'a::real-normed-field poly
  assumes degree p > degree q
  shows ((λx. poly q x / poly p x) ⟶ 0 ) at-infinity
proof −
  define pp where pp ≡ (λx. x^(degree p) / poly p x)
  define qq where qq ≡ (λx. poly q x/x^(degree q))
  define dd where dd ≡ (λx::'a. 1/x^(degree p − degree q))
  have ∀ F x in at-infinity. poly q x / poly p x = qq x ∗ pp x ∗ dd x
  proof (rule eventually-at-infinityI[of 1])
   fix x::'a assume norm x≥1
   then have x≠0 by auto
   then show poly q x / poly p x = qq x ∗ pp x ∗ dd x
    unfolding qq-def pp-def dd-def using assms
    by (auto simp add:field-simps divide-simps power-diff)
  qed
  moreover have ((λx. qq x ∗ pp x ∗ dd x) ⟶ 0) at-infinity
  proof −
   have (qq ⟶ lead-coeff q) at-infinity
    unfolding qq-def using poly-divide-tendsto-aux[of q] .
   moreover have (pp ⟶ 1/lead-coeff p) at-infinity
   proof −
    have p≠0 using assms by auto
    then show ?thesis
     unfolding pp-def using poly-divide-tendsto-aux[of p]
     apply (drule-tac tendsto-inverse)
     by (auto simp add:inverse-eq-divide)
   qed
```

```
    moreover have (dd ⟶ 0) at-infinity
      unfolding dd-def
      apply (rule tendsto-divide-0)
      by (auto intro!: filterlim-power-at-infinity simp add:assms)
    ultimately show ?thesis by (auto intro:tendsto-eq-intros)
  qed
  ultimately show ?thesis by (auto dest:tendsto-cong)
qed

lemma poly-eventually-not-zero:
  fixes p::real poly
  assumes p≠0
  shows eventually (λx. poly p x ≠ 0) at-infinity
proof (rule eventually-at-infinityI[of Max (norm ` {x. poly p x = 0}) + 1])
  fix x::real assume §: Max (norm ` {x. poly p x = 0}) + 1 ≤ norm x
  have False when poly p x = 0
  proof −
    define S where S=norm `{x. poly p x = 0}
    have norm x∈S
      using that unfolding S-def by auto
    moreover have finite S
      using ‹p≠0› poly-roots-finite unfolding S-def by blast
    ultimately have norm x≤Max S
      by simp
    moreover have Max S + 1 ≤ norm x
      using § unfolding S-def by simp
    ultimately show False by argo
  qed
  then show poly p x ≠ 0 by auto
qed

no-notation cCons (infixr ‹##› 65)

end
```

5 A formalization of formal power series

```
theory Formal-Power-Series
imports
  Complex-Main
  Euclidean-Algorithm
  Primes
  HOL−Library.FuncSet
  HOL−Library.Multiset
begin
```

5.1 The type of formal power series

```
typedef 'a fps = {f :: nat ⇒ 'a. True}
```

morphisms *fps-nth Abs-fps*
by *simp*

notation *fps-nth* (**infixl** ‹$› *75*)

lemma *expand-fps-eq*: $p = q \longleftrightarrow (\forall n.\ p\ \$\ n = q\ \$\ n)$
by (*simp add*: *fps-nth-inject* [*symmetric*] *fun-eq-iff*)

lemmas *fps-eq-iff* = *expand-fps-eq*

lemma *fps-ext*: $(\bigwedge n.\ p\ \$\ n = q\ \$\ n) \Longrightarrow p = q$
by (*simp add*: *expand-fps-eq*)

lemma *fps-nth-Abs-fps* [*simp*]: *Abs-fps f* $ *n* = *f n*
by (*simp add*: *Abs-fps-inverse*)

Definition of the basic elements 0 and 1 and the basic operations of addition, negation and multiplication.

instantiation *fps* :: (*zero*) *zero*
begin
definition *fps-zero-def*: *0* = *Abs-fps* (λ*n*. *0*)
instance ..
end

lemma *fps-zero-nth* [*simp*]: *0* $ *n* = *0*
unfolding *fps-zero-def* **by** *simp*

lemma *fps-nonzero-nth*: $f \neq 0 \longleftrightarrow (\exists\ n.\ f\ \$\ n \neq 0)$
by (*simp add*: *expand-fps-eq*)

lemma *fps-nonzero-nth-minimal*: $f \neq 0 \longleftrightarrow (\exists n.\ f\ \$\ n \neq 0 \wedge (\forall m < n.\ f\ \$\ m = 0))$
(**is** *?lhs* ⟷ *?rhs*)
proof
let *?n* = *LEAST n*. *f* $ *n* ≠ *0*
show *?rhs* **if** *?lhs*
proof −
from *that* **have** ∃ *n*. *f* $ *n* ≠ *0*
by (*simp add*: *fps-nonzero-nth*)
then have *f* $ *?n* ≠ *0*
by (*rule LeastI-ex*)
moreover have ∀ *m*<*?n*. *f* $ *m* = *0*
by (*auto dest*: *not-less-Least*)
ultimately show *?thesis* **by** *metis*
qed
qed (*auto simp*: *expand-fps-eq*)

lemma *fps-nonzeroI*: *f*$*n* ≠ *0* ⟹ *f* ≠ *0*
by *auto*

instantiation *fps* :: ({*one*, *zero*}) *one*
begin
  definition *fps-one-def*: *1* = *Abs-fps* (λ*n*. *if n* = *0 then 1 else 0*)
  instance ..
end

lemma *fps-one-nth* [*simp*]: *1* $ *n* = (*if n* = *0 then 1 else 0*)
  unfolding *fps-one-def* **by** *simp*

instantiation *fps* :: (*plus*) *plus*
begin
  definition *fps-plus-def*: (+) = (λ*f g*. *Abs-fps* (λ*n*. *f* $ *n* + *g* $ *n*))
  instance ..
end

lemma *fps-add-nth* [*simp*]: (*f* + *g*) $ *n* = *f* $ *n* + *g* $ *n*
  unfolding *fps-plus-def* **by** *simp*

instantiation *fps* :: (*minus*) *minus*
begin
  definition *fps-minus-def*: (−) = (λ*f g*. *Abs-fps* (λ*n*. *f* $ *n* − *g* $ *n*))
  instance ..
end

lemma *fps-sub-nth* [*simp*]: (*f* − *g*) $ *n* = *f* $ *n* − *g* $ *n*
  unfolding *fps-minus-def* **by** *simp*

instantiation *fps* :: (*uminus*) *uminus*
begin
  definition *fps-uminus-def*: *uminus* = (λ*f*. *Abs-fps* (λ*n*. − (*f* $ *n*)))
  instance ..
end

lemma *fps-neg-nth* [*simp*]: (− *f*) $ *n* = − (*f* $ *n*)
  unfolding *fps-uminus-def* **by** *simp*

lemma *fps-neg-0* [*simp*]: −(*0*::*'a*::*group-add fps*) = *0*
  by (*rule iffD2*, *rule fps-eq-iff*, *auto*)

instantiation *fps* :: ({*comm-monoid-add*, *times*}) *times*
begin
  definition *fps-times-def*: (∗) = (λ*f g*. *Abs-fps* (λ*n*. ∑ *i*=*0*..*n*. *f* $ *i* ∗ *g* $ (*n* − *i*)))
  instance ..
end

lemma *fps-mult-nth*: (*f* ∗ *g*) $ *n* = (∑ *i*=*0*..*n*. *f*$*i* ∗ *g*$(*n* − *i*))
  unfolding *fps-times-def* **by** *simp*

lemma *fps-mult-nth-0* [*simp*]: $(f * g)\ \$\ 0 = f\ \$\ 0 * g\ \$\ 0$
unfolding *fps-times-def* **by** *simp*

lemma *fps-mult-nth-1*: $(f * g)\ \$\ 1 = f\$0 * g\$1 + f\$1 * g\$0$
by (*simp add*: *fps-mult-nth*)

lemma *fps-mult-nth-1'* [*simp*]: $(f * g)\ \$\ Suc\ 0 = f\$0 * g\$Suc\ 0 + f\$Suc\ 0 * g\$0$
by (*simp add*: *fps-mult-nth*)

lemmas *mult-nth-0* = *fps-mult-nth-0*
lemmas *mult-nth-1* = *fps-mult-nth-1*

instance *fps* :: ({*comm-monoid-add*, *mult-zero*}) *mult-zero*
proof
fix $a :: 'a\ fps$
show $0 * a = 0$ **by** (*simp add*: *fps-ext fps-mult-nth*)
show $a * 0 = 0$ **by** (*simp add*: *fps-ext fps-mult-nth*)
qed

declare *atLeastAtMost-iff* [*presburger*]
declare *Bex-def* [*presburger*]
declare *Ball-def* [*presburger*]

lemma *mult-delta-left*:
fixes $x\ y :: 'a::mult\text{-}zero$
shows $(if\ b\ then\ x\ else\ 0) * y = (if\ b\ then\ x * y\ else\ 0)$
by *simp*

lemma *mult-delta-right*:
fixes $x\ y :: 'a::mult\text{-}zero$
shows $x * (if\ b\ then\ y\ else\ 0) = (if\ b\ then\ x * y\ else\ 0)$
by *simp*

lemma *fps-one-mult*:
fixes $f :: 'a::\{comm\text{-}monoid\text{-}add, mult\text{-}zero, monoid\text{-}mult\}\ fps$
shows $1 * f = f$
and $f * 1 = f$
by (*simp-all add*: *fps-ext fps-mult-nth mult-delta-left mult-delta-right*)

5.2 Subdegrees

definition *subdegree* :: $('a::zero)\ fps \Rightarrow nat$ **where**
$subdegree\ f = (if\ f = 0\ then\ 0\ else\ LEAST\ n.\ f\$n \neq 0)$

lemma *subdegreeI*:
assumes $f\ \$\ d \neq 0$ **and** $\bigwedge i.\ i < d \Longrightarrow f\ \$\ i = 0$
shows $subdegree\ f = d$
by (*smt* (*verit*) *LeastI-ex assms fps-zero-nth linorder-cases not-less-Least subde-*

gree-def)

lemma *nth-subdegree-nonzero* [*simp,intro*]: $f \neq 0 \Longrightarrow f\ \$\ subdegree\ f \neq 0$
using *fps-nonzero-nth-minimal subdegreeI* **by** *blast*

lemma *nth-less-subdegree-zero* [*dest*]: $n < subdegree\ f \Longrightarrow f\ \$\ n = 0$
by (*metis fps-nonzero-nth-minimal fps-zero-nth subdegreeI*)

lemma *subdegree-geI*:
assumes $f \neq 0$ $\bigwedge i.\ i < n \Longrightarrow f\$i = 0$
shows *subdegree* $f \geq n$
by (*meson assms leI nth-subdegree-nonzero*)

lemma *subdegree-greaterI*:
assumes $f \neq 0$ $\bigwedge i.\ i \leq n \Longrightarrow f\$i = 0$
shows *subdegree* $f > n$
by (*meson assms leI nth-subdegree-nonzero*)

lemma *subdegree-leI*:
$f\ \$\ n \neq 0 \Longrightarrow subdegree\ f \leq n$
using *linorder-not-less* **by** *blast*

lemma *subdegree-0* [*simp*]: *subdegree* $0 = 0$
by (*simp add*: *subdegree-def*)

lemma *subdegree-1* [*simp*]: *subdegree* $1 = 0$
by (*metis fps-one-nth nth-subdegree-nonzero subdegree-0*)

lemma *subdegree-eq-0-iff*: *subdegree* $f = 0 \longleftrightarrow f = 0 \vee f\ \$\ 0 \neq 0$
using *nth-subdegree-nonzero subdegree-leI* **by** *fastforce*

lemma *subdegree-eq-0* [*simp*]: $f\ \$\ 0 \neq 0 \Longrightarrow subdegree\ f = 0$
by (*simp add*: *subdegree-eq-0-iff*)

lemma *nth-subdegree-zero-iff* [*simp*]: $f\ \$\ subdegree\ f = 0 \longleftrightarrow f = 0$
by (*cases* $f = 0$) *auto*

lemma *fps-nonzero-subdegree-nonzeroI*: $subdegree\ f > 0 \Longrightarrow f \neq 0$
by *auto*

lemma *subdegree-uminus* [*simp*]:
subdegree $(-(f::('a::group\text{-}add)\ fps)) = subdegree\ f$
proof (*cases* $f=0$)
case *False* **thus** *?thesis* **by** (*force intro*: *subdegreeI*)
qed *simp*

lemma *subdegree-minus-commute* [*simp*]:
fixes $f :: 'a::group\text{-}add\ fps$
shows *subdegree* $(f-g) = subdegree\ (g - f)$

proof (*cases g−f=0*)
case *True* **then show** *?thesis*
by (*metis fps-sub-nth nth-subdegree-nonzero right-minus-eq*)
next
case *False* **show** *?thesis*
using *nth-subdegree-nonzero*[*OF False*] **by** (*fastforce intro*: *subdegreeI*)
qed

lemma *subdegree-add-ge′*:
fixes *f g* :: *′a*::*monoid-add fps*
assumes *f* + *g* ≠ *0*
shows *subdegree* (*f* + *g*) ≥ *min* (*subdegree f*) (*subdegree g*)
using *assms*
by (*force intro*: *subdegree-geI*)

lemma *subdegree-add-ge*:
assumes *f* ≠ −(*g* :: (*′a* :: *group-add*) *fps*)
shows *subdegree* (*f* + *g*) ≥ *min* (*subdegree f*) (*subdegree g*)
proof (*rule subdegree-add-ge′*)
have *f* + *g* = *0* ⟹ *False*
proof−
assume *fg*: *f* + *g* = *0*
have ⋀*n*. *f* $ *n* = − *g* $ *n*
by (*metis add-eq-0-iff equation-minus-iff fg fps-add-nth fps-neg-nth fps-zero-nth*)
with *assms* **show** *False* **by** (*auto intro*: *fps-ext*)
qed
thus *f* + *g* ≠ *0* **by** *fast*
qed

lemma *subdegree-add-eq1*:
assumes *f* ≠ *0*
and *subdegree f* < *subdegree* (*g* :: *′a*::*monoid-add fps*)
shows *subdegree* (*f* + *g*) = *subdegree f*
using *assms* **by**(*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-add-eq2*:
assumes *g* ≠ *0*
and *subdegree g* < *subdegree* (*f* :: *′a* :: *monoid-add fps*)
shows *subdegree* (*f* + *g*) = *subdegree g*
using *assms* **by** (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-diff-eq1*:
assumes *f* ≠ *0*
and *subdegree f* < *subdegree* (*g* :: *′a* :: *group-add fps*)
shows *subdegree* (*f* − *g*) = *subdegree f*
using *assms* **by** (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-diff-eq1-cancel*:
assumes *f* ≠ *0*

and *subdegree f < subdegree* (*g* :: *'a* :: *cancel-comm-monoid-add fps*)
shows *subdegree* (*f − g*) = *subdegree f*
using *assms* **by** (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-diff-eq2*:
assumes *g ≠ 0*
and *subdegree g < subdegree* (*f* :: *'a* :: *group-add fps*)
shows *subdegree* (*f − g*) = *subdegree g*
using *assms* **by** (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *subdegree-diff-ge* [*simp*]:
assumes *f ≠* (*g* :: *'a* :: *group-add fps*)
shows *subdegree* (*f − g*) ≥ *min* (*subdegree f*) (*subdegree g*)
proof−
have *f ≠ −* (*− g*)
using *assms expand-fps-eq* **by** *fastforce*
moreover have *f* + *− g* = *f − g* **by** (*simp add*: *fps-ext*)
ultimately show *?thesis*
using *subdegree-add-ge*[*of f −g*] **by** *simp*
qed

lemma *subdegree-diff-ge′*:
fixes *f g* :: *'a* :: *comm-monoid-diff fps*
assumes *f − g ≠ 0*
shows *subdegree* (*f − g*) ≥ *subdegree f*
using *assms* **by** (*auto intro*: *subdegree-geI simp*: *nth-less-subdegree-zero*)

lemma *nth-subdegree-mult-left* [*simp*]:
fixes *f g* :: (*'a* :: {*mult-zero,comm-monoid-add*}) *fps*
shows (*f ∗ g*) $ (*subdegree f*) = *f* $ *subdegree f ∗ g* $ *0*
by (*cases subdegree f*) (*simp-all add*: *fps-mult-nth nth-less-subdegree-zero*)

lemma *nth-subdegree-mult-right* [*simp*]:
fixes *f g* :: (*'a* :: {*mult-zero,comm-monoid-add*}) *fps*
shows (*f ∗ g*) $ (*subdegree g*) = *f* $ *0 ∗ g* $ *subdegree g*
by (*cases subdegree g*) (*simp-all add*: *fps-mult-nth nth-less-subdegree-zero sum.atLeast-Suc-atMost*)

lemma *nth-subdegree-mult* [*simp*]:
fixes *f g* :: (*'a* :: {*mult-zero,comm-monoid-add*}) *fps*
shows (*f ∗ g*) $ (*subdegree f* + *subdegree g*) = *f* $ *subdegree f ∗ g* $ *subdegree g*
proof−
let *?n* = *subdegree f* + *subdegree g*
have (*f ∗ g*) $ *?n* = (∑ *i*=*0*..*?n*. *f*$*i ∗ g*$(*?n−i*))
by (*simp add*: *fps-mult-nth*)
also have ... = (∑ *i*=*0*..*?n*. *if i* = *subdegree f then f*$*i ∗ g*$(*?n−i*) *else 0*)
proof (*intro sum.cong*)
fix *x* **assume** *x*: *x* ∈ {*0*..*?n*}
hence *x* = *subdegree f* ∨ *x < subdegree f* ∨ *?n − x < subdegree g* **by** *auto*

thus $f\ \$\ x * g\ \$\ (?n - x) = (\mathit{if}\ x = \mathit{subdegree}\ f\ \mathit{then}\ f\ \$\ x * g\ \$\ (?n - x)\ \mathit{else}\ 0)$
by (*elim disjE conjE*) *auto*
qed *auto*
also have $\ldots = f\ \$\ \mathit{subdegree}\ f * g\ \$\ \mathit{subdegree}\ g$ **by** *simp*
finally show *?thesis* **.**
qed

lemma *fps-mult-nth-eq0*:
fixes $f\ g :: 'a::\{\mathit{comm\text{-}monoid\text{-}add}, \mathit{mult\text{-}zero}\}\ \mathit{fps}$
assumes $n < \mathit{subdegree}\ f + \mathit{subdegree}\ g$
shows $(f*g)\ \$\ n = 0$
proof−
have $\bigwedge i.\ i \in \{0..n\} \Longrightarrow f\$i * g\$(n - i) = 0$
proof−
fix i **assume** i: $i \in \{0..n\}$
show $f\$i * g\$(n - i) = 0$
proof (*cases* $i < \mathit{subdegree}\ f \vee n - i < \mathit{subdegree}\ g$)
case *False* **with** *assms i* **show** *?thesis* **by** *auto*
qed (*auto simp*: *nth-less-subdegree-zero*)
qed
thus $(f * g)\ \$\ n = 0$ **by** (*simp add*: *fps-mult-nth*)
qed

lemma *fps-mult-subdegree-ge*:
fixes $f\ g :: 'a::\{\mathit{comm\text{-}monoid\text{-}add}, \mathit{mult\text{-}zero}\}\ \mathit{fps}$
assumes $f*g \neq 0$
shows $\mathit{subdegree}\ (f*g) \geq \mathit{subdegree}\ f + \mathit{subdegree}\ g$
using *assms fps-mult-nth-eq0*
by (*intro subdegree-geI*) *simp*

lemma *subdegree-mult'*:
fixes $f\ g :: 'a::\{\mathit{comm\text{-}monoid\text{-}add}, \mathit{mult\text{-}zero}\}\ \mathit{fps}$
assumes $f\ \$\ \mathit{subdegree}\ f * g\ \$\ \mathit{subdegree}\ g \neq 0$
shows $\mathit{subdegree}\ (f*g) = \mathit{subdegree}\ f + \mathit{subdegree}\ g$
proof−
from *assms* **have** $(f * g)\ \$\ (\mathit{subdegree}\ f + \mathit{subdegree}\ g) \neq 0$ **by** *simp*
hence $f*g \neq 0$ **by** *fastforce*
hence $\mathit{subdegree}\ (f*g) \geq \mathit{subdegree}\ f + \mathit{subdegree}\ g$ **using** *fps-mult-subdegree-ge*
by *fast*
moreover from *assms* **have** $\mathit{subdegree}\ (f*g) \leq \mathit{subdegree}\ f + \mathit{subdegree}\ g$
by (*intro subdegree-leI*) *simp*
ultimately show *?thesis* **by** *simp*
qed

lemma *subdegree-mult* [*simp*]:
fixes $f\ g :: 'a :: \{\mathit{semiring\text{-}no\text{-}zero\text{-}divisors}\}\ \mathit{fps}$
assumes $f \neq 0\ g \neq 0$
shows $\mathit{subdegree}\ (f * g) = \mathit{subdegree}\ f + \mathit{subdegree}\ g$

using *assms*
by (*intro subdegree-mult'*) *simp*

lemma *fps-mult-nth-conv-upto-subdegree-left*:
fixes *f g* :: ('*a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows (*f* ∗ *g*) $ *n* = (∑ *i*=*subdegree f*..*n*. *f* $ *i* ∗ *g* $ (*n* − *i*))
proof (*cases subdegree f* ≤ *n*)
case *True*
hence {*0*..*n*} = {*0*..<*subdegree f*} ∪ {*subdegree f*..*n*} **by** *auto*
moreover have {*0*..<*subdegree f*} ∩ {*subdegree f*..*n*} = {} **by** *auto*
ultimately show *?thesis*
using *nth-less-subdegree-zero*[*of* - *f*]
by (*simp add*: *fps-mult-nth sum.union-disjoint*)
qed (*simp add*: *fps-mult-nth nth-less-subdegree-zero*)

lemma *fps-mult-nth-conv-upto-subdegree-right*:
fixes *f g* :: ('*a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows (*f* ∗ *g*) $ *n* = (∑ *i*=*0*..*n* − *subdegree g*. *f* $ *i* ∗ *g* $ (*n* − *i*))
proof−
have {*0*..*n*} = {*0*..*n* − *subdegree g*} ∪ {*n* − *subdegree g*<..*n*} **by** *auto*
moreover have {*0*..*n* − *subdegree g*} ∩ {*n* − *subdegree g*<..*n*} = {} **by** *auto*
moreover have ∀ *i*∈{*n* − *subdegree g*<..*n*}. *g* $ (*n* − *i*) = *0*
using *nth-less-subdegree-zero*[*of* - *g*] **by** *auto*
ultimately show *?thesis* **by** (*simp add*: *fps-mult-nth sum.union-disjoint*)
qed

lemma *fps-mult-nth-conv-inside-subdegrees*:
fixes *f g* :: ('*a* :: {*mult-zero*,*comm-monoid-add*}) *fps*
shows (*f* ∗ *g*) $ *n* = (∑ *i*=*subdegree f*..*n* − *subdegree g*. *f* $ *i* ∗ *g* $ (*n* − *i*))
proof (*cases subdegree f* ≤ *n* − *subdegree g*)
case *True*
hence {*subdegree f*..*n*} = {*subdegree f*..*n* − *subdegree g*} ∪ {*n* − *subdegree g*<..*n*}
by *auto*
moreover have {*subdegree f*..*n* − *subdegree g*} ∩ {*n* − *subdegree g*<..*n*} = {}
by *auto*
moreover have ∀ *i*∈{*n* − *subdegree g*<..*n*}. *f* $ *i* ∗ *g* $ (*n* − *i*) = *0*
using *nth-less-subdegree-zero*[*of* - *g*] **by** *auto*
ultimately show *?thesis*
using *fps-mult-nth-conv-upto-subdegree-left*[*of f g n*]
by (*simp add*: *sum.union-disjoint*)
next
case *False*
hence *1*: *subdegree f* > *n* − *subdegree g* **by** *simp*
show *?thesis*
proof (*cases f*∗*g* = *0*)
case *False*
with *1* **have** *n* < *subdegree* (*f*∗*g*) **using** *fps-mult-subdegree-ge*[*of f g*] **by** *simp*
with *1* **show** *?thesis* **by** *auto*
qed (*simp add*: *1*)

qed

lemma *fps-mult-nth-outside-subdegrees*:
fixes $f\ g :: ('a :: \{mult\text{-}zero, comm\text{-}monoid\text{-}add\})\ fps$
shows $n < subdegree\ f \Longrightarrow (f * g)\ \$\ n = 0$
and $n < subdegree\ g \Longrightarrow (f * g)\ \$\ n = 0$
by (*auto simp*: *fps-mult-nth-conv-inside-subdegrees*)

5.3 Ring structure

instance $fps :: (semigroup\text{-}add)\ semigroup\text{-}add$
proof
fix $a\ b\ c :: 'a\ fps$
show $a + b + c = a + (b + c)$
by (*simp add*: *fps-ext add.assoc*)
qed

instance $fps :: (ab\text{-}semigroup\text{-}add)\ ab\text{-}semigroup\text{-}add$
proof
fix $a\ b :: 'a\ fps$
show $a + b = b + a$
by (*simp add*: *fps-ext add.commute*)
qed

instance $fps :: (monoid\text{-}add)\ monoid\text{-}add$
proof
fix $a :: 'a\ fps$
show $0 + a = a$ **by** (*simp add*: *fps-ext*)
show $a + 0 = a$ **by** (*simp add*: *fps-ext*)
qed

instance $fps :: (comm\text{-}monoid\text{-}add)\ comm\text{-}monoid\text{-}add$
proof
fix $a :: 'a\ fps$
show $0 + a = a$ **by** (*simp add*: *fps-ext*)
qed

instance $fps :: (cancel\text{-}semigroup\text{-}add)\ cancel\text{-}semigroup\text{-}add$
proof
fix $a\ b\ c :: 'a\ fps$
show $b = c$ **if** $a + b = a + c$
using *that* **by** (*simp add*: *expand-fps-eq*)
show $b = c$ **if** $b + a = c + a$
using *that* **by** (*simp add*: *expand-fps-eq*)
qed

instance $fps :: (cancel\text{-}ab\text{-}semigroup\text{-}add)\ cancel\text{-}ab\text{-}semigroup\text{-}add$
proof
fix $a\ b\ c :: 'a\ fps$

```
  show a + b − a = b
    by (simp add: expand-fps-eq)
  show a − b − c = a − (b + c)
    by (simp add: expand-fps-eq diff-diff-eq)
qed

instance fps :: (cancel-comm-monoid-add) cancel-comm-monoid-add ..

instance fps :: (group-add) group-add
proof
  fix a b :: 'a fps
  show − a + a = 0 by (simp add: fps-ext)
  show a + − b = a − b by (simp add: fps-ext)
qed

instance fps :: (ab-group-add) ab-group-add
proof
  fix a b :: 'a fps
  show − a + a = 0 by (simp add: fps-ext)
  show a − b = a + − b by (simp add: fps-ext)
qed

instance fps :: (zero-neq-one) zero-neq-one
  by standard (simp add: expand-fps-eq)

lemma fps-mult-assoc-lemma:
  fixes k :: nat
    and f :: nat ⇒ nat ⇒ nat ⇒ 'a::comm-monoid-add
  shows (∑j=0..k. ∑i=0..j. f i (j − i) (n − j)) =
        (∑j=0..k. ∑i=0..k − j. f j i (n − j − i))
  by (induct k) (simp-all add: Suc-diff-le sum.distrib add.assoc)

instance fps :: (semiring-0) semiring-0
proof
  fix a b c :: 'a fps
  show (a + b) * c = a * c + b * c
    by (simp add: expand-fps-eq fps-mult-nth distrib-right sum.distrib)
  show a * (b + c) = a * b + a * c
    by (simp add: expand-fps-eq fps-mult-nth distrib-left sum.distrib)
  show (a * b) * c = a * (b * c)
  proof (rule fps-ext)
    fix n :: nat
    have (∑j=0..n. ∑i=0..j. a$i * b$(j − i) * c$(n − j)) =
          (∑j=0..n. ∑i=0..n − j. a$j * b$i * c$(n − j − i))
      by (rule fps-mult-assoc-lemma)
    then show ((a * b) * c) $ n = (a * (b * c)) $ n
      by (simp add: fps-mult-nth sum-distrib-left sum-distrib-right mult.assoc)
  qed
qed
```

instance *fps* :: (*semiring-0-cancel*) *semiring-0-cancel* **..**

lemma *fps-mult-commute-lemma*:
  fixes *n* :: *nat*
    and *f* :: *nat* ⇒ *nat* ⇒ *'a*::*comm-monoid-add*
  shows (∑ *i*=*0*..*n*. *f i* (*n* − *i*)) = (∑ *i*=*0*..*n*. *f* (*n* − *i*) *i*)
  by (*rule sum.reindex-bij-witness*[**where** *i*=(−) *n* **and** *j*=(−) *n*]) *auto*

instance *fps* :: (*comm-semiring-0*) *comm-semiring-0*
proof
  fix *a b c* :: *'a fps*
  show *a* ∗ *b* = *b* ∗ *a*
  proof (*rule fps-ext*)
    fix *n* :: *nat*
    have (∑ *i*=*0*..*n*. *a*$*i* ∗ *b*$(*n* − *i*)) = (∑ *i*=*0*..*n*. *a*$(*n* − *i*) ∗ *b*$*i*)
      by (*rule fps-mult-commute-lemma*)
    then show (*a* ∗ *b*) $ *n* = (*b* ∗ *a*) $ *n*
      by (*simp add*: *fps-mult-nth mult.commute*)
  qed
qed (*simp add*: *distrib-right*)

instance *fps* :: (*comm-semiring-0-cancel*) *comm-semiring-0-cancel* **..**

instance *fps* :: (*semiring-1*) *semiring-1*
proof
  fix *a* :: *'a fps*
  show *1* ∗ *a* = *a a* ∗ *1* = *a* **by** (*simp-all add*: *fps-one-mult*)
qed

instance *fps* :: (*comm-semiring-1*) *comm-semiring-1*
  by *standard simp*

instance *fps* :: (*semiring-1-cancel*) *semiring-1-cancel* **..**

lemma *fps-square-nth*: (*f*^*2*) $ *n* = (∑ *k*≤*n*. *f* $ *k* ∗ *f* $ (*n* − *k*))
  by (*simp add*: *power2-eq-square fps-mult-nth atLeast0AtMost*)

lemma *fps-sum-nth*: *sum f S* $ *n* = *sum* (λ*k*. (*f k*) $ *n*) *S*
proof (*cases finite S*)
  case *True*
  then show *?thesis* **by** (*induct set*: *finite*) *auto*
next
  case *False*
  then show *?thesis* **by** *simp*
qed

definition *fps-const c* = *Abs-fps* (λ*n*. *if n* = *0 then c else 0*)

lemma *fps-nth-fps-const* [*simp*]: *fps-const c $ n* = (*if n* = *0 then c else 0*)
unfolding *fps-const-def* **by** *simp*

lemma *fps-const-0-eq-0* [*simp*]: *fps-const 0* = *0*
by (*simp add*: *fps-ext*)

lemma *fps-const-nonzero-eq-nonzero*: *c* ≠ *0* ⟹ *fps-const c* ≠ *0*
using *fps-nonzeroI*[*of fps-const c 0*] **by** *simp*

lemma *fps-const-eq-0-iff* [*simp*]: *fps-const c* = *0* ⟷ *c* = *0*
by (*auto simp*: *fps-eq-iff*)

lemma *fps-const-1-eq-1* [*simp*]: *fps-const 1* = *1*
by (*simp add*: *fps-ext*)

lemma *fps-const-eq-1-iff* [*simp*]: *fps-const c* = *1* ⟷ *c* = *1*
by (*auto simp*: *fps-eq-iff*)

lemma *subdegree-fps-const* [*simp*]: *subdegree* (*fps-const c*) = *0*
by (*cases c* = *0*) (*auto intro*!: *subdegreeI*)

lemma *fps-const-neg* [*simp*]: − (*fps-const* (*c*::′*a*::*group-add*)) = *fps-const* (− *c*)
by (*simp add*: *fps-ext*)

lemma *fps-const-add* [*simp*]: *fps-const* (*c*::′*a*::*monoid-add*) + *fps-const d* = *fps-const* (*c* + *d*)
by (*simp add*: *fps-ext*)

lemma *fps-const-add-left*: *fps-const* (*c*::′*a*::*monoid-add*) + *f* =
Abs-fps (λ*n*. *if n* = *0 then c* + *f*$*0 else f*$*n*)
by (*simp add*: *fps-ext*)

lemma *fps-const-add-right*: *f* + *fps-const* (*c*::′*a*::*monoid-add*) =
Abs-fps (λ*n*. *if n* = *0 then f*$*0* + *c else f*$*n*)
by (*simp add*: *fps-ext*)

lemma *fps-const-sub* [*simp*]: *fps-const* (*c*::′*a*::*group-add*) − *fps-const d* = *fps-const* (*c* − *d*)
by (*simp add*: *fps-ext*)

lemmas *fps-const-minus* = *fps-const-sub*

lemma *fps-const-mult*[*simp*]:
fixes *c d* :: ′*a*::{*comm-monoid-add*,*mult-zero*}
shows *fps-const c* ∗ *fps-const d* = *fps-const* (*c* ∗ *d*)
by (*simp add*: *fps-eq-iff fps-mult-nth sum.neutral*)

lemma *fps-const-mult-left*:

fps-const (*c*::'*a*::{*comm-monoid-add*,*mult-zero*}) * *f* = *Abs-fps* (λ*n*. *c* * *f*$*n*)
unfolding *fps-eq-iff fps-mult-nth*
by (*simp add*: *fps-const-def mult-delta-left*)

lemma *fps-const-mult-right*:
f * *fps-const* (*c*::'*a*::{*comm-monoid-add*,*mult-zero*}) = *Abs-fps* (λ*n*. *f*$*n* * *c*)
unfolding *fps-eq-iff fps-mult-nth*
by (*simp add*: *fps-const-def mult-delta-right*)

lemma *fps-mult-left-const-nth* [*simp*]:
(*fps-const* (*c*::'*a*::{*comm-monoid-add*,*mult-zero*}) * *f*)$*n* = *c* * *f*$*n*
by (*simp add*: *fps-mult-nth mult-delta-left*)

lemma *fps-mult-right-const-nth* [*simp*]:
(*f* * *fps-const* (*c*::'*a*::{*comm-monoid-add*,*mult-zero*}))$*n* = *f*$*n* * *c*
by (*simp add*: *fps-mult-nth mult-delta-right*)

lemma *fps-const-power* [*simp*]: *fps-const c* ^ *n* = *fps-const* (*c* ^ *n*)
by (*induct n*) *auto*

instance *fps* :: (*ring*) *ring* **..**

instance *fps* :: (*comm-ring*) *comm-ring* **..**

instance *fps* :: (*ring-1*) *ring-1* **..**

instance *fps* :: (*comm-ring-1*) *comm-ring-1* **..**

instance *fps* :: (*semiring-no-zero-divisors*) *semiring-no-zero-divisors*
proof
fix *a b* :: '*a fps*
assume *a* ≠ *0* **and** *b* ≠ *0*
hence (*a* * *b*) $ (*subdegree a* + *subdegree b*) ≠ *0* **by** *simp*
thus *a* * *b* ≠ *0* **using** *fps-nonzero-nth* **by** *fast*
qed

instance *fps* :: (*semiring-1-no-zero-divisors*) *semiring-1-no-zero-divisors* **..**

instance *fps* :: ({*cancel-semigroup-add*,*semiring-no-zero-divisors-cancel*})
semiring-no-zero-divisors-cancel
proof
fix *a b c* :: '*a fps*
show (*a* * *c* = *b* * *c*) = (*c* = *0* ∨ *a* = *b*)
proof
assume *ab*: *a* * *c* = *b* * *c*
have *c* ≠ *0* ⟹ *a* = *b*
proof (*rule fps-ext*)
fix *n*

```
      assume c: c ≠ 0
      show a $ n = b $ n
      proof (induct n rule: nat-less-induct)
        case (1 n)
        with ab c show ?case
          using fps-mult-nth-conv-upto-subdegree-right[of a c subdegree c + n]
                fps-mult-nth-conv-upto-subdegree-right[of b c subdegree c + n]
          by    (cases n) auto
      qed
    qed
    thus c = 0 ∨ a = b by fast
  qed auto
  show (c * a = c * b) = (c = 0 ∨ a = b)
  proof
    assume ab: c * a = c * b
    have c ≠ 0 ⟹ a = b
    proof (rule fps-ext)
      fix n
      assume c: c ≠ 0
      show a $ n = b $ n
      proof (induct n rule: nat-less-induct)
        case (1 n)
        moreover have ∀ i∈{Suc (subdegree c)..subdegree c + n}. subdegree c + n
− i < n by auto
        ultimately show ?case
          using ab c fps-mult-nth-conv-upto-subdegree-left[of c a subdegree c + n]
                fps-mult-nth-conv-upto-subdegree-left[of c b subdegree c + n]
          by    (simp add: sum.atLeast-Suc-atMost)
      qed
    qed
    thus c = 0 ∨ a = b by fast
  qed auto
qed

instance fps :: (ring-no-zero-divisors) ring-no-zero-divisors ..

instance fps :: (ring-1-no-zero-divisors) ring-1-no-zero-divisors ..

instance fps :: (idom) idom ..

lemma fps-of-nat: fps-const (of-nat c) = of-nat c
  by (induction c) (simp-all add: fps-const-add [symmetric] del: fps-const-add)

lemma fps-of-int: fps-const (of-int c) = of-int c
  by (induction c) (simp-all add: fps-const-minus [symmetric] fps-of-nat fps-const-neg
[symmetric]
                                  del: fps-const-minus fps-const-neg)

lemma semiring-char-fps [simp]: CHAR('a :: comm-semiring-1 fps) = CHAR('a)
```

by (*rule CHAR-eqI*) (*auto simp flip*: *fps-of-nat simp*: *of-nat-eq-0-iff-char-dvd*)

instance *fps* :: ({*semiring-prime-char*,*comm-semiring-1*}) *semiring-prime-char*
by (*rule semiring-prime-charI*) *auto*
instance *fps* :: ({*comm-semiring-prime-char*,*comm-semiring-1*}) *comm-semiring-prime-char*
by *standard*
instance *fps* :: ({*comm-ring-prime-char*,*comm-semiring-1*}) *comm-ring-prime-char*
by *standard*
instance *fps* :: ({*idom-prime-char*,*comm-semiring-1*}) *idom-prime-char*
by *standard*

lemma *fps-numeral-fps-const*: *numeral k* = *fps-const* (*numeral k*)
by (*induct k*) (*simp-all only*: *numeral.simps fps-const-1-eq-1 fps-const-add* [*symmetric*])

lemmas *numeral-fps-const* = *fps-numeral-fps-const*

lemma *neg-numeral-fps-const*:
(− *numeral k* :: *'a* :: *ring-1 fps*) = *fps-const* (− *numeral k*)
by (*simp add*: *numeral-fps-const*)

lemma *fps-numeral-nth*: *numeral n* $ *i* = (*if i* = *0 then numeral n else 0*)
by (*simp add*: *numeral-fps-const*)

lemma *fps-numeral-nth-0* [*simp*]: *numeral n* $ *0* = *numeral n*
by (*simp add*: *numeral-fps-const*)

lemma *subdegree-numeral* [*simp*]: *subdegree* (*numeral n*) = *0*
by (*simp add*: *numeral-fps-const*)

lemma *fps-nth-of-nat* [*simp*]:
(*of-nat c*) $ *n* = (*if n=0 then of-nat c else 0*)
by (*simp add*: *fps-of-nat*[*symmetric*])

lemma *fps-nth-of-int* [*simp*]:
(*of-int c*) $ *n* = (*if n=0 then of-int c else 0*)
by (*simp add*: *fps-of-int*[*symmetric*])

lemma *fps-mult-of-nat-nth* [*simp*]:
shows (*of-nat k* ∗ *f*) $ *n* = *of-nat k* ∗ *f*$*n*
and (*f* ∗ *of-nat k*) $ *n* = *f*$*n* ∗ *of-nat k*
by (*simp-all add*: *fps-of-nat*[*symmetric*])

lemma *fps-mult-of-int-nth* [*simp*]:
shows (*of-int k* ∗ *f*) $ *n* = *of-int k* ∗ *f*$*n*
and (*f* ∗ *of-int k*) $ *n* = *f*$*n* ∗ *of-int k*
by (*simp-all add*: *fps-of-int*[*symmetric*])

lemma *numeral-neq-fps-zero* [*simp*]: (*numeral f* :: *'a* :: *field-char-0 fps*) ≠ *0*
proof

```
  assume numeral f = (0 :: 'a fps)
  from arg-cong[of - - λF. F $ 0, OF this] show False by simp
qed

instance fps :: (semiring-char-0) semiring-char-0
proof
  show inj (of-nat :: nat ⇒ 'a fps)
  proof
    fix m n :: nat
    assume of-nat m = (of-nat n :: 'a fps)
    hence fps-nth (of-nat m) 0 = (fps-nth (of-nat n) 0 :: 'a)
      by (simp only: )
    thus m = n
      by simp
  qed
qed

lemma subdegree-power-ge:
  f^n ≠ 0 ⟹ subdegree (f^n) ≥ n * subdegree f
proof (induct n)
  case (Suc n) thus ?case using fps-mult-subdegree-ge by fastforce
qed simp

lemma fps-pow-nth-below-subdegree:
  k < n * subdegree f ⟹ (f^n) $ k = 0
proof (cases f^n = 0)
  case False
  assume k < n * subdegree f
  with False have k < subdegree (f^n) using subdegree-power-ge[of f n] by simp
  thus (f^n) $ k = 0 by auto
qed simp

lemma fps-pow-base [simp]:
  (f ^ n) $ (n * subdegree f) = (f $ subdegree f) ^ n
proof (induct n)
  case (Suc n)
  show ?case
  proof (cases Suc n * subdegree f < subdegree f + subdegree (f^n))
    case True with Suc show ?thesis
      by (auto simp: fps-mult-nth-eq0 distrib-right)
  next
    case False
    hence ∀ i∈{Suc (subdegree f)..Suc n * subdegree f − subdegree (f ^ n)}.
          f ^ n $ (Suc n * subdegree f − i) = 0
     by (auto simp: fps-pow-nth-below-subdegree)
   with False Suc show ?thesis
     using fps-mult-nth-conv-inside-subdegrees[of f f^n Suc n * subdegree f]
           sum.atLeast-Suc-atMost[of
             subdegree f
```

```
            Suc n * subdegree f - subdegree (f ^ n)
            λi. f $ i * f ^ n $ (Suc n * subdegree f - i)
          ]
    by    simp
  qed
qed simp

lemma subdegree-power-eqI:
  fixes f :: 'a::semiring-1 fps
  shows (f $ subdegree f) ^ n ≠ 0 ⟹ subdegree (f ^ n) = n * subdegree f
proof (induct n)
  case (Suc n)
  from Suc have 1: subdegree (f ^ n) = n * subdegree f by fastforce
  with Suc(2) have f $ subdegree f * f ^ n $ subdegree (f ^ n) ≠ 0 by simp
  with 1 show ?case using subdegree-mult'[of f f^n] by simp
qed simp

lemma subdegree-power [simp]:
  subdegree ((f :: ('a :: semiring-1-no-zero-divisors) fps) ^ n) = n * subdegree f
  by (cases f = 0; induction n) simp-all

lemma subdegree-prod:
  fixes f :: 'a ⇒ 'b :: idom fps
  assumes ⋀x. x ∈ A ⟹ f x ≠ 0
  shows   subdegree (∏ x∈A. f x) = (∑ x∈A. subdegree (f x))
  using assms by (induction A rule: infinite-finite-induct) auto


lemma minus-one-power-iff: (- (1::'a::ring-1)) ^ n = (if even n then 1 else - 1)
  by (induct n) auto

definition fps-X = Abs-fps (λn. if n = 1 then 1 else 0)

lemma subdegree-fps-X [simp]: subdegree (fps-X :: ('a :: zero-neq-one) fps) = 1
  by (auto intro!: subdegreeI simp: fps-X-def)

lemma fps-X-mult-nth [simp]:
  fixes f :: 'a::{comm-monoid-add,mult-zero,monoid-mult} fps
  shows (fps-X * f) $ n = (if n = 0 then 0 else f $ (n - 1))
proof (cases n)
  case (Suc m)
  moreover have (fps-X * f) $ Suc m = f $ (Suc m - 1)
  proof (cases m)
    case 0 thus ?thesis using fps-mult-nth-1[of fps-X f] by (simp add: fps-X-def)
  next
   case (Suc k) thus ?thesis by (simp add: fps-mult-nth fps-X-def sum.atLeast-Suc-atMost)
  qed
  ultimately show ?thesis by simp
qed (simp add: fps-X-def)
```

lemma *fps-X-mult-right-nth* [*simp*]:
fixes *a* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows (*a* ∗ *fps-X*) $ *n* = (*if n* = *0 then 0 else a* $ (*n* − *1*))
proof (*cases n*)
case (*Suc m*)
moreover have (*a* ∗ *fps-X*) $ *Suc m* = *a* $ (*Suc m* − *1*)
proof (*cases m*)
case *0* **thus** *?thesis* **using** *fps-mult-nth-1*[*of a fps-X*] **by** (*simp add*: *fps-X-def*)
next
case (*Suc k*)
hence (*a* ∗ *fps-X*) $ *Suc m* = (∑ *i*=*0*..*k*. *a*$*i* ∗ *fps-X*$(*Suc m* − *i*)) + *a*$(*Suc k*)
by (*simp add*: *fps-mult-nth fps-X-def*)
moreover have ∀ *i*∈{*0*..*k*}. *a*$*i* ∗ *fps-X*$(*Suc m* − *i*) = *0* **by** (*auto simp*: *Suc fps-X-def*)
ultimately show *?thesis* **by** (*simp add*: *Suc*)
qed
ultimately show *?thesis* **by** *simp*
qed (*simp add*: *fps-X-def*)

lemma *fps-mult-fps-X-commute*:
fixes *a* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *fps-X* ∗ *a* = *a* ∗ *fps-X*
by (*simp add*: *fps-eq-iff*)

lemma *fps-mult-fps-X-power-commute*: *fps-X* ^ *k* ∗ *a* = *a* ∗ *fps-X* ^ *k*
proof (*induct k*)
case (*Suc k*)
hence *fps-X* ^ *Suc k* ∗ *a* = *a* ∗ *fps-X* ∗ *fps-X* ^ *k*
by (*simp add*: *mult.assoc fps-mult-fps-X-commute*[*symmetric*])
thus *?case* **by** (*simp add*: *mult.assoc*)
qed *simp*

lemma *fps-subdegree-mult-fps-X*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
assumes *f* ≠ *0*
shows *subdegree* (*fps-X* ∗ *f*) = *subdegree f* + *1*
and *subdegree* (*f* ∗ *fps-X*) = *subdegree f* + *1*
proof−
show *subdegree* (*fps-X* ∗ *f*) = *subdegree f* + *1*
proof (*intro subdegreeI*)
fix *i* :: *nat* **assume** *i*: *i* < *subdegree f* + *1*
show (*fps-X* ∗ *f*) $ *i* = *0*
proof (*cases i*=*0*)
case *False* **with** *i* **show** *?thesis* **by** (*simp add*: *nth-less-subdegree-zero*)
next
case *True* **thus** *?thesis* **using** *fps-X-mult-nth*[*of f i*] **by** *simp*
qed

qed (*simp add*: *assms*)
thus *subdegree* (*f* ∗ *fps-X*) = *subdegree f* + *1*
by (*simp add*: *fps-mult-fps-X-commute*)
qed

lemma *fps-mult-fps-X-nonzero*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
assumes *f* ≠ *0*
shows *fps-X* ∗ *f* ≠ *0*
and *f* ∗ *fps-X* ≠ *0*
using *assms fps-subdegree-mult-fps-X*[*of f*]
fps-nonzero-subdegree-nonzeroI[*of fps-X* ∗ *f*]
fps-nonzero-subdegree-nonzeroI[*of f* ∗ *fps-X*]
by *auto*

lemma *fps-mult-fps-X-power-nonzero*:
assumes *f* ≠ *0*
shows *fps-X* ^ *n* ∗ *f* ≠ *0*
and *f* ∗ *fps-X* ^ *n* ≠ *0*
proof −
show *fps-X* ^ *n* ∗ *f* ≠ *0*
by (*induct n*) (*simp-all add*: *assms mult.assoc fps-mult-fps-X-nonzero*(*1*))
thus *f* ∗ *fps-X* ^ *n* ≠ *0*
by (*simp add*: *fps-mult-fps-X-power-commute*)
qed

lemma *fps-X-power-iff*: *fps-X* ^ *n* = *Abs-fps* (λ*m*. *if m* = *n then 1 else 0*)
by (*induction n*) (*auto simp*: *fps-eq-iff*)

lemma *fps-X-nth*[*simp*]: *fps-X*$*n* = (*if n* = *1 then 1 else 0*)
by (*simp add*: *fps-X-def*)

lemma *fps-X-power-nth*[*simp*]: (*fps-X*^*k*) $*n* = (*if n* = *k then 1 else 0*)
by (*simp add*: *fps-X-power-iff*)

lemma *fps-X-power-subdegree*: *subdegree* (*fps-X*^*n*) = *n*
by (*auto intro*: *subdegreeI*)

lemma *fps-X-power-mult-nth*:
(*fps-X*^*k* ∗ *f*) $ *n* = (*if n* < *k then 0 else f* $ (*n* − *k*))
by (*cases n*<*k*)
(*simp-all add*: *fps-mult-nth-conv-upto-subdegree-left fps-X-power-subdegree sum.atLeast-Suc-atMost*)

lemma *fps-X-power-mult-right-nth*:
(*f* ∗ *fps-X*^*k*) $ *n* = (*if n* < *k then 0 else f* $ (*n* − *k*))
using *fps-mult-fps-X-power-commute*[*of k f*] *fps-X-power-mult-nth*[*of k f*] **by** *simp*

lemma *fps-subdegree-mult-fps-X-power*:

assumes $f \neq 0$
shows *subdegree* (*fps-X* ^ $n * f$) = *subdegree* $f + n$
and *subdegree* ($f *$ *fps-X* ^ n) = *subdegree* $f + n$
proof −
from *assms* **show** *subdegree* (*fps-X* ^ $n * f$) = *subdegree* $f + n$
by (*induct n*)
(*simp-all add*: *algebra-simps fps-subdegree-mult-fps-X*(*1*) *fps-mult-fps-X-power-nonzero*(*1*))
thus *subdegree* ($f *$ *fps-X* ^ n) = *subdegree* $f + n$
by (*simp add*: *fps-mult-fps-X-power-commute*)
qed

lemma *fps-mult-fps-X-plus-1-nth*:
((*1*+*fps-X*)∗*a*) $n = (*if* n = 0 *then* (a$n :: 'a::*semiring-1*) *else* a$n + a$(n − 1))
proof (*cases n*)
case *0*
then show *?thesis*
by (*simp add*: *fps-mult-nth*)
next
case (*Suc m*)
have ((*1* + *fps-X*)∗*a*) $ n = *sum* (λ*i*. (*1* + *fps-X*) $ *i* ∗ *a* $ (*n* − *i*)) {*0*..*n*}
by (*simp add*: *fps-mult-nth*)
also have ... = *sum* (λ*i*. (*1*+*fps-X*)$*i* ∗ a$(*n*−*i*)) {*0*.. *1*}
unfolding *Suc* **by** (*rule sum.mono-neutral-right*) *auto*
also have ... = (*if n* = *0 then a*$*n else a*$*n* + *a*$(*n* − *1*))
by (*simp add*: *Suc*)
finally show *?thesis* **.**
qed

lemma *fps-mult-right-fps-X-plus-1-nth*:
fixes *a* :: 'a :: *semiring-1 fps*
shows (*a*∗(*1*+*fps-X*)) $ *n* = (*if n* = *0 then a*$*n else a*$*n* + *a*$(*n* − *1*))
using *fps-mult-fps-X-plus-1-nth*
by (*simp add*: *distrib-left fps-mult-fps-X-commute distrib-right*)

lemma *fps-X-neq-fps-const* [*simp*]: (*fps-X* :: 'a :: *zero-neq-one fps*) ≠ *fps-const c*
proof
assume (*fps-X*::'a *fps*) = *fps-const* (*c*::'a)
hence *fps-X*$*1* = (*fps-const* (*c*::'a))$*1* **by** (*simp only*:)
thus *False* **by** *auto*
qed

lemma *fps-X-neq-zero* [*simp*]: (*fps-X* :: 'a :: *zero-neq-one fps*) ≠ *0*
by (*simp only*: *fps-const-0-eq-0*[*symmetric*] *fps-X-neq-fps-const*) *simp*

lemma *fps-X-neq-one* [*simp*]: (*fps-X* :: 'a :: *zero-neq-one fps*) ≠ *1*
by (*simp only*: *fps-const-1-eq-1*[*symmetric*] *fps-X-neq-fps-const*) *simp*

lemma *fps-X-neq-numeral* [*simp*]: *fps-X* ≠ *numeral c*

by (*simp only*: *numeral-fps-const fps-X-neq-fps-const*) *simp*

lemma *fps-X-pow-eq-fps-X-pow-iff* [*simp*]: *fps-X* ^ *m* = *fps-X* ^ *n* ⟷ *m* = *n*
proof
assume (*fps-X* :: *'a fps*) ^ *m* = *fps-X* ^ *n*
hence (*fps-X* :: *'a fps*) ^ *m* $ *m* = *fps-X* ^ *n* $ *m* **by** (*simp only*:)
thus *m* = *n* **by** (*simp split*: *if-split-asm*)
qed *simp-all*

5.4 Shifting and slicing

definition *fps-shift* :: *nat* ⇒ *'a fps* ⇒ *'a fps* **where**
fps-shift n f = *Abs-fps* (λ*i*. *f* $ (*i* + *n*))

lemma *fps-shift-nth* [*simp*]: *fps-shift n f* $ *i* = *f* $ (*i* + *n*)
by (*simp add*: *fps-shift-def*)

lemma *fps-shift-0* [*simp*]: *fps-shift 0 f* = *f*
by (*intro fps-ext*) (*simp add*: *fps-shift-def*)

lemma *fps-shift-zero* [*simp*]: *fps-shift n 0* = *0*
by (*intro fps-ext*) (*simp add*: *fps-shift-def*)

lemma *fps-shift-one*: *fps-shift n 1* = (*if n* = *0 then 1 else 0*)
by (*intro fps-ext*) (*simp add*: *fps-shift-def*)

lemma *fps-shift-fps-const*: *fps-shift n* (*fps-const c*) = (*if n* = *0 then fps-const c else 0*)
by (*intro fps-ext*) (*simp add*: *fps-shift-def*)

lemma *fps-shift-numeral*: *fps-shift n* (*numeral c*) = (*if n* = *0 then numeral c else 0*)
by (*simp add*: *numeral-fps-const fps-shift-fps-const*)

lemma *fps-shift-fps-X* [*simp*]:
n ≥ *1* ⟹ *fps-shift n fps-X* = (*if n* = *1 then 1 else 0*)
by (*intro fps-ext*) (*auto simp*: *fps-X-def*)

lemma *fps-shift-fps-X-power* [*simp*]:
n ≤ *m* ⟹ *fps-shift n* (*fps-X* ^ *m*) = *fps-X* ^ (*m* − *n*)
by (*intro fps-ext*) *auto*

lemma *fps-shift-subdegree* [*simp*]:
n ≤ *subdegree f* ⟹ *subdegree* (*fps-shift n f*) = *subdegree f* − *n*
by (*cases f*=*0*) (*auto intro*: *subdegreeI simp*: *nth-less-subdegree-zero*)

lemma *fps-shift-fps-shift*:
fps-shift (*m* + *n*) *f* = *fps-shift m* (*fps-shift n f*)
by (*rule fps-ext*) (*simp add*: *add-ac*)

lemma *fps-shift-fps-shift-reorder*:
fps-shift m (fps-shift n f) = fps-shift n (fps-shift m f)
using *fps-shift-fps-shift*[*of m n f*] *fps-shift-fps-shift*[*of n m f*] **by** (*simp add*: *add.commute*)

lemma *fps-shift-rev-shift*:
m ≤ n ⟹ fps-shift n (Abs-fps (λk. if k<m then 0 else f $ (k−m))) = fps-shift (n−m) f
m > n ⟹ fps-shift n (Abs-fps (λk. if k<m then 0 else f $ (k−m))) =
Abs-fps (λk. if k<m−n then 0 else f $ (k−(m−n)))
proof −
assume *m ≤ n*
thus *fps-shift n (Abs-fps (λk. if k<m then 0 else f $ (k−m))) = fps-shift (n−m) f*
by (*intro fps-ext*) *auto*
next
assume *mn*: *m > n*
hence ⋀*k*. *k ≥ m−n ⟹ k+n−m = k − (m−n)* **by** *auto*
thus
fps-shift n (Abs-fps (λk. if k<m then 0 else f $ (k−m))) =
Abs-fps (λk. if k<m−n then 0 else f $ (k−(m−n)))
by (*intro fps-ext*) *auto*
qed

lemma *fps-shift-add*:
fps-shift n (f + g) = fps-shift n f + fps-shift n g
by (*simp add*: *fps-eq-iff*)

lemma *fps-shift-diff*:
fps-shift n (f − g) = fps-shift n f − fps-shift n g
by (*auto intro*: *fps-ext*)

lemma *fps-shift-uminus*:
fps-shift n (−f) = − fps-shift n f
by (*auto intro*: *fps-ext*)

lemma *fps-shift-mult*:
assumes *n ≤ subdegree (g :: 'b :: {comm-monoid-add, mult-zero} fps)*
shows *fps-shift n (h∗g) = h ∗ fps-shift n g*
proof−
have *case1*: ⋀*a b*::'*b fps*. *1 ≤ subdegree b ⟹ fps-shift 1 (a∗b) = a ∗ fps-shift 1 b*
proof (*rule fps-ext*)
fix *a b* :: '*b fps*
and *n* :: *nat*
assume *b*: *1 ≤ subdegree b*
have ⋀*i*. *i ≤ n ⟹ n + 1 − i = (n−i) + 1*
by (*simp add*: *algebra-simps*)

```
    with b show fps-shift 1 (a*b) $ n = (a * fps-shift 1 b) $ n
      by (simp add: fps-mult-nth nth-less-subdegree-zero)
  qed
  have n ≤ subdegree g ⟹ fps-shift n (h*g) = h * fps-shift n g
  proof (induct n)
    case (Suc n)
    have fps-shift (Suc n) (h*g) = fps-shift 1 (fps-shift n (h*g))
      by (simp add: fps-shift-fps-shift[symmetric])
    also have ... = h * (fps-shift 1 (fps-shift n g))
      using Suc case1 by force
    finally show ?case by (simp add: fps-shift-fps-shift[symmetric])
  qed simp
  with assms show ?thesis by fast
qed

lemma fps-shift-mult-right-noncomm:
  assumes n ≤ subdegree (g :: 'b :: {comm-monoid-add, mult-zero} fps)
  shows fps-shift n (g*h) = fps-shift n g * h
proof−
  have case1: ⋀a b::'b fps. 1 ≤ subdegree a ⟹ fps-shift 1 (a*b) = fps-shift 1 a *
b
  proof (rule fps-ext)
    fix a b :: 'b fps
    and n :: nat
    assume 1 ≤ subdegree a
    hence fps-shift 1 (a*b) $ n = (∑ i=Suc 0..Suc n. a$i * b$(n+1−i))
      using sum.atLeast-Suc-atMost[of 0 n+1 λi. a$i * b$(n+1−i)]
      by    (simp add: fps-mult-nth nth-less-subdegree-zero)
    thus fps-shift 1 (a*b) $ n = (fps-shift 1 a * b) $ n
      using sum.shift-bounds-cl-Suc-ivl[of λi. a$i * b$(n+1−i) 0 n]
      by    (simp add: fps-mult-nth)
  qed
  have n ≤ subdegree g ⟹ fps-shift n (g*h) = fps-shift n g * h
  proof (induct n)
    case (Suc n)
    have fps-shift (Suc n) (g*h) = fps-shift 1 (fps-shift n (g*h))
      by (simp add: fps-shift-fps-shift[symmetric])
    also have ... = (fps-shift 1 (fps-shift n g)) * h
      using Suc case1 by force
    finally show ?case by (simp add: fps-shift-fps-shift[symmetric])
  qed simp
  with assms show ?thesis by fast
qed

lemma fps-shift-mult-right:
  assumes n ≤ subdegree (g :: 'b :: comm-semiring-0 fps)
  shows   fps-shift n (g*h) = h * fps-shift n g
  by      (simp add: assms fps-shift-mult-right-noncomm mult.commute)
```

lemma *fps-shift-mult-both*:
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fps*
assumes $m \leq$ *subdegree f n* $\leq$ *subdegree g*
shows *fps-shift m f* $*$ *fps-shift n g* = *fps-shift* $(m+n)$ $(f*g)$
using *assms*
by (*simp add*: *fps-shift-mult fps-shift-mult-right-noncomm fps-shift-fps-shift*)

lemma *fps-shift-subdegree-zero-iff* [*simp*]:
fps-shift (*subdegree f*) *f* = *0* $\longleftrightarrow$ *f* = *0*
by (*subst* (*1*) *nth-subdegree-zero-iff*[*symmetric*], *cases f* = *0*)
(*simp-all del*: *nth-subdegree-zero-iff*)

lemma *fps-shift-times-fps-X*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *1* $\leq$ *subdegree f* $\Longrightarrow$ *fps-shift 1 f* $*$ *fps-X* = *f*
by (*intro fps-ext*) (*simp add*: *nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X'* [*simp*]:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *fps-shift 1* (*f* $*$ *fps-X*) = *f*
by (*intro fps-ext*) (*simp add*: *nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X''*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fps*
shows *1* $\leq$ *n* $\Longrightarrow$ *fps-shift n* (*f* $*$ *fps-X*) = *fps-shift* (*n* $-$ *1*) *f*
by (*intro fps-ext*) (*simp add*: *nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X-power*:
n $\leq$ *subdegree f* $\Longrightarrow$ *fps-shift n f* $*$ *fps-X* ^ *n* = *f*
by (*intro fps-ext*) (*simp add*: *fps-X-power-mult-right-nth nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X-power'* [*simp*]:
fps-shift n (*f* $*$ *fps-X*^*n*) = *f*
by (*intro fps-ext*) (*simp add*: *fps-X-power-mult-right-nth nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X-power''*:
m $\leq$ *n* $\Longrightarrow$ *fps-shift n* (*f* $*$ *fps-X*^*m*) = *fps-shift* (*n* $-$ *m*) *f*
by (*intro fps-ext*) (*simp add*: *fps-X-power-mult-right-nth nth-less-subdegree-zero*)

lemma *fps-shift-times-fps-X-power'''*:
m $>$ *n* $\Longrightarrow$ *fps-shift n* (*f* $*$ *fps-X*^*m*) = *f* $*$ *fps-X*^(*m* $-$ *n*)
proof (*cases f*=*0*)
case *False*
assume *m*: *m*>*n*
hence *m* = *n* + (*m*−*n*) **by** *auto*
with *False m* **show** *?thesis*
using *power-add*[*of fps-X*::*'a fps n m*−*n*]
fps-shift-mult-right-noncomm[*of n f* $*$ *fps-X*^*n fps-X*^(*m*−*n*)]
by (*simp add*: *mult.assoc fps-subdegree-mult-fps-X-power*(*2*))

qed *simp*

lemma *subdegree-decompose*:
f = *fps-shift* (*subdegree f*) *f* ∗ *fps-X* ^ *subdegree f*
by (*rule fps-ext*) (*auto simp*: *fps-X-power-mult-right-nth*)

lemma *subdegree-decompose′*:
n ≤ *subdegree f* ⟹ *f* = *fps-shift n f* ∗ *fps-X*^*n*
by (*rule fps-ext*) (*auto simp*: *fps-X-power-mult-right-nth intro*!: *nth-less-subdegree-zero*)

instantiation *fps* :: (*zero*) *unit-factor*
begin
definition *fps-unit-factor-def* [*simp*]:
unit-factor f = *fps-shift* (*subdegree f*) *f*
instance ..
end

lemma *fps-unit-factor-zero-iff*: *unit-factor* (*f*::′*a*::*zero fps*) = *0* ⟷ *f* = *0*
by *simp*

lemma *fps-unit-factor-nth-0*: *f* ≠ *0* ⟹ *unit-factor f* $ *0* ≠ *0*
by *simp*

lemma *fps-X-unit-factor*: *unit-factor* (*fps-X* :: ′*a* :: *zero-neq-one fps*) = *1*
by (*intro fps-ext*) *auto*

lemma *fps-X-power-unit-factor*: *unit-factor* (*fps-X* ^ *n*) = *1*
proof−
define *X* :: ′*a fps* **where** *X* ≡ *fps-X*
hence *unit-factor* (*X*^*n*) = *fps-shift n* (*X*^*n*)
by (*simp add*: *fps-X-power-subdegree*)
moreover have *fps-shift n* (*X*^*n*) = *1*
by (*auto intro*: *fps-ext simp*: *fps-X-power-iff X-def*)
ultimately show *?thesis* **by** (*simp add*: *X-def*)
qed

lemma *fps-unit-factor-decompose*:
f = *unit-factor f* ∗ *fps-X* ^ *subdegree f*
by (*simp add*: *subdegree-decompose*)

lemma *fps-unit-factor-decompose′*:
f = *fps-X* ^ *subdegree f* ∗ *unit-factor f*
using *fps-unit-factor-decompose* **by** (*simp add*: *fps-mult-fps-X-power-commute*)

lemma *fps-unit-factor-uminus*:
unit-factor (−*f*) = − *unit-factor* (*f*::′*a*::*group-add fps*)
by (*simp add*: *fps-shift-uminus*)

lemma *fps-unit-factor-shift*:

assumes $n \leq$ *subdegree f*
shows *unit-factor* (*fps-shift n f*) = *unit-factor f*
by (*simp add*: *assms fps-shift-fps-shift*[*symmetric*])

lemma *fps-unit-factor-mult-fps-X*:
fixes *f* :: *'a*::{*comm-monoid-add*,*monoid-mult*,*mult-zero*} *fps*
shows *unit-factor* (*fps-X* ∗ *f*) = *unit-factor f*
and *unit-factor* (*f* ∗ *fps-X*) = *unit-factor f*
proof −
show *unit-factor* (*fps-X* ∗ *f*) = *unit-factor f*
by (*cases f=0*) (*auto intro*: *fps-ext simp*: *fps-subdegree-mult-fps-X*(*1*))
thus *unit-factor* (*f* ∗ *fps-X*) = *unit-factor f* **by** (*simp add*: *fps-mult-fps-X-commute*)
qed

lemma *fps-unit-factor-mult-fps-X-power*:
shows *unit-factor* (*fps-X* ^ *n* ∗ *f*) = *unit-factor f*
and *unit-factor* (*f* ∗ *fps-X* ^ *n*) = *unit-factor f*
proof −
show *unit-factor* (*fps-X* ^ *n* ∗ *f*) = *unit-factor f*
proof (*induct n*)
case (*Suc m*) **thus** *?case*
using *fps-unit-factor-mult-fps-X*(*1*)[*of fps-X* ^ *m* ∗ *f*] **by** (*simp add*: *mult.assoc*)
qed *simp*
thus *unit-factor* (*f* ∗ *fps-X* ^ *n*) = *unit-factor f*
by (*simp add*: *fps-mult-fps-X-power-commute*)
qed

lemma *fps-unit-factor-mult-unit-factor*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
shows *unit-factor* (*f* ∗ *unit-factor g*) = *unit-factor* (*f* ∗ *g*)
and *unit-factor* (*unit-factor f* ∗ *g*) = *unit-factor* (*f* ∗ *g*)
proof −
show *unit-factor* (*f* ∗ *unit-factor g*) = *unit-factor* (*f* ∗ *g*)
proof (*cases f*∗*g* = *0*)
case *False* **thus** *?thesis*
using *fps-mult-subdegree-ge*[*of f g*] *fps-unit-factor-shift*[*of subdegree g f*∗*g*]
by (*simp add*: *fps-shift-mult*)
next
case *True*
moreover have *f* ∗ *unit-factor g* = *fps-shift* (*subdegree g*) (*f*∗*g*)
by (*simp add*: *fps-shift-mult*)
ultimately show *?thesis* **by** *simp*
qed
show *unit-factor* (*unit-factor f* ∗ *g*) = *unit-factor* (*f* ∗ *g*)
proof (*cases f*∗*g* = *0*)
case *False* **thus** *?thesis*
using *fps-mult-subdegree-ge*[*of f g*] *fps-unit-factor-shift*[*of subdegree f f*∗*g*]
by (*simp add*: *fps-shift-mult-right-noncomm*)
next

```
    case True
    moreover have unit-factor f * g = fps-shift (subdegree f) (f*g)
      by (simp add: fps-shift-mult-right-noncomm)
    ultimately show ?thesis by simp
  qed
qed

lemma fps-unit-factor-mult-both-unit-factor:
  fixes f g :: 'a::{comm-monoid-add,mult-zero} fps
  shows unit-factor (unit-factor f * unit-factor g) = unit-factor (f * g)
  using fps-unit-factor-mult-unit-factor(1)[of unit-factor f g]
        fps-unit-factor-mult-unit-factor(2)[of f g]
  by    simp

lemma fps-unit-factor-mult':
  fixes   f g :: 'a::{comm-monoid-add,mult-zero} fps
  assumes f $ subdegree f * g $ subdegree g ≠ 0
  shows   unit-factor (f * g) = unit-factor f * unit-factor g
  using   assms
  by      (simp add: subdegree-mult' fps-shift-mult-both)

lemma fps-unit-factor-mult:
  fixes f g :: 'a::semiring-no-zero-divisors fps
  shows unit-factor (f * g) = unit-factor f * unit-factor g
  using fps-unit-factor-mult'[of f g]
  by    (cases f=0 ∨ g=0) auto

definition fps-cutoff n f = Abs-fps (λi. if i < n then f$i else 0)

lemma fps-cutoff-nth [simp]: fps-cutoff n f $ i = (if i < n then f$i else 0)
  unfolding fps-cutoff-def by simp

lemma fps-cutoff-zero-iff: fps-cutoff n f = 0 ⟷ (f = 0 ∨ n ≤ subdegree f)
proof
  assume A: fps-cutoff n f = 0
  thus f = 0 ∨ n ≤ subdegree f
  proof (cases f = 0)
    assume f ≠ 0
    with A have n ≤ subdegree f
      by (intro subdegree-geI) (simp-all add: fps-eq-iff split: if-split-asm)
    thus ?thesis ..
  qed simp
qed (auto simp: fps-eq-iff intro: nth-less-subdegree-zero)

lemma fps-cutoff-0 [simp]: fps-cutoff 0 f = 0
  by (simp add: fps-eq-iff)

lemma fps-cutoff-zero [simp]: fps-cutoff n 0 = 0
  by (simp add: fps-eq-iff)
```

lemma *fps-cutoff-one*: *fps-cutoff n 1 = (if n = 0 then 0 else 1)*
by (*simp add*: *fps-eq-iff*)

lemma *fps-cutoff-fps-const*: *fps-cutoff n* (*fps-const c*) = (*if n = 0 then 0 else fps-const c*)
by (*simp add*: *fps-eq-iff*)

lemma *fps-cutoff-numeral*: *fps-cutoff n* (*numeral c*) = (*if n = 0 then 0 else numeral c*)
by (*simp add*: *numeral-fps-const fps-cutoff-fps-const*)

lemma *fps-shift-cutoff*:
fps-shift n f ∗ *fps-X*^*n* + *fps-cutoff n f* = *f*
by (*simp add*: *fps-eq-iff fps-X-power-mult-right-nth*)

lemma *fps-shift-cutoff'*:
fps-X^*n* ∗ *fps-shift n f* + *fps-cutoff n f* = *f*
by (*simp add*: *fps-eq-iff fps-X-power-mult-nth*)

lemma *fps-cutoff-left-mult-nth*:
k < *n* ⟹ (*fps-cutoff n f* ∗ *g*) $ *k* = (*f* ∗ *g*) $ *k*
by (*simp add*: *fps-mult-nth*)

lemma *fps-cutoff-add*: *fps-cutoff n* (*f* + *g* :: *'a* :: *monoid-add fps*) = *fps-cutoff n f* + *fps-cutoff n g*
by (*auto simp*: *fps-eq-iff*)

lemma *fps-cutoff-diff*: *fps-cutoff n* (*f* − *g* :: *'a* :: *group-add fps*) = *fps-cutoff n f* − *fps-cutoff n g*
by (*auto simp*: *fps-eq-iff*)

lemma *fps-cutoff-uminus*: *fps-cutoff n* (−*f* :: *'a* :: *group-add fps*) = −*fps-cutoff n f*
by (*auto simp*: *fps-eq-iff*)

lemma *fps-cutoff-right-mult-nth*:
assumes *k* < *n*
shows (*f* ∗ *fps-cutoff n g*) $ *k* = (*f* ∗ *g*) $ *k*
proof−
from *assms* **have** ∀ *i*∈{*0*..*k*}. *fps-cutoff n g* $ (*k* − *i*) = *g* $ (*k* − *i*) **by** *auto*
thus *?thesis* **by** (*simp add*: *fps-mult-nth*)
qed

lemma *fps-cutoff-eq-fps-cutoff-iff*:
fps-cutoff n f = *fps-cutoff n g* ⟷ (∀ *k*<*n*. *fps-nth f k* = *fps-nth g k*)
by (*subst fps-eq-iff*) *auto*

lemma *fps-conv-fps-X-power-mult-fps-shift*:

assumes $f = 0 \vee$ *subdegree* $f \geq n$
shows $f =$ *fps-X* $\hat{}\ n *$ *fps-shift* $n\ f$
proof −
have $f =$ *fps-X* $\hat{}\ n *$ *fps-shift* $n\ f +$ *fps-cutoff* $n\ f$
by (*auto simp*: *fps-eq-iff fps-X-power-mult-nth*)
also have *fps-cutoff* $n\ f = 0$
by (*subst fps-cutoff-zero-iff*) (*use assms* **in** *auto*)
finally show *?thesis* **by** *simp*
qed

5.5 Metrizability

instantiation *fps* :: ({*minus*,*zero*}) *dist*
begin

definition
dist-fps-def: *dist* (*a* :: *'a fps*) *b* = (*if a = b then 0 else inverse* (*2* $\hat{}$ *subdegree* (*a* − *b*)))

lemma *dist-fps-ge0*: *dist* (*a* :: *'a fps*) *b* ≥ *0*
by (*simp add*: *dist-fps-def*)

instance ..

end

instantiation *fps* :: (*group-add*) *metric-space*
begin

definition *uniformity-fps-def* [*code del*]:
(*uniformity* :: (*'a fps* × *'a fps*) *filter*) = (*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-fps-def'* [*code del*]:
open (*U* :: *'a fps set*) ⟷ (∀ *x*∈*U*. *eventually* (λ(*x'*, *y*). *x'* = *x* ⟶ *y* ∈ *U*) *uniformity*)

lemma *dist-fps-sym*: *dist* (*a* :: *'a fps*) *b* = *dist b a*
by (*simp add*: *dist-fps-def*)

instance
proof
show *th*: *dist a b* = *0* ⟷ *a* = *b* **for** *a b* :: *'a fps*
by (*simp add*: *dist-fps-def split*: *if-split-asm*)
then have *th'*[*simp*]: *dist a a* = *0* **for** *a* :: *'a fps* **by** *simp*

fix *a b c* :: *'a fps*
consider *a* = *b* | *c* = *a* ∨ *c* = *b* | *a* ≠ *b a* ≠ *c b* ≠ *c* **by** *blast*
then show *dist a b* ≤ *dist a c* + *dist b c*

proof *cases*
case *1*
then show *?thesis* **by** (*simp add*: *dist-fps-def*)
next
case *2*
then show *?thesis*
by (*cases c = a*) (*simp-all add*: *th dist-fps-sym*)
next
case *neq*: *3*
have *False* **if** *dist a b > dist a c + dist b c*
proof −
let *?n = subdegree* (*a − b*)
from *neq* **have** *dist a b > 0 dist b c > 0* **and** *dist a c > 0* **by** (*simp-all add*: *dist-fps-def*)
with *that* **have** *dist a b > dist a c* **and** *dist a b > dist b c* **by** *simp-all*
with *neq* **have** *?n < subdegree* (*a − c*) **and** *?n < subdegree* (*b − c*)
by (*simp-all add*: *dist-fps-def field-simps*)
hence (*a − c*) $ *?n = 0* **and** (*b − c*) $ *?n = 0*
by (*simp-all only*: *nth-less-subdegree-zero*)
hence (*a − b*) $ *?n = 0* **by** *simp*
moreover from *neq* **have** (*a − b*) $ *?n ≠ 0* **by** (*intro nth-subdegree-nonzero*) *simp-all*
ultimately show *False* **by** *contradiction*
qed
thus *?thesis* **by** (*auto simp add*: *not-le*[*symmetric*])
qed
qed (*rule open-fps-def' uniformity-fps-def*)+

end

declare *uniformity-Abort*[**where** *'a='a* :: *group-add fps*, *code*]

lemma *open-fps-def*: *open* (*S* :: *'a*::*group-add fps set*) = (∀ *a* ∈ *S*. ∃ *r*. *r >0* ∧ {*y*. *dist y a < r*} ⊆ *S*)
unfolding *open-dist subset-eq* **by** *simp*

Topology

5.6 The topology of formal power series

A set of formal power series is open iff for any power series f in it, there exists some number n such that all power series that agree with f on the first n components are also in it.

lemma *open-fps-iff*:
open A ⟷ (∀ *F*∈*A*. ∃ *n*. {*G*. *fps-cutoff n G = fps-cutoff n F*} ⊆ *A*)
proof
assume *open A*
show ∀ *F*∈*A*. ∃ *n*. {*G*. *fps-cutoff n G = fps-cutoff n F*} ⊆ *A*

```
  proof
    fix F :: 'a fps
    assume F: F ∈ A
    with ‹open A› obtain e where e: e > 0 ⋀G. dist G F < e ⟹ G ∈ A
      by (force simp: open-fps-def)
    thm dist-fps-def
    have filterlim (λn. (1/2)^n :: real) (nhds 0) at-top
      by (intro LIMSEQ-realpow-zero) auto
    from order-tendstoD(2)[OF this e(1)] have eventually (λn. 1 / 2 ^ n < e)
at-top
      by (simp add: power-divide)
    then obtain n where n: 1 / 2 ^ n < e
      by (auto simp: eventually-sequentially)
    show ∃n. {G. fps-cutoff n G = fps-cutoff n F} ⊆ A
    proof (rule exI[of - n], safe)
      fix G assume ∗: fps-cutoff n G = fps-cutoff n F
      show G ∈ A
      proof (cases G = F)
        case False
        hence dist G F = inverse (2 ^ subdegree (G − F))
          by (auto simp: dist-fps-def)
        also have subdegree (G − F) ≥ n
        proof (rule subdegree-geI)
          fix i assume i < n
          hence fps-nth (G − F) i = fps-nth (fps-cutoff n G − fps-cutoff n F) i
            by (auto simp: fps-eq-iff)
          also from ∗ have ... = 0
            by simp
          finally show fps-nth (G − F) i = 0 .
        qed (use False in auto)
        hence inverse (2 ^ subdegree (G − F) :: real) ≤ inverse (2 ^ n)
          by (intro le-imp-inverse-le power-increasing) auto
        also have ... < e
          using n by (simp add: field-simps)
        finally show G ∈ A
          using e(2)[of G] by auto
      qed (use ‹F ∈ A› in auto)
    qed
  qed
next
  assume ∗: ∀F∈A. ∃n. {G. fps-cutoff n G = fps-cutoff n F} ⊆ A
  show open A
    unfolding open-fps-def
  proof safe
    fix F assume F: F ∈ A
    with ∗ obtain n where n: ⋀G. fps-cutoff n G = fps-cutoff n F ⟹ G ∈ A
      by blast
    show ∃r>0. {G. dist G F < r} ⊆ A
    proof (rule exI[of - 1 / 2 ^ n], safe)
```

```
    fix G assume dist: dist G F < 1 / 2 ^ n
    show G ∈ A
    proof (cases G = F)
      case False
      hence dist G F = inverse (2 ^ subdegree (F − G))
        by (simp add: dist-fps-def)
      with dist have n < subdegree (F − G)
        by (auto simp: field-simps)
      hence fps-nth (F − G) i = 0 if i ≤ n for i
        using that nth-less-subdegree-zero[of i F − G] by simp
      hence fps-cutoff n G = fps-cutoff n F
        by (auto simp: fps-eq-iff fps-cutoff-def)
      thus G ∈ A
        by (rule n)
    qed (use ‹F ∈ A› in auto)
  qed auto
 qed
qed

lemma open-fps-cutoff: open {H. fps-cutoff N H = fps-cutoff N G}
 unfolding open-fps-iff
proof safe
 fix F assume F: fps-cutoff N F = fps-cutoff N G
 show ∃ n. {G. fps-cutoff n G = fps-cutoff n F}
          ⊆ {H. fps-cutoff N H = fps-cutoff N G}
   by (rule exI[of - N]) (use F in ‹auto simp: fps-eq-iff›)
qed

lemma eventually-fps-nth-eq-nhds-fps-strong:
 eventually (λg. ∀ k≤n. fps-nth g k = fps-nth f k) (nhds f)
proof −
 have eventually (λg. g ∈ {g. fps-cutoff (n+1) g = fps-cutoff (n+1) f}) (nhds
f)
   by (rule eventually-nhds-in-open, rule open-fps-cutoff) auto
 thus ?thesis
   by eventually-elim (auto simp: fps-cutoff-eq-fps-cutoff-iff)
qed

lemma eventually-fps-nth-eq-nhds-fps: eventually (λg. fps-nth g k = fps-nth f k)
(nhds f)
 using eventually-fps-nth-eq-nhds-fps-strong[of k] by eventually-elim auto
```

A family of formal power series f_x tends to a limit series g at some filter F iff for any $N \geq 0$, the set of x for which f_x and G agree on the first N coefficients is in F.

For a sequence $(f_i)_{n\geq 0}$ this means that $f_i \longrightarrow G$ iff for any $N \geq 0$, f_x and G agree for all but finitely many x.

```
lemma tendsto-fps-iff:
 filterlim f (nhds (g :: 'a :: group-add fps)) F ⟷
```

```
    (∀ n. eventually (λx. fps-nth (f x) n = fps-nth g n) F)
proof safe
  assume lim: filterlim f (nhds (g :: 'a :: group-add fps)) F
  show eventually (λx. fps-nth (f x) n = fps-nth g n) F for n
  proof −
    define S where S = {H. fps-cutoff (n+1) H = fps-cutoff (n+1) g}
    have S: open S g ∈ S
      unfolding S-def using open-fps-cutoff[of n+1 g] by (auto simp: S-def)
    from lim and S have eventually (λx. f x ∈ S) F
      using topological-tendstoD by blast
    thus eventually (λx. fps-nth (f x) n = fps-nth g n) F
      by eventually-elim (auto simp: S-def fps-cutoff-eq-fps-cutoff-iff)
  qed
next
  assume ∗: ∀ n. eventually (λx. fps-nth (f x) n = fps-nth g n) F
  show filterlim f (nhds (g :: 'a :: group-add fps)) F
  proof (rule topological-tendstoI)
    fix S :: 'a fps set
    assume S: open S g ∈ S
    then obtain N where N: {H. fps-cutoff N H = fps-cutoff N g} ⊆ S
      unfolding open-fps-iff by blast
    have eventually (λx. ∀ n∈{..<N}. fps-nth (f x) n = fps-nth g n) F
      by (subst eventually-ball-finite-distrib) (use ∗ in auto)
    hence eventually (λx. f x ∈ {H. fps-cutoff N H = fps-cutoff N g}) F
      by eventually-elim (auto simp: fps-cutoff-eq-fps-cutoff-iff)
    thus eventually (λx. f x ∈ S) F
      by eventually-elim (use N in auto)
  qed
qed

lemma tendsto-fpsI:
  assumes ⋀n. eventually (λx. fps-nth (f x) n = fps-nth G n) F
  shows   filterlim f (nhds (G :: 'a :: group-add fps)) F
  unfolding tendsto-fps-iff using assms by blast
```

The infinite sums and justification of the notation in textbooks.

```
lemma reals-power-lt-ex:
  fixes x y :: real
  assumes xp: x > 0
    and y1: y > 1
  shows ∃ k>0. (1/y)^k < x
proof −
  have yp: y > 0
    using y1 by simp
  from reals-Archimedean2[of max 0 (− log y x) + 1]
  obtain k :: nat where k: real k > max 0 (− log y x) + 1
    by blast
  from k have kp: k > 0
    by simp
```

```
  from k have real k > - log y x
    by simp
  then have ln y * real k > - ln x
    unfolding log-def
    using ln-gt-zero-iff[OF yp] y1
    by (simp add: minus-divide-left field-simps del: minus-divide-left[symmetric])
  then have ln y * real k + ln x > 0
    by simp
  then have exp (real k * ln y + ln x) > exp 0
    by (simp add: ac-simps)
  then have y ^ k * x > 1
    unfolding exp-zero exp-add exp-of-nat-mult exp-ln [OF xp] exp-ln [OF yp]
    by simp
  then have x > (1 / y)^k using yp
    by (simp add: field-simps)
  then show ?thesis
    using kp by blast
qed

lemma fps-sum-rep-nth: (sum (λi. fps-const(a$i)*fps-X^i) {0..m})$n = (if n ≤
m then a$n else 0)
  by (simp add: fps-sum-nth if-distrib cong del: if-weak-cong)

lemma fps-notation: (λn. sum (λi. fps-const(a$i) * fps-X^i) {0..n}) ⟶ a
  (is ?s ⟶ a)
proof −
  have ∃ n0. ∀ n ≥ n0. dist (?s n) a < r if r > 0 for r
  proof −
    obtain n0 where n0: (1/2)^n0 < r n0 > 0
      using reals-power-lt-ex[OF ‹r > 0›, of 2] by auto
    show ?thesis
    proof −
      have dist (?s n) a < r if nn0: n ≥ n0 for n
      proof −
        from that have thnn0: (1/2)^n ≤ (1/2 :: real)^n0
          by (simp add: field-split-simps)
        show ?thesis
        proof (cases ?s n = a)
          case True
          then show ?thesis
            unfolding dist-eq-0-iff[of ?s n a, symmetric]
            using ‹r > 0› by (simp del: dist-eq-0-iff)
        next
          case False
          from False have dth: dist (?s n) a = (1/2)^subdegree (?s n − a)
            by (simp add: dist-fps-def field-simps)
          from False have kn: subdegree (?s n − a) > n
            by (intro subdegree-greaterI) (simp-all add: fps-sum-rep-nth)
          then have dist (?s n) a < (1/2)^n
```

by (*simp add*: *field-simps dist-fps-def*)
also have ... $\leq (1/2)\hat{}n0$
using *nn0* **by** (*simp add*: *field-split-simps*)
also have ... $< r$
using *n0* **by** *simp*
finally show *?thesis* **.**
qed
qed
then show *?thesis* **by** *blast*
qed
qed
then show *?thesis*
unfolding *lim-sequentially* **by** *blast*
qed

5.7 Division

declare *sum.cong*[*fundef-cong*]

fun *fps-left-inverse-constructor* ::
'a::{*comm-monoid-add*,*times*,*uminus*} *fps* ⇒ *'a* ⇒ *nat* ⇒ *'a*
where
fps-left-inverse-constructor f a 0 = *a*
| *fps-left-inverse-constructor f a* (*Suc n*) =
− *sum* (λ*i*. *fps-left-inverse-constructor f a i* ∗ *f*$(*Suc n* − *i*)) {*0*..*n*} ∗ *a*

— This will construct a left inverse for f in case that $x * f \$ 0 = 1$
abbreviation *fps-left-inverse* ≡ (λ*f x*. *Abs-fps* (*fps-left-inverse-constructor f x*))

fun *fps-right-inverse-constructor* ::
'a::{*comm-monoid-add*,*times*,*uminus*} *fps* ⇒ *'a* ⇒ *nat* ⇒ *'a*
where
fps-right-inverse-constructor f a 0 = *a*
| *fps-right-inverse-constructor f a n* =
− *a* ∗ *sum* (λ*i*. *f*$*i* ∗ *fps-right-inverse-constructor f a* (*n* − *i*)) {*1*..*n*}

— This will construct a right inverse for f in case that $f \$ 0 * y = 1$
abbreviation *fps-right-inverse* ≡ (λ*f y*. *Abs-fps* (*fps-right-inverse-constructor f y*))

instantiation *fps* :: ({*comm-monoid-add*,*inverse*,*times*,*uminus*}) *inverse*
begin

— For backwards compatibility.
abbreviation *natfun-inverse*:: *'a fps* ⇒ *nat* ⇒ *'a*
where *natfun-inverse f* ≡ *fps-right-inverse-constructor f* (*inverse* (*f*$*0*))

definition *fps-inverse-def*: *inverse f* = *Abs-fps* (*natfun-inverse f*)
— With scalars from a (possibly non-commutative) ring, this defines a right inverse. Furthermore, if scalars are of class *mult-zero* and satisfy condition *inverse 0* = *0*,

then this will evaluate to zero when the zeroth term is zero.

definition *fps-divide-def*: *f div g* = *fps-shift* (*subdegree g*) (*f* ∗ *inverse* (*unit-factor g*))
— If scalars are of class *mult-zero* and satisfy condition *inverse 0 = 0*, then div by zero will equal zero.

instance ..

end

lemma *fps-lr-inverse-0-iff*:
(*fps-left-inverse f x*) $ *0* = *0* ⟷ *x* = *0*
(*fps-right-inverse f x*) $ *0* = *0* ⟷ *x* = *0*
by *auto*

lemma *fps-inverse-0-iff'*: (*inverse f*) $ *0* = *0* ⟷ *inverse* (*f* $ *0*) = *0*
by (*simp add*: *fps-inverse-def fps-lr-inverse-0-iff*(*2*))

lemma *fps-inverse-0-iff*[*simp*]: (*inverse f*) $ *0* = (*0*::*'a*::*division-ring*) ⟷ *f* $ *0* = *0*
by (*simp add*: *fps-inverse-0-iff'*)

lemma *fps-lr-inverse-nth-0*:
(*fps-left-inverse f x*) $ *0* = *x* (*fps-right-inverse f x*) $ *0* = *x*
by *auto*

lemma *fps-inverse-nth-0* [*simp*]: (*inverse f*) $ *0* = *inverse* (*f* $ *0*)
by (*simp add*: *fps-inverse-def*)

lemma *fps-lr-inverse-starting0*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*} *fps*
and *g* :: *'b*::{*ab-group-add*,*mult-zero*} *fps*
shows *fps-left-inverse f 0* = *0*
and *fps-right-inverse g 0* = *0*
proof−
show *fps-left-inverse f 0* = *0*
proof (*rule fps-ext*)
fix *n* **show** *fps-left-inverse f 0* $ *n* = *0* $ *n*
by (*cases n*) (*simp-all add*: *fps-inverse-def*)
qed
show *fps-right-inverse g 0* = *0*
proof (*rule fps-ext*)
fix *n* **show** *fps-right-inverse g 0* $ *n* = *0* $ *n*
by (*cases n*) (*simp-all add*: *fps-inverse-def*)
qed
qed

lemma *fps-lr-inverse-eq0-imp-starting0*:

fps-left-inverse f x = 0 ⟹ x = 0
fps-right-inverse f x = 0 ⟹ x = 0
proof−
assume *A*: *fps-left-inverse f x = 0*
have *0 = fps-left-inverse f x $ 0* **by** (*subst A*) *simp*
thus *x = 0* **by** *simp*
next
assume *A*: *fps-right-inverse f x = 0*
have *0 = fps-right-inverse f x $ 0* **by** (*subst A*) *simp*
thus *x = 0* **by** *simp*
qed

lemma *fps-lr-inverse-eq-0-iff*:
fixes *x* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
and *y* :: *'b*::{*ab-group-add*,*mult-zero*}
shows *fps-left-inverse f x = 0 ⟷ x = 0*
and *fps-right-inverse g y = 0 ⟷ y = 0*
using *fps-lr-inverse-starting0 fps-lr-inverse-eq0-imp-starting0*
by *auto*

lemma *fps-inverse-eq-0-iff'*:
fixes *f* :: *'a*::{*ab-group-add*,*inverse*,*mult-zero*} *fps*
shows *inverse f = 0 ⟷ inverse* (*f $ 0*) *= 0*
by (*simp add*: *fps-inverse-def fps-lr-inverse-eq-0-iff*(*2*))

lemma *fps-inverse-eq-0-iff*[*simp*]: *inverse f* = (*0*:: (*'a*::*division-ring*) *fps*) ⟷ *f* $ *0 = 0*
using *fps-inverse-eq-0-iff'*[*of f*] **by** *simp*

lemmas *fps-inverse-eq-0'* = *iffD2*[*OF fps-inverse-eq-0-iff'*]
lemmas *fps-inverse-eq-0* = *iffD2*[*OF fps-inverse-eq-0-iff*]

lemma *fps-const-lr-inverse*:
fixes *a* :: *'a*::{*ab-group-add*,*mult-zero*}
and *b* :: *'b*::{*comm-monoid-add*,*mult-zero*,*uminus*}
shows *fps-left-inverse* (*fps-const a*) *x = fps-const x*
and *fps-right-inverse* (*fps-const b*) *y = fps-const y*
proof−
show *fps-left-inverse* (*fps-const a*) *x = fps-const x*
proof (*rule fps-ext*)
fix *n* **show** *fps-left-inverse* (*fps-const a*) *x $ n = fps-const x $ n*
by (*cases n*) *auto*
qed
show *fps-right-inverse* (*fps-const b*) *y = fps-const y*
proof (*rule fps-ext*)
fix *n* **show** *fps-right-inverse* (*fps-const b*) *y $ n = fps-const y $ n*
by (*cases n*) *auto*
qed
qed

lemma *fps-const-inverse*:
fixes $a :: 'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}$
shows $inverse\ (fps\text{-}const\ a) = fps\text{-}const\ (inverse\ a)$
unfolding *fps-inverse-def*
by (*simp add*: *fps-const-lr-inverse*(2))

lemma *fps-lr-inverse-zero*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus\}$
shows *fps-left-inverse 0 x = fps-const x*
and *fps-right-inverse 0 y = fps-const y*
using *fps-const-lr-inverse*[*of 0*]
by *simp-all*

lemma *fps-inverse-zero-conv-fps-const*:
$inverse\ (0::'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus,inverse\}\ fps) = fps\text{-}const\ (inverse\ 0)$
using *fps-lr-inverse-zero*(2)[*of inverse* (0::'a)] **by** (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-zero'*:
assumes $inverse\ (0::'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}) = 0$
shows $inverse\ (0::'a\ fps) = 0$
by (*simp add*: *assms fps-inverse-zero-conv-fps-const*)

lemma *fps-inverse-zero* [*simp*]:
$inverse\ (0::'a::division\text{-}ring\ fps) = 0$
by (*rule fps-inverse-zero'*[*OF inverse-zero*])

lemma *fps-lr-inverse-one*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,one\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus,one\}$
shows *fps-left-inverse 1 x = fps-const x*
and *fps-right-inverse 1 y = fps-const y*
using *fps-const-lr-inverse*[*of 1*]
by *simp-all*

lemma *fps-lr-inverse-one-one*:
$fps\text{-}left\text{-}inverse\ 1\ 1 = (1::'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,one\}\ fps)$
$fps\text{-}right\text{-}inverse\ 1\ 1 = (1::'b::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus,one\}\ fps)$
by (*simp-all add*: *fps-lr-inverse-one*)

lemma *fps-inverse-one'*:
assumes $inverse\ (1::'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus,one\}) = 1$
shows $inverse\ (1 :: 'a\ fps) = 1$
using *assms fps-lr-inverse-one-one*(2)
by (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-one* [*simp*]: $inverse\ (1 :: 'a :: division\text{-}ring\ fps) = 1$

by (*rule fps-inverse-one'*[*OF inverse-1*])

lemma *fps-lr-inverse-minus*:
fixes *f* :: *'a*::*ring-1 fps*
shows *fps-left-inverse* (−*f*) (−*x*) = − *fps-left-inverse f x*
and *fps-right-inverse* (−*f*) (−*x*) = − *fps-right-inverse f x*
proof−

show *fps-left-inverse* (−*f*) (−*x*) = − *fps-left-inverse f x*
proof (*intro fps-ext*)
fix *n* **show** *fps-left-inverse* (−*f*) (−*x*) $ *n* = − *fps-left-inverse f x* $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*) **thus** *?case* **by** (*cases n*) (*simp-all add*: *sum-negf algebra-simps*)
qed
qed

show *fps-right-inverse* (−*f*) (−*x*) = − *fps-right-inverse f x*
proof (*intro fps-ext*)
fix *n* **show** *fps-right-inverse* (−*f*) (−*x*) $ *n* = − *fps-right-inverse f x* $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*) **show** *?case*
proof (*cases n*)
case (*Suc m*)
with *1* **have**
∀ *i*∈{*1*..*Suc m*}. *fps-right-inverse* (−*f*) (−*x*) $ (*Suc m* − *i*) =
− *fps-right-inverse f x* $ (*Suc m* − *i*)
by *auto*
with *Suc* **show** *?thesis* **by** (*simp add*: *sum-negf algebra-simps*)
qed *simp*
qed
qed

qed

lemma *fps-inverse-minus* [*simp*]: *inverse* (−*f*) = −*inverse* (*f* :: *'a* :: *division-ring fps*)
by (*simp add*: *fps-inverse-def fps-lr-inverse-minus*(*2*))

lemma *fps-left-inverse*:
fixes *f* :: *'a*::*ring-1 fps*
assumes *f0*: *x* ∗ *f*$*0* = *1*
shows *fps-left-inverse f x* ∗ *f* = *1*
proof (*rule fps-ext*)
fix *n* **show** (*fps-left-inverse f x* ∗ *f*) $ *n* = *1* $ *n*
by (*cases n*) (*simp-all add*: *f0 fps-mult-nth mult.assoc*)
qed

lemma *fps-right-inverse*:
fixes *f* :: *'a*::*ring-1 fps*

```
  assumes f0: f$0 * y = 1
  shows   f * fps-right-inverse f y = 1
proof (rule fps-ext)
  fix n
  show (f * fps-right-inverse f y) $ n = 1 $ n
  proof (cases n)
    case (Suc k)
    moreover from Suc have fps-right-inverse f y $ n =
          - y * sum (λi. f$i * fps-right-inverse-constructor f y (n - i)) {1..n}
      by simp
    hence
      (f * fps-right-inverse f y) $ n =
        - 1 * sum (λi. f$i * fps-right-inverse-constructor f y (n - i)) {1..n} +
        sum (λi. f$i * (fps-right-inverse-constructor f y (n - i))) {1..n}
     by (simp add: fps-mult-nth sum.atLeast-Suc-atMost mult.assoc f0[symmetric])
    thus (f * fps-right-inverse f y) $ n = 1 $ n by (simp add: Suc)
  qed (simp add: f0 fps-inverse-def)
qed
```

It is possible in a ring for an element to have a left inverse but not a right inverse, or vice versa. But when an element has both, they must be the same.

```
lemma fps-left-inverse-eq-fps-right-inverse:
  fixes   f :: 'a::ring-1 fps
  assumes f0: x * f$0 = 1 f $ 0 * y = 1
  — These assumptions imply that x equals y, but no need to assume that.
  shows   fps-left-inverse f x = fps-right-inverse f y
proof−
  from f0(2) have f * fps-right-inverse f y = 1
     by (simp add: fps-right-inverse)
  hence fps-left-inverse f x * f * fps-right-inverse f y = fps-left-inverse f x
    by (simp add: mult.assoc)
  moreover from f0(1) have
    fps-left-inverse f x * f * fps-right-inverse f y = fps-right-inverse f y
    by (simp add: fps-left-inverse)
  ultimately show ?thesis by simp
qed

lemma fps-left-inverse-eq-fps-right-inverse-comm:
  fixes   f :: 'a::comm-ring-1 fps
  assumes f0: x * f$0 = 1
  shows   fps-left-inverse f x = fps-right-inverse f x
  using   assms fps-left-inverse-eq-fps-right-inverse[of x f x]
  by      (simp add: mult.commute)

lemma fps-left-inverse':
  fixes   f :: 'a::ring-1 fps
  assumes x * f$0 = 1 f$0 * y = 1
  — These assumptions imply x equals y, but no need to assume that.
```

shows *fps-right-inverse f y * f = 1*
using *assms fps-left-inverse-eq-fps-right-inverse[of x f y] fps-left-inverse[of x f]*
by *simp*

lemma *fps-right-inverse'*:
fixes *f :: 'a::ring-1 fps*
assumes *x * f$0 = 1 f$0 * y = 1*
— These assumptions imply x equals y, but no need to assume that.
shows *f * fps-left-inverse f x = 1*
using *assms fps-left-inverse-eq-fps-right-inverse[of x f y] fps-right-inverse[of f y]*
by *simp*

lemma *inverse-mult-eq-1* [*intro*]:
assumes *f$0 ≠ (0::'a::division-ring)*
shows *inverse f * f = 1*
using *fps-left-inverse'[of inverse (f$0)]*
by (*simp add*: *assms fps-inverse-def*)

lemma *inverse-mult-eq-1'*:
assumes *f$0 ≠ (0::'a::division-ring)*
shows *f * inverse f = 1*
using *assms fps-right-inverse*
by (*force simp*: *fps-inverse-def*)

lemma *fps-mult-left-inverse-unit-factor*:
fixes *f :: 'a::ring-1 fps*
assumes *x * f $ subdegree f = 1*
shows *fps-left-inverse (unit-factor f) x * f = fps-X ^ subdegree f*
proof−
have
*fps-left-inverse (unit-factor f) x * f =*
*fps-left-inverse (unit-factor f) x * unit-factor f * fps-X ^ subdegree f*
using *fps-unit-factor-decompose[of f]* **by** (*simp add*: *mult.assoc*)
with *assms* **show** *?thesis* **by** (*simp add*: *fps-left-inverse*)
qed

lemma *fps-mult-right-inverse-unit-factor*:
fixes *f :: 'a::ring-1 fps*
assumes *f $ subdegree f * y = 1*
shows *f * fps-right-inverse (unit-factor f) y = fps-X ^ subdegree f*
proof−
have
*f * fps-right-inverse (unit-factor f) y =*
*fps-X ^ subdegree f * (unit-factor f * fps-right-inverse (unit-factor f) y)*
using *fps-unit-factor-decompose'[of f]* **by** (*simp add*: *mult.assoc[symmetric]*)
with *assms* **show** *?thesis* **by** (*simp add*: *fps-right-inverse*)
qed

lemma *fps-mult-right-inverse-unit-factor-divring*:

(*f* :: 'a::*division-ring fps*) ≠ *0* ⟹ *f* * *inverse* (*unit-factor f*) = *fps-X* ^ *subdegree f*
using *fps-mult-right-inverse-unit-factor*[*of f*]
by (*simp add*: *fps-inverse-def*)

lemma *fps-left-inverse-idempotent-ring1*:
fixes *f* :: 'a::*ring-1 fps*
assumes *x* * *f*$*0* = *1* *y* * *x* = *1*
— These assumptions imply *y* equals *f*$*0*, but no need to assume that.
shows *fps-left-inverse* (*fps-left-inverse f x*) *y* = *f*
proof−
from *assms*(*1*) **have**
fps-left-inverse (*fps-left-inverse f x*) *y* * *fps-left-inverse f x* * *f* =
fps-left-inverse (*fps-left-inverse f x*) *y*
by (*simp add*: *fps-left-inverse mult.assoc*)
moreover from *assms*(*2*) **have**
fps-left-inverse (*fps-left-inverse f x*) *y* * *fps-left-inverse f x* = *1*
by (*simp add*: *fps-left-inverse*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fps-left-inverse-idempotent-comm-ring1*:
fixes *f* :: 'a::*comm-ring-1 fps*
assumes *x* * *f*$*0* = *1*
shows *fps-left-inverse* (*fps-left-inverse f x*) (*f*$*0*) = *f*
using *assms fps-left-inverse-idempotent-ring1*[*of x f f*$*0*]
by (*simp add*: *mult.commute*)

lemma *fps-right-inverse-idempotent-ring1*:
fixes *f* :: 'a::*ring-1 fps*
assumes *f*$*0* * *x* = *1* *x* * *y* = *1*
— These assumptions imply *y* equals *f*$*0*, but no need to assume that.
shows *fps-right-inverse* (*fps-right-inverse f x*) *y* = *f*
proof−
from *assms*(*1*) **have** *f* * (*fps-right-inverse f x* * *fps-right-inverse* (*fps-right-inverse f x*) *y*) =
fps-right-inverse (*fps-right-inverse f x*) *y*
by (*simp add*: *fps-right-inverse mult.assoc*[*symmetric*])
moreover from *assms*(*2*) **have**
fps-right-inverse f x * *fps-right-inverse* (*fps-right-inverse f x*) *y* = *1*
by (*simp add*: *fps-right-inverse*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fps-right-inverse-idempotent-comm-ring1*:
fixes *f* :: 'a::*comm-ring-1 fps*
assumes *f*$*0* * *x* = *1*
shows *fps-right-inverse* (*fps-right-inverse f x*) (*f*$*0*) = *f*
using *assms fps-right-inverse-idempotent-ring1*[*of f x f*$*0*]

```
  by      (simp add: mult.commute)

lemma fps-inverse-idempotent[intro, simp]:
  f$0 ≠ (0::'a::division-ring) ⟹ inverse (inverse f) = f
  using fps-right-inverse-idempotent-ring1[of f]
  by    (simp add: fps-inverse-def)

lemma fps-lr-inverse-unique-ring1:
  fixes   f g :: 'a :: ring-1 fps
  assumes fg: f * g = 1 g$0 * f$0 = 1
  shows   fps-left-inverse g (f$0) = f
  and     fps-right-inverse f (g$0) = g
proof−

  show fps-left-inverse g (f$0) = f
  proof (intro fps-ext)
    fix n show fps-left-inverse g (f$0) $ n = f $ n
    proof (induct n rule: nat-less-induct)
      case (1 n) show ?case
      proof (cases n)
        case (Suc k)
        hence ∀ i∈{0..k}. fps-left-inverse g (f$0) $ i = f $ i using 1 by simp
        hence fps-left-inverse g (f$0) $ Suc k = f $ Suc k − 1 $ Suc k * f$0
         by (simp add: fps-mult-nth fg(1)[symmetric] distrib-right mult.assoc fg(2))
        with Suc show ?thesis by simp
      qed simp
    qed
  qed

  show fps-right-inverse f (g$0) = g
  proof (intro fps-ext)
    fix n show fps-right-inverse f (g$0) $ n = g $ n
    proof (induct n rule: nat-less-induct)
      case (1 n) show ?case
      proof (cases n)
        case (Suc k)
        hence ∀ i∈{1..Suc k}. fps-right-inverse f (g$0) $ (Suc k − i) = g $ (Suc k
− i)
          using 1 by auto
        hence
          fps-right-inverse f (g$0) $ Suc k = 1 * g $ Suc k − g$0 * 1 $ Suc k
        by (simp add: fps-mult-nth fg(1)[symmetric] algebra-simps fg(2)[symmetric]
sum.atLeast-Suc-atMost)
        with Suc show ?thesis by simp
      qed simp
    qed
  qed

qed
```

lemma *fps-lr-inverse-unique-divring*:
fixes *f g* :: *'a* ::*division-ring fps*
assumes *fg*: *f* ∗ *g* = *1*
shows *fps-left-inverse g* (*f*$*0*) = *f*
and *fps-right-inverse f* (*g*$*0*) = *g*
proof−
from *fg* **have** *f*$*0* ∗ *g*$*0* = *1* **using** *fps-mult-nth-0*[*of f g*] **by** *simp*
hence *g*$*0* ∗ *f*$*0* = *1* **using** *inverse-unique*[*of f*$*0*] *left-inverse*[*of f*$*0*] **by** *force*
thus *fps-left-inverse g* (*f*$*0*) = *f fps-right-inverse f* (*g*$*0*) = *g*
using *fg fps-lr-inverse-unique-ring1* **by** *auto*
qed

lemma *fps-inverse-unique*:
fixes *f g* :: *'a* :: *division-ring fps*
assumes *fg*: *f* ∗ *g* = *1*
shows *inverse f* = *g*
proof −
from *fg* **have** *if0*: *inverse* (*f*$*0*) = *g*$*0 f*$*0* ≠ *0*
using *inverse-unique*[*of f*$*0*] *fps-mult-nth-0*[*of f g*] **by** *auto*
with *fg* **have** *fps-right-inverse f* (*g*$*0*) = *g*
using *left-inverse*[*of f*$*0*] **by** (*intro fps-lr-inverse-unique-ring1*(*2*)) *simp-all*
with *if0*(*1*) **show** *?thesis* **by** (*simp add*: *fps-inverse-def*)
qed

lemma *inverse-fps-numeral*:
inverse (*numeral n* :: (*'a* :: *field-char-0*) *fps*) = *fps-const* (*inverse* (*numeral n*))
by (*intro fps-inverse-unique fps-ext*) (*simp-all add*: *fps-numeral-nth*)

lemma *inverse-fps-of-nat*:
inverse (*of-nat n* :: *'a* :: {*semiring-1*,*times*,*uminus*,*inverse*} *fps*) =
fps-const (*inverse* (*of-nat n*))
by (*simp add*: *fps-of-nat fps-const-inverse*[*symmetric*])

lemma *fps-lr-inverse-mult-ring1*:
fixes *f g* :: *'a*::*ring-1 fps*
assumes *x*: *x* ∗ *f*$*0* = *1 f*$*0* ∗ *x* = *1*
and *y*: *y* ∗ *g*$*0* = *1 g*$*0* ∗ *y* = *1*
shows *fps-left-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fps-left-inverse g y* ∗ *fps-left-inverse f x*
and *fps-right-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fps-right-inverse g y* ∗ *fps-right-inverse*
f x
proof −
define *h* **where** *h* ≡ *fps-left-inverse g y* ∗ *fps-left-inverse f x*
hence *h0*: *h*$*0* = *y*∗*x* **by** *simp*
have *fps-left-inverse* (*f*∗*g*) (*h*$*0*) = *h*
proof (*intro fps-lr-inverse-unique-ring1*(*1*))
from *h-def*
have *h* ∗ (*f* ∗ *g*) = *fps-left-inverse g y* ∗ (*fps-left-inverse f x* ∗ *f*) ∗ *g*
by (*simp add*: *mult.assoc*)

```
    thus h * (f * g) = 1
      using fps-left-inverse[OF x(1)] fps-left-inverse[OF y(1)] by simp
    from h-def have (f*g)$0 * h$0 = f$0 * 1 * x
      by (simp add: mult.assoc y(2)[symmetric])
    with x(2) show (f * g) $ 0 * h $ 0 = 1 by simp
  qed
  with h-def
    show  fps-left-inverse (f * g) (y*x) = fps-left-inverse g y * fps-left-inverse f x
    by    simp
next
  define h where h ≡ fps-right-inverse g y * fps-right-inverse f x
  hence h0: h$0 = y*x by simp
  have fps-right-inverse (f*g) (h$0) = h
  proof (intro fps-lr-inverse-unique-ring1(2))
    from h-def
      have  f * g * h = f * (g * fps-right-inverse g y) * fps-right-inverse f x
      by    (simp add: mult.assoc)
    thus f * g * h = 1
      using fps-right-inverse[OF x(2)] fps-right-inverse[OF y(2)] by simp
    from h-def have h$0 * (f*g)$0 = y * 1 * g$0
      by (simp add: mult.assoc x(1)[symmetric])
    with y(1) show h$0 * (f*g)$0  = 1 by simp
  qed
  with h-def
    show  fps-right-inverse (f * g) (y*x) = fps-right-inverse g y * fps-right-inverse
f x
    by    simp
qed

lemma fps-lr-inverse-mult-divring:
  fixes f g :: 'a::division-ring fps
  shows fps-left-inverse (f * g) (inverse ((f*g)$0)) =
        fps-left-inverse g (inverse (g$0)) * fps-left-inverse f (inverse (f$0))
  and   fps-right-inverse (f * g) (inverse ((f*g)$0)) =
        fps-right-inverse g (inverse (g$0)) * fps-right-inverse f (inverse (f$0))
proof−
  show fps-left-inverse (f * g) (inverse ((f*g)$0)) =
        fps-left-inverse g (inverse (g$0)) * fps-left-inverse f (inverse (f$0))
  proof (cases f$0 = 0 ∨ g$0 = 0)
    case True
    hence fps-left-inverse (f * g) (inverse ((f*g)$0)) = 0
      by (simp add: fps-lr-inverse-eq-0-iff(1))
    moreover from True have
      fps-left-inverse g (inverse (g$0)) * fps-left-inverse f (inverse (f$0)) = 0
      by (auto simp: fps-lr-inverse-eq-0-iff(1))
    ultimately show ?thesis by simp
  next
    case False
    hence fps-left-inverse (f * g) (inverse (g$0) * inverse (f$0)) =
```

```
        fps-left-inverse g (inverse (g$0)) * fps-left-inverse f (inverse (f$0))
    by (intro fps-lr-inverse-mult-ring1(1)) simp-all
   with False show ?thesis by (simp add: nonzero-inverse-mult-distrib)
  qed
  show fps-right-inverse (f * g) (inverse ((f*g)$0)) =
       fps-right-inverse g (inverse (g$0)) * fps-right-inverse f (inverse (f$0))
  proof (cases f$0 = 0 ∨ g$0 = 0)
   case True
   from True have fps-right-inverse (f * g) (inverse ((f*g)$0)) = 0
    by (simp add: fps-lr-inverse-eq-0-iff(2))
   moreover from True have
    fps-right-inverse g (inverse (g$0)) * fps-right-inverse f (inverse (f$0)) = 0
    by (auto simp: fps-lr-inverse-eq-0-iff(2))
   ultimately show ?thesis by simp
  next
   case False
   hence fps-right-inverse (f * g) (inverse (g$0) * inverse (f$0)) =
         fps-right-inverse g (inverse (g$0)) * fps-right-inverse f (inverse (f$0))
    by (intro fps-lr-inverse-mult-ring1(2)) simp-all
   with False show ?thesis by (simp add: nonzero-inverse-mult-distrib)
  qed
qed

lemma fps-inverse-mult-divring:
  inverse (f * g) = inverse g * inverse (f :: 'a::division-ring fps)
  using fps-lr-inverse-mult-divring(2) by (simp add: fps-inverse-def)

lemma fps-inverse-mult: inverse (f * g :: 'a::field fps) = inverse f * inverse g
  by (simp add: fps-inverse-mult-divring)

lemma inverse-prod-fps: inverse (prod f A) = (∏x∈A. inverse (f x) :: 'a :: field
fps)
  by (induction A rule: infinite-finite-induct) (auto simp: fps-inverse-mult)

lemma fps-lr-inverse-gp-ring1:
  fixes   ones ones-inv :: 'a :: ring-1 fps
  defines ones ≡ Abs-fps (λn. 1)
  and     ones-inv ≡ Abs-fps (λn. if n=0 then 1 else if n=1 then − 1 else 0)
  shows   fps-left-inverse ones 1 = ones-inv
  and     fps-right-inverse ones 1 = ones-inv
proof−
  show fps-left-inverse ones 1 = ones-inv
  proof (rule fps-ext)
   fix n
   show fps-left-inverse ones 1 $ n = ones-inv $ n
   proof (induct n rule: nat-less-induct)
    case (1 n) show ?case
    proof (cases n)
     case (Suc m)
```

```
      have m: n = Suc m by fact
      moreover have fps-left-inverse ones 1 $ Suc m = ones-inv $ Suc m
      proof (cases m)
        case (Suc k) thus ?thesis
       using Suc m 1 by (simp add: ones-def ones-inv-def sum.atLeast-Suc-atMost)
      qed (simp add: ones-def ones-inv-def)
      ultimately show ?thesis by simp
    qed (simp add: ones-inv-def)
  qed
  moreover have fps-right-inverse ones 1 = fps-left-inverse ones 1
    by (auto intro: fps-left-inverse-eq-fps-right-inverse[symmetric] simp: ones-def)
  ultimately show fps-right-inverse ones 1 = ones-inv by simp
qed

lemma fps-lr-inverse-gp-ring1':
  fixes   ones :: 'a :: ring-1 fps
  defines ones ≡ Abs-fps (λn. 1)
  shows   fps-left-inverse ones 1 = 1 − fps-X
  and     fps-right-inverse ones 1 = 1 − fps-X
proof−
  define ones-inv :: 'a :: ring-1 fps
    where ones-inv ≡ Abs-fps (λn. if n=0 then 1 else if n=1 then − 1 else 0)
  hence fps-left-inverse ones 1 = ones-inv
  and   fps-right-inverse ones 1 = ones-inv
    using ones-def fps-lr-inverse-gp-ring1 by auto
  thus fps-left-inverse ones 1 = 1 − fps-X
  and   fps-right-inverse ones 1 = 1 − fps-X
    by (auto intro: fps-ext simp: ones-inv-def)
qed

lemma fps-inverse-gp:
  inverse (Abs-fps(λn. (1::'a::division-ring))) =
    Abs-fps (λn. if n= 0 then 1 else if n=1 then − 1 else 0)
  using fps-lr-inverse-gp-ring1(2) by (simp add: fps-inverse-def)

lemma fps-inverse-gp': inverse (Abs-fps (λn. 1::'a::division-ring)) = 1 − fps-X
  by (simp add: fps-inverse-def fps-lr-inverse-gp-ring1'(2))

lemma fps-lr-inverse-one-minus-fps-X:
  fixes   ones :: 'a :: ring-1 fps
  defines ones ≡ Abs-fps (λn. 1)
  shows fps-left-inverse (1 − fps-X) 1 = ones
  and   fps-right-inverse (1 − fps-X) 1 = ones
proof−
  have fps-left-inverse ones 1 = 1 − fps-X
    using fps-lr-inverse-gp-ring1'(1) by (simp add: ones-def)
  thus fps-left-inverse (1 − fps-X) 1 = ones
    using fps-left-inverse-idempotent-ring1[of 1 ones 1] by (simp add: ones-def)
```

```
  have fps-right-inverse ones 1 = 1 − fps-X
    using fps-lr-inverse-gp-ring1'(2) by (simp add: ones-def)
  thus fps-right-inverse (1 − fps-X) 1 = ones
    using fps-right-inverse-idempotent-ring1[of ones 1 1] by (simp add: ones-def)
qed

lemma fps-inverse-one-minus-fps-X:
  fixes   ones :: 'a :: division-ring fps
  defines ones ≡ Abs-fps (λn. 1)
  shows   inverse (1 − fps-X) = ones
  by      (simp add: fps-inverse-def assms fps-lr-inverse-one-minus-fps-X(2))

lemma fps-lr-one-over-one-minus-fps-X-squared:
  shows   fps-left-inverse ((1 − fps-X)^2) (1::'a::ring-1) = Abs-fps (λn. of-nat
(n+1))
          fps-right-inverse ((1 − fps-X)^2) (1::'a) = Abs-fps (λn. of-nat (n+1))
proof−
  define  f invf2 :: 'a fps
    where f ≡ (1 − fps-X)
    and   invf2 ≡ Abs-fps (λn. of-nat (n+1))

  have f2-nth-simps:
    f^2 $ 1 = − of-nat 2 f^2 $ 2 = 1 ⋀n. n>2 ⟹ f^2 $ n = 0
     by (simp-all add: power2-eq-square f-def fps-mult-nth sum.atLeast-Suc-atMost)

  show fps-left-inverse (f^2) 1 = invf2
  proof (intro fps-ext)
    fix n show fps-left-inverse (f^2) 1 $ n = invf2 $ n
    proof (induct n rule: nat-less-induct)
      case (1 t)
      hence induct-assm:
        ⋀m. m < t ⟹ fps-left-inverse (f^2) 1 $ m = invf2 $ m
        by fast
      show ?case
      proof (cases t)
        case (Suc m)
        have m: t = Suc m by fact
        moreover have fps-left-inverse (f^2) 1 $ Suc m = invf2 $ Suc m
        proof (cases m)
          case 0 thus ?thesis using f2-nth-simps(1) by (simp add: invf2-def)
        next
          case (Suc l)
          have l: m = Suc l by fact
         moreover have fps-left-inverse (f^2) 1 $ Suc (Suc l) = invf2 $ Suc (Suc
l)
          proof (cases l)
        case 0 thus ?thesis using f2-nth-simps(1,2) by (simp add: Suc-1[symmetric]
invf2-def)
          next
```

```
        case (Suc k)
        from Suc l m
          have A: fps-left-inverse (f^2) 1 $ Suc (Suc k) = invf2 $ Suc (Suc k)
          and  B: fps-left-inverse (f^2) 1 $ Suc k = invf2 $ Suc k
          using induct-assm[of Suc k] induct-assm[of Suc (Suc k)]
          by    auto
        have times2: ⋀a::nat. 2*a = a + a by simp
        have ∀ i∈{0..k}. (f^2)$(Suc (Suc (Suc k)) - i) = 0
          using f2-nth-simps(3) by auto
        hence
          fps-left-inverse (f^2) 1 $ Suc (Suc (Suc k)) =
            fps-left-inverse (f^2) 1 $ Suc (Suc k) * of-nat 2 -
            fps-left-inverse (f^2) 1 $ Suc k
          using sum.ub-add-nat f2-nth-simps(1,2) by simp
        also have ... = of-nat (2 * Suc (Suc (Suc k))) - of-nat (Suc (Suc k))
          by (subst A, subst B) (simp add: invf2-def mult.commute)
        also have ... = of-nat (Suc (Suc (Suc k)) + 1)
          by (subst times2[of Suc (Suc (Suc k))]) simp
        finally have
          fps-left-inverse (f^2) 1 $ Suc (Suc (Suc k)) = invf2 $ Suc (Suc (Suc
k))
          by (simp add: invf2-def)
        with Suc show ?thesis by simp
      qed
      ultimately show ?thesis by simp
    qed
    ultimately show ?thesis by simp
  qed (simp add: invf2-def)
 qed
qed

 moreover have fps-right-inverse (f^2) 1 = fps-left-inverse (f^2) 1
  by  (auto
       intro: fps-left-inverse-eq-fps-right-inverse[symmetric]
       simp: f-def power2-eq-square
      )
 ultimately show fps-right-inverse (f^2) 1 = invf2
  by simp

qed

lemma fps-one-over-one-minus-fps-X-squared':
 assumes inverse (1::'a::{ring-1,inverse}) = 1
 shows   inverse ((1 - fps-X)^2 :: 'a fps) = Abs-fps (λn. of-nat (n+1))
 using   assms fps-lr-one-over-one-minus-fps-X-squared(2)
 by      (simp add: fps-inverse-def power2-eq-square)

lemma fps-one-over-one-minus-fps-X-squared:
 inverse ((1 - fps-X)^2 :: 'a :: division-ring fps) = Abs-fps (λn. of-nat (n+1))
```

by (*rule fps-one-over-one-minus-fps-X-squared'*[*OF inverse-1*])

lemma *fps-lr-inverse-fps-X-plus1*:
fps-left-inverse (*1* + *fps-X*) (*1*::*'a*::*ring-1*) = *Abs-fps* (λ*n*. (−*1*)^*n*)
fps-right-inverse (*1* + *fps-X*) (*1*::*'a*) = *Abs-fps* (λ*n*. (−*1*)^*n*)
proof−

show *fps-left-inverse* (*1* + *fps-X*) (*1*::*'a*) = *Abs-fps* (λ*n*. (−*1*)^*n*)
proof (*rule fps-ext*)
fix *n* **show** *fps-left-inverse* (*1* + *fps-X*) (*1*::*'a*) $ *n* = *Abs-fps* (λ*n*. (−*1*)^*n*) $ *n*
proof (*induct n rule*: *nat-less-induct*)
case (*1 n*) **show** *?case*
proof (*cases n*)
case (*Suc m*)
have *m*: *n* = *Suc m* **by** *fact*
from *Suc 1* **have**
A: *fps-left-inverse* (*1* + *fps-X*) (*1*::*'a*) $ *n* =
− (∑ *i*=*0*..*m*. (− *1*)^*i* ∗ (*1* + *fps-X*) $ (*Suc m* − *i*))
by *simp*
show *?thesis*
proof (*cases m*)
case (*Suc l*)
have ∀ *i*∈{*0*..*l*}. ((*1*::*'a fps*) + *fps-X*) $ (*Suc* (*Suc l*) − *i*) = *0* **by** *auto*
with *Suc A m* **show** *?thesis* **by** *simp*
qed (*simp add*: *m A*)
qed *simp*
qed

moreover have
fps-right-inverse (*1* + *fps-X*) (*1*::*'a*) = *fps-left-inverse* (*1* + *fps-X*) *1*
by (*intro fps-left-inverse-eq-fps-right-inverse*[*symmetric*]) *simp-all*
ultimately show *fps-right-inverse* (*1* + *fps-X*) (*1*::*'a*) = *Abs-fps* (λ*n*. (−*1*)^*n*)
by *simp*

qed

lemma *fps-inverse-fps-X-plus1'*:
assumes *inverse* (*1*::*'a*::{*ring-1*,*inverse*}) = *1*
shows *inverse* (*1* + *fps-X*) = *Abs-fps* (λ*n*. (− (*1*::*'a*)) ^ *n*)
using *assms fps-lr-inverse-fps-X-plus1*(*2*)
by (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-fps-X-plus1*:
inverse (*1* + *fps-X*) = *Abs-fps* (λ*n*. (− (*1*::*'a*::*division-ring*)) ^ *n*)
by (*rule fps-inverse-fps-X-plus1'*[*OF inverse-1*])

lemma *subdegree-lr-inverse*:

fixes $x :: 'a::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus\}$
and $y :: 'b::\{ab\text{-}group\text{-}add,mult\text{-}zero\}$
shows *subdegree* (*fps-left-inverse f x*) = *0*
and *subdegree* (*fps-right-inverse g y*) = *0*
proof−
show *subdegree* (*fps-left-inverse f x*) = *0*
using *fps-lr-inverse-eq-0-iff*(*1*) *subdegree-eq-0-iff* **by** *fastforce*
show *subdegree* (*fps-right-inverse g y*) = *0*
using *fps-lr-inverse-eq-0-iff*(*2*) *subdegree-eq-0-iff* **by** *fastforce*
qed

lemma *subdegree-inverse* [*simp*]:
fixes $f :: 'a::\{ab\text{-}group\text{-}add,inverse,mult\text{-}zero\}$ *fps*
shows *subdegree* (*inverse f*) = *0*
using *subdegree-lr-inverse*(*2*)
by (*simp add*: *fps-inverse-def*)

lemma *fps-right-inverse-constructor-rec*:
n > *0* ⟹ *fps-right-inverse-constructor f a n* =
−*a* ∗ *sum* (λ*i*. *fps-nth f i* ∗ *fps-right-inverse-constructor f a* (*n* − *i*))
{*1*..*n*}
by (*cases n*) *auto*

lemma *fps-right-inverse-constructor-cong*:
assumes ⋀*k*. *k* ≤ *n* ⟹ *fps-nth f k* = *fps-nth g k*
shows *fps-right-inverse-constructor f c n* = *fps-right-inverse-constructor g c n*
using *assms*
proof (*induction n rule*: *less-induct*)
case (*less n*)
show *?case*
proof (*cases n* > *0*)
case *n*: *True*
have *fps-right-inverse-constructor f c n* =
−*c* ∗ *sum* (λ*i*. *fps-nth f i* ∗ *fps-right-inverse-constructor f c* (*n* − *i*))
{*1*..*n*}
by (*subst fps-right-inverse-constructor-rec*) (*use n* **in** *auto*)
also have *sum* (λ*i*. *fps-nth f i* ∗ *fps-right-inverse-constructor f c* (*n* − *i*)) {*1*..*n*}
=
sum (λ*i*. *fps-nth g i* ∗ *fps-right-inverse-constructor g c* (*n* − *i*)) {*1*..*n*}
by (*intro sum.cong refl arg-cong2*[*of* - - - - (∗)] *less*) (*use assms* **in** *auto*)
also have −*c* ∗ ... = *fps-right-inverse-constructor g c n*
by (*subst* (*2*) *fps-right-inverse-constructor-rec*) (*use n* **in** *auto*)
finally show *?thesis* **.**
qed *auto*
qed

lemma *fps-cutoff-inverse*:
fixes $f :: 'a :: field\ fps$
assumes *fps-nth f 0* ≠ *0*

```
  shows   fps-cutoff n (inverse (fps-cutoff n f)) = fps-cutoff n (inverse f)
proof (cases n = 0)
  case True
  show ?thesis
    by (simp add: True)
next
  case False
  show ?thesis
  proof (subst fps-cutoff-eq-fps-cutoff-iff, safe)
    fix k assume k < n
    have fps-nth (inverse (fps-cutoff n f)) k =
          fps-right-inverse-constructor (fps-cutoff n f) (inverse (fps-nth f 0)) k
      using False by (simp add: fps-inverse-def)
    also have ... = fps-right-inverse-constructor f (inverse (fps-nth f 0)) k
      by (rule fps-right-inverse-constructor-cong) (use ‹k < n› in auto)
    also have ... = fps-nth (inverse f) k
      using False by (simp add: fps-inverse-def)
    finally show fps-nth (inverse (fps-cutoff n f)) k = fps-nth (inverse f) k .
  qed
qed

lemma tendsto-inverse-fps-aux:
  fixes f :: 'a :: field fps
  assumes fps-nth f 0 ≠ 0
  shows   ((λf. inverse f) ⟶ inverse f) (at f)
  unfolding tendsto-fps-iff
proof
  fix n :: nat
  have eventually (λg. ∀k≤n. fps-nth g k = fps-nth f k) (nhds f)
    by (rule eventually-fps-nth-eq-nhds-fps-strong)
  hence eventually (λg. ∀k≤n. fps-nth g k = fps-nth f k) (at f)
    using eventually-nhds-conv-at by blast
  thus eventually (λg. fps-nth (inverse g) n = fps-nth (inverse f) n) (at f)
  proof eventually-elim
    case (elim g)
    from elim have fps-nth g 0 = fps-nth f 0
      by auto
    with assms have [simp]: fps-nth g 0 ≠ 0
      by simp
   have fps-cutoff (n+1) (inverse f) = fps-cutoff (n+1) (inverse (fps-cutoff (n+1)
f))
      by (rule fps-cutoff-inverse [symmetric]) fact
    also have fps-cutoff (n+1) f = fps-cutoff (n+1) g
      by (subst fps-cutoff-eq-fps-cutoff-iff) (use elim in auto)
    also have fps-cutoff (n+1) (inverse ...) = fps-cutoff (n+1) (inverse g)
      by (rule fps-cutoff-inverse) auto
    finally show ?case
      by (subst (asm) fps-cutoff-eq-fps-cutoff-iff) auto
  qed
```

qed

lemma *tendsto-inverse-fps* [*tendsto-intros*]:
fixes $g :: 'a :: field\ fps$
assumes $(f \longrightarrow g)\ F$
assumes $fps\text{-}nth\ g\ 0 \neq 0$
shows $((\lambda x.\ inverse\ (f\ x)) \longrightarrow inverse\ g)\ F$
by (*rule tendsto-compose*[*OF tendsto-inverse-fps-aux assms*(*1*)]) *fact*

lemma *fps-div-zero* [*simp*]:
$0\ div\ (g :: 'a :: \{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}\ fps) = 0$
by (*simp add*: *fps-divide-def*)

lemma *fps-div-by-zero'*:
fixes $g :: 'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus\}\ fps$
assumes $inverse\ (0::'a) = 0$
shows $g\ div\ 0 = 0$
by (*simp add*: *fps-divide-def assms fps-inverse-zero'*)

lemma *fps-div-by-zero* [*simp*]: $(g::'a::division\text{-}ring\ fps)\ div\ 0 = 0$
by (*rule fps-div-by-zero'*[*OF inverse-zero*])

lemma *fps-divide-unit'*: $subdegree\ g = 0 \Longrightarrow f\ div\ g = f * inverse\ g$
by (*simp add*: *fps-divide-def*)

lemma *fps-divide-unit*: $g\$0 \neq 0 \Longrightarrow f\ div\ g = f * inverse\ g$
by (*intro fps-divide-unit'*) (*simp add*: *subdegree-eq-0-iff*)

lemma *fps-divide-nth-0'*:
$subdegree\ (g::'a::division\text{-}ring\ fps) = 0 \Longrightarrow (f\ div\ g)\ \$\ 0 = f\ \$\ 0\ /\ (g\ \$\ 0)$
by (*simp add*: *fps-divide-unit' divide-inverse*)

lemma *fps-divide-nth-0* [*simp*]:
$g\ \$\ 0 \neq 0 \Longrightarrow (f\ div\ g)\ \$\ 0 = f\ \$\ 0\ /\ (g\ \$\ 0 :: - :: division\text{-}ring)$
by (*simp add*: *fps-divide-nth-0'*)

lemma *fps-divide-nth-below*:
fixes $f\ g :: 'a::\{comm\text{-}monoid\text{-}add,uminus,mult\text{-}zero,inverse\}\ fps$
shows $n < subdegree\ f - subdegree\ g \Longrightarrow (f\ div\ g)\ \$\ n = 0$
by (*simp add*: *fps-divide-def fps-mult-nth-eq0*)

lemma *fps-divide-nth-base*:
fixes $f\ g :: 'a::division\text{-}ring\ fps$
assumes $subdegree\ g \leq subdegree\ f$
shows $(f\ div\ g)\ \$\ (subdegree\ f - subdegree\ g) = f\ \$\ subdegree\ f * inverse\ (g\ \$$
$subdegree\ g)$
by (*simp add*: *assms fps-divide-def fps-divide-unit'*)

lemma *fps-divide-subdegree-ge*:

fixes *f g* :: *'a*::{*comm-monoid-add*,*uminus*,*mult-zero*,*inverse*} *fps*
assumes *f / g* ≠ *0*
shows *subdegree* (*f / g*) ≥ *subdegree f* − *subdegree g*
by (*intro subdegree-geI*) (*simp-all add*: *assms fps-divide-nth-below*)

lemma *fps-divide-subdegree*:
fixes *f g* :: *'a*::*division-ring fps*
assumes *f* ≠ *0 g* ≠ *0 subdegree g* ≤ *subdegree f*
shows *subdegree* (*f / g*) = *subdegree f* − *subdegree g*
proof (*intro antisym*)
from *assms* **have** *1*: (*f div g*) $ (*subdegree f* − *subdegree g*) ≠ *0*
using *fps-divide-nth-base*[*of g f*] **by** *simp*
thus *subdegree* (*f / g*) ≤ *subdegree f* − *subdegree g* **by** (*intro subdegree-leI*) *simp*
from *1* **have** *f / g* ≠ *0* **by** (*auto intro*: *fps-nonzeroI*)
thus *subdegree f* − *subdegree g* ≤ *subdegree* (*f / g*) **by** (*rule fps-divide-subdegree-ge*)
qed

lemma *fps-divide-shift-numer*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
assumes *n* ≤ *subdegree f*
shows *fps-shift n f / g* = *fps-shift n* (*f*/*g*)
using *assms fps-shift-mult-right-noncomm*[*of n f inverse* (*unit-factor g*)]
fps-shift-fps-shift-reorder[*of subdegree g n f* ∗ *inverse* (*unit-factor g*)]
by (*simp add*: *fps-divide-def*)

lemma *fps-divide-shift-denom*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*
assumes *n* ≤ *subdegree g subdegree g* ≤ *subdegree f*
shows *f / fps-shift n g* = *Abs-fps* (λ*k*. *if k*<*n then 0 else* (*f*/*g*) $ (*k*−*n*))
proof (*intro fps-ext*)
fix *k*
from *assms*(*1*) **have** *LHS*:
(*f / fps-shift n g*) $ *k* = (*f* ∗ *inverse* (*unit-factor g*)) $ (*k* + (*subdegree g* − *n*))
using *fps-unit-factor-shift*[*of n g*]
by (*simp add*: *fps-divide-def*)
show (*f / fps-shift n g*) $ *k* = *Abs-fps* (λ*k*. *if k*<*n then 0 else* (*f*/*g*) $ (*k*−*n*)) $ *k*
proof (*cases k*<*n*)
case *True* **with** *assms LHS* **show** *?thesis* **using** *fps-mult-nth-eq0*[*of* - *f*] **by** *simp*
next
case *False*
hence (*f*/*g*) $ (*k*−*n*) = (*f* ∗ *inverse* (*unit-factor g*)) $ ((*k*−*n*) + *subdegree g*)
by (*simp add*: *fps-divide-def*)
with *False LHS assms*(*1*) **show** *?thesis* **by** *auto*
qed
qed

lemma *fps-divide-unit-factor-numer*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fps*

shows *unit-factor f / g = fps-shift (subdegree f) (f/g)*
by *(simp add: fps-divide-shift-numer)*

lemma *fps-divide-unit-factor-denom*:
fixes *f g :: 'a::{inverse,comm-monoid-add,uminus,mult-zero} fps*
assumes *subdegree g ≤ subdegree f*
shows
f / unit-factor g = Abs-fps (λk. if k<subdegree g then 0 else (f/g) $ (k−subdegree g))
by *(simp add: assms fps-divide-shift-denom)*

lemma *fps-divide-unit-factor-both'*:
fixes *f g :: 'a::{inverse,comm-monoid-add,uminus,mult-zero} fps*
assumes *subdegree g ≤ subdegree f*
shows *unit-factor f / unit-factor g = fps-shift (subdegree f − subdegree g) (f / g)*
using *assms fps-divide-unit-factor-numer[of f unit-factor g]*
fps-divide-unit-factor-denom[of g f]
fps-shift-rev-shift(1)[of subdegree g subdegree f f/g]
by *simp*

lemma *fps-divide-unit-factor-both*:
fixes *f g :: 'a::division-ring fps*
assumes *subdegree g ≤ subdegree f*
shows *unit-factor f / unit-factor g = unit-factor (f / g)*
using *assms fps-divide-unit-factor-both'[of g f] fps-divide-subdegree[of f g]*
by *(cases f=0 ∨ g=0) auto*

lemma *fps-divide-self*:
(f::'a::division-ring fps) ≠ 0 ⟹ f / f = 1
using *fps-mult-right-inverse-unit-factor-divring[of f]*
by *(simp add: fps-divide-def)*

lemma *fps-divide-add*:
fixes *f g h :: 'a::{semiring-0,inverse,uminus} fps*
shows *(f + g) / h = f / h + g / h*
by *(simp add: fps-divide-def algebra-simps fps-shift-add)*

lemma *fps-divide-diff*:
fixes *f g h :: 'a::{ring,inverse} fps*
shows *(f − g) / h = f / h − g / h*
by *(simp add: fps-divide-def algebra-simps fps-shift-diff)*

lemma *fps-divide-uminus*:
fixes *f g h :: 'a::{ring,inverse} fps*
shows *(− f) / g = − (f / g)*
by *(simp add: fps-divide-def algebra-simps fps-shift-uminus)*

lemma *fps-divide-uminus'*:

fixes *f g h* :: *'a*::*division-ring fps*
shows *f* / (− *g*) = − (*f* / *g*)
by (*simp add*: *fps-divide-def fps-unit-factor-uminus fps-shift-uminus*)

lemma *fps-divide-times*:
fixes *f g h* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fps*
assumes *subdegree h* ≤ *subdegree g*
shows (*f* ∗ *g*) / *h* = *f* ∗ (*g* / *h*)
using *assms fps-mult-subdegree-ge*[*of g inverse* (*unit-factor h*)]
fps-shift-mult[*of subdegree h g* ∗ *inverse* (*unit-factor h*) *f*]
by (*fastforce simp add*: *fps-divide-def mult.assoc*)

lemma *fps-divide-times2*:
fixes *f g h* :: *'a*::{*comm-semiring-0*,*inverse*,*uminus*} *fps*
assumes *subdegree h* ≤ *subdegree f*
shows (*f* ∗ *g*) / *h* = (*f* / *h*) ∗ *g*
using *assms fps-divide-times*[*of h f g*]
by (*simp add*: *mult.commute*)

lemma *fps-times-divide-eq*:
fixes *f g* :: *'a*::*field fps*
assumes *g* ≠ *0* **and** *subdegree f* ≥ *subdegree g*
shows *f div g* ∗ *g* = *f*
using *assms fps-divide-times2*[*of g f g*]
by (*simp add*: *fps-divide-times fps-divide-self*)

lemma *fps-divide-times-eq*:
(*g* :: *'a*::*division-ring fps*) ≠ *0* ⟹ (*f* ∗ *g*) *div g* = *f*
by (*simp add*: *fps-divide-times fps-divide-self*)

lemma *fps-divide-by-mult'*:
fixes *f g h* :: *'a* :: *division-ring fps*
assumes *subdegree h* ≤ *subdegree f*
shows *f* / (*g* ∗ *h*) = *f* / *h* / *g*
proof (*cases f*=*0* ∨ *g*=*0* ∨ *h*=*0*)
case *False* **with** *assms* **show** *?thesis*
using *fps-unit-factor-mult*[*of g h*]
by (*auto simp*:
fps-divide-def fps-shift-fps-shift fps-inverse-mult-divring mult.assoc
fps-shift-mult-right-noncomm
)
qed *auto*

lemma *fps-divide-by-mult*:
fixes *f g h* :: *'a* :: *field fps*
assumes *subdegree g* ≤ *subdegree f*
shows *f* / (*g* ∗ *h*) = *f* / *g* / *h*
proof−
have *f* / (*g* ∗ *h*) = *f* / (*h* ∗ *g*) **by** (*simp add*: *mult.commute*)

also have ... = *f* / *g* / *h* **using** *fps-divide-by-mult'*[*OF assms*] **by** *simp*
finally show *?thesis* **by** *simp*
qed

lemma *fps-divide-cancel*:
fixes *f g h* :: *'a* :: *division-ring fps*
shows *h* ≠ *0* ⟹ (*f* ∗ *h*) *div* (*g* ∗ *h*) = *f div g*
by (*cases f=0*)
(*auto simp*: *fps-divide-by-mult' fps-divide-times-eq*)

lemma *fps-divide-1'*:
fixes *a* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows *a* / *1* = *a*
using *assms fps-inverse-one' fps-one-mult*(*2*)[*of a*]
by (*force simp*: *fps-divide-def*)

lemma *fps-divide-1* [*simp*]: (*a* :: *'a*::*division-ring fps*) / *1* = *a*
by (*rule fps-divide-1'*[*OF inverse-1*])

lemma *fps-divide-X'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / *fps-X* = *fps-shift 1 f*
using *assms fps-one-mult*(*2*)[*of f*]
by (*simp add*: *fps-divide-def fps-X-unit-factor fps-inverse-one'*)

lemma *fps-divide-X* [*simp*]: *a* / *fps-X* = *fps-shift 1* (*a*::*'a*::*division-ring fps*)
by (*rule fps-divide-X'*[*OF inverse-1*])

lemma *fps-divide-X-power'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / (*fps-X* ^ *n*) = *fps-shift n f*
using *fps-inverse-one'*[*OF assms*] *fps-one-mult*(*2*)[*of f*]
by (*simp add*: *fps-divide-def fps-X-power-subdegree*)

lemma *fps-divide-X-power* [*simp*]: *a* / (*fps-X* ^ *n*) = *fps-shift n* (*a*::*'a*::*division-ring fps*)
by (*rule fps-divide-X-power'*[*OF inverse-1*])

lemma *fps-divide-shift-denom-conv-times-fps-X-power*:
fixes *f g* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *n* ≤ *subdegree g subdegree g* ≤ *subdegree f*
shows *f* / *fps-shift n g* = *f* / *g* ∗ *fps-X* ^ *n*
using *assms*
by (*intro fps-ext*) (*simp-all add*: *fps-divide-shift-denom fps-X-power-mult-right-nth*)

lemma *fps-divide-unit-factor-denom-conv-times-fps-X-power*:
fixes *f g* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *subdegree g* ≤ *subdegree f*
shows *f* / *unit-factor g* = *f* / *g* ∗ *fps-X* ^ *subdegree g*
by (*simp add*: *assms fps-divide-shift-denom-conv-times-fps-X-power*)

lemma *fps-shift-altdef'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows *fps-shift n f* = *f div fps-X*^*n*
using *assms*
by (*simp add*:
fps-divide-def fps-X-power-subdegree fps-X-power-unit-factor fps-inverse-one'
)

lemma *fps-shift-altdef*:
fps-shift n f = (*f* :: *'a* :: *division-ring fps*) *div fps-X*^*n*
by (*rule fps-shift-altdef'*[*OF inverse-1*])

lemma *fps-div-fps-X-power-nth'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows (*f div fps-X*^*n*) $ *k* = *f* $ (*k* + *n*)
using *assms*
by (*simp add*: *fps-shift-altdef'* [*symmetric*])

lemma *fps-div-fps-X-power-nth*: ((*f* :: *'a* :: *division-ring fps*) *div fps-X*^*n*) $ *k* = *f* $ (*k* + *n*)
by (*rule fps-div-fps-X-power-nth'*[*OF inverse-1*])

lemma *fps-div-fps-X-nth'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fps*
assumes *inverse* (*1*::*'a*) = *1*
shows (*f div fps-X*) $ *k* = *f* $ *Suc k*
using *assms fps-div-fps-X-power-nth'*[*of f 1*]
by *simp*

lemma *fps-div-fps-X-nth*: ((*f* :: *'a* :: *division-ring fps*) *div fps-X*) $ *k* = *f* $ *Suc k*
by (*rule fps-div-fps-X-nth'*[*OF inverse-1*])

lemma *divide-fps-const'*:
fixes *c* :: *'a* :: {*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*}
shows *f* / *fps-const c* = *f* ∗ *fps-const* (*inverse c*)
by (*simp add*: *fps-divide-def fps-const-inverse*)

lemma *divide-fps-const* [*simp*]:
fixes *c* :: *'a* :: {*comm-semiring-0*,*inverse*,*uminus*}
shows *f* / *fps-const c* = *fps-const* (*inverse c*) ∗ *f*

by (*simp add*: *divide-fps-const' mult.commute*)

lemma *fps-const-divide*: *fps-const* (*x* :: - :: *division-ring*) / *fps-const y* = *fps-const* (*x* / *y*)
by (*simp add*: *fps-divide-def fps-const-inverse divide-inverse*)

lemma *fps-numeral-divide-divide*:
x / *numeral b* / *numeral c* = (*x* / *numeral* (*b* * *c*) :: 'a :: *field fps*)
by (*simp add*: *fps-divide-by-mult*[*symmetric*])

lemma *fps-numeral-mult-divide*:
numeral b * *x* / *numeral c* = (*numeral b* / *numeral c* * *x* :: 'a :: *field fps*)
by (*simp add*: *fps-divide-times2*)

lemmas *fps-numeral-simps* =
fps-numeral-divide-divide fps-numeral-mult-divide inverse-fps-numeral neg-numeral-fps-const

lemma *fps-is-left-unit-iff-zeroth-is-left-unit*:
fixes *f* :: 'a :: *ring-1 fps*
shows (∃ *g*. *1* = *f* * *g*) ⟷ (∃ *k*. *1* = *f*$*0* * *k*)
proof
assume ∃ *g*. *1* = *f* * *g*
then obtain *g* **where** *1* = *f* * *g* **by** *fast*
hence *1* = *f*$*0* * *g*$*0* **using** *fps-mult-nth-0*[*of f g*] **by** *simp*
thus ∃ *k*. *1* = *f*$*0* * *k* **by** *auto*
next
assume ∃ *k*. *1* = *f*$*0* * *k*
then obtain *k* **where** *1* = *f*$*0* * *k* **by** *fast*
hence *1* = *f* * *fps-right-inverse f k*
using *fps-right-inverse* **by** *simp*
thus ∃ *g*. *1* = *f* * *g* **by** *fast*
qed

lemma *fps-is-right-unit-iff-zeroth-is-right-unit*:
fixes *f* :: 'a :: *ring-1 fps*
shows (∃ *g*. *1* = *g* * *f*) ⟷ (∃ *k*. *1* = *k* * *f*$*0*)
proof
assume ∃ *g*. *1* = *g* * *f*
then obtain *g* **where** *1* = *g* * *f* **by** *fast*
hence *1* = *g*$*0* * *f*$*0* **using** *fps-mult-nth-0*[*of g f*] **by** *simp*
thus ∃ *k*. *1* = *k* * *f*$*0* **by** *auto*
next
assume ∃ *k*. *1* = *k* * *f*$*0*
then obtain *k* **where** *1* = *k* * *f*$*0* **by** *fast*
hence *1* = *fps-left-inverse f k* * *f*
using *fps-left-inverse* **by** *simp*
thus ∃ *g*. *1* = *g* * *f* **by** *fast*
qed

lemma *fps-is-unit-iff* [*simp*]: (*f* :: *'a* :: *field fps*) *dvd 1* ⟷ *f* $ *0* ≠ *0*
proof
  assume *f dvd 1*
  then obtain *g* **where** *1* = *f* ∗ *g* **by** (*elim dvdE*)
  from *this*[*symmetric*] **have** (*f*∗*g*) $ *0* = *1* **by** *simp*
  thus *f* $ *0* ≠ *0* **by** *auto*
next
  assume *A*: *f* $ *0* ≠ *0*
  thus *f dvd 1* **by** (*simp add*: *inverse-mult-eq-1*[*OF A*, *symmetric*])
qed

lemma *subdegree-eq-0-left*:
  fixes *f* :: *'a*::{*comm-monoid-add*,*zero-neq-one*,*mult-zero*} *fps*
  assumes ∃ *g*. *1* = *f* ∗ *g*
  shows *subdegree f* = *0*
proof (*intro subdegree-eq-0*)
  from *assms* **obtain** *g* **where** *1* = *f* ∗ *g* **by** *fast*
  hence *f*$*0* ∗ *g*$*0* = *1* **using** *fps-mult-nth-0*[*of f g*] **by** *simp*
  thus *f*$*0* ≠ *0* **by** *auto*
qed

lemma *subdegree-eq-0-right*:
  fixes *f* :: *'a*::{*comm-monoid-add*,*zero-neq-one*,*mult-zero*} *fps*
  assumes ∃ *g*. *1* = *g* ∗ *f*
  shows *subdegree f* = *0*
proof (*intro subdegree-eq-0*)
  from *assms* **obtain** *g* **where** *1* = *g* ∗ *f* **by** *fast*
  hence *g*$*0* ∗ *f*$*0* = *1* **using** *fps-mult-nth-0*[*of g f*] **by** *simp*
  thus *f*$*0* ≠ *0* **by** *auto*
qed

lemma *subdegree-eq-0'* [*simp*]: (*f* :: *'a* :: *field fps*) *dvd 1* ⟹ *subdegree f* = *0*
  by *simp*

lemma *fps-dvd1-left-trivial-unit-factor*:
  fixes *f* :: *'a*::{*comm-monoid-add*, *zero-neq-one*, *mult-zero*} *fps*
  assumes ∃ *g*. *1* = *f* ∗ *g*
  shows *unit-factor f* = *f*
  using *assms subdegree-eq-0-left*
  by *fastforce*

lemma *fps-dvd1-right-trivial-unit-factor*:
  fixes *f* :: *'a*::{*comm-monoid-add*, *zero-neq-one*, *mult-zero*} *fps*
  assumes ∃ *g*. *1* = *g* ∗ *f*
  shows *unit-factor f* = *f*
  using *assms subdegree-eq-0-right*
  by *fastforce*

lemma *fps-dvd1-trivial-unit-factor*:

(*f* :: *'a*::*comm-semiring-1 fps*) *dvd 1* ⟹ *unit-factor f* = *f*
unfolding *dvd-def* **by** (*rule fps-dvd1-left-trivial-unit-factor*) *simp*

lemma *fps-unit-dvd-left*:
fixes *f* :: *'a* :: *division-ring fps*
assumes *f* $ *0* ≠ *0*
shows ∃ *g*. *1* = *f* ∗ *g*
using *assms fps-is-left-unit-iff-zeroth-is-left-unit right-inverse*
by *fastforce*

lemma *fps-unit-dvd-right*:
fixes *f* :: *'a* :: *division-ring fps*
assumes *f* $ *0* ≠ *0*
shows ∃ *g*. *1* = *g* ∗ *f*
using *assms fps-is-right-unit-iff-zeroth-is-right-unit left-inverse*
by *fastforce*

lemma *fps-unit-dvd* [*simp*]: (*f* $ *0* :: *'a* :: *field*) ≠ *0* ⟹ *f dvd g*
using *fps-unit-dvd-left dvd-trans*[*of f 1*] **by** *simp*

lemma *dvd-left-imp-subdegree-le*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
assumes ∃ *k*. *g* = *f* ∗ *k g* ≠ *0*
shows *subdegree f* ≤ *subdegree g*
using *assms fps-mult-subdegree-ge*
by *fastforce*

lemma *dvd-right-imp-subdegree-le*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
assumes ∃ *k*. *g* = *k* ∗ *f g* ≠ *0*
shows *subdegree f* ≤ *subdegree g*
using *assms fps-mult-subdegree-ge*
by *fastforce*

lemma *dvd-imp-subdegree-le*:
f dvd g ⟹ *g* ≠ *0* ⟹ *subdegree f* ≤ *subdegree g*
using *dvd-left-imp-subdegree-le* **by** *fast*

lemma *subdegree-le-imp-dvd-left-ring1*:
fixes *f g* :: *'a* :: *ring-1 fps*
assumes ∃ *y*. *f* $ *subdegree f* ∗ *y* = *1 subdegree f* ≤ *subdegree g*
shows ∃ *k*. *g* = *f* ∗ *k*
proof−
define *h* :: *'a fps* **where** *h* ≡ *fps-X* ^ (*subdegree g* − *subdegree f*)
from *assms*(*1*) **obtain** *y* **where** *f* $ *subdegree f* ∗ *y* = *1* **by** *fast*
hence *unit-factor f* $ *0* ∗ *y* = *1* **by** *simp*
from *this* **obtain** *k* **where** *1* = *unit-factor f* ∗ *k*
using *fps-is-left-unit-iff-zeroth-is-left-unit*[*of unit-factor f*] **by** *auto*
hence *fps-X* ^ *subdegree f* = *fps-X* ^ *subdegree f* ∗ *unit-factor f* ∗ *k*

by (*simp add*: *mult.assoc*)
moreover have *fps-X* ^ *subdegree f* ∗ *unit-factor f* = *f*
by (*rule fps-unit-factor-decompose'*[*symmetric*])
ultimately have
fps-X ^ (*subdegree f* + (*subdegree g* − *subdegree f*)) = *f* ∗ *k* ∗ *h*
by (*simp add*: *power-add h-def*)
hence *g* = *f* ∗ (*k* ∗ *h* ∗ *unit-factor g*)
using *fps-unit-factor-decompose'*[*of g*]
by (*simp add*: *assms*(*2*) *mult.assoc*)
thus *?thesis* **by** *fast*
qed

lemma *subdegree-le-imp-dvd-left-divring*:
fixes *f g* :: *'a* :: *division-ring fps*
assumes *f* ≠ *0 subdegree f* ≤ *subdegree g*
shows ∃ *k*. *g* = *f* ∗ *k*
proof (*intro subdegree-le-imp-dvd-left-ring1*)
from *assms*(*1*) **have** *f* $ *subdegree f* ≠ *0* **by** *simp*
thus ∃ *y*. *f* $ *subdegree f* ∗ *y* = *1* **using** *right-inverse* **by** *blast*
qed (*rule assms*(*2*))

lemma *subdegree-le-imp-dvd-right-ring1*:
fixes *f g* :: *'a* :: *ring-1 fps*
assumes ∃ *x*. *x* ∗ *f* $ *subdegree f* = *1 subdegree f* ≤ *subdegree g*
shows ∃ *k*. *g* = *k* ∗ *f*
proof−
define *h* :: *'a fps* **where** *h* ≡ *fps-X* ^ (*subdegree g* − *subdegree f*)
from *assms*(*1*) **obtain** *x* **where** *x* ∗ *f* $ *subdegree f* = *1* **by** *fast*
hence *x* ∗ *unit-factor f* $ *0* = *1* **by** *simp*
from *this* **obtain** *k* **where** *1* = *k* ∗ *unit-factor f*
using *fps-is-right-unit-iff-zeroth-is-right-unit*[*of unit-factor f*] **by** *auto*
hence *fps-X* ^ *subdegree f* = *k* ∗ (*unit-factor f* ∗ *fps-X* ^ *subdegree f*)
by (*simp add*: *mult.assoc*[*symmetric*])
moreover have *unit-factor f* ∗ *fps-X* ^ *subdegree f* = *f*
by (*rule fps-unit-factor-decompose*[*symmetric*])
ultimately have *fps-X* ^ (*subdegree g* − *subdegree f* + *subdegree f*) = *h* ∗ *k* ∗ *f*
by (*simp add*: *power-add h-def mult.assoc*)
hence *g* = *unit-factor g* ∗ *h* ∗ *k* ∗ *f*
using *fps-unit-factor-decompose*[*of g*]
by (*simp add*: *assms*(*2*) *mult.assoc*)
thus *?thesis* **by** *fast*
qed

lemma *subdegree-le-imp-dvd-right-divring*:
fixes *f g* :: *'a* :: *division-ring fps*
assumes *f* ≠ *0 subdegree f* ≤ *subdegree g*
shows ∃ *k*. *g* = *k* ∗ *f*
proof (*intro subdegree-le-imp-dvd-right-ring1*)
from *assms*(*1*) **have** *f* $ *subdegree f* ≠ *0* **by** *simp*

thus $\exists x.\ x * f\ \$\ subdegree\ f = 1$ **using** *left-inverse* **by** *blast*
qed (*rule assms*(*2*))

lemma *fps-dvd-iff*:
assumes ($f :: 'a :: field\ fps$) $\neq 0\ g \neq 0$
shows $f\ dvd\ g \longleftrightarrow subdegree\ f \leq subdegree\ g$
proof
assume $subdegree\ f \leq subdegree\ g$
with *assms* **show** $f\ dvd\ g$
using *subdegree-le-imp-dvd-left-divring*
by (*auto intro*: *dvdI*)
qed (*simp add*: *assms dvd-imp-subdegree-le*)

lemma *subdegree-div'*:
fixes $p\ q :: 'a::division\text{-}ring\ fps$
assumes $\exists k.\ p = k * q$
shows $subdegree\ (p\ div\ q) = subdegree\ p - subdegree\ q$
proof (*cases* $p = 0$)
case *False*
from *assms*(*1*) **obtain** k **where** k: $p = k * q$ **by** *blast*
with *False* **have** $subdegree\ (p\ div\ q) = subdegree\ k$ **by** (*simp add*: *fps-divide-times-eq*)
moreover have $k\ \$\ subdegree\ k * q\ \$\ subdegree\ q \neq 0$
proof
assume $k\ \$\ subdegree\ k * q\ \$\ subdegree\ q = 0$
hence $k\ \$\ subdegree\ k * q\ \$\ subdegree\ q * inverse\ (q\ \$\ subdegree\ q) = 0$ **by** *simp*
with *False* k **show** *False* **by** (*simp add*: *mult.assoc*)
qed
ultimately show *?thesis* **by** (*simp add*: k *subdegree-mult'*)
qed *simp*

lemma *subdegree-div*:
fixes $p\ q :: 'a :: field\ fps$
assumes $q\ dvd\ p$
shows $subdegree\ (p\ div\ q) = subdegree\ p - subdegree\ q$
using *assms*
unfolding *dvd-def*
by (*auto intro*: *subdegree-div'*)

lemma *subdegree-div-unit'*:
fixes $p\ q :: 'a :: \{ab\text{-}group\text{-}add,mult\text{-}zero,inverse\}\ fps$
assumes $q\ \$\ 0 \neq 0\ p\ \$\ subdegree\ p * inverse\ (q\ \$\ 0) \neq 0$
shows $subdegree\ (p\ div\ q) = subdegree\ p$
using *assms subdegree-mult'*[*of p inverse q*]
by (*auto simp add*: *fps-divide-unit*)

lemma *subdegree-div-unit''*:
fixes $p\ q :: 'a :: \{ring\text{-}no\text{-}zero\text{-}divisors,inverse\}\ fps$
assumes $q\ \$\ 0 \neq 0\ inverse\ (q\ \$\ 0) \neq 0$

shows *subdegree* (*p div q*) = *subdegree p*
by (*cases p* = *0*) (*auto intro*: *subdegree-div-unit' simp*: *assms*)

lemma *subdegree-div-unit*:
fixes *p q* :: *'a* :: *division-ring fps*
assumes *q* $ *0* ≠ *0*
shows *subdegree* (*p div q*) = *subdegree p*
by (*intro subdegree-div-unit''*) (*simp-all add*: *assms*)

instantiation *fps* :: ({*comm-semiring-1*,*inverse*,*uminus*}) *modulo*
begin

definition *fps-mod-def*:
f mod g = (*if g* = *0 then f else*
let h = *unit-factor g in fps-cutoff* (*subdegree g*) (*f* ∗ *inverse h*) ∗ *h*)

instance ..

end

lemma *fps-mod-zero* [*simp*]:
(*f*::*'a*::{*comm-semiring-1*,*inverse*,*uminus*} *fps*) *mod* *0* = *f*
by (*simp add*: *fps-mod-def*)

lemma *fps-mod-eq-zero*:
assumes *g* ≠ *0* **and** *subdegree f* ≥ *subdegree g*
shows *f mod g* = *0*
proof (*cases f* ∗ *inverse* (*unit-factor g*) = *0*)
case *False*
have *fps-cutoff* (*subdegree g*) (*f* ∗ *inverse* (*unit-factor g*)) = *0*
using *False assms*(*2*) *fps-mult-subdegree-ge fps-cutoff-zero-iff* **by** *force*
with *assms*(*1*) **show** *?thesis* **by** (*simp add*: *fps-mod-def Let-def*)
qed (*simp add*: *assms fps-mod-def*)

lemma *fps-mod-unit* [*simp*]: *g*$*0* ≠ *0* ⟹ *f mod g* = *0*
by (*intro fps-mod-eq-zero*) *auto*

lemma *subdegree-mod*:
assumes *subdegree* (*f*::*'a*::*field fps*) < *subdegree g*
shows *subdegree* (*f mod g*) = *subdegree f*
proof (*cases f* = *0*)
case *False*
with *assms* **show** *?thesis*
by (*intro subdegreeI*)
(*auto simp*: *inverse-mult-eq-1 fps-mod-def Let-def fps-cutoff-left-mult-nth mult.assoc*)
qed (*simp add*: *fps-mod-def*)

instance *fps* :: (*field*) *idom-modulo*

proof

fix *f g* :: *'a fps*

define *n* **where** *n* = *subdegree g*
define *h* **where** *h* = *f* ∗ *inverse* (*unit-factor g*)

show *f div g* ∗ *g* + *f mod g* = *f*
proof (*cases g* = *0*)
case *False*
with *n-def h-def* **have**
f div g ∗ *g* + *f mod g* = (*fps-shift n h* ∗ *fps-X* ^ *n* + *fps-cutoff n h*) ∗ *unit-factor g*
by (*simp add*: *fps-divide-def fps-mod-def Let-def subdegree-decompose algebra-simps*)
with *False* **show** *?thesis*
by (*simp add*: *fps-shift-cutoff h-def inverse-mult-eq-1*)
qed *auto*

qed (*rule fps-divide-times-eq*, *simp-all add*: *fps-divide-def*)

instantiation *fps* :: (*field*) *normalization-semidom-multiplicative*
begin

definition *fps-normalize-def* [*simp*]:
normalize f = (*if f* = *0 then 0 else fps-X* ^ *subdegree f*)

instance proof
fix *f g* :: *'a fps*
assume *is-unit f*
thus *unit-factor* (*f* ∗ *g*) = *f* ∗ *unit-factor g*
using *fps-unit-factor-mult*[*of f g*] **by** *simp*
next
fix *f g* :: *'a fps*
show *unit-factor f* ∗ *normalize f* = *f*
by (*simp add*: *fps-shift-times-fps-X-power*)
next
fix *f g* :: *'a fps*
show *unit-factor* (*f* ∗ *g*) = *unit-factor f* ∗ *unit-factor g*
using *fps-unit-factor-mult*[*of f g*] **by** *simp*
qed (*simp-all add*: *fps-divide-def Let-def*)

end

5.8 Computing reciprocals via Hensel lifting

lemma *inverse-fps-hensel-lifting*:
fixes *F G* :: *'a* :: *field fps* **and** *n* :: *nat*
assumes *G-eq*: *fps-cutoff n G* = *fps-cutoff n* (*inverse F*)

assumes *unit*: *fps-nth F 0* $\neq$ *0*
shows *fps-cutoff* (*2*∗*n*) (*inverse F*) = *fps-cutoff* (*2*∗*n*) (*G* ∗ (*2* − *F* ∗ *G*))
proof −
define *R* **where** *R* = *inverse F* − *G*
have *eq*: *G* = *inverse F* − *R*
by (*simp add*: *R-def*)
from *assms* **have** *fps-cutoff n R* = *0*
by (*simp add*: *R-def fps-cutoff-diff*)
hence *R*: *R* = *0* ∨ *subdegree R* ≥ *n*
by (*simp add*: *fps-cutoff-zero-iff*)

have *G* ∗ (*2* − *F* ∗ *G*) − *inverse F* =
inverse F + *F* ∗ *inverse F* ∗ *R* ∗ *2* − *F* ∗ R^2 − *R* ∗ *2* − *F* ∗ *inverse F* ∗ *inverse F*
by (*simp add*: *eq algebra-simps power2-eq-square*)
also have *F* ∗ *inverse F* = *1*
using *unit* **by** (*simp add*: *inverse-mult-eq-1* ′)
also have *inverse F* + *1* ∗ *R* ∗ *2* − *F* ∗ R^2 − *R* ∗ *2* − *1* ∗ *inverse F* = −*F* ∗ R^2
by (*simp add*: *algebra-simps*)
finally have *fps-cutoff* (*2*∗*n*) (*G* ∗ (*2* − *F* ∗ *G*) − *inverse F*) = *fps-cutoff* (*2*∗*n*) (−*F* ∗ R^2)
by (*rule arg-cong*)
also have ... = *0*
proof (*cases* −*F* ∗ R^2 = *0*)
case *False*
have *2* ∗ *n* ≤ *subdegree* (−*F* ∗ R^2)
using *False R unit* **by** *simp*
thus *?thesis*
by (*simp add*: *fps-cutoff-zero-iff*)
qed *auto*
finally show *?thesis*
by (*simp add*: *fps-cutoff-diff*)
qed

lemma *inverse-fps-hensel-lifting* ′:
fixes *F G* :: ′*a* :: *field fps* **and** *n* :: *nat*
assumes *G-eq*: *fps-cutoff n G* = *fps-cutoff n* (*inverse F*)
assumes *unit*: *fps-nth F 0* $\neq$ *0*
defines *P* ≡ *fps-shift n* (*F* ∗ *G* − *1*)
shows *fps-cutoff* (*2*∗*n*) (*inverse F*) = *fps-cutoff* (*2*∗*n*) (*G* ∗ (*1* − *fps-X* ^ *n* ∗ *P*))
proof −
define *R* **where** *R* = *inverse F* − *G*
have *eq*: *G* = *inverse F* − *R*
by (*simp add*: *R-def*)
from *assms* **have** *fps-cutoff n R* = *0*
by (*simp add*: *R-def fps-cutoff-diff*)
hence *R*: *R* = *0* ∨ *subdegree R* ≥ *n*

```
    by (simp add: fps-cutoff-zero-iff)

  have FG-eq: F * G = 1 + fps-X ^ n * P
  proof (cases F * G - 1 = 0)
    case False
    have eq: F * G - 1 = F * (G - inverse F)
      using unit by (simp add: inverse-mult-eq-1' ring-distribs)
    have subdegree (F * (G - inverse F)) ≥ n
    proof -
      have fps-cutoff n (G - inverse F) = 0
        using G-eq by (simp add: fps-cutoff-diff)
      hence n ≤ subdegree (G - inverse F)
        using False unfolding eq by (simp add: fps-cutoff-zero-iff)
      also have subdegree (G - inverse F) = subdegree (F * (G - inverse F))
        by (subst subdegree-mult) (use unit False in ‹auto simp: eq›)
      finally have n ≤ subdegree (F * (G - inverse F)) .
      thus ?thesis
        by blast
    qed
    hence F * G - 1 = fps-X ^ n * P
      unfolding eq P-def by (intro fps-conv-fps-X-power-mult-fps-shift) auto
    thus ?thesis
      by (simp add: algebra-simps)
  qed (auto simp: P-def)

  have G * (1 - fps-X ^ n * P) - inverse F = G * (2 - F * G) - inverse F
    by (auto simp: FG-eq)
  also have G * (2 - F * G) - inverse F =
        inverse F + F * inverse F * R * 2 - F * R^2 - R * 2 - F * inverse F *
inverse F
    by (simp add: eq algebra-simps power2-eq-square)
  also have F * inverse F = 1
    using unit by (simp add: inverse-mult-eq-1')
  also have inverse F + 1 * R * 2 - F * R^2 - R * 2 - 1 * inverse F = -F *
R^2
    by (simp add: algebra-simps)
  finally have fps-cutoff (2*n) (G * (1 - fps-X ^ n * P) - inverse F) = fps-cutoff
(2*n) (-F * R^2)
    by (rule arg-cong)
  also have ... = 0
  proof (cases -F * R^2 = 0)
    case False
    have 2 * n ≤ subdegree (-F * R^2)
      using False R unit by simp
    thus ?thesis
      by (simp add: fps-cutoff-zero-iff)
  qed auto
  finally show ?thesis
    by (simp add: fps-cutoff-diff)
```

qed

5.9 Euclidean division

instantiation *fps* :: (*field*) *euclidean-ring-cancel*
begin

definition *fps-euclidean-size-def*:
euclidean-size f = (*if f* = *0 then 0 else 2* ^ *subdegree f*)

instance proof
fix *f g* :: *'a fps* **assume** [*simp*]: *g* ≠ *0*
show *euclidean-size f* ≤ *euclidean-size* (*f* ∗ *g*)
by (*cases f* = *0*) (*simp-all add*: *fps-euclidean-size-def*)
show *euclidean-size* (*f mod g*) < *euclidean-size g*
proof (*cases f* = *0*)
case *True*
then show *?thesis*
by (*simp add*: *fps-euclidean-size-def*)
next
case *False*
then show *?thesis*
using *le-less-linear*[*of subdegree g subdegree f*]
by (*force simp add*: *fps-mod-eq-zero fps-euclidean-size-def subdegree-mod*)
qed
next
fix *f g h* :: *'a fps* **assume** [*simp*]: *h* ≠ *0*
show (*h* ∗ *f*) *div* (*h* ∗ *g*) = *f div g*
by (*simp add*: *fps-divide-cancel mult.commute*)
show (*f* + *g* ∗ *h*) *div h* = *g* + *f div h*
by (*simp add*: *fps-divide-add fps-divide-times-eq*)
qed (*simp add*: *fps-euclidean-size-def*)

end

instance *fps* :: (*field*) *normalization-euclidean-semiring* **..**

instantiation *fps* :: (*field*) *euclidean-ring-gcd*
begin
definition *fps-gcd-def*: (*gcd* :: *'a fps* ⇒ -) = *Euclidean-Algorithm.gcd*
definition *fps-lcm-def*: (*lcm* :: *'a fps* ⇒ -) = *Euclidean-Algorithm.lcm*
definition *fps-Gcd-def*: (*Gcd* :: *'a fps set* ⇒ -) = *Euclidean-Algorithm.Gcd*
definition *fps-Lcm-def*: (*Lcm* :: *'a fps set* ⇒ -) = *Euclidean-Algorithm.Lcm*
instance by *standard* (*simp-all add*: *fps-gcd-def fps-lcm-def fps-Gcd-def fps-Lcm-def*)
end

lemma *fps-gcd*:
assumes [*simp*]: *f* ≠ *0 g* ≠ *0*
shows *gcd f g* = *fps-X* ^ *min* (*subdegree f*) (*subdegree g*)

proof −
let *?m* = *min* (*subdegree f*) (*subdegree g*)
show *gcd f g* = *fps-X* ^ *?m*
proof (*rule sym*, *rule gcdI*)
fix *d* **assume** *d dvd f d dvd g*
thus *d dvd fps-X* ^ *?m* **by** (*cases d* = *0*) (*simp-all add*: *fps-dvd-iff*)
qed (*simp-all add*: *fps-dvd-iff*)
qed

lemma *fps-gcd-altdef*: *gcd f g* =
(*if f* = *0* ∧ *g* = *0 then 0 else*
if f = *0 then fps-X* ^ *subdegree g else*
if g = *0 then fps-X* ^ *subdegree f else*
fps-X ^ *min* (*subdegree f*) (*subdegree g*))
by (*simp add*: *fps-gcd*)

lemma *fps-lcm*:
assumes [*simp*]: *f* ≠ *0 g* ≠ *0*
shows *lcm f g* = *fps-X* ^ *max* (*subdegree f*) (*subdegree g*)
proof −
let *?m* = *max* (*subdegree f*) (*subdegree g*)
show *lcm f g* = *fps-X* ^ *?m*
proof (*rule sym*, *rule lcmI*)
fix *d* **assume** *f dvd d g dvd d*
thus *fps-X* ^ *?m dvd d* **by** (*cases d* = *0*) (*simp-all add*: *fps-dvd-iff*)
qed (*simp-all add*: *fps-dvd-iff*)
qed

lemma *fps-lcm-altdef*: *lcm f g* =
(*if f* = *0* ∨ *g* = *0 then 0 else fps-X* ^ *max* (*subdegree f*) (*subdegree g*))
by (*simp add*: *fps-lcm*)

lemma *fps-Gcd*:
assumes *A* − {*0*} ≠ {}
shows *Gcd A* = *fps-X* ^ (*INF f*∈*A*−{*0*}. *subdegree f*)
proof (*rule sym*, *rule GcdI*)
fix *f* **assume** *f* ∈ *A*
thus *fps-X* ^ (*INF f*∈*A* − {*0*}. *subdegree f*) *dvd f*
by (*cases f* = *0*) (*auto simp*: *fps-dvd-iff intro*!: *cINF-lower*)
next
fix *d* **assume** *d*: ⋀*f*. *f* ∈ *A* ⟹ *d dvd f*
from *assms* **obtain** *f* **where** *f* ∈ *A* − {*0*} **by** *auto*
with *d*[*of f*] **have** [*simp*]: *d* ≠ *0* **by** *auto*
from *d assms* **have** *subdegree d* ≤ (*INF f*∈*A*−{*0*}. *subdegree f*)
by (*intro cINF-greatest*) (*simp-all add*: *fps-dvd-iff*[*symmetric*])
with *d assms* **show** *d dvd fps-X* ^ (*INF f*∈*A*−{*0*}. *subdegree f*) **by** (*simp add*: *fps-dvd-iff*)
qed *simp-all*

lemma *fps-Gcd-altdef*: *Gcd A* =
(*if A* ⊆ {*0*} *then 0 else fps-X* ^ (*INF f*∈*A*−{*0*}. *subdegree f*))
using *fps-Gcd* **by** *auto*

lemma *fps-Lcm*:
assumes *A* ≠ {} *0* ∉ *A bdd-above* (*subdegree*‘*A*)
shows *Lcm A* = *fps-X* ^ (*SUP f*∈*A*. *subdegree f*)
proof (*rule sym*, *rule LcmI*)
fix *f* **assume** *f* ∈ *A*
moreover from *assms*(*3*) **have** *bdd-above* (*subdegree* ‘ *A*) **by** *auto*
ultimately show *f dvd fps-X* ^ (*SUP f*∈*A*. *subdegree f*) **using** *assms*(*2*)
by (*cases f* = *0*) (*auto simp*: *fps-dvd-iff intro*!: *cSUP-upper*)
next
fix *d* **assume** *d*: ⋀*f*. *f* ∈ *A* ⟹ *f dvd d*
from *assms* **obtain** *f* **where** *f*: *f* ∈ *A f* ≠ *0* **by** *auto*
show *fps-X* ^ (*SUP f*∈*A*. *subdegree f*) *dvd d*
proof (*cases d* = *0*)
assume *d* ≠ *0*
moreover from *d* **have** ⋀*f*. *f* ∈ *A* ⟹ *f* ≠ *0* ⟹ *f dvd d* **by** *blast*
ultimately have *subdegree d* ≥ (*SUP f*∈*A*. *subdegree f*) **using** *assms*
by (*intro cSUP-least*) (*auto simp*: *fps-dvd-iff*)
with ‹*d* ≠ *0*› **show** *?thesis* **by** (*simp add*: *fps-dvd-iff*)
qed *simp-all*
qed *simp-all*

lemma *fps-Lcm-altdef*:
Lcm A =
(*if 0* ∈ *A* ∨ ¬*bdd-above* (*subdegree*‘*A*) *then 0 else*
if A = {} *then 1 else fps-X* ^ (*SUP f*∈*A*. *subdegree f*))
proof (*cases bdd-above* (*subdegree*‘*A*))
assume *unbounded*: ¬*bdd-above* (*subdegree*‘*A*)
have *Lcm A* = *0*
proof (*rule ccontr*)
assume *Lcm A* ≠ *0*
from *unbounded* **obtain** *f* **where** *f*: *f* ∈ *A subdegree* (*Lcm A*) < *subdegree f*
unfolding *bdd-above-def* **by** (*auto simp*: *not-le*)
moreover from *f* **and** ‹*Lcm A* ≠ *0*› **have** *subdegree f* ≤ *subdegree* (*Lcm A*)
by (*intro dvd-imp-subdegree-le dvd-Lcm*) *simp-all*
ultimately show *False* **by** *simp*
qed
with *unbounded* **show** *?thesis* **by** *simp*
qed (*simp-all add*: *fps-Lcm Lcm-eq-0-I*)

5.10 Formal Derivatives

definition *fps-deriv f* = *Abs-fps* (λ*n*. *of-nat* (*n* + *1*) ∗ *f* $ (*n* + *1*))

lemma *fps-deriv-nth*[*simp*]: *fps-deriv f* $ *n* = *of-nat* (*n* + *1*) ∗ *f* $ (*n* + *1*)
by (*simp add*: *fps-deriv-def*)

lemma *fps-0th-higher-deriv*:
(*fps-deriv* ^^ *n*) *f* $ *0* = *fact n* * *f* $ *n*
by (*induction n arbitrary*: *f*)
(*simp-all add*: *funpow-Suc-right mult-of-nat-commute algebra-simps del*: *funpow.simps*)

lemma *fps-deriv-mult*[*simp*]:
fps-deriv (*f* * *g*) = *f* * *fps-deriv g* + *fps-deriv f* * *g*
proof (*intro fps-ext*)
fix *n*
have *LHS*: *fps-deriv* (*f* * *g*) $ *n* = (∑ *i*=*0*..*Suc n*. *of-nat* (*n*+*1*) * *f*$*i* * *g*$(*Suc n* − *i*))
by (*simp add*: *fps-mult-nth sum-distrib-left algebra-simps*)

have ∀ *i*∈{*1*..*n*}. *n* − (*i* − *1*) = *n* − *i* + *1* **by** *auto*
moreover have
(∑ *i*=*0*..*n*. *of-nat* (*i*+*1*) * *f*$(*i*+*1*) * *g*$(*n* − *i*)) =
(∑ *i*=*1*..*Suc n*. *of-nat i* * *f*$*i* * *g*$(*n* − (*i* − *1*)))
by (*intro sum.reindex-bij-witness*[**where** *i*=λ*x*. *x*−*1* **and** *j*=λ*x*. *x*+*1*]) *auto*
ultimately have
(*f* * *fps-deriv g* + *fps-deriv f* * *g*) $ *n* =
of-nat (*Suc n*) * *f*$*0* * *g*$(*Suc n*) +
(∑ *i*=*1*..*n*. (*of-nat* (*n* − *i* + *1*) + *of-nat i*) * *f* $ *i* * *g* $ (*n* − *i* + *1*)) +
of-nat (*Suc n*) * *f*$(*Suc n*) * *g*$*0*
by (*simp add*: *fps-mult-nth algebra-simps mult-of-nat-commute sum.atLeast-Suc-atMost sum.distrib*)
moreover have
∀ *i*∈{*1*..*n*}.
(*of-nat* (*n* − *i* + *1*) + *of-nat i*) * *f* $ *i* * *g* $ (*n* − *i* + *1*) =
of-nat (*n* + *1*) * *f* $ *i* * *g* $ (*Suc n* − *i*)
proof
fix *i* **assume** *i*: *i* ∈ {*1*..*n*}
from *i* **have** *of-nat* (*n* − *i* + *1*) + (*of-nat i* :: *'a*) = *of-nat* (*n* + *1*)
using *of-nat-add*[*of n*−*i*+*1 i*,*symmetric*] **by** *simp*
moreover from *i* **have** *Suc n* − *i* = *n* − *i* + *1* **by** *auto*
ultimately show (*of-nat* (*n* − *i* + *1*) + *of-nat i*) * *f* $ *i* * *g* $ (*n* − *i* + *1*) =
of-nat (*n* + *1*) * *f* $ *i* * *g* $ (*Suc n* − *i*)
by *simp*
qed
ultimately have
(*f* * *fps-deriv g* + *fps-deriv f* * *g*) $ *n* =
(∑ *i*=*0*..*Suc n*. *of-nat* (*Suc n*) * *f* $ *i* * *g* $ (*Suc n* − *i*))
by (*simp add*: *sum.atLeast-Suc-atMost*)
with *LHS* **show** *fps-deriv* (*f* * *g*) $ *n* = (*f* * *fps-deriv g* + *fps-deriv f* * *g*) $ *n*
by *simp*
qed

lemma *fps-deriv-fps-X*[*simp*]: *fps-deriv fps-X* = *1*

by (*simp add*: *fps-deriv-def fps-X-def fps-eq-iff*)

lemma *fps-deriv-neg*[*simp*]:
fps-deriv (− (*f*:: *'a*::*ring-1 fps*)) = − (*fps-deriv f*)
by (*simp add*: *fps-eq-iff fps-deriv-def*)

lemma *fps-deriv-add*[*simp*]: *fps-deriv* (*f* + *g*) = *fps-deriv f* + *fps-deriv g*
by (*auto intro*: *fps-ext simp*: *algebra-simps*)

lemma *fps-deriv-sub*[*simp*]:
fps-deriv ((*f*:: *'a*::*ring-1 fps*) − *g*) = *fps-deriv f* − *fps-deriv g*
using *fps-deriv-add* [*of f* − *g*] **by** *simp*

lemma *fps-deriv-const*[*simp*]: *fps-deriv* (*fps-const c*) = *0*
by (*simp add*: *fps-ext fps-deriv-def fps-const-def*)

lemma *fps-deriv-of-nat* [*simp*]: *fps-deriv* (*of-nat n*) = *0*
by (*simp add*: *fps-of-nat* [*symmetric*])

lemma *fps-deriv-of-int* [*simp*]: *fps-deriv* (*of-int n*) = *0*
by (*simp add*: *fps-of-int* [*symmetric*])

lemma *fps-deriv-numeral* [*simp*]: *fps-deriv* (*numeral n*) = *0*
by (*simp add*: *numeral-fps-const*)

lemma *fps-deriv-mult-const-left*[*simp*]:
fps-deriv (*fps-const c* ∗ *f*) = *fps-const c* ∗ *fps-deriv f*
by *simp*

lemma *fps-deriv-linear*[*simp*]:
fps-deriv (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) =
fps-const a ∗ *fps-deriv f* + *fps-const b* ∗ *fps-deriv g*
by *simp*

lemma *fps-deriv-0*[*simp*]: *fps-deriv 0* = *0*
by (*simp add*: *fps-deriv-def fps-eq-iff*)

lemma *fps-deriv-1*[*simp*]: *fps-deriv 1* = *0*
by (*simp add*: *fps-deriv-def fps-eq-iff*)

lemma *fps-deriv-mult-const-right*[*simp*]:
fps-deriv (*f* ∗ *fps-const c*) = *fps-deriv f* ∗ *fps-const c*
by *simp*

lemma *fps-deriv-sum*:
fps-deriv (*sum f S*) = *sum* (λ*i*. *fps-deriv* (*f i*)) *S*
proof (*cases finite S*)
case *False*
then show *?thesis* **by** *simp*

next
  case *True*
  show *?thesis* **by** (*induct rule*: *finite-induct* [*OF True*]) *simp-all*
qed

lemma *fps-deriv-eq-0-iff* [*simp*]:
  fps-deriv f = 0 ⟷ f = fps-const (*f$0* :: *'a*::{*semiring-no-zero-divisors*,*semiring-char-0*})
proof
  assume *f*: *fps-deriv f = 0*
  show *f = fps-const* (*f$0*)
  proof (*intro fps-ext*)
    fix *n* **show** *f $ n = fps-const* (*f$0*) *$ n*
    proof (*cases n*)
      case (*Suc m*)
      have (*of-nat* (*Suc m*) :: *'a*) *≠ 0* **by** (*rule of-nat-neq-0*)
      with *f Suc* **show** *?thesis* **using** *fps-deriv-nth*[*of f*] **by** *auto*
    qed *simp*
  qed
next
  show *f = fps-const* (*f$0*) *⟹ fps-deriv f = 0* **using** *fps-deriv-const*[*of f$0*] **by** *simp*
qed

lemma *fps-deriv-eq-iff*:
  fixes *f g* :: *'a*::{*ring-1-no-zero-divisors*,*semiring-char-0*} *fps*
  shows *fps-deriv f = fps-deriv g ⟷* (*f = fps-const*(*f$0 − g$0*) *+ g*)
proof −
  have *fps-deriv f = fps-deriv g ⟷ fps-deriv* (*f − g*) *= 0*
    using *fps-deriv-sub*[*of f g*]
    by *simp*
  also have *... ⟷ f − g = fps-const* ((*f − g*) *$ 0*)
    unfolding *fps-deriv-eq-0-iff* **..**
  finally show *?thesis*
    by (*simp add*: *field-simps*)
qed

lemma *fps-deriv-eq-iff-ex*:
  fixes *f g* :: *'a*::{*ring-1-no-zero-divisors*,*semiring-char-0*} *fps*
  shows (*fps-deriv f = fps-deriv g*) *⟷* (∃ *c*. *f = fps-const c + g*)
  by (*auto simp*: *fps-deriv-eq-iff*)

fun *fps-nth-deriv* :: *nat ⇒ 'a*::*semiring-1 fps ⇒ 'a fps*
where
  fps-nth-deriv 0 f = f
| *fps-nth-deriv* (*Suc n*) *f = fps-nth-deriv n* (*fps-deriv f*)

lemma *fps-nth-deriv-commute*: *fps-nth-deriv* (*Suc n*) *f = fps-deriv* (*fps-nth-deriv n f*)

by (*induct n arbitrary*: *f*) *auto*

lemma *fps-nth-deriv-linear*[*simp*]:
fps-nth-deriv n (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) =
fps-const a ∗ *fps-nth-deriv n f* + *fps-const b* ∗ *fps-nth-deriv n g*
by (*induct n arbitrary*: *f g*) *auto*

lemma *fps-nth-deriv-neg*[*simp*]:
fps-nth-deriv n (− (*f* :: *'a*::*ring-1 fps*)) = − (*fps-nth-deriv n f*)
by (*induct n arbitrary*: *f*) *simp-all*

lemma *fps-nth-deriv-add*[*simp*]:
fps-nth-deriv n ((*f* :: *'a*::*ring-1 fps*) + *g*) = *fps-nth-deriv n f* + *fps-nth-deriv n g*
using *fps-nth-deriv-linear*[*of n 1 f 1 g*] **by** *simp*

lemma *fps-nth-deriv-sub*[*simp*]:
fps-nth-deriv n ((*f* :: *'a*::*ring-1 fps*) − *g*) = *fps-nth-deriv n f* − *fps-nth-deriv n g*
using *fps-nth-deriv-add* [*of n f* − *g*] **by** *simp*

lemma *fps-nth-deriv-0*[*simp*]: *fps-nth-deriv n 0* = *0*
by (*induct n*) *simp-all*

lemma *fps-nth-deriv-1*[*simp*]: *fps-nth-deriv n 1* = (*if n* = *0 then 1 else 0*)
by (*induct n*) *simp-all*

lemma *fps-nth-deriv-const*[*simp*]:
fps-nth-deriv n (*fps-const c*) = (*if n* = *0 then fps-const c else 0*)
by (*cases n*) *simp-all*

lemma *fps-nth-deriv-mult-const-left*[*simp*]:
fps-nth-deriv n (*fps-const c* ∗ *f*) = *fps-const c* ∗ *fps-nth-deriv n f*
using *fps-nth-deriv-linear*[*of n c f 0 0*] **by** *simp*

lemma *fps-nth-deriv-mult-const-right*[*simp*]:
fps-nth-deriv n (*f* ∗ *fps-const c*) = *fps-nth-deriv n f* ∗ *fps-const c*
by (*induct n arbitrary*: *f*) *auto*

lemma *fps-nth-deriv-sum*:
fps-nth-deriv n (*sum f S*) = *sum* (λ*i*. *fps-nth-deriv n* (*f i* :: *'a*::*ring-1 fps*)) *S*
proof (*cases finite S*)
case *True*
show *?thesis* **by** (*induct rule*: *finite-induct* [*OF True*]) *simp-all*
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *fps-deriv-maclauren-0*:
(*fps-nth-deriv k* (*f* :: *'a*::*comm-semiring-1 fps*)) $ *0* = *of-nat* (*fact k*) ∗ *f* $ *k*

by (*induct k arbitrary*: *f*) (*simp-all add*: *field-simps*)

lemma *fps-deriv-lr-inverse*:
fixes *x y* :: *'a*::*ring-1*
assumes *x* ∗ *f*$*0* = *1 f*$*0* ∗ *y* = *1*
— These assumptions imply *x* equals *y*, but no need to assume that.
shows *fps-deriv* (*fps-left-inverse f x*) =
− *fps-left-inverse f x* ∗ *fps-deriv f* ∗ *fps-left-inverse f x*
and *fps-deriv* (*fps-right-inverse f y*) =
− *fps-right-inverse f y* ∗ *fps-deriv f* ∗ *fps-right-inverse f y*
proof−

define *L* **where** *L* ≡ *fps-left-inverse f x*
hence *fps-deriv* (*L* ∗ *f*) = *0* **using** *fps-left-inverse*[*OF assms*(*1*)] **by** *simp*
with *assms* **show** *fps-deriv L* = − *L* ∗ *fps-deriv f* ∗ *L*
using *fps-right-inverse'*[*OF assms*]
by (*simp add*: *minus-unique mult.assoc L-def*)

define *R* **where** *R* ≡ *fps-right-inverse f y*
hence *fps-deriv* (*f* ∗ *R*) = *0* **using** *fps-right-inverse*[*OF assms*(*2*)] **by** *simp*
hence *1*: *f* ∗ *fps-deriv R* + *fps-deriv f* ∗ *R* = *0* **by** *simp*
have *R* ∗ *f* ∗ *fps-deriv R* = − *R* ∗ *fps-deriv f* ∗ *R*
using *iffD2*[*OF eq-neg-iff-add-eq-0*, *OF 1*] **by** (*simp add*: *mult.assoc*)
thus *fps-deriv R* = − *R* ∗ *fps-deriv f* ∗ *R*
using *fps-left-inverse'*[*OF assms*] **by** (*simp add*: *R-def*)

qed

lemma *fps-deriv-lr-inverse-comm*:
fixes *x* :: *'a*::*comm-ring-1*
assumes *x* ∗ *f*$*0* = *1*
shows *fps-deriv* (*fps-left-inverse f x*) = − *fps-deriv f* ∗ (*fps-left-inverse f x*)^2
and *fps-deriv* (*fps-right-inverse f x*) = − *fps-deriv f* ∗ (*fps-right-inverse f x*)^2
using *assms fps-deriv-lr-inverse*[*of x f x*]
by (*simp-all add*: *mult.commute power2-eq-square*)

lemma *fps-inverse-deriv-divring*:
fixes *a* :: *'a*::*division-ring fps*
assumes *a*$*0* ≠ *0*
shows *fps-deriv* (*inverse a*) = − *inverse a* ∗ *fps-deriv a* ∗ *inverse a*
using *assms fps-deriv-lr-inverse*(*2*)[*of inverse* (*a*$*0*) *a inverse* (*a*$*0*)]
by (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-deriv*:
fixes *a* :: *'a*::*field fps*
assumes *a*$*0* ≠ *0*
shows *fps-deriv* (*inverse a*) = − *fps-deriv a* ∗ (*inverse a*)^2
using *assms fps-deriv-lr-inverse-comm*(*2*)[*of inverse* (*a*$*0*) *a*]
by (*simp add*: *fps-inverse-def*)

lemma *fps-inverse-deriv'*:
fixes $a :: 'a$::*field fps*
assumes *a0*: $a \$ 0 \neq 0$
shows *fps-deriv* (*inverse a*) = $-$ *fps-deriv a* / a^2
using *fps-inverse-deriv*[*OF a0*] *a0*
by (*simp add*: *fps-divide-unit power2-eq-square fps-inverse-mult*)

lemma *fps-divide-deriv*:
assumes *b dvd* (*a* :: $'a$ :: *field fps*)
shows *fps-deriv* (*a* / *b*) = (*fps-deriv a* ∗ *b* − *a* ∗ *fps-deriv b*) / *b*^2
proof −
have *eq-divide-imp*: $c \neq 0 \Longrightarrow a * c = b \Longrightarrow a = b$ *div c* **for** *a b c* :: $'a$ :: *field fps*
by (*drule sym*) (*simp add*: *mult.assoc*)
from *assms* **have** *a* = *a* / *b* ∗ *b* **by** *simp*
also have *fps-deriv* (*a* / *b* ∗ *b*) = *fps-deriv* (*a* / *b*) ∗ *b* + *a* / *b* ∗ *fps-deriv b* **by** *simp*
finally have *fps-deriv* (*a* / *b*) ∗ *b*^2 = *fps-deriv a* ∗ *b* − *a* ∗ *fps-deriv b* **using** *assms*
by (*simp add*: *power2-eq-square algebra-simps*)
thus *?thesis* **by** (*cases b* = *0*) (*simp-all add*: *eq-divide-imp*)
qed

lemma *fps-nth-deriv-fps-X*[*simp*]: *fps-nth-deriv n fps-X* = (*if n* = *0 then fps-X else if n*=*1 then 1 else 0*)
by (*cases n*) *simp-all*

5.11 Powers

lemma *fps-power-zeroth*: (*a*^*n*) $ *0* = (*a*$*0*)^*n*
by (*induct n*) *auto*

lemma *fps-power-zeroth-eq-one*: *a*$*0* = *1* ⟹ *a*^*n* $ *0* = *1*
by (*simp add*: *fps-power-zeroth*)

lemma *fps-power-first*:
fixes $a :: 'a$::*comm-semiring-1 fps*
shows (*a*^*n*) $ *1* = *of-nat n* ∗ (*a*$*0*)^(*n*−*1*) ∗ *a*$*1*
proof (*cases n*)
case (*Suc m*)
have (*a* ^ *Suc m*) $ *1* = *of-nat* (*Suc m*) ∗ (*a*$*0*)^(*Suc m* − *1*) ∗ *a*$*1*
proof (*induct m*)
case (*Suc k*)
hence (*a* ^ *Suc* (*Suc k*)) $ *1* =
a$*0* ∗ *of-nat* (*Suc k*) ∗ (*a* $ *0*)^*k* ∗ *a*$*1* + *a*$*1* ∗ ((*a*$*0*)^(*Suc k*))
using *fps-mult-nth-1*[*of a*] **by** (*simp add*: *fps-power-zeroth*[*symmetric*] *mult.assoc*)
thus *?case* **by** (*simp add*: *algebra-simps*)

qed *simp*
with *Suc* **show** *?thesis* **by** *simp*
qed *simp*

lemma *fps-power-first-eq*: $a\ \$\ 0 = 1 \Longrightarrow a\hat{}n\ \$\ 1 = of\text{-}nat\ n * a\$1$
proof (*induct n*)
case (*Suc n*)
show *?case* **unfolding** *power-Suc fps-mult-nth*
using *Suc.hyps*[*OF* ‹$a\$0 = 1$›] ‹$a\$0 = 1$› *fps-power-zeroth-eq-one*[*OF* ‹$a\$0=1$›]
by (*simp add*: *algebra-simps*)
qed *simp*

lemma *fps-power-first-eq'*:
assumes $a\ \$\ 1 = 1$
shows $a\hat{}n\ \$\ 1 = of\text{-}nat\ n * (a\$0)\hat{}(n-1)$
proof (*cases n*)
case (*Suc m*)
from *assms* **have** $(a\ \hat{}\ Suc\ m)\ \$\ 1 = of\text{-}nat\ (Suc\ m) * (a\$0)\hat{}(Suc\ m - 1)$
using *fps-mult-nth-1*[*of a*]
by (*induct m*)
(*simp-all add*: *algebra-simps mult-of-nat-commute fps-power-zeroth*)
with *Suc* **show** *?thesis* **by** *simp*
qed *simp*

lemmas *startsby-one-power* = *fps-power-zeroth-eq-one*

lemma *startsby-zero-power*: $a\ \$\ 0 = 0 \Longrightarrow n > 0 \Longrightarrow a\hat{}n\ \$0 = 0$
by (*simp add*: *fps-power-zeroth zero-power*)

lemma *startsby-power*: $a\ \$0 = v \Longrightarrow a\hat{}n\ \$0 = v\hat{}n$
by (*simp add*: *fps-power-zeroth*)

lemma *startsby-nonzero-power*:
fixes $a :: 'a{::}semiring\text{-}1\text{-}no\text{-}zero\text{-}divisors\ fps$
shows $a\ \$\ 0 \neq 0 \Longrightarrow a\hat{}n\ \$\ 0 \neq 0$
by (*simp add*: *startsby-power*)

lemma *startsby-zero-power-iff*[*simp*]:
$a\hat{}n\ \$0 = (0{::}'a{::}semiring\text{-}1\text{-}no\text{-}zero\text{-}divisors) \longleftrightarrow n \neq 0 \wedge a\$0 = 0$
proof
show $a\ \hat{}\ n\ \$\ 0 = 0 \Longrightarrow n \neq 0 \wedge a\ \$\ 0 = 0$
proof
assume *a*: $a\hat{}n\ \$\ 0 = 0$
thus $a\ \$\ 0 = 0$ **using** *startsby-nonzero-power* **by** *auto*
have $n = 0 \Longrightarrow a\hat{}n\ \$\ 0 = 1$ **by** *simp*
with *a* **show** $n \neq 0$ **by** *fastforce*
qed
show $n \neq 0 \wedge a\ \$\ 0 = 0 \Longrightarrow a\ \hat{}\ n\ \$\ 0 = 0$
by (*cases n*) *auto*

qed

lemma *startsby-zero-power-prefix*:
assumes *a0*: $a \$ 0 = 0$
shows $\forall n < k.\ a \hat{} k \$ n = 0$
proof (*induct k rule*: *nat-less-induct*, *clarify*)
case (*1 k*)
fix $j :: nat$ **assume** *j*: $j < k$
show $a \hat{} k \$ j = 0$
proof (*cases k*)
case *0* **with** *j* **show** *?thesis* **by** *simp*
next
case (*Suc i*)
with *1 j* **have** $\forall m \in \{0<..j\}.\ a \hat{} i \$ (j - m) = 0$ **by** *auto*
with *Suc a0* **show** *?thesis* **by** (*simp add*: *fps-mult-nth sum.atLeast-Suc-atMost*)
qed
qed

lemma *startsby-zero-sum-depends*:
assumes *a0*: $a \$0 = 0$
and *kn*: $n \geq k$
shows $sum\ (\lambda i.\ (a \hat{} i)\$k)\ \{0\ ..\ n\} = sum\ (\lambda i.\ (a \hat{} i)\$k)\ \{0\ ..\ k\}$
proof (*intro strip sum.mono-neutral-right*)
show $\bigwedge i.\ i \in \{0..n\} - \{0..k\} \Longrightarrow a \hat{} i \$ k = 0$
by (*simp add*: *a0 startsby-zero-power-prefix*)
qed (*use kn* **in** *auto*)

lemma *startsby-zero-power-nth-same*:
assumes *a0*: $a\$0 = 0$
shows $a \hat{} n \$ n = (a\$1) \hat{} n$
proof (*induct n*)
case (*Suc n*)
have $\forall i \in \{Suc\ 1..Suc\ n\}.\ a \hat{} n \$ (Suc\ n - i) = 0$
using *a0 startsby-zero-power-prefix*[*of a n*] **by** *auto*
thus *?case*
using *a0 Suc sum.atLeast-Suc-atMost*[*of 0 Suc n* $\lambda i.\ a \$ i * a \hat{} n \$ (Suc\ n - i)$]
sum.atLeast-Suc-atMost[*of 1 Suc n* $\lambda i.\ a \$ i * a \hat{} n \$ (Suc\ n - i)$]
by (*simp add*: *fps-mult-nth*)
qed *simp*

lemma *fps-lr-inverse-power*:
fixes $a :: 'a::ring\text{-}1\ fps$
assumes $x * a\$0 = 1\ a\$0 * x = 1$
shows *fps-left-inverse* $(a \hat{} n)\ (x \hat{} n) = fps\text{-}left\text{-}inverse\ a\ x \hat{} n$
and *fps-right-inverse* $(a \hat{} n)\ (x \hat{} n) = fps\text{-}right\text{-}inverse\ a\ x \hat{} n$
proof−

from *assms* **have** *xn*: $\bigwedge n.\ x \hat{} n * (a \hat{} n \$ 0) = 1\ \bigwedge n.\ (a \hat{} n \$ 0) * x \hat{} n = 1$

```
    by (simp-all add: left-right-inverse-power fps-power-zeroth)

  show fps-left-inverse (a^n) (x^n) = fps-left-inverse a x ^ n
  proof (induct n)
    case 0
    then show ?case by (simp add: fps-lr-inverse-one-one(1))
  next
    case (Suc n)
    with assms show ?case
      using xn fps-lr-inverse-mult-ring1(1)[of x a x^n a^n]
      by    (simp add: power-Suc2[symmetric])
  qed

  moreover have fps-right-inverse (a^n) (x^n) = fps-left-inverse (a^n) (x^n)
    using xn by (intro fps-left-inverse-eq-fps-right-inverse[symmetric])
  moreover have fps-right-inverse a x = fps-left-inverse a x
    using assms by (intro fps-left-inverse-eq-fps-right-inverse[symmetric])
  ultimately show fps-right-inverse (a^n) (x^n) = fps-right-inverse a x ^ n
    by simp

qed

lemma fps-inverse-power:
  fixes a :: 'a::division-ring fps
  shows inverse (a^n) = inverse a ^ n
proof (cases n=0 a$0 = 0 rule: case-split[case-product case-split])
  case False-True
  hence LHS: inverse (a^n) = 0 and RHS: inverse a ^ n = 0
    by (simp-all add: startsby-zero-power)
  show ?thesis using trans-sym[OF LHS RHS] by fast
next
  case False-False
  from False-False(2) show ?thesis
    by  (simp add:
       fps-inverse-def fps-power-zeroth power-inverse fps-lr-inverse-power(2)[symmetric]
        )
qed auto

lemma fps-deriv-power':
  fixes a :: 'a::comm-semiring-1 fps
  shows fps-deriv (a ^ n) = (of-nat n) * fps-deriv a * a ^ (n - 1)
proof (cases n)
  case (Suc m)
  moreover have fps-deriv (a^Suc m) = of-nat (Suc m) * fps-deriv a * a^m
    by (induct m) (simp-all add: algebra-simps)
  ultimately show ?thesis by simp
qed simp

lemma fps-deriv-power:
```

fixes $a :: 'a::comm\text{-}semiring\text{-}1\ fps$
shows $fps\text{-}deriv\ (a\ \hat{}\ n) = fps\text{-}const\ (of\text{-}nat\ n) * fps\text{-}deriv\ a * a\ \hat{}\ (n - 1)$
by (*simp add*: *fps-deriv-power' fps-of-nat*)

5.12 Finite and infinite products

lemma *fps-prod-nth'*:
assumes *finite A*
shows $fps\text{-}nth\ (\prod x{\in}A.\ f\ x)\ n = (\sum X{\in}multisets\text{-}of\text{-}size\ A\ n.\ \prod x{\in}A.\ fps\text{-}nth\ (f\ x)\ (count\ X\ x))$
using *assms*
proof (*induction A arbitrary*: *n rule*: *finite-induct*)
case (*insert a A n*)
note [*simp*] = ‹$a \notin A$›
note [*intro*, *simp*] = ‹*finite A*›
have $(\sum X{\in}multisets\text{-}of\text{-}size\ (insert\ a\ A)\ n.\ \prod x{\in}insert\ a\ A.\ fps\text{-}nth\ (f\ x)\ (count\ X\ x)) =$
$(\sum (m,X){\in}(SIGMA\ m{:}\{0..n\}.\ multisets\text{-}of\text{-}size\ A\ (n{-}m)).$
$\prod x{\in}insert\ a\ A.\ fps\text{-}nth\ (f\ x)\ (count\ (X + replicate\text{-}mset\ m\ a)\ x))$
by (*subst sum.reindex-bij-betw*[*OF bij-betw-multisets-of-size-insert*, *symmetric*])
(*simp-all add*: *case-prod-unfold*)
also have $\ldots = (\sum m{=}0..n.\ \sum X{\in}multisets\text{-}of\text{-}size\ A\ (n{-}m).$
$\prod x{\in}insert\ a\ A.\ fps\text{-}nth\ (f\ x)\ (count\ (X + replicate\text{-}mset\ m\ a)\ x))$
by (*rule sum.Sigma* [*symmetric*]) *auto*
also have $\ldots = (\sum m{=}0..n.\ fps\text{-}nth\ (f\ a)\ m * fps\text{-}nth\ (\prod x{\in}A.\ f\ x)\ (n - m))$
proof (*rule sum.cong*)
fix *m*
assume *m*: $m \in \{0..n\}$
have $(\sum X{\in}multisets\text{-}of\text{-}size\ A\ (n{-}m).$
$\prod x{\in}insert\ a\ A.\ fps\text{-}nth\ (f\ x)\ (count\ (X + replicate\text{-}mset\ m\ a)\ x)) =$
$(\sum X{\in}multisets\text{-}of\text{-}size\ A\ (n{-}m).\ fps\text{-}nth\ (f\ a)\ (count\ X\ a + m) *$
$(\prod x{\in}A.\ fps\text{-}nth\ (f\ x)\ (count\ (X + replicate\text{-}mset\ m\ a)\ x)))$
by *simp*
also have $\ldots = (\sum X{\in}multisets\text{-}of\text{-}size\ A\ (n{-}m).\ fps\text{-}nth\ (f\ a)\ m *$
$(\prod x{\in}A.\ fps\text{-}nth\ (f\ x)\ (count\ (X + replicate\text{-}mset\ m\ a)\ x)))$
by (*intro sum.cong arg-cong2*[*of - - - - (*)*] *arg-cong2*[*of - - - - fps-nth*] *refl*)
(*auto simp*: *multisets-of-size-def simp flip*: *not-in-iff*)
also have $\ldots = fps\text{-}nth\ (f\ a)\ m * (\sum X{\in}multisets\text{-}of\text{-}size\ A\ (n{-}m).$
$(\prod x{\in}A.\ fps\text{-}nth\ (f\ x)\ (count\ (X + replicate\text{-}mset\ m\ a)\ x)))$
by (*simp add*: *sum-distrib-left*)
also have $(\sum X{\in}multisets\text{-}of\text{-}size\ A\ (n{-}m).\ \prod x{\in}A.\ fps\text{-}nth\ (f\ x)\ (count\ (X + replicate\text{-}mset\ m\ a)\ x)) =$
$(\sum X{\in}multisets\text{-}of\text{-}size\ A\ (n{-}m).\ \prod x{\in}A.\ fps\text{-}nth\ (f\ x)\ (count\ X\ x))$
by (*intro sum.cong prod.cong*) *auto*
also have $\ldots = fps\text{-}nth\ (\prod x{\in}A.\ f\ x)\ (n - m)$
by (*rule insert.IH* [*symmetric*])
finally show $(\sum X{\in}multisets\text{-}of\text{-}size\ A\ (n{-}m).\ \prod x{\in}insert\ a\ A.\ fps\text{-}nth\ (f\ x)\ (count\ (X + replicate\text{-}mset\ m\ a)\ x)) =$
$fps\text{-}nth\ (f\ a)\ m * fps\text{-}nth\ (\prod x{\in}A.\ f\ x)\ (n - m)$.

```
  qed auto
  also have ... = fps-nth (∏ x∈insert a A. f x) n
    by (simp add: fps-mult-nth)
  finally show ?case ..
qed auto

theorem tendsto-prod-fps:
  fixes f :: nat ⇒ 'a :: {idom, t2-space} fps
  assumes [simp]: ⋀k. f k ≠ 0
  assumes g: ⋀n k. k > g n ⟹ subdegree (f k - 1) > n
  defines P ≡ Abs-fps (λn. (∑ X∈multisets-of-size {..g n} n. ∏ i≤g n. fps-nth (f
i) (count X i)))
  shows   (λn. ∏ k≤n. f k) ⟶ P
proof (rule tendsto-fpsI)
  fix n :: nat
  show eventually (λN. fps-nth (prod f {..N}) n = fps-nth P n) at-top
    using eventually-ge-at-top[of g n]
  proof eventually-elim
    case (elim N)
     have fps-nth (prod f {..N}) n = (∑ X∈multisets-of-size {..N} n. ∏ x≤N.
fps-nth (f x) (count X x))
      by (subst fps-prod-nth') auto
    also have ... = (∑ X | X ∈ multisets-of-size {..N} n ∧ (∀ x≤N. fps-nth (f x)
(count X x) ≠ 0).
                    ∏ x≤N. fps-nth (f x) (count X x))
      by (intro sum.mono-neutral-right) auto

    also have {X. X ∈ multisets-of-size {..N} n ∧ (∀ x≤N. fps-nth (f x) (count X
x) ≠ 0)} =
              {X. X ∈ multisets-of-size {..g n} n ∧ (∀ x≤N. fps-nth (f x) (count X
x) ≠ 0)}
      (is ?lhs = ?rhs)
    proof (intro equalityI subsetI)
      fix X assume X ∈ ?rhs
      thus X ∈ ?lhs using elim multisets-of-size-mono[of {..g n} {..N}] by auto
    next
      fix X assume X ∈ ?lhs
     hence X: set-mset X ⊆ {..N} size X = n ⋀x. x ≤ N ⟹ fps-nth (f x) (count
X x) ≠ 0
        by (auto simp: multisets-of-size-def)
      have set-mset X ⊆ {..g n}
      proof
        fix x assume *: x ∈ set-mset X
        show x ∈ {..g n}
        proof (rule ccontr)
          assume x ∉ {..g n}
          hence x: x > g n x ≤ N
            using X(1) * by auto
          have count X x ≤ n
```

using X x *count-le-size*[*of* X x] **by** (*auto simp*: *Pi-def*)
also have $n < subdegree\ (f\ x - 1)$
by (*rule g*) (*use x* **in** *auto*)
finally have *fps-nth* $(f\ x - 1)$ (*count* X x) $= 0$
by *blast*
hence *fps-nth* $(f\ x)$ (*count* X x) $= 0$
using $*$ **by** *simp*
moreover have *fps-nth* $(f\ x)$ (*count* X x) $\neq 0$
by (*intro X*(*3*)) (*use x* **in** *auto*)
ultimately show *False* **by** *contradiction*
qed
qed
thus $X \in$ *?rhs* **using** X
by (*auto simp*: *multisets-of-size-def*)
qed

also have $(\sum X \mid X \in$ *multisets-of-size* $\{..g\ n\}\ n \wedge (\forall x{\leq}N.$ *fps-nth* $(f\ x)$ (*count* $X\ x) \neq 0).$
$\prod x{\leq}N.$ *fps-nth* $(f\ x)$ (*count* $X\ x)) =$
$(\sum X \mid X \in$ *multisets-of-size* $\{..g\ n\}\ n \wedge (\forall x{\leq}N.$ *fps-nth* $(f\ x)$ (*count* $X\ x) \neq 0).$
$\prod i{\leq}g\ n.$ *fps-nth* $(f\ i)$ (*count* $X\ i))$
proof (*intro sum.cong prod.mono-neutral-right ballI*)
fix $X\ i$
assume *i*: $i \in \{..N\} - \{..g\ n\}$
assume $X \in \{X.\ X \in$ *multisets-of-size* $\{..g\ n\}\ n \wedge (\forall x{\leq}N.$ *fps-nth* $(f\ x)$ $(count\ X\ x) \neq 0)\}$
hence *h*: $X \in$ *multisets-of-size* $\{..g\ n\}\ n \bigwedge x.\ x \leq N \Longrightarrow$ *fps-nth* $(f\ x)$ (*count* $X\ x) \neq 0$
by *blast+*
have $i \notin\# X$
using $i\ h$ **unfolding** *multisets-of-size-def* **by** *auto*
have $n < subdegree\ (f\ i - 1)$
by (*intro g*) (*use i* **in** *auto*)
moreover have *count* $X\ i \leq n$
using $\langle i \notin\# X\rangle$ **by** (*simp add*: *not-in-iff*)
ultimately have *fps-nth* $(f\ i - 1)$ (*count* $X\ i) = 0$
by (*intro nth-less-subdegree-zero*) *auto*
thus *fps-nth* $(f\ i)$ (*count* $X\ i) = 1$
using *h*(*2*) $i\ \langle i \notin\# X\rangle$ **by** (*auto split*: *if-splits*)
qed (*use elim* **in** *auto*)

also have $(\sum X \mid X \in$ *multisets-of-size* $\{..g\ n\}\ n \wedge (\forall x{\leq}N.$ *fps-nth* $(f\ x)$ (*count* $X\ x) \neq 0).$
$\prod i{\leq}g\ n.$ *fps-nth* $(f\ i)$ (*count* $X\ i)) =$
$(\sum X \in$ *multisets-of-size* $\{..g\ n\}\ n.\ \prod i{\leq}g\ n.$ *fps-nth* $(f\ i)$ (*count* $X\ i))$
proof (*intro sum.mono-neutral-left ballI*)
fix X **assume** $X \in$ *multisets-of-size* $\{..g\ n\}\ n -$
$\{X{\in}$*multisets-of-size* $\{..g\ n\}\ n.\ \forall x{\leq}N.$ *fps-nth* $(f\ x)$ (*count* $X\ x)$

```
≠ 0}
      then obtain i
        where h: X ∈ multisets-of-size {..g n} n
        and i: i ≤ N fps-nth (f i) (count X i) = 0
        by blast
      have ¬(i > g n)
      proof
        assume i': i > g n
        hence count X i = 0
          using h by (auto simp: multisets-of-size-def simp flip: not-in-iff)
        have subdegree (f i − 1) > n
          by (intro g) (use i' in auto)
        hence subdegree (f i − 1) > 0
          by simp
        hence fps-nth (f i − 1) 0 = 0
          by blast
        hence fps-nth (f i) (count X i) = 1
          using ‹count X i = 0› by simp
        thus False using i by simp
      qed
      thus  (∏ x≤g n. fps-nth (f x) (count X x)) = 0
        using i by auto
    qed auto

    also have ... = fps-nth P n
      by (simp add: P-def)
    finally show fps-nth (∏ k≤N. f k) n = fps-nth P n .
  qed
qed
```

5.13 Integration

```
definition fps-integral :: 'a::{semiring-1,inverse} fps ⇒ 'a ⇒ 'a fps
  where fps-integral a a0 =
        Abs-fps (λn. if n=0 then a0 else inverse (of-nat n) * a$(n − 1))

abbreviation fps-integral0 a ≡ fps-integral a 0

lemma fps-integral-nth-0-Suc [simp]:
  fixes a :: 'a::{semiring-1,inverse} fps
  shows fps-integral a a0 $ 0 = a0
  and   fps-integral a a0 $ Suc n = inverse (of-nat (Suc n)) * a $ n
  by    (auto simp: fps-integral-def)

lemma fps-integral-conv-plus-const:
  fps-integral a a0 = fps-integral a 0 + fps-const a0
  unfolding fps-integral-def by (intro fps-ext) simp

lemma fps-deriv-fps-integral:
```

fixes $a :: 'a::\{division\text{-}ring, ring\text{-}char\text{-}0\}$ *fps*
shows *fps-deriv* (*fps-integral a a0*) = *a*
proof (*intro fps-ext*)
fix *n*
have (*of-nat* (*Suc n*) :: *'a*) ≠ *0* **by** (*rule of-nat-neq-0*)
hence *of-nat* (*Suc n*) * *inverse* (*of-nat* (*Suc n*) :: *'a*) = *1* **by** *simp*
moreover have
fps-deriv (*fps-integral a a0*) $ *n* = *of-nat* (*Suc n*) * *inverse* (*of-nat* (*Suc n*)) * *a* $ *n*
by (*simp add*: *mult.assoc*)
ultimately show *fps-deriv* (*fps-integral a a0*) $ *n* = *a* $ *n* **by** *simp*
qed

lemma *fps-integral0-deriv*:
fixes $a :: 'a::\{division\text{-}ring, ring\text{-}char\text{-}0\}$ *fps*
shows *fps-integral0* (*fps-deriv a*) = *a* − *fps-const* (*a$0*)
proof (*intro fps-ext*)
fix *n*
show *fps-integral0* (*fps-deriv a*) $ *n* = (*a* − *fps-const* (*a$0*)) $ *n*
proof (*cases n*)
case (*Suc m*)
have (*of-nat* (*Suc m*) :: *'a*) ≠ *0* **by** (*rule of-nat-neq-0*)
hence *inverse* (*of-nat* (*Suc m*) :: *'a*) * *of-nat* (*Suc m*) = *1* **by** *simp*
moreover have
fps-integral0 (*fps-deriv a*) $ *Suc m* =
inverse (*of-nat* (*Suc m*)) * *of-nat* (*Suc m*) * *a* $ (*Suc m*)
by (*simp add*: *mult.assoc*)
ultimately show *?thesis* **using** *Suc* **by** *simp*
qed *simp*
qed

lemma *fps-integral-deriv*:
fixes $a :: 'a::\{division\text{-}ring, ring\text{-}char\text{-}0\}$ *fps*
shows *fps-integral* (*fps-deriv a*) (*a$0*) = *a*
using *fps-integral-conv-plus-const*[*of fps-deriv a a$0*]
by (*simp add*: *fps-integral0-deriv*)

lemma *fps-integral0-zero*:
fps-integral0 (*0*::*'a*::\{*semiring-1*,*inverse*\} *fps*) = *0*
by (*intro fps-ext*) (*simp add*: *fps-integral-def*)

lemma *fps-integral0-fps-const'*:
fixes $c :: 'a::\{semiring\text{-}1, inverse\}$
assumes *inverse* (*1*::*'a*) = *1*
shows *fps-integral0* (*fps-const c*) = *fps-const c* * *fps-X*
proof (*intro fps-ext*)
fix *n*
show *fps-integral0* (*fps-const c*) $ *n* = (*fps-const c* * *fps-X*) $ *n*
by (*cases n*) (*simp-all add*: *assms mult-delta-right*)

qed

lemma *fps-integral0-fps-const*:
fixes *c* :: *'a*::*division-ring*
shows *fps-integral0* (*fps-const c*) = *fps-const c* ∗ *fps-X*
by (*rule fps-integral0-fps-const'*[*OF inverse-1*])

lemma *fps-integral0-one'*:
assumes *inverse* (*1*::*'a*::{*semiring-1*,*inverse*}) = *1*
shows *fps-integral0* (*1*::*'a fps*) = *fps-X*
using *assms fps-integral0-fps-const'*[*of 1*::*'a*]
by *simp*

lemma *fps-integral0-one*:
fps-integral0 (*1*::*'a*::*division-ring fps*) = *fps-X*
by (*rule fps-integral0-one'*[*OF inverse-1*])

lemma *fps-integral0-fps-const-mult-left*:
fixes *a* :: *'a*::*division-ring fps*
shows *fps-integral0* (*fps-const c* ∗ *a*) = *fps-const c* ∗ *fps-integral0 a*
proof (*intro fps-ext*)
fix *n*
show *fps-integral0* (*fps-const c* ∗ *a*) $ *n* = (*fps-const c* ∗ *fps-integral0 a*) $ *n*
using *mult-inverse-of-nat-commute*[*of n c*, *symmetric*]
mult.assoc[*of inverse* (*of-nat n*) *c a*$(*n*−*1*)]
mult.assoc[*of c inverse* (*of-nat n*) *a*$(*n*−*1*)]
by (*simp add*: *fps-integral-def*)
qed

lemma *fps-integral0-fps-const-mult-right*:
fixes *a* :: *'a*::{*semiring-1*,*inverse*} *fps*
shows *fps-integral0* (*a* ∗ *fps-const c*) = *fps-integral0 a* ∗ *fps-const c*
by (*intro fps-ext*) (*simp add*: *fps-integral-def algebra-simps*)

lemma *fps-integral0-neg*:
fixes *a* :: *'a*::{*ring-1*,*inverse*} *fps*
shows *fps-integral0* (−*a*) = − *fps-integral0 a*
using *fps-integral0-fps-const-mult-right*[*of a* −*1*]
by (*simp add*: *fps-const-neg*[*symmetric*])

lemma *fps-integral0-add*:
fps-integral0 (*a*+*b*) = *fps-integral0 a* + *fps-integral0 b*
by (*intro fps-ext*) (*simp add*: *fps-integral-def algebra-simps*)

lemma *fps-integral0-linear*:
fixes *a b* :: *'a*::*division-ring*
shows *fps-integral0* (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) =
fps-const a ∗ *fps-integral0 f* + *fps-const b* ∗ *fps-integral0 g*
by (*simp add*: *fps-integral0-add fps-integral0-fps-const-mult-left*)

lemma *fps-integral0-linear2*:
fps-integral0 (*f* ∗ *fps-const a* + *g* ∗ *fps-const b*) =
fps-integral0 f ∗ *fps-const a* + *fps-integral0 g* ∗ *fps-const b*
by (*simp add*: *fps-integral0-add fps-integral0-fps-const-mult-right*)

lemma *fps-integral-linear*:
fixes *a b a0 b0* :: *'a*::*division-ring*
shows
fps-integral (*fps-const a* ∗ *f* + *fps-const b* ∗ *g*) (*a*∗*a0* + *b*∗*b0*) =
fps-const a ∗ *fps-integral f a0* + *fps-const b* ∗ *fps-integral g b0*
using *fps-integral-conv-plus-const*[*of*
fps-const a ∗ *f* + *fps-const b* ∗ *g*
a∗*a0* + *b*∗*b0*
]
fps-integral-conv-plus-const[*of f a0*] *fps-integral-conv-plus-const*[*of g b0*]
by (*simp add*: *fps-integral0-linear algebra-simps*)

lemma *fps-integral0-sub*:
fixes *a b* :: *'a*::{*ring-1*,*inverse*} *fps*
shows *fps-integral0* (*a*−*b*) = *fps-integral0 a* − *fps-integral0 b*
using *fps-integral0-linear2*[*of a 1 b* −*1*]
by (*simp add*: *fps-const-neg*[*symmetric*])

lemma *fps-integral0-of-nat*:
fps-integral0 (*of-nat n* :: *'a*::*division-ring fps*) = *of-nat n* ∗ *fps-X*
using *fps-integral0-fps-const*[*of of-nat n* :: *'a*] **by** (*simp add*: *fps-of-nat*)

lemma *fps-integral0-sum*:
fps-integral0 (*sum f S*) = *sum* (λ*i*. *fps-integral0* (*f i*)) *S*
proof (*cases finite S*)
case *True* **show** *?thesis*
by (*induct rule*: *finite-induct* [*OF True*])
(*simp-all add*: *fps-integral0-zero fps-integral0-add*)
qed (*simp add*: *fps-integral0-zero*)

lemma *fps-integral0-by-parts*:
fixes *a b* :: *'a*::{*division-ring*,*ring-char-0*} *fps*
shows
fps-integral0 (*a* ∗ *b*) =
a ∗ *fps-integral0 b* − *fps-integral0* (*fps-deriv a* ∗ *fps-integral0 b*)
proof−
have *fps-integral0* (*fps-deriv* (*a* ∗ *fps-integral0 b*)) = *a* ∗ *fps-integral0 b*
using *fps-integral0-deriv*[*of* (*a* ∗ *fps-integral0 b*)] **by** *simp*
moreover have
fps-integral0 (*a* ∗ *b*) =
fps-integral0 (*fps-deriv* (*a* ∗ *fps-integral0 b*)) −
fps-integral0 (*fps-deriv a* ∗ *fps-integral0 b*)
by (*auto simp*: *fps-deriv-fps-integral fps-integral0-sub*[*symmetric*])

ultimately show *?thesis* **by** *simp*
qed

lemma *fps-integral0-fps-X*:
fps-integral0 (*fps-X*::*'a*::{*semiring-1*,*inverse*} *fps*) =
fps-const (*inverse* (*of-nat* 2)) ∗ *fps-X*^{2}
by (*intro fps-ext*) (*auto simp*: *fps-integral-def*)

lemma *fps-integral0-fps-X-power*:
fps-integral0 ((*fps-X*::*'a*::{*semiring-1*,*inverse*} *fps*) ^ *n*) =
fps-const (*inverse* (*of-nat* (*Suc n*))) ∗ *fps-X* ^ *Suc n*
proof (*intro fps-ext*)
fix *k* **show**
fps-integral0 ((*fps-X*::*'a fps*) ^ *n*) $ *k* =
(*fps-const* (*inverse* (*of-nat* (*Suc n*))) ∗ *fps-X* ^ *Suc n*) $ *k*
by (*cases k*) *simp-all*
qed

5.14 Composition

definition *fps-compose* :: *'a*::*semiring-1 fps* ⇒ *'a fps* ⇒ *'a fps* (**infixl** ‹*oo*› *55*)
where *a oo b* = *Abs-fps* (λ*n*. *sum* (λ*i*. *a*$*i* ∗ (*b*^*i*$*n*)) {*0*..*n*})

lemma *fps-compose-nth*: (*a oo b*)$*n* = *sum* (λ*i*. *a*$*i* ∗ (*b*^*i*$*n*)) {*0*..*n*}
by (*simp add*: *fps-compose-def*)

lemma *fps-compose-nth-0* [*simp*]: (*f oo g*) $ *0* = *f* $ *0*
by (*simp add*: *fps-compose-nth*)

lemma *fps-compose-fps-X*[*simp*]: *a oo fps-X* = (*a* :: *'a*::*comm-ring-1 fps*)
by (*simp add*: *fps-ext fps-compose-def mult-delta-right*)

lemma *fps-const-compose*[*simp*]: *fps-const* (*a*::*'a*::*comm-ring-1*) *oo b* = *fps-const a*
by (*simp add*: *fps-eq-iff fps-compose-nth mult-delta-left*)

lemma *numeral-compose*[*simp*]: (*numeral k* :: *'a*::*comm-ring-1 fps*) *oo b* = *numeral k*
unfolding *numeral-fps-const* **by** *simp*

lemma *neg-numeral-compose*[*simp*]: (− *numeral k* :: *'a*::*comm-ring-1 fps*) *oo b* = − *numeral k*
unfolding *neg-numeral-fps-const* **by** *simp*

lemma *fps-X-fps-compose-startby0*[*simp*]: *a*$*0* = *0* ⟹ *fps-X oo a* = (*a* :: *'a*::*comm-ring-1 fps*)
by (*simp add*: *fps-eq-iff fps-compose-def mult-delta-left not-le*)

5.15 Rules from Herbert Wilf's Generatingfunctionology

5.15.1 Rule 1

lemma *fps-power-mult-eq-shift*:
*fps-X^Suc k * Abs-fps (λn. a (n + Suc k)) =*
*Abs-fps a − sum (λi. fps-const (a i :: 'a::comm-ring-1) * fps-X^i) {0 .. k}*
(**is** *?lhs = ?rhs*)
proof −
have *?lhs $ n = ?rhs $ n* **for** *n :: nat*
proof −
have *?lhs $ n = (if n < Suc k then 0 else a n)*
unfolding *fps-X-power-mult-nth* **by** *auto*
also have ... = *?rhs $ n*
proof (*induct k*)
case *0*
then show *?case*
by (*simp add*: *fps-sum-nth*)
next
case (*Suc k*)
have (*Abs-fps a − sum (λi. fps-const (a i :: 'a) * fps-X^i) {0 .. Suc k}*)*$n =*
(*Abs-fps a − sum (λi. fps-const (a i :: 'a) * fps-X^i) {0 .. k} −*
*fps-const (a (Suc k)) * fps-X^ Suc k) $ n*
by (*simp add*: *field-simps*)
also have ... = (*if n < Suc k then 0 else a n*) − (*fps-const (a (Suc k)) * fps-X^ Suc k*)*$n*
using *Suc.hyps*[*symmetric*] **unfolding** *fps-sub-nth* **by** *simp*
also have ... = (*if n < Suc (Suc k) then 0 else a n*)
unfolding *fps-X-power-mult-right-nth*
by (*simp add*: *not-less le-less-Suc-eq*)
finally show *?case*
by *simp*
qed
finally show *?thesis* **.**
qed
then show *?thesis*
by (*simp add*: *fps-eq-iff*)
qed

5.15.2 Rule 2

definition *fps-XD = (*)* *fps-X* ∘ *fps-deriv*

lemma *fps-XD-add*[*simp*]:*fps-XD (a + b) = fps-XD a + fps-XD (b :: 'a::comm-ring-1 fps)*
by (*simp add*: *fps-XD-def field-simps*)

lemma *fps-XD-mult-const*[*simp*]:*fps-XD (fps-const (c::'a::comm-ring-1) * a) = fps-const c * fps-XD a*
by (*simp add*: *fps-XD-def field-simps*)

lemma *fps-XD-linear*[*simp*]: *fps-XD* (*fps-const c* ∗ *a* + *fps-const d* ∗ *b*) =
fps-const c ∗ *fps-XD a* + *fps-const d* ∗ *fps-XD* (*b* :: *'a*::*comm-ring-1 fps*)
by *simp*

lemma *fps-XDN-linear*:
(*fps-XD* ^^ *n*) (*fps-const c* ∗ *a* + *fps-const d* ∗ *b*) =
fps-const c ∗ (*fps-XD* ^^ *n*) *a* + *fps-const d* ∗ (*fps-XD* ^^ *n*) (*b* :: *'a*::*comm-ring-1 fps*)
by (*induct n*) *simp-all*

lemma *fps-mult-fps-X-deriv-shift*: *fps-X*∗ *fps-deriv a* = *Abs-fps* (λ*n*. *of-nat n*∗ *a*$*n*)
by (*simp add*: *fps-eq-iff*)

lemma *fps-mult-fps-XD-shift*:
(*fps-XD* ^^ *k*) (*a* :: *'a*::*comm-ring-1 fps*) = *Abs-fps* (λ*n*. (*of-nat n* ^ *k*) ∗ *a*$*n*)
by (*induct k arbitrary*: *a*) (*simp-all add*: *fps-XD-def fps-eq-iff field-simps del*: *One-nat-def*)

5.15.3 Rule 3

Rule 3 is trivial and is given by `fps_times_def`.

5.15.4 Rule 5 — summation and "division" by $1 - X$

lemma *fps-divide-fps-X-minus1-sum-lemma*:
a = ((*1*::*'a*::*ring-1 fps*) − *fps-X*) ∗ *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0*..*n*})
proof (*rule fps-ext*)
define *f g* :: *'a fps*
where *f* ≡ *1* − *fps-X*
and *g* ≡ *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0*..*n*})
fix *n* **show** *a* $ *n*= (*f* ∗ *g*) $ *n*
proof (*cases n*)
case (*Suc m*)
hence (*f* ∗ *g*) $ *n* = *g* $ *Suc m* − *g* $ *m*
using *fps-mult-nth*[*of f g Suc m*]
sum.atLeast-Suc-atMost[*of 0 Suc m* λ*i*. *f* $ *i* ∗ *g* $ (*Suc m* − *i*)]
sum.atLeast-Suc-atMost[*of 1 Suc m* λ*i*. *f* $ *i* ∗ *g* $ (*Suc m* − *i*)]
by (*simp add*: *f-def*)
with *Suc* **show** *?thesis* **by** (*simp add*: *g-def*)
qed (*simp add*: *f-def g-def*)
qed

lemma *fps-divide-fps-X-minus1-sum-ring1*:
assumes *inverse 1* = (*1*::*'a*::{*ring-1*,*inverse*})
shows *a* /((*1*::*'a fps*) − *fps-X*) = *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {*0*..*n*})
proof−
from *assms* **have** *a* /((*1*::*'a fps*) − *fps-X*) = *a* ∗ *Abs-fps* (λ*n*. *1*)
by (*simp add*: *fps-divide-def fps-inverse-def fps-lr-inverse-one-minus-fps-X*(*2*))

thus *?thesis* **by** (*auto intro*: *fps-ext simp*: *fps-mult-nth*)
qed

lemma *fps-divide-fps-X-minus1-sum*:
a /((1::'*a*::*division-ring fps*) − *fps-X*) = *Abs-fps* (λ*n*. *sum* (λ*i*. *a* $ *i*) {0..*n*})
using *fps-divide-fps-X-minus1-sum-ring1*[*of a*] **by** *simp*

5.15.5 Rule 4 in its more general form

This generalizes Rule 3 for an arbitrary finite product of FPS, also the relevant instance of powers of a FPS.

definition *natpermute n k* = {*l* :: *nat list*. *length l* = *k* ∧ *sum-list l* = *n*}

lemma *natlist-trivial-1*: *natpermute n 1* = {[*n*]}
proof −
have ⟦*length xs* = *1*; *n* = *sum-list xs*⟧ ⟹ *xs* = [*sum-list xs*] **for** *xs*
by (*cases xs*) *auto*
then show *?thesis*
by (*auto simp add*: *natpermute-def*)
qed

lemma *natlist-trivial-Suc0* [*simp*]: *natpermute n* (*Suc 0*) = {[*n*]}
using *natlist-trivial-1* **by** *force*

lemma *append-natpermute-less-eq*:
assumes *xs* @ *ys* ∈ *natpermute n k*
shows *sum-list xs* ≤ *n*
and *sum-list ys* ≤ *n*
proof −
from *assms* **have** *sum-list* (*xs* @ *ys*) = *n*
by (*simp add*: *natpermute-def*)
then have *sum-list xs* + *sum-list ys* = *n*
by *simp*
then show *sum-list xs* ≤ *n* **and** *sum-list ys* ≤ *n*
by *simp-all*
qed

lemma *natpermute-split*:
assumes *h* ≤ *k*
shows *natpermute n k* =
(⋃ *m* ∈{0..*n*}. {*l1* @ *l2* |*l1 l2*. *l1* ∈ *natpermute m h* ∧ *l2* ∈ *natpermute* (*n* − *m*) (*k* − *h*)})
(**is** *?L* = *?R* **is** - = (⋃ *m* ∈{0..*n*}. *?S m*))
proof
show *?R* ⊆ *?L*
proof
fix *l*
assume *l*: *l* ∈ *?R*
from *l* **obtain** *m xs ys* **where** *h*: *m* ∈ {0..*n*}

```
    and xs: xs ∈ natpermute m h
    and ys: ys ∈ natpermute (n − m) (k − h)
    and leq: l = xs@ys by blast
  from xs have xs': sum-list xs = m
    by (simp add: natpermute-def)
  from ys have ys': sum-list ys = n − m
    by (simp add: natpermute-def)
  show l ∈ ?L using leq xs ys h
    using assms by (force simp add: natpermute-def)
 qed
 show ?L ⊆ ?R
 proof
  fix l
  assume l: l ∈ natpermute n k
  let ?xs = take h l
  let ?ys = drop h l
  let ?m = sum-list ?xs
  from l have ls: sum-list (?xs @ ?ys) = n
    by (simp add: natpermute-def)
  have xs: ?xs ∈ natpermute ?m h using l assms
    by (simp add: natpermute-def)
  have l-take-drop: sum-list l = sum-list (take h l @ drop h l)
    by simp
  then have ys: ?ys ∈ natpermute (n − ?m) (k − h)
   using l assms ls by (auto simp add: natpermute-def simp del: append-take-drop-id)
  from ls have m: ?m ∈ {0..n}
    by (simp add: l-take-drop del: append-take-drop-id)
  have sum-list (take h l) ≤ sum-list l
    using l-take-drop ls m by presburger
  with xs ys ls l show l ∈ ?R
    by simp (metis append-take-drop-id m)
 qed
qed

lemma natpermute-0: natpermute n 0 = (if n = 0 then {[]} else {})
 by (auto simp add: natpermute-def)

lemma natpermute-0'[simp]: natpermute 0 k = (if k = 0 then {[]} else {replicate
k 0})
 by (auto simp add: set-replicate-conv-if natpermute-def replicate-length-same)

lemma natpermute-finite: finite (natpermute n k)
proof (induct k arbitrary: n)
 case 0
 then show ?case
   by (simp add: natpermute-0)
next
 case (Suc k)
 then show ?case
```

```
    using natpermute-split [of k Suc k] finite-UN-I by simp
qed

lemma natpermute-contain-maximal:
  {xs ∈ natpermute n (k + 1). n ∈ set xs} = (⋃ i∈{0 .. k}. {(replicate (k + 1) 0)
[i:=n]})
  (is ?A = ?B)
proof
  show ?A ⊆ ?B
  proof
    fix xs
    assume xs ∈ ?A
    then have H: xs ∈ natpermute n (k + 1) and n: n ∈ set xs
      by blast+
    then obtain i where i: i ∈ {0.. k} xs!i = n
      unfolding in-set-conv-nth by (auto simp add: less-Suc-eq-le natpermute-def)
    have eqs: ({0..k} − {i}) ∪ {i} = {0..k}
      using i by auto
    have f: finite({0..k} − {i}) finite {i}
      by auto
    have d: ({0..k} − {i}) ∩ {i} = {}
      using i by auto
    from H have n = sum (nth xs) {0..k}
    by (auto simp add: natpermute-def atLeastLessThanSuc-atLeastAtMost sum-list-sum-nth)
    also have ... = n + sum (nth xs) ({0..k} − {i})
      unfolding sum.union-disjoint[OF f d, unfolded eqs] using i by simp
    finally have zxs: ∀ j∈ {0..k} − {i}. xs!j = 0
      by auto
    from H have xsl: length xs = k+1
      by (simp add: natpermute-def)
    from i have i': i < length (replicate (k+1) 0)   i < k+1
      unfolding length-replicate by presburger+
    have xs = (replicate (k+1) 0) [i := n]
    proof (rule nth-equalityI)
      show length xs = length ((replicate (k + 1) 0)[i := n])
        by (metis length-list-update length-replicate xsl)
      show xs ! j = (replicate (k + 1) 0)[i := n] ! j if j < length xs for j
      proof (cases j = i)
        case True
        then show ?thesis
          by (metis i'(1) i(2) nth-list-update)
      next
        case False
        with that show ?thesis
          by (simp add: xsl zxs del: replicate.simps split: nat.split)
      qed
    qed
    then show xs ∈ ?B using i by blast
  qed
```

```
  show ?B ⊆ ?A
  proof
    fix xs
    assume xs ∈ ?B
    then obtain i where i: i ∈ {0..k} and xs: xs = (replicate (k + 1) 0) [i:=n]
      by auto
    have nxs: n ∈ set xs
      unfolding xs using set-update-memI i
     by (metis Suc-eq-plus1 atLeast0AtMost atMost-iff le-simps(2) length-replicate)
    have xsl: length xs = k + 1
      by (simp only: xs length-replicate length-list-update)
    have sum-list xs = sum (nth xs) {0..<k+1}
      unfolding sum-list-sum-nth xsl ..
    also have ... = sum (λj. if j = i then n else 0) {0..< k+1}
      by (rule sum.cong) (simp-all add: xs del: replicate.simps)
    also have ... = n using i by simp
    finally have xs ∈ natpermute n (k + 1)
      using xsl unfolding natpermute-def mem-Collect-eq by blast
    then show xs ∈ ?A
      using nxs by blast
  qed
qed
```

The general form.

```
lemma fps-prod-nth:
  fixes m :: nat
    and a :: nat ⇒ 'a::comm-ring-1 fps
  shows (prod a {0 .. m}) $ n =
    sum (λv. prod (λj. (a j) $ (v!j)) {0..m}) (natpermute n (m+1))
  (is ?P m n)
proof (induct m arbitrary: n rule: nat-less-induct)
  fix m n assume H: ∀ m' < m. ∀ n. ?P m' n
  show ?P m n
  proof (cases m)
    case 0
    then show ?thesis
      by simp
  next
    case (Suc k)
    then have km: k < m by arith
    have u0: {0 .. k} ∪ {m} = {0..m}
      using Suc by (simp add: set-eq-iff) presburger
    have f0: finite {0 .. k} finite {m} by auto
    have d0: {0 .. k} ∩ {m} = {} using Suc by auto
    have (prod a {0 .. m}) $ n = (prod a {0 .. k} * a m) $ n
      unfolding prod.union-disjoint[OF f0 d0, unfolded u0] by simp
    also have ... = (∑ i = 0..n. (∑ v∈natpermute i (k + 1).
          (∏ j = 0..k. a j $ v ! j) * a m $ (n − i)))
      unfolding fps-mult-nth H[rule-format, OF km] sum-distrib-right ..
```

also have ... = ($\sum i = 0..n.$
$\sum v \in (\lambda l1.\ l1\ @\ [n - i])\ `\ natpermute\ i\ (Suc\ k).$
$(\prod j = 0..k.\ a\ j\ \$\ v\ !\ j) * a\ (Suc\ k)\ \$\ v\ !\ Suc\ k)$
by (*intro sum.cong* [*OF refl sym*] *sum.reindex-cong*) (*auto simp*: *inj-on-def natpermute-def nth-append Suc*)
also have ... = ($\sum v \in (\bigcup x \in \{0..n\}.\ \{l1\ @\ [n - x]\ |l1.\ l1 \in natpermute\ x\ (Suc\ k)\}$).
$(\prod j = 0..k.\ a\ j\ \$\ v\ !\ j) * a\ (Suc\ k)\ \$\ v\ !\ Suc\ k)$
by (*subst sum.UNION-disjoint*) (*auto simp add*: *natpermute-finite setcompr-eq-image*)
also have ... = ($\sum v \in natpermute\ n\ (m + 1).\ \prod j \in \{0..m\}.\ a\ j\ \$\ v\ !\ j$)
using *natpermute-split*[*of m m + 1*] **by** (*simp add*: *Suc*)
finally show *?thesis* **.**
qed
qed

The special form for powers.

lemma *fps-power-nth-Suc*:
fixes *m* :: *nat*
and *a* :: *'a*::*comm-ring-1 fps*
shows (*a* ^ *Suc m*)$*n* = *sum* (λ*v*. *prod* (λ*j*. *a* $ (*v*!*j*)) {*0*..*m*}) (*natpermute n* (*m*+*1*))
proof −
have *th0*: *a*^*Suc m* = *prod* (λ*i*. *a*) {*0*..*m*}
by (*simp add*: *prod-constant*)
show *?thesis* **unfolding** *th0 fps-prod-nth* **..**
qed

lemma *fps-power-nth*:
fixes *m* :: *nat*
and *a* :: *'a*::*comm-ring-1 fps*
shows (*a* ^*m*)$*n* =
(*if m*=*0 then 1*$*n else sum* (λ*v*. *prod* (λ*j*. *a* $ (*v*!*j*)) {*0*..*m* − *1*}) (*natpermute n m*))
by (*cases m*) (*simp-all add*: *fps-power-nth-Suc del*: *power-Suc*)

lemmas *fps-nth-power-0* = *fps-power-zeroth*

lemma *natpermute-max-card*:
assumes *n0*: *n* ≠ *0*
shows *card* {*xs* ∈ *natpermute n* (*k* + *1*). *n* ∈ *set xs*} = *k* + *1*
unfolding *natpermute-contain-maximal*
proof −
let *?A* = λ*i*. {(*replicate* (*k* + *1*) *0*)[*i* := *n*]}
let *?K* = {*0* ..*k*}
have *fK*: *finite ?K*
by *simp*
have *fAK*: ∀ *i*∈*?K*. *finite* (*?A i*)
by *auto*
have *d*: ∀ *i*∈ *?K*. ∀ *j*∈ *?K*. *i* ≠ *j* ⟶

```
    {(replicate (k + 1) 0)[i := n]} ∩ {(replicate (k + 1) 0)[j := n]} = {}
  proof clarify
    fix i j
    assume i: i ∈ ?K and j: j ∈ ?K and ij: i ≠ j
    have False if eq: (replicate (k+1) 0)[i:=n] = (replicate (k+1) 0)[j:= n]
    proof −
      have (replicate (k+1) 0) [i:=n] ! i = n
        using i by (simp del: replicate.simps)
      moreover
      have (replicate (k+1) 0) [j:=n] ! i = 0
        using i ij by (simp del: replicate.simps)
      ultimately show ?thesis
        using eq n0 by (simp del: replicate.simps)
    qed
    then show {(replicate (k + 1) 0)[i := n]} ∩ {(replicate (k + 1) 0)[j := n]} =
{}
      by auto
  qed
  from card-UN-disjoint[OF fK fAK d]
  show card (⋃ i∈{0..k}. {(replicate (k + 1) 0)[i := n]}) = k + 1
    by simp
qed

lemma fps-power-Suc-nth:
  fixes f :: 'a :: comm-ring-1 fps
  assumes k: k > 0
  shows (f ^ Suc m) $ k =
          of-nat (Suc m) * (f $ k * (f $ 0) ^ m) +
          (∑ v∈{v∈natpermute k (m+1). k ∉ set v}. ∏ j = 0..m. f $ v ! j)
proof −
  define A B
    where A = {v∈natpermute k (m+1). k ∈ set v}
      and  B = {v∈natpermute k (m+1). k ∉ set v}
  have [simp]: finite A finite B A ∩ B = {} by (auto simp: A-def B-def natper-
mute-finite)

  from natpermute-max-card[of k m] k have card-A: card A = m + 1 by (simp
add: A-def)
  {
    fix v assume v: v ∈ A
    from v have [simp]: length v = Suc m by (simp add: A-def natpermute-def)
    from v have ∃j. j ≤ m ∧ v ! j = k
      by (auto simp: set-conv-nth A-def natpermute-def less-Suc-eq-le)
    then obtain j where j: j ≤ m v ! j = k by auto

    from v have k = sum-list v by (simp add: A-def natpermute-def)
    also have ... = (∑ i=0..m. v ! i)
    by (simp add: sum-list-sum-nth atLeastLessThanSuc-atLeastAtMost del: sum.op-ivl-Suc)
    also from j have {0..m} = insert j ({0..m}−{j}) by auto
```

```
    also from j have (∑ i∈.... v ! i) = k + (∑ i∈{0..m}−{j}. v ! i)
      by (subst sum.insert) simp-all
    finally have (∑ i∈{0..m}−{j}. v ! i) = 0 by simp
    hence zero: v ! i = 0 if i ∈ {0..m}−{j} for i using that
      by (subst (asm) sum-eq-0-iff) auto

    from j have {0..m} = insert j ({0..m} − {j}) by auto
    also from j have (∏ i∈.... f $ (v ! i)) = f $ k * (∏ i∈{0..m} − {j}. f $ (v !
i))
      by (subst prod.insert) auto
    also have (∏ i∈{0..m} − {j}. f $ (v ! i)) = (∏ i∈{0..m} − {j}. f $ 0)
      by (intro prod.cong) (simp-all add: zero)
    also from j have ... = (f $ 0) ^ m by (subst prod-constant) simp-all
    finally have (∏ j = 0..m. f $ (v ! j)) = f $ k * (f $ 0) ^ m .
  } note A = this

  have (f ^ Suc m) $ k = (∑ v∈natpermute k (m + 1). ∏ j = 0..m. f $ v ! j)
    by (rule fps-power-nth-Suc)
  also have natpermute k (m+1) = A ∪ B unfolding A-def B-def by blast
  also have (∑ v∈.... ∏ j = 0..m. f $ (v ! j)) =
              (∑ v∈A. ∏ j = 0..m. f $ (v ! j)) + (∑ v∈B. ∏ j = 0..m. f $ (v ! j))
    by (intro sum.union-disjoint) simp-all
  also have (∑ v∈A. ∏ j = 0..m. f $ (v ! j)) = of-nat (Suc m) * (f $ k * (f $ 0)
^ m)
    by (simp add: A card-A)
  finally show ?thesis by (simp add: B-def)
qed

lemma fps-power-Suc-eqD:
  fixes f g :: 'a :: {idom,semiring-char-0} fps
  assumes f ^ Suc m = g ^ Suc m f $ 0 = g $ 0 f $ 0 ≠ 0
  shows   f = g
proof (rule fps-ext)
  fix k :: nat
  show f $ k = g $ k
  proof (induction k rule: less-induct)
    case (less k)
    show ?case
    proof (cases k = 0)
      case False
      let ?h = λf. (∑ v | v ∈ natpermute k (m + 1) ∧ k ∉ set v. ∏ j = 0..m. f $
v ! j)
      from False fps-power-Suc-nth[of k f m] fps-power-Suc-nth[of k g m]
        have f $ k * (of-nat (Suc m) * (f $ 0) ^ m) + ?h f =
              g $ k * (of-nat (Suc m) * (f $ 0) ^ m) + ?h g using assms
        by (simp add: mult-ac del: power-Suc of-nat-Suc)
      also have v ! i < k if v ∈ {v∈natpermute k (m+1). k ∉ set v} i ≤ m for v i
        using that elem-le-sum-list[of i v] unfolding natpermute-def
        by (auto simp: set-conv-nth dest!: spec[of - i])
```

```
    hence ?h f = ?h g
      by (intro sum.cong refl prod.cong less lessI) (simp add: natpermute-def)
    finally have f $ k * (of-nat (Suc m) * (f $ 0) ^ m) = g $ k * (of-nat (Suc
m) * (f $ 0) ^ m)
      by simp
    with assms show f $ k = g $ k
      by (subst (asm) mult-right-cancel) (auto simp del: of-nat-Suc)
  qed (simp-all add: assms)
 qed
qed

lemma fps-power-Suc-eqD':
 fixes f g :: 'a :: {idom,semiring-char-0} fps
 assumes f ^ Suc m = g ^ Suc m f $ subdegree f = g $ subdegree g
 shows   f = g
proof (cases f = 0)
 case False
 have Suc m * subdegree f = subdegree (f ^ Suc m)
   by (rule subdegree-power [symmetric])
 also have f ^ Suc m = g ^ Suc m by fact
 also have subdegree ... = Suc m * subdegree g by (rule subdegree-power)
 finally have [simp]: subdegree f = subdegree g
   by (subst (asm) Suc-mult-cancel1)
 have fps-shift (subdegree f) f * fps-X ^ subdegree f = f
   by (rule subdegree-decompose [symmetric])
 also have ... ^ Suc m = g ^ Suc m by fact
 also have g = fps-shift (subdegree g) g * fps-X ^ subdegree g
   by (rule subdegree-decompose)
 also have subdegree f = subdegree g by fact
 finally have fps-shift (subdegree g) f ^ Suc m = fps-shift (subdegree g) g ^ Suc
m
   by (simp add: algebra-simps power-mult-distrib del: power-Suc)
 hence fps-shift (subdegree g) f = fps-shift (subdegree g) g
   by (rule fps-power-Suc-eqD) (insert assms False, auto)
 with subdegree-decompose[of f] subdegree-decompose[of g] show ?thesis by simp
qed (insert assms, simp-all)

lemma fps-power-eqD':
 fixes f g :: 'a :: {idom,semiring-char-0} fps
 assumes f ^ m = g ^ m f $ subdegree f = g $ subdegree g m > 0
 shows   f = g
 using fps-power-Suc-eqD'[of f m-1 g] assms by simp

lemma fps-power-eqD:
 fixes f g :: 'a :: {idom,semiring-char-0} fps
 assumes f ^ m = g ^ m f $ 0 = g $ 0 f $ 0 ≠ 0 m > 0
 shows   f = g
 by (rule fps-power-eqD'[of f m g]) (insert assms, simp-all)
```

```
lemma fps-compose-inj-right:
  assumes a0: a$0 = (0::'a::idom)
    and a1: a$1 ≠ 0
  shows (b oo a = c oo a) ⟷ b = c
  (is ?lhs ⟷ ?rhs)
proof
  show ?lhs if ?rhs using that by simp
  show ?rhs if ?lhs
  proof −
    have b$n = c$n for n
    proof (induct n rule: nat-less-induct)
      fix n
      assume H: ∀ m<n. b$m = c$m
      show b$n = c$n
      proof (cases n)
        case 0
        from ‹?lhs› have (b oo a)$n = (c oo a)$n
          by simp
        then show ?thesis
          using 0 by (simp add: fps-compose-nth)
      next
        case (Suc n1)
        have f: finite {0 .. n1} finite {n} by simp-all
        have eq: {0 .. n1} ∪ {n} = {0 .. n} using Suc by auto
        have d: {0 .. n1} ∩ {n} = {} using Suc by auto
        have seq: (∑ i = 0..n1. b $ i * a ^ i $ n) = (∑ i = 0..n1. c $ i * a ^ i $
n)
          using H Suc by auto
        have th0: (b oo a) $n = (∑ i = 0..n1. c $ i * a ^ i $ n) + b$n * (a$1)^n
          unfolding fps-compose-nth sum.union-disjoint[OF f d, unfolded eq] seq
          using startsby-zero-power-nth-same[OF a0]
          by simp
        have th1: (c oo a) $n = (∑ i = 0..n1. c $ i * a ^ i $ n) + c$n * (a$1)^n
          unfolding fps-compose-nth sum.union-disjoint[OF f d, unfolded eq]
          using startsby-zero-power-nth-same[OF a0]
          by simp
        from ‹?lhs›[unfolded fps-eq-iff, rule-format, of n] th0 th1 a1
        show ?thesis by auto
      qed
    qed
    then show ?rhs by (simp add: fps-eq-iff)
  qed
qed
```

5.16 Radicals

```
declare prod.cong [fundef-cong]

function radical :: (nat ⇒ 'a ⇒ 'a) ⇒ nat ⇒ 'a::field fps ⇒ nat ⇒ 'a
```

```
where
  radical r 0 a 0 = 1
| radical r 0 a (Suc n) = 0
| radical r (Suc k) a 0 = r (Suc k) (a$0)
| radical r (Suc k) a (Suc n) =
    (a$ Suc n − sum (λxs. prod (λj. radical r (Suc k) a (xs ! j)) {0..k})
      {xs. xs ∈ natpermute (Suc n) (Suc k) ∧ Suc n ∉ set xs}) /
    (of-nat (Suc k) * (radical r (Suc k) a 0)^k)
  by pat-completeness auto

termination radical
proof
  let ?R = measure (λ(r, k, a, n). n)
  {
    show wf ?R by auto
  next
    fix r :: nat ⇒ 'a ⇒ 'a
    and a :: 'a fps
    and k n xs i
    assume xs: xs ∈ {xs ∈ natpermute (Suc n) (Suc k). Suc n ∉ set xs} and i: i
∈ {0..k}
    have False if c: Suc n ≤ xs ! i
    proof −
      from xs i have xs !i ≠ Suc n
        by (simp add: in-set-conv-nth natpermute-def)
      with c have c': Suc n < xs!i by arith
      have fths: finite {0 ..< i} finite {i} finite {i+1..<Suc k}
        by simp-all
      have d: {0 ..< i} ∩ ({i} ∪ {i+1 ..< Suc k}) = {} {i} ∩ {i+1..< Suc k} =
{}
        by auto
      have eqs: {0..<Suc k} = {0 ..< i} ∪ ({i} ∪ {i+1 ..< Suc k})
        using i by auto
      from xs have Suc n = sum-list xs
        by (simp add: natpermute-def)
      also have ... = sum (nth xs) {0..<Suc k} using xs
        by (simp add: natpermute-def sum-list-sum-nth)
      also have ... = xs!i + sum (nth xs) {0..<i} + sum (nth xs) {i+1..<Suc k}
       unfolding eqs sum.union-disjoint[OF fths(1) finite-UnI[OF fths(2,3)] d(1)]
        unfolding sum.union-disjoint[OF fths(2) fths(3) d(2)]
        by simp
      finally show ?thesis using c' by simp
    qed
    then show ((r, Suc k, a, xs!i), r, Suc k, a, Suc n) ∈ ?R
      using not-less by auto
  next
    fix r :: nat ⇒ 'a ⇒ 'a
    and a :: 'a fps
    and k n
```

show $((r, Suc\ k, a, 0), r, Suc\ k, a, Suc\ n) \in ?R$ **by** *simp*
}
qed

definition *fps-radical r n a* = *Abs-fps* (*radical r n a*)

lemma *radical-0* [*simp*]: $\bigwedge n.\ 0 < n \Longrightarrow radical\ r\ 0\ a\ n = 0$
using *radical.elims* **by** *blast*

lemma *fps-radical0*[*simp*]: *fps-radical r 0 a* = *1*
by (*auto simp add*: *fps-eq-iff fps-radical-def*)

lemma *fps-radical-nth-0*[*simp*]: *fps-radical r n a $ 0* = (*if n* = *0 then 1 else r n* (*a$0*))
by (*cases n*) (*simp-all add*: *fps-radical-def*)

lemma *fps-radical-power-nth*[*simp*]:
assumes *r*: $(r\ k\ (a\$0)) \hat{}\ k = a\$0$
shows $fps\text{-}radical\ r\ k\ a \hat{}\ k\ \$\ 0 = (if\ k = 0\ then\ 1\ else\ a\$0)$
proof (*cases k*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc h*)
have *eq1*: $fps\text{-}radical\ r\ k\ a \hat{}\ k\ \$\ 0 = (\prod j{\in}\{0..h\}.\ fps\text{-}radical\ r\ k\ a\ \$\ (replicate\ k\ 0)\ !\ j)$
unfolding *fps-power-nth Suc* **by** *simp*
also have ... = $(\prod j{\in}\{0..h\}.\ r\ k\ (a\$0))$
proof (*rule prod.cong* [*OF refl*])
show $fps\text{-}radical\ r\ k\ a\ \$\ replicate\ k\ 0\ !\ j = r\ k\ (a\ \$\ 0)$ **if** $j \in \{0..h\}$ **for** *j*
proof −
have $j < Suc\ h$
using *that* **by** *presburger*
then show *?thesis*
by (*metis Suc fps-radical-nth-0 nth-replicate old.nat.distinct*(*2*))
qed
qed
also have ... = *a$0*
using *r Suc* **by** *simp*
finally show *?thesis*
using *Suc* **by** *simp*
qed

lemma *power-radical*:
fixes *a*:: *'a*::*field-char-0 fps*
assumes *a0*: $a\$0 \neq 0$
shows $(r\ (Suc\ k)\ (a\$0)) \hat{}\ Suc\ k = a\$0 \longleftrightarrow (fps\text{-}radical\ r\ (Suc\ k)\ a) \hat{}\ (Suc\ k) = a$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)

```
proof
  let ?r = fps-radical r (Suc k) a
  show ?rhs if r0: ?lhs
  proof −
    from a0 r0 have r00: r (Suc k) (a$0) ≠ 0 by auto
    have ?r ^ Suc k $ z = a$z for z
    proof (induct z rule: nat-less-induct)
      fix n
      assume H: ∀ m<n. ?r ^ Suc k $ m = a$m
      show ?r ^ Suc k $ n = a $n
      proof (cases n)
        case 0
        then show ?thesis
          using fps-radical-power-nth[of r Suc k a, OF r0] by simp
      next
        case (Suc n1)
        then have n ≠ 0 by simp
        let ?Pnk = natpermute n (k + 1)
        let ?Pnkn = {xs ∈ ?Pnk. n ∈ set xs}
        let ?Pnknn = {xs ∈ ?Pnk. n ∉ set xs}
        have eq: ?Pnkn ∪ ?Pnknn = ?Pnk by blast
        have d: ?Pnkn ∩ ?Pnknn = {} by blast
        have f: finite ?Pnkn finite ?Pnknn
          using finite-Un[of ?Pnkn ?Pnknn, unfolded eq]
          by (metis natpermute-finite)+
        let ?f = λv. ∏j∈{0..k}. ?r $ v ! j
        have sum ?f ?Pnkn = sum (λv. ?r $ n * r (Suc k) (a $ 0) ^ k) ?Pnkn
        proof (rule sum.cong)
          fix v assume v: v ∈ {xs ∈ natpermute n (k + 1). n ∈ set xs}
          let ?ths = (∏j∈{0..k}. fps-radical r (Suc k) a $ v ! j) =
            fps-radical r (Suc k) a $ n * r (Suc k) (a $ 0) ^ k
          from v obtain i where i: i ∈ {0..k} v = (replicate (k+1) 0) [i:= n]
            unfolding natpermute-contain-maximal by auto
          have (∏j∈{0..k}. fps-radical r (Suc k) a $ v ! j) =
                (∏j∈{0..k}. if j = i then fps-radical r (Suc k) a $ n else r (Suc k)
(a$0))
              using i r0 by (auto simp del: replicate.simps intro: prod.cong)
          also have ... = (fps-radical r (Suc k) a $ n) * r (Suc k) (a$0) ^ k
            using i r0 by (simp add: prod-gen-delta)
          finally show ?ths .
        qed rule
        then have sum ?f ?Pnkn = of-nat (k+1) * ?r $ n * r (Suc k) (a $ 0) ^ k
          by (simp add: natpermute-max-card[OF ‹n ≠ 0›, simplified])
        also have ... = a$n − sum ?f ?Pnknn
         unfolding Suc using r00 a0 by (simp add: field-simps fps-radical-def del:
of-nat-Suc)
        finally have fn: sum ?f ?Pnkn = a$n − sum ?f ?Pnknn .
        have (?r ^ Suc k)$n = sum ?f ?Pnkn + sum ?f ?Pnknn
          unfolding fps-power-nth-Suc sum.union-disjoint[OF f d, unfolded eq] ..
```

```
      also have ... = a$n unfolding fn by simp
      finally show ?thesis .
    qed
  qed
  then show ?thesis using r0 by (simp add: fps-eq-iff)
 qed
 show ?lhs if ?rhs
 proof −
  from that have ((fps-radical r (Suc k) a) ^ (Suc k))$0 = a$0
   by simp
  then show ?thesis
   unfolding fps-power-nth-Suc
   by (simp add: prod-constant del: replicate.simps)
 qed
qed

lemma radical-unique:
 assumes r0: (r (Suc k) (b$0)) ^ Suc k = b$0
  and a0: r (Suc k) (b$0 ::'a::field-char-0) = a$0
  and b0: b$0 ≠ 0
 shows a^(Suc k) = b ⟷ a = fps-radical r (Suc k) b
  (is ?lhs ⟷ ?rhs is - ⟷ a = ?r)
proof
 show ?lhs if ?rhs
  using that using power-radical[OF b0, of r k, unfolded r0] by simp
 show ?rhs if ?lhs
 proof −
  have r00: r (Suc k) (b$0) ≠ 0 using b0 r0 by auto
  have ceq: card {0..k} = Suc k by simp
  from a0 have a0r0: a$0 = ?r$0 by simp
  have a $ n = ?r $ n for n
  proof (induct n rule: nat-less-induct)
   fix n
   assume h: ∀ m<n. a$m = ?r $m
   show a$n = ?r $ n
   proof (cases n)
    case 0
    then show ?thesis using a0 by simp
   next
    case (Suc n1)
    have fK: finite {0..k} by simp
    have nz: n ≠ 0 using Suc by simp
    let ?Pnk = natpermute n (Suc k)
    let ?Pnkn = {xs ∈ ?Pnk. n ∈ set xs}
    let ?Pnknn = {xs ∈ ?Pnk. n ∉ set xs}
    have eq: ?Pnkn ∪ ?Pnknn = ?Pnk by blast
    have d: ?Pnkn ∩ ?Pnknn = {} by blast
    have f: finite ?Pnkn finite ?Pnknn
     using finite-Un[of ?Pnkn ?Pnknn, unfolded eq]
```

```
    by (metis natpermute-finite)+
  let ?f = λv. ∏j∈{0..k}. ?r $ v ! j
  let ?g = λv. ∏j∈{0..k}. a $ v ! j
  have sum ?g ?Pnkn = sum (λv. a $ n * (?r$0)^k) ?Pnkn
  proof (rule sum.cong)
    fix v
    assume v: v ∈ {xs ∈ natpermute n (Suc k). n ∈ set xs}
    let ?ths = (∏j∈{0..k}. a $ v ! j) = a $ n * (?r$0)^k
    from v obtain i where i: i ∈ {0..k} v = (replicate (k+1) 0) [i:= n]
      unfolding Suc-eq-plus1 natpermute-contain-maximal
      by (auto simp del: replicate.simps)
   have (∏j∈{0..k}. a $ v ! j) = (∏j∈{0..k}. if j = i then a $ n else r (Suc
k) (b$0))
      using i a0 by (auto simp del: replicate.simps intro: prod.cong)
    also have ... = a $ n * (?r $ 0)^k
      using i by (simp add: prod-gen-delta)
    finally show ?ths .
  qed rule
  then have th0: sum ?g ?Pnkn = of-nat (k+1) * a $ n * (?r $ 0)^k
    by (simp add: natpermute-max-card[OF nz, simplified])
  have th1: sum ?g ?Pnknn = sum ?f ?Pnknn
  proof (rule sum.cong, rule refl, rule prod.cong, simp)
    fix xs i
    assume xs: xs ∈ ?Pnknn and i: i ∈ {0..k}
    have False if c: n ≤ xs ! i
    proof −
      from xs i have xs ! i ≠ n
        by (simp add: in-set-conv-nth natpermute-def)
      with c have c': n < xs!i by arith
      have fths: finite {0 ..< i} finite {i} finite {i+1..<Suc k}
        by simp-all
       have d: {0 ..< i} ∩ ({i} ∪ {i+1 ..< Suc k}) = {} {i} ∩ {i+1..< Suc
k} = {}
        by auto
      have eqs: {0..<Suc k} = {0 ..< i} ∪ ({i} ∪ {i+1 ..< Suc k})
        using i by auto
      from xs have n = sum-list xs
        by (simp add: natpermute-def)
      also have ... = sum (nth xs) {0..<Suc k}
        using xs by (simp add: natpermute-def sum-list-sum-nth)
     also have ... = xs!i + sum (nth xs) {0..<i} + sum (nth xs) {i+1..<Suc
k}
       unfolding eqs  sum.union-disjoint[OF fths(1) finite-UnI[OF fths(2,3)]
d(1)]
        unfolding sum.union-disjoint[OF fths(2) fths(3) d(2)]
        by simp
      finally show ?thesis using c' by simp
    qed
    then have thn: xs!i < n by presburger
```

```
      from h[rule-format, OF thn] show a$(xs !i) = ?r$(xs!i) .
    qed
    have th00: ⋀x::'a. of-nat (Suc k) * (x * inverse (of-nat (Suc k))) = x
      by (simp add: field-simps del: of-nat-Suc)
    from ‹?lhs› have b$n = a^Suc k $ n
      by (simp add: fps-eq-iff)
    also have a ^ Suc k$n = sum ?g ?Pnkn + sum ?g ?Pnknn
      unfolding fps-power-nth-Suc
      using sum.union-disjoint[OF f d, unfolded Suc-eq-plus1[symmetric],
        unfolded eq, of ?g] by simp
    also have ... = of-nat (k+1) * a $ n * (?r $ 0)^k + sum ?f ?Pnknn
      unfolding th0 th1 ..
    finally have §: of-nat (k+1) * a $ n * (?r $ 0)^k = b$n - sum ?f ?Pnknn
      by simp
    have a$n = (b$n - sum ?f ?Pnknn) / (of-nat (k+1) * (?r $ 0)^k)
      apply (rule eq-divide-imp)
      using r00 § by (simp-all add: ac-simps del: of-nat-Suc)
    then show ?thesis
      unfolding fps-radical-def Suc
      by (simp del: of-nat-Suc)
  qed
 qed
 then show ?rhs by (simp add: fps-eq-iff)
 qed
qed


lemma radical-power:
 assumes r0: r (Suc k) ((a$0) ^ Suc k) = a$0
   and a0: (a$0 :: 'a::field-char-0) ≠ 0
 shows (fps-radical r (Suc k) (a ^ Suc k)) = a
proof −
 let ?ak = a^ Suc k
 have ak0: ?ak $ 0 = (a$0) ^ Suc k
   by (simp add: fps-nth-power-0 del: power-Suc)
 from r0 have th0: r (Suc k) (a ^ Suc k $ 0) ^ Suc k = a ^ Suc k $ 0
   using ak0 by auto
 from r0 ak0 have th1: r (Suc k) (a ^ Suc k $ 0) = a $ 0
   by auto
 from ak0 a0 have ak00: ?ak $ 0 ≠0
   by auto
 from radical-unique[of r k ?ak a, OF th0 th1 ak00] show ?thesis
   by metis
qed

lemma fps-deriv-radical':
 fixes a :: 'a::field-char-0 fps
 assumes r0: (r (Suc k) (a$0)) ^ Suc k = a$0
   and a0: a$0 ≠ 0
```

```
  shows fps-deriv (fps-radical r (Suc k) a) =
    fps-deriv a / ((of-nat (Suc k)) * (fps-radical r (Suc k) a) ^ k)
proof -
  let ?r = fps-radical r (Suc k) a
  let ?w = (of-nat (Suc k)) * ?r ^ k
  from a0 r0 have r0': r (Suc k) (a$0) ≠ 0
    by auto
  from r0' have w0: ?w $ 0 ≠ 0
    by (simp del: of-nat-Suc)
  note th0 = inverse-mult-eq-1[OF w0]
  let ?iw = inverse ?w
  from iffD1[OF power-radical[of a r], OF a0 r0]
  have fps-deriv (?r ^ Suc k) = fps-deriv a
    by simp
  then have fps-deriv ?r * ?w = fps-deriv a
    by (simp add: fps-deriv-power' ac-simps del: power-Suc)
  then have ?iw * fps-deriv ?r * ?w = ?iw * fps-deriv a
    by simp
  with a0 r0 have fps-deriv ?r * (?iw * ?w) = fps-deriv a / ?w
    by (subst fps-divide-unit) (auto simp del: of-nat-Suc)
  then show ?thesis unfolding th0 by simp
qed

lemma fps-deriv-radical:
  fixes a :: 'a::field-char-0 fps
  assumes r0: (r (Suc k) (a$0)) ^ Suc k = a$0
    and a0: a$0 ≠ 0
  shows fps-deriv (fps-radical r (Suc k) a) =
    fps-deriv a / (fps-const (of-nat (Suc k)) * (fps-radical r (Suc k) a) ^ k)
  using fps-deriv-radical'[of r k a, OF r0 a0]
  by (simp add: fps-of-nat[symmetric])

lemma radical-mult-distrib:
  fixes a :: 'a::field-char-0 fps
  assumes k: k > 0
    and ra0: r k (a $ 0) ^ k = a $ 0
    and rb0: r k (b $ 0) ^ k = b $ 0
    and a0: a $ 0 ≠ 0
    and b0: b $ 0 ≠ 0
  shows r k ((a * b) $ 0) = r k (a $ 0) * r k (b $ 0) ⟷
    fps-radical r k (a * b) = fps-radical r k a * fps-radical r k b
    (is ?lhs ⟷ ?rhs)
proof
  show ?rhs if r0': ?lhs
  proof -
    from r0' have r0: (r k ((a * b) $ 0)) ^ k = (a * b) $ 0
      by (simp add: fps-mult-nth ra0 rb0 power-mult-distrib)
    show ?thesis
    proof (cases k)
```

```
    case 0
    then show ?thesis using r0' by simp
   next
    case (Suc h)
    let ?ra = fps-radical r (Suc h) a
    let ?rb = fps-radical r (Suc h) b
    have th0: r (Suc h) ((a * b) $ 0) = (fps-radical r (Suc h) a * fps-radical r
(Suc h) b) $ 0
     using r0' Suc by (simp add: fps-mult-nth)
    have ab0: (a*b) $ 0 ≠ 0
     using a0 b0 by (simp add: fps-mult-nth)
    from radical-unique[of r h a*b fps-radical r (Suc h) a * fps-radical r (Suc h)
b, OF r0[unfolded Suc] th0 ab0, symmetric]
    iffD1[OF power-radical[of - r], OF a0 ra0[unfolded Suc]] iffD1[OF power-radical[of
- r], OF b0 rb0[unfolded Suc]] Suc r0'
    show ?thesis
     by (auto simp add: power-mult-distrib simp del: power-Suc)
   qed
  qed
  show ?lhs if ?rhs
  proof −
   from that have (fps-radical r k (a * b)) $ 0 = (fps-radical r k a * fps-radical r
k b) $ 0
    by simp
   then show ?thesis
    using k by (simp add: fps-mult-nth)
  qed
qed



lemma radical-divide:
  fixes a :: 'a::field-char-0 fps
  assumes kp: k > 0
   and ra0: (r k (a $ 0)) ^ k = a $ 0
   and rb0: (r k (b $ 0)) ^ k = b $ 0
   and a0: a$0 ≠ 0
   and b0: b$0 ≠ 0
  shows r k ((a $ 0) / (b$0)) = r k (a$0) / r k (b $ 0) ⟷
   fps-radical r k (a/b) = fps-radical r k a / fps-radical r k b
  (is ?lhs = ?rhs)
proof
  let ?r = fps-radical r k
  from kp obtain h where k: k = Suc h
   by (cases k) auto
  have ra0': r k (a$0) ≠ 0 using a0 ra0 k by auto
  have rb0': r k (b$0) ≠ 0 using b0 rb0 k by auto

  show ?lhs if ?rhs
```

proof −
from *that* **have** *?r* (*a*/*b*) $ *0* = (*?r a* / *?r b*)$*0*
by *simp*
then show *?thesis*
using *k a0 b0 rb0′* **by** (*simp add*: *fps-divide-unit fps-mult-nth fps-inverse-def divide-inverse*)
qed
show *?rhs* **if** *?lhs*
proof −
from *a0 b0* **have** *ab0*[*simp*]: (*a*/*b*)$*0* = *a*$*0* / *b*$*0*
by (*simp add*: *fps-divide-def fps-mult-nth divide-inverse fps-inverse-def*)
have *th0*: *r k* ((*a*/*b*)$*0*) ^ *k* = (*a*/*b*)$*0*
by (*simp add*: ‹*?lhs*› *power-divide ra0 rb0*)
from *a0 b0 ra0′ rb0′ kp* ‹*?lhs*›
have *th1*: *r k* ((*a* / *b*) $ *0*) = (*fps-radical r k a* / *fps-radical r k b*) $ *0*
by (*simp add*: *fps-divide-unit fps-mult-nth fps-inverse-def divide-inverse*)
from *a0 b0 ra0′ rb0′ kp* **have** *ab0′*: (*a* / *b*) $ *0* ≠ *0*
by (*simp add*: *fps-divide-unit fps-mult-nth fps-inverse-def nonzero-imp-inverse-nonzero*)
note *tha*[*simp*] = *iffD1*[*OF power-radical*[**where** *r*=*r* **and** *k*=*h*], *OF a0 ra0*[*unfolded k*], *unfolded k*[*symmetric*]]
note *thb*[*simp*] = *iffD1*[*OF power-radical*[**where** *r*=*r* **and** *k*=*h*], *OF b0 rb0*[*unfolded k*], *unfolded k*[*symmetric*]]
from *b0 rb0′* **have** *th2*: (*?r a* / *?r b*)^*k* = *a*/*b*
by (*simp add*: *fps-divide-unit power-mult-distrib fps-inverse-power*[*symmetric*])

from *iffD1*[*OF radical-unique*[**where** *r*=*r* **and** *a*=*?r a* / *?r b* **and** *b*=*a*/*b* **and** *k*=*h*], *symmetric*, *unfolded k*[*symmetric*], *OF th0 th1 ab0′ th2*]
show *?thesis* **.**
qed
qed

lemma *radical-inverse*:
fixes *a* :: *'a*::*field-char-0 fps*
assumes *k*: *k* > *0*
and *ra0*: *r k* (*a* $ *0*) ^ *k* = *a* $ *0*
and *r1*: (*r k 1*)^*k* = *1*
and *a0*: *a*$*0* ≠ *0*
shows *r k* (*inverse* (*a* $ *0*)) = *r k 1* / (*r k* (*a* $ *0*)) ⟷
fps-radical r k (*inverse a*) = *fps-radical r k 1* / *fps-radical r k a*
using *radical-divide*[**where** *k*=*k* **and** *r*=*r* **and** *a*=*1* **and** *b*=*a*, *OF k*] *ra0 r1 a0*
by (*simp add*: *divide-inverse fps-divide-def*)

5.17 Chain rule

lemma *fps-compose-deriv*:
fixes *a* :: *'a*::*idom fps*
assumes *b0*: *b*$*0* = *0*
shows *fps-deriv* (*a oo b*) = ((*fps-deriv a*) *oo b*) ∗ *fps-deriv b*
proof −

```
  have (fps-deriv (a oo b))$n = (((fps-deriv a) oo b) * (fps-deriv b)) $n for n
  proof −
    have (fps-deriv (a oo b))$n = sum (λi. a $ i * (fps-deriv (b^i))$n) {0.. Suc n}
      by (simp add: fps-compose-def field-simps sum-distrib-left del: of-nat-Suc)
    also have ... = sum (λi. a$i * ((fps-const (of-nat i)) * (fps-deriv b * (b^(i −
1))))$n) {0.. Suc n}
      by (simp add: field-simps fps-deriv-power del: fps-mult-left-const-nth of-nat-Suc)
    also have ... = sum (λi. of-nat i * a$i * (((b^(i − 1)) * fps-deriv b))$n) {0..
Suc n}
      unfolding fps-mult-left-const-nth  by (simp add: field-simps)
    also have ... = sum (λi. of-nat i * a$i * (sum (λj. (b^(i − 1))$j * (fps-deriv
b)$(n − j)) {0..n})) {0.. Suc n}
      unfolding fps-mult-nth ..
    also have ... = sum (λi. of-nat i * a$i * (sum (λj. (b^(i − 1))$j * (fps-deriv
b)$(n − j)) {0..n})) {1.. Suc n}
      by (intro sum.mono-neutral-right) (auto simp add: mult-delta-left not-le)
     also have ... = sum (λi. of-nat (i + 1) * a$(i+1) * (sum (λj. (b^ i)$j *
of-nat (n − j + 1) * b$(n − j + 1)) {0..n})) {0.. n}
      unfolding fps-deriv-nth
      by (rule sum.reindex-cong [of Suc]) (simp-all add: mult.assoc)
    finally have th0: (fps-deriv (a oo b))$n =
      sum (λi. of-nat (i + 1) * a$(i+1) * (sum (λj. (b^ i)$j * of-nat (n − j + 1)
* b$(n − j + 1)) {0..n})) {0.. n} .

    have (((fps-deriv a) oo b) * (fps-deriv b))$n = sum (λi. (fps-deriv b)$ (n − i)
* ((fps-deriv a) oo b)$i) {0..n}
      unfolding fps-mult-nth by (simp add: ac-simps)
    also have ... = sum (λi. sum (λj. of-nat (n − i +1) * b$(n − i + 1) * of-nat
(j + 1) * a$(j+1) * (b^j)$i) {0..n}) {0..n}
      unfolding fps-deriv-nth fps-compose-nth sum-distrib-left mult.assoc
       by (auto simp: subset-eq b0 startsby-zero-power-prefix sum.mono-neutral-left
intro: sum.cong)
     also have ... = sum (λi. of-nat (i + 1) * a$(i+1) * (sum (λj. (b^ i)$j *
of-nat (n − j + 1) * b$(n − j + 1)) {0..n})) {0.. n}
      unfolding sum-distrib-left
      by (subst sum.swap) (force intro: sum.cong)
    finally show ?thesis
      unfolding th0 by simp
  qed
  then show ?thesis by (simp add: fps-eq-iff)
qed

lemma fps-poly-sum-fps-X:
  assumes ∀ i > n. a$i = 0
  shows a = sum (λi. fps-const (a$i) * fps-X^i) {0..n} (is a = ?r)
proof −
  have a$i = ?r$i for i
    unfolding fps-sum-nth fps-mult-left-const-nth fps-X-power-nth
    by (simp add: mult-delta-right assms)
```

then show *?thesis*
unfolding *fps-eq-iff* **by** *blast*
qed

5.18 Compositional inverses

fun *compinv* :: *'a fps* ⇒ *nat* ⇒ *'a*::*field*
where
compinv a 0 = *fps-X*$*0*
| *compinv a* (*Suc n*) =
(*fps-X*$ *Suc n* − *sum* (λ*i*. (*compinv a i*) ∗ (*a*^*i*)$*Suc n*) {*0* .. *n*}) / (*a*$*1*) ^ *Suc n*

definition *fps-inv a* = *Abs-fps* (*compinv a*)

lemma *fps-inv*:
assumes *a0*: *a*$*0* = *0*
and *a1*: *a*$*1* ≠ *0*
shows *fps-inv a oo a* = *fps-X*
proof −
let *?i* = *fps-inv a oo a*
have *?i* $*n* = *fps-X*$*n* **for** *n*
proof (*induct n rule*: *nat-less-induct*)
fix *n*
assume *h*: ∀ *m*<*n*. *?i*$*m* = *fps-X*$*m*
show *?i* $ *n* = *fps-X*$*n*
proof (*cases n*)
case *0*
then show *?thesis* **using** *a0*
by (*simp add*: *fps-compose-nth fps-inv-def*)
next
case (*Suc n1*)
have *?i* $ *n* = *sum* (λ*i*. (*fps-inv a* $ *i*) ∗ (*a*^*i*)$*n*) {*0* .. *n1*} + *fps-inv a* $ *Suc n1* ∗ (*a* $ *1*)^ *Suc n1*
by (*simp only*: *fps-compose-nth*) (*simp add*: *Suc startsby-zero-power-nth-same* [*OF a0*] *del*: *power-Suc*)
also have . . . = *sum* (λ*i*. (*fps-inv a* $ *i*) ∗ (*a*^*i*)$*n*) {*0* .. *n1*} +
(*fps-X*$ *Suc n1* − *sum* (λ*i*. (*fps-inv a* $ *i*) ∗ (*a*^*i*)$*n*) {*0* .. *n1*})
using *a0 a1 Suc* **by** (*simp add*: *fps-inv-def*)
also have . . . = *fps-X*$*n* **using** *Suc* **by** *simp*
finally show *?thesis* **.**
qed
qed
then show *?thesis*
by (*simp add*: *fps-eq-iff*)
qed

fun *gcompinv* :: *'a fps* ⇒ *'a fps* ⇒ *nat* ⇒ *'a*::*field*

```
where
  gcompinv b a 0 = b$0
| gcompinv b a (Suc n) =
    (b$ Suc n − sum (λi. (gcompinv b a i) ∗ (a^i)$Suc n) {0 .. n}) / (a$1) ^ Suc
n

definition fps-ginv b a = Abs-fps (gcompinv b a)

lemma fps-ginv:
  assumes a0: a$0 = 0
    and a1: a$1 ≠ 0
  shows fps-ginv b a oo a = b
proof −
  let ?i = fps-ginv b a oo a
  have ?i $n = b$n for n
  proof (induct n rule: nat-less-induct)
    fix n
    assume h: ∀ m<n. ?i$m = b$m
    show ?i $ n = b$n
    proof (cases n)
      case 0
      then show ?thesis using a0
        by (simp add: fps-compose-nth fps-ginv-def)
    next
      case (Suc n1)
      have ?i $ n = sum (λi. (fps-ginv b a $ i) ∗ (a^i)$n) {0 .. n1} + fps-ginv b
a $ Suc n1 ∗ (a $ 1)^ Suc n1
        by (simp only: fps-compose-nth) (simp add: Suc startsby-zero-power-nth-same
[OF a0] del: power-Suc)
      also have ... = sum (λi. (fps-ginv b a $ i) ∗ (a^i)$n) {0 .. n1} +
        (b$ Suc n1 − sum (λi. (fps-ginv b a $ i) ∗ (a^i)$n) {0 .. n1})
        using a0 a1 Suc by (simp add: fps-ginv-def)
      also have ... = b$n using Suc by simp
      finally show ?thesis .
    qed
  qed
  then show ?thesis
    by (simp add: fps-eq-iff)
qed

lemma fps-inv-ginv: fps-inv = fps-ginv fps-X
proof −
  have compinv x n = gcompinv fps-X x n for n and x :: 'a fps
  proof (induction n rule: nat-less-induct)
    case (1 n)
    then show ?case
      by (cases n) auto
  qed
  then show ?thesis
```

by (*auto simp add*: *fun-eq-iff fps-eq-iff fps-inv-def fps-ginv-def*)
qed

lemma *fps-compose-1*[*simp*]: *1 oo a = 1*
by (*simp add*: *fps-eq-iff fps-compose-nth mult-delta-left*)

lemma *fps-compose-0*[*simp*]: *0 oo a = 0*
by (*simp add*: *fps-eq-iff fps-compose-nth*)

lemma *fps-compose-0-right*[*simp*]: *a oo 0 = fps-const* (*a $ 0*)
by (*simp add*: *fps-eq-iff fps-compose-nth power-0-left sum.neutral*)

lemma *fps-compose-add-distrib*: (*a + b*) *oo c* = (*a oo c*) + (*b oo c*)
by (*simp add*: *fps-eq-iff fps-compose-nth field-simps sum.distrib*)

lemma *fps-compose-sum-distrib*: (*sum f S*) *oo a* = *sum* (*λi. f i oo a*) *S*
proof (*cases finite S*)
case *True*
show *?thesis*
proof (*rule finite-induct*[*OF True*])
show *sum f* {} *oo a* = (∑ *i*∈{}. *f i oo a*)
by *simp*
next
fix *x F*
assume *fF*: *finite F*
and *xF*: *x* ∉ *F*
and *h*: *sum f F oo a* = *sum* (*λi. f i oo a*) *F*
show *sum f* (*insert x F*) *oo a* = *sum* (*λi. f i oo a*) (*insert x F*)
using *fF xF h* **by** (*simp add*: *fps-compose-add-distrib*)
qed
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *convolution-eq*:
sum (*λi. a* (*i* :: *nat*) ∗ *b* (*n* − *i*)) {*0* .. *n*} =
sum (*λ*(*i*,*j*). *a i* ∗ *b j*) {(*i*,*j*). *i* ≤ *n* ∧ *j* ≤ *n* ∧ *i* + *j* = *n*}
by (*rule sum.reindex-bij-witness*[**where** *i*=*fst* **and** *j*=*λi.* (*i*, *n* − *i*)]) *auto*

lemma *product-composition-lemma*:
assumes *c0*: *c$0* = (*0*::*'a*::*idom*)
and *d0*: *d$0* = *0*
shows ((*a oo c*) ∗ (*b oo d*))$*n* =
sum (*λ*(*k*,*m*). *a*$*k* ∗ *b*$*m* ∗ (*c*^*k* ∗ *d*^*m*) $ *n*) {(*k*,*m*). *k* + *m* ≤ *n*} (**is** *?l* = *?r*)
proof −
let *?S* = {(*k*::*nat*, *m*::*nat*). *k* + *m* ≤ *n*}
have *s*: *?S* ⊆ {*0*..*n*} × {*0*..*n*} **by** (*simp add*: *subset-eq*)
have *f*: *finite* {(*k*::*nat*, *m*::*nat*). *k* + *m* ≤ *n*}

by (*auto intro*: *finite-subset*[*OF s*])
have *?r* = (∑ (*k*, *m*) ∈ {(*k*, *m*). *k* + *m* ≤ *n*}. ∑ *j* = *0*..*n*. *a* $ *k* * *b* $ *m* * (*c* ^ *k* $ *j* * *d* ^ *m* $ (*n* − *j*)))
by (*simp add*: *fps-mult-nth sum-distrib-left*)
also have ... = (∑ *i* = *0*..*n*. ∑ (*k*,*m*)∈{(*k*,*m*). *k*+*m* ≤ *n*}. *a* $ *k* * *c* ^ *k* $ *i* * *b* $ *m* * *d* ^ *m* $ (*n*−*i*))
unfolding *sum.swap* [**where** *A* = {*0*..*n*}] **by** (*auto simp add*: *field-simps intro*: *sum.cong*)
also have ... = (∑ *i* = *0*..*n*.
∑ *q* = *0*..*i*. ∑ *j* = *0*..*n* − *i*. *a* $ *q* * *c* ^ *q* $ *i* * (*b* $ *j* * *d* ^ *j* $ (*n* − *i*)))
apply (*rule sum.cong* [*OF refl*])
apply (*simp add*: *sum.cartesian-product mult.assoc*)
apply (*rule sum.mono-neutral-right*[*OF f*], *force*)
by *clarsimp* (*meson c0 d0 leI startsby-zero-power-prefix*)
also have ... = *?l*
by (*simp add*: *fps-mult-nth fps-compose-nth sum-product*)
finally show *?thesis* **by** *simp*
qed

lemma *sum-pair-less-iff*:
sum (λ((*k*::*nat*),*m*). *a k* * *b m* * *c* (*k* + *m*)) {(*k*,*m*). *k* + *m* ≤ *n*} =
sum (λ*s*. *sum* (λ*i*. *a i* * *b* (*s* − *i*) * *c s*) {*0*..*s*}) {*0*..*n*}
(**is** *?l* = *?r*)
proof −
have *th0*: {(*k*, *m*). *k* + *m* ≤ *n*} = (⋃ *s*∈{*0*..*n*}. ⋃ *i*∈{*0*..*s*}. {(*i*, *s* − *i*)})
by *auto*
show *?l* = *?r*
unfolding *th0*
by (*simp add*: *sum.UNION-disjoint eq-diff-iff disjoint-iff*)
qed

lemma *fps-compose-mult-distrib-lemma*:
assumes *c0*: *c*$*0* = (*0*::*'a*::*idom*)
shows ((*a oo c*) * (*b oo c*))$*n* = *sum* (λ*s*. *sum* (λ*i*. *a*$*i* * *b*$(*s* − *i*) * (*c*^*s*) $ *n*) {*0*..*s*}) {*0*..*n*}
unfolding *product-composition-lemma*[*OF c0 c0*] *power-add*[*symmetric*]
unfolding *sum-pair-less-iff*[**where** *a* = λ*k*. *a*$*k* **and** *b*=λ*m*. *b*$*m* **and** *c*=λ*s*. (*c* ^ *s*)$*n* **and** *n* = *n*] **..**

lemma *fps-compose-mult-distrib*:
assumes *c0*: *c* $ *0* = (*0*::*'a*::*idom*)
shows (*a* * *b*) *oo c* = (*a oo c*) * (*b oo c*)
proof (*clarsimp simp add*: *fps-eq-iff fps-compose-mult-distrib-lemma* [*OF c0*])
show (*a* * *b oo c*) $ *n* = (∑ *s* = *0*..*n*. ∑ *i* = *0*..*s*. *a* $ *i* * *b* $ (*s* − *i*) * *c* ^ *s* $ *n*) **for** *n*
by (*simp add*: *fps-compose-nth fps-mult-nth sum-distrib-right*)
qed

lemma *fps-compose-prod-distrib*:
assumes *c0*: *c$0 = (0::'a::idom)*
shows *prod a S oo c = prod (λk. a k oo c) S*
proof (*induct S rule*: *infinite-finite-induct*)
next
case (*insert*)
then show *?case*
by (*simp add*: *fps-compose-mult-distrib*[*OF c0*])
qed *auto*

lemma *fps-compose-divide*:
assumes [*simp*]: *g dvd f h $ 0 = 0*
shows *fps-compose f h = fps-compose (f / g :: 'a :: field fps) h * fps-compose g h*
proof −
have *f = (f / g) * g* **by** *simp*
also have *fps-compose ... h = fps-compose (f / g) h * fps-compose g h*
by (*subst fps-compose-mult-distrib*) *simp-all*
finally show *?thesis* **.**
qed

lemma *fps-compose-divide-distrib*:
assumes *g dvd f h $ 0 = 0 fps-compose g h ≠ 0*
shows *fps-compose (f / g :: 'a :: field fps) h = fps-compose f h / fps-compose g h*
using *fps-compose-divide*[*OF assms*(*1*,*2*)] *assms*(*3*) **by** *simp*

lemma *fps-compose-power*:
assumes *c0*: *c$0 = (0::'a::idom)*
shows *(a oo c)^n = a^n oo c*
proof (*cases n*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc m*)
have *(∏ n = 0..m. a) oo c = (∏ n = 0..m. a oo c)*
using *c0 fps-compose-prod-distrib* **by** *blast*
moreover have *th0*: *a^n = prod (λk. a) {0..m} (a oo c) ^ n = prod (λk. a oo c) {0..m}*
by (*simp-all add*: *prod-constant Suc*)
ultimately show *?thesis*
by *presburger*
qed

lemma *fps-compose-uminus*: *− (a::'a::ring-1 fps) oo c = − (a oo c)*
by (*simp add*: *fps-eq-iff fps-compose-nth field-simps sum-negf*[*symmetric*])

lemma *fps-compose-sub-distrib*: *(a − b) oo (c::'a::ring-1 fps) = (a oo c) − (b oo*

```
c)
  using fps-compose-add-distrib [of a − b c] by (simp add: fps-compose-uminus)

lemma fps-X-fps-compose: fps-X oo a = Abs-fps (λn. if n = 0 then (0::'a::comm-ring-1)
else a$n)
  by (simp add: fps-eq-iff fps-compose-nth mult-delta-left)

lemma fps-compose-eq-0-iff:
  fixes F G :: 'a :: idom fps
  assumes fps-nth G 0 = 0
  shows fps-compose F G = 0 ⟷ F = 0 ∨ (G = 0 ∧ fps-nth F 0 = 0)
proof safe
  assume ∗: fps-compose F G = 0 F ≠ 0
  have fps-nth (fps-compose F G) 0 = fps-nth F 0
    by simp
  also have fps-compose F G = 0
    by (simp add: ∗)
  finally show fps-nth F 0 = 0
    by simp
  show G = 0
  proof (rule ccontr)
    assume G ≠ 0
    hence subdegree G > 0 using assms
      using subdegree-eq-0-iff by blast
    define N where N = subdegree F ∗ subdegree G
    have fps-nth (fps-compose F G) N = (∑ i = 0..N. fps-nth F i ∗ fps-nth (G ^
i) N)
      unfolding fps-compose-def by (simp add: N-def)
    also have ... = (∑ i∈{subdegree F}. fps-nth F i ∗ fps-nth (G ^ i) N)
    proof (intro sum.mono-neutral-right ballI)
      fix i assume i: i ∈ {0..N} − {subdegree F}
      show fps-nth F i ∗ fps-nth (G ^ i) N = 0
      proof (cases i subdegree F rule: linorder-cases)
        assume i > subdegree F
        hence fps-nth (G ^ i) N = 0
          using i ‹subdegree G > 0› by (intro fps-pow-nth-below-subdegree) (auto
simp: N-def)
        thus ?thesis by simp
      qed (use i in ‹auto simp: N-def›)
    qed (use ‹subdegree G > 0› in ‹auto simp: N-def›)
    also have ... = fps-nth F (subdegree F) ∗ fps-nth (G ^ subdegree F) N
      by simp
    also have ... ≠ 0
      using ‹G ≠ 0› ‹F ≠ 0› by (auto simp: N-def)
    finally show False using ∗ by auto
  qed
qed auto

lemma subdegree-fps-compose [simp]:
```

fixes $F\ G :: 'a :: idom\ fps$
assumes [*simp*]: *fps-nth G 0 = 0*
shows *subdegree* (*fps-compose F G*) = *subdegree F* ∗ *subdegree G*
proof (*cases G = 0*; *cases F = 0*)
assume [*simp*]: $G \neq 0\ F \neq 0$
define m **where** m = *subdegree F*
define F' **where** F' = *fps-shift m F*
have *F-eq*: $F = F' * fps\text{-}X\ \hat{}\ m$
unfolding *F'-def* **by** (*simp add*: *fps-shift-times-fps-X-power m-def*)
have [*simp*]: $F' \neq 0$
using ‹$F \neq 0$› **unfolding** *F-eq* **by** *auto*
have *subdegree* (*fps-compose F G*) = *subdegree* (*fps-compose F' G*) + *m* ∗ *subdegree G*
by (*simp add*: *F-eq fps-compose-mult-distrib fps-compose-eq-0-iff flip*: *fps-compose-power*)
also have *subdegree* (*fps-compose F' G*) = *0*
by (*intro subdegree-eq-0*) (*auto simp*: *F'-def m-def*)
finally show *?thesis* **by** (*simp add*: *m-def*)
qed *auto*

lemma *fps-inverse-compose*:
assumes *b0*: (*b$0* :: 'a::*field*) = *0*
and *a0*: *a$0* ≠ *0*
shows *inverse a oo b* = *inverse* (*a oo b*)
proof −
let *?ia* = *inverse a*
let *?ab* = *a oo b*
let *?iab* = *inverse ?ab*

from *a0* **have** *ia0*: *?ia $ 0* ≠ *0* **by** *simp*
from *a0* **have** *ab0*: *?ab $ 0* ≠ *0* **by** (*simp add*: *fps-compose-def*)
have (*?ia oo b*) ∗ (*a oo b*) = *1*
unfolding *fps-compose-mult-distrib*[*OF b0*, *symmetric*]
unfolding *inverse-mult-eq-1*[*OF a0*]
fps-compose-1 **..**

then have (*?ia oo b*) ∗ (*a oo b*) ∗ *?iab* = *1* ∗ *?iab* **by** *simp*
then have (*?ia oo b*) ∗ (*?iab* ∗ (*a oo b*)) = *?iab* **by** *simp*
then show *?thesis* **unfolding** *inverse-mult-eq-1*[*OF ab0*] **by** *simp*
qed

lemma *fps-divide-compose*:
assumes *c0*: (*c$0* :: 'a::*field*) = *0*
and *b0*: *b$0* ≠ *0*
shows (*a*/*b*) *oo c* = (*a oo c*) / (*b oo c*)
using *b0 c0* **by** (*simp add*: *fps-divide-unit fps-inverse-compose fps-compose-mult-distrib*)

lemma *gp*:
assumes *a0*: *a$0* = (*0*::'a::*field*)
shows (*Abs-fps* (λ*n*. *1*)) *oo a* = *1*/(*1* − *a*)

```
    (is ?one oo a = -)
proof −
  have o0: ?one $ 0 ≠ 0 by simp
  have th0: (1 − fps-X) $ 0 ≠ (0::'a) by simp
  from fps-inverse-gp[where ?'a = 'a]
  have inverse ?one = 1 − fps-X by (simp add: fps-eq-iff)
  then have inverse (inverse ?one) = inverse (1 − fps-X) by simp
  then have th: ?one = 1/(1 − fps-X) unfolding fps-inverse-idempotent[OF o0]
    by (simp add: fps-divide-def)
  show ?thesis
    unfolding th
    unfolding fps-divide-compose[OF a0 th0]
    fps-compose-1 fps-compose-sub-distrib fps-X-fps-compose-startby0[OF a0] ..
qed

lemma fps-compose-radical:
  assumes b0: b$0 = (0::'a::field-char-0)
    and ra0: r (Suc k) (a$0) ^ Suc k = a$0
    and a0: a$0 ≠ 0
  shows fps-radical r (Suc k)  a oo b = fps-radical r (Suc k) (a oo b)
proof −
  let ?r = fps-radical r (Suc k)
  let ?ab = a oo b
  have ab0: ?ab $ 0 = a$0
    by (simp add: fps-compose-def)
  from ab0 a0 ra0 have rab0: ?ab $ 0 ≠ 0 r (Suc k) (?ab $ 0) ^ Suc k = ?ab $ 0
    by simp-all
  have th00: r (Suc k) ((a oo b) $ 0) = (fps-radical r (Suc k) a oo b) $ 0
    by (simp add: ab0 fps-compose-def)
  have th0: (?r a oo b) ^ (Suc k) = a  oo b
    unfolding fps-compose-power[OF b0]
    unfolding iffD1[OF power-radical[of a r k], OF a0 ra0]  ..
  from iffD1[OF radical-unique[where r=r and k=k and b= ?ab and a = ?r a
oo b, OF rab0(2) th00 rab0(1)], OF th0]
  show ?thesis  .
qed

lemma fps-const-mult-apply-left: fps-const c * (a oo b) = (fps-const c * a) oo b
  by (simp add: fps-eq-iff fps-compose-nth sum-distrib-left mult.assoc)

lemma fps-const-mult-apply-right:
  (a oo b) * fps-const (c::'a::comm-semiring-1) = (fps-const c * a) oo b
  by (simp add: fps-const-mult-apply-left mult.commute)

lemma fps-compose-assoc:
  assumes c0: c$0 = (0::'a::idom)
    and b0: b$0 = 0
  shows a oo (b oo c) = a oo b oo c (is ?l = ?r)
proof −
```

```
  have ?l$n = ?r$n for n
  proof −
    have ?l$n = (sum (λi. (fps-const (a$i) ∗ b^i) oo c) {0..n})$n
    by (simp add: fps-compose-nth fps-compose-power[OF c0] fps-const-mult-apply-left
        sum-distrib-left mult.assoc fps-sum-nth)
    also have ... = ((sum (λi. fps-const (a$i) ∗ b^i) {0..n}) oo c)$n
      by (simp add: fps-compose-sum-distrib)
    also have ... = (∑ i = 0..n. ∑ j = 0..n. a $ j ∗ (b ^ j $ i ∗ c ^ i $ n))
      by (simp add: fps-compose-nth fps-sum-nth sum-distrib-right mult.assoc)
    also have ... = (∑ i = 0..n. ∑ j = 0..i. a $ j ∗ (b ^ j $ i ∗ c ^ i $ n))
     by (intro sum.cong [OF refl] sum.mono-neutral-right; simp add: b0 startsby-zero-power-prefix)
    also have ... = ?r$n
      by (simp add: fps-compose-nth  sum-distrib-right mult.assoc)
    finally show ?thesis .
  qed
  then show ?thesis
    by (simp add: fps-eq-iff)
qed


lemma fps-X-power-compose:
  assumes a0: a$0=0
  shows fps-X^k oo a = (a::'a::idom fps)^k
  (is ?l = ?r)
proof (cases k)
  case 0
  then show ?thesis by simp
next
  case (Suc h)
  have ?l $ n = ?r $n for n
  proof −
    consider k > n | k ≤ n by arith
    then show ?thesis
    proof cases
      case 1
      then show ?thesis
        using a0 startsby-zero-power-prefix[OF a0] Suc
        by (simp add: fps-compose-nth del: power-Suc)
    next
      case 2
      then show ?thesis
        by (simp add: fps-compose-nth mult-delta-left)
    qed
  qed
  then show ?thesis
    unfolding fps-eq-iff by blast
qed

lemma fps-inv-right:
```

```
  assumes a0: a$0 = 0
    and a1: a$1 ≠ 0
  shows a oo fps-inv a = fps-X
proof −
  let ?ia = fps-inv a
  let ?iaa = a oo fps-inv a
  have th0: ?ia $ 0 = 0
    by (simp add: fps-inv-def)
  have th1: ?iaa $ 0 = 0
    using a0 a1 by (simp add: fps-inv-def fps-compose-nth)
  have th2: fps-X$0 = 0
    by simp
  from fps-inv[OF a0 a1] have a oo (fps-inv a oo a) = a oo fps-X
    by simp
  then have (a oo fps-inv a) oo a = fps-X oo a
     by (simp add: fps-compose-assoc[OF a0 th0] fps-X-fps-compose-startby0[OF
a0])
  with fps-compose-inj-right[OF a0 a1] show ?thesis
    by simp
qed

lemma fps-inv-deriv:
  assumes a0: a$0 = (0::'a::field)
    and a1: a$1 ≠ 0
  shows fps-deriv (fps-inv a) = inverse (fps-deriv a oo fps-inv a)
proof −
  let ?ia = fps-inv a
  let ?d = fps-deriv a oo ?ia
  let ?dia = fps-deriv ?ia
  have ia0: ?ia$0 = 0
    by (simp add: fps-inv-def)
  have th0: ?d$0 ≠ 0
    using a1 by (simp add: fps-compose-nth)
  from fps-inv-right[OF a0 a1] have ?d * ?dia = 1
    by (simp add: fps-compose-deriv[OF ia0, of a, symmetric] )
  then have inverse ?d * ?d * ?dia = inverse ?d * 1
    by simp
  with inverse-mult-eq-1 [OF th0] show ?dia = inverse ?d
    by simp
qed

lemma fps-inv-idempotent:
  assumes a0: a$0 = 0
    and a1: a$1 ≠ 0
  shows fps-inv (fps-inv a) = a
proof −
  let ?r = fps-inv
  have ra0: ?r a $ 0 = 0
    by (simp add: fps-inv-def)
```

```
  from a1 have ra1: ?r a $ 1 ≠ 0
    by (simp add: fps-inv-def field-simps)
  have fps-X0: fps-X$0 = 0
    by simp
  from fps-inv[OF ra0 ra1] have ?r (?r a) oo ?r a = fps-X .
  then have ?r (?r a) oo ?r a oo a = fps-X oo a
    by simp
  then have ?r (?r a) oo (?r a oo a) = a
    unfolding fps-X-fps-compose-startby0[OF a0]
    unfolding fps-compose-assoc[OF a0 ra0, symmetric] .
  then show ?thesis
    unfolding fps-inv[OF a0 a1] by simp
qed

lemma fps-ginv-ginv:
  assumes a0: a$0 = 0
    and a1: a$1 ≠ 0
    and c0: c$0 = 0
    and  c1: c$1 ≠ 0
  shows fps-ginv b (fps-ginv c a) = b oo a oo fps-inv c
proof −
  let ?r = fps-ginv
  from c0 have rca0: ?r c a $0 = 0
    by (simp add: fps-ginv-def)
  from a1 c1 have rca1: ?r c a $ 1 ≠ 0
    by (simp add: fps-ginv-def field-simps)
  from fps-ginv[OF rca0 rca1]
  have ?r b (?r c a) oo ?r c a = b .
  then have ?r b (?r c a) oo ?r c a oo a = b oo a
    by simp
  then have ?r b (?r c a) oo (?r c a oo a) = b oo a
    by (simp add: a0 fps-compose-assoc rca0)
  then have ?r b (?r c a) oo c = b oo a
    unfolding fps-ginv[OF a0 a1] .
  then have ?r b (?r c a) oo c oo fps-inv c= b oo a oo fps-inv c
    by simp
  then have ?r b (?r c a) oo (c oo fps-inv c) = b oo a oo fps-inv c
    by (metis c0 c1 fps-compose-assoc fps-compose-nth-0 fps-inv fps-inv-right)
  then show ?thesis
    unfolding fps-inv-right[OF c0 c1] by simp
qed

lemma fps-ginv-deriv:
  assumes a0:a$0 = (0::'a::field)
    and a1: a$1 ≠ 0
  shows fps-deriv (fps-ginv b a) = (fps-deriv b / fps-deriv a) oo fps-ginv fps-X a
proof −
  let ?ia = fps-ginv b a
  let ?ifps-Xa = fps-ginv fps-X a
```

let *?d = fps-deriv*
let *?dia = ?d ?ia*
have *ifps-Xa0*: *?ifps-Xa $ 0 = 0*
by (*simp add*: *fps-ginv-def*)
have *da0*: *?d a $ 0 ≠ 0*
using *a1* **by** *simp*
from *fps-ginv*[*OF a0 a1*, *of b*] **have** *?d* (*?ia oo a*) = *fps-deriv b*
by *simp*
then have (*?d ?ia oo a*) ∗ *?d a* = *?d b*
unfolding *fps-compose-deriv*[*OF a0*] **.**
then have (*?d ?ia oo a*) ∗ *?d a* ∗ *inverse* (*?d a*) = *?d b* ∗ *inverse* (*?d a*)
by *simp*
with *a1* **have** (*?d ?ia oo a*) ∗ (*inverse* (*?d a*) ∗ *?d a*) = *?d b* / *?d a*
by (*simp add*: *fps-divide-unit*)
then have (*?d ?ia oo a*) *oo ?ifps-Xa* = (*?d b* / *?d a*) *oo ?ifps-Xa*
unfolding *inverse-mult-eq-1*[*OF da0*] **by** *simp*
then have *?d ?ia oo* (*a oo ?ifps-Xa*) = (*?d b* / *?d a*) *oo ?ifps-Xa*
unfolding *fps-compose-assoc*[*OF ifps-Xa0 a0*] **.**
then show *?thesis* **unfolding** *fps-inv-ginv*[*symmetric*]
unfolding *fps-inv-right*[*OF a0 a1*] **by** *simp*
qed

lemma *fps-compose-linear*:
fps-compose (*f* :: *'a* :: *comm-ring-1 fps*) (*fps-const c* ∗ *fps-X*) = *Abs-fps* (λ*n*. *c*^*n* ∗ *f* $ *n*)
by (*simp add*: *fps-eq-iff fps-compose-def power-mult-distrib*
if-distrib cong: *if-cong*)

lemma *fps-compose-uminus'*:
fps-compose f (−*fps-X* :: *'a* :: *comm-ring-1 fps*) = *Abs-fps* (λ*n*. (−1)^*n* ∗ *f* $ *n*)
using *fps-compose-linear*[*of f* −1]
by (*simp only*: *fps-const-neg* [*symmetric*] *fps-const-1-eq-1*) *simp*
lemma *fps-nth-compose-linear* [*simp*]:
fixes *f* :: *'a* :: *comm-ring-1 fps*
shows *fps-nth* (*fps-compose f* (*fps-const c* ∗ *fps-X*)) *n* = *c* ^ *n* ∗ *fps-nth f n*
proof −
have *fps-nth* (*fps-compose f* (*fps-const c* ∗ *fps-X*)) *n* =
(∑ *i*∈{*n*}. *fps-nth f i* ∗ *fps-nth* ((*fps-const c* ∗ *fps-X*) ^ *i*) *n*)
unfolding *fps-compose-nth*
by (*intro sum.mono-neutral-cong-right*) (*auto simp*: *power-mult-distrib*)
also have ... = *c* ^ *n* ∗ *fps-nth f n*
by (*simp add*: *power-mult-distrib*)
finally show *?thesis* **.**
qed

5.19 Elementary series

5.19.1 Exponential series

definition *fps-exp x* = *Abs-fps* (λ*n*. *x*^*n* / *of-nat* (*fact n*))

lemma *fps-exp-deriv*[*simp*]: *fps-deriv* (*fps-exp a*) = *fps-const* (*a*::'*a*::*field-char-0*) ∗ *fps-exp a*
(**is** *?l* = *?r*)
proof −
have *?l*$*n* = *?r* $ *n* **for** *n*
using *of-nat-neq-0* **by** (*auto simp add*: *fps-exp-def divide-simps*)
then show *?thesis*
by (*simp add*: *fps-eq-iff*)
qed

lemma *fps-exp-unique-ODE*:
fps-deriv a = *fps-const c* ∗ *a* ⟷ *a* = *fps-const* (*a*$0) ∗ *fps-exp* (*c*::'*a*::*field-char-0*)
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?rhs* **if** *?lhs*
proof −
from *that* **have** *th*: ⋀*n*. *a* $ *Suc n* = *c* ∗ *a*$*n* / *of-nat* (*Suc n*)
by (*simp add*: *fps-deriv-def fps-eq-iff field-simps del*: *of-nat-Suc*)
have *th'*: *a*$*n* = *a*$*0* ∗ *c* ^ *n*/ (*fact n*) **for** *n*
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case *Suc*
then show *?case*
by (*simp add*: *th divide-simps*)
qed
show *?thesis*
by (*auto simp add*: *fps-eq-iff fps-const-mult-left fps-exp-def intro*: *th'*)
qed
show *?lhs* **if** *?rhs*
using *that* **by** (*metis fps-exp-deriv fps-deriv-mult-const-left mult.left-commute*)
qed

lemma *fps-exp-add-mult*: *fps-exp* (*a* + *b*) = *fps-exp* (*a*::'*a*::*field-char-0*) ∗ *fps-exp b* (**is** *?l* = *?r*)
proof −
have *fps-deriv ?r* = *fps-const* (*a* + *b*) ∗ *?r*
by (*simp add*: *fps-const-add*[*symmetric*] *field-simps del*: *fps-const-add*)
then have *?r* = *?l*
by (*simp only*: *fps-exp-unique-ODE*) (*simp add*: *fps-mult-nth fps-exp-def*)
then show *?thesis* **..**
qed

lemma *fps-exp-nth*[*simp*]: *fps-exp a* $ *n* = *a*^*n* / *of-nat* (*fact n*)
by (*simp add*: *fps-exp-def*)

lemma *fps-exp-0*[*simp*]: *fps-exp* (*0*::'*a*::*field*) = *1*

by (*simp add*: *fps-eq-iff power-0-left*)

lemma *fps-exp-neg*: *fps-exp* (− *a*) = *inverse* (*fps-exp* (*a*::′*a*::*field-char-0*))
proof −
from *fps-exp-add-mult*[*of a* − *a*] **have** *th0*: *fps-exp a* ∗ *fps-exp* (− *a*) = *1* **by** *simp*
from *fps-inverse-unique*[*OF th0*] **show** *?thesis* **by** *simp*
qed

lemma *fps-exp-nth-deriv*[*simp*]:
fps-nth-deriv n (*fps-exp* (*a*::′*a*::*field-char-0*)) = (*fps-const a*)^*n* ∗ (*fps-exp a*)
by (*induct n*) *auto*

lemma *fps-X-compose-fps-exp*[*simp*]: *fps-X oo fps-exp* (*a*::′*a*::*field*) = *fps-exp a* − *1*
by (*simp add*: *fps-eq-iff fps-X-fps-compose*)

lemma *fps-inv-fps-exp-compose*:
assumes *a*: *a* ≠ *0*
shows *fps-inv* (*fps-exp a* − *1*) *oo* (*fps-exp a* − *1*) = *fps-X*
and (*fps-exp a* − *1*) *oo fps-inv* (*fps-exp a* − *1*) = *fps-X*
proof −
let *?b* = *fps-exp a* − *1*
have *b0*: *?b* $ *0* = *0*
by *simp*
have *b1*: *?b* $ *1* ≠ *0*
by (*simp add*: *a*)
from *fps-inv*[*OF b0 b1*] **show** *fps-inv* (*fps-exp a* − *1*) *oo* (*fps-exp a* − *1*) = *fps-X*
.
from *fps-inv-right*[*OF b0 b1*] **show** (*fps-exp a* − *1*) *oo fps-inv* (*fps-exp a* − *1*) = *fps-X* .
qed

lemma *fps-exp-power-mult*: (*fps-exp* (*c*::′*a*::*field-char-0*))^*n* = *fps-exp* (*of-nat n* ∗ *c*)
by (*induct n*) (*simp-all add*: *field-simps fps-exp-add-mult*)

lemma *radical-fps-exp*:
assumes *r*: *r* (*Suc k*) *1* = *1*
shows *fps-radical r* (*Suc k*) (*fps-exp* (*c*::′*a*::*field-char-0*)) = *fps-exp* (*c* / *of-nat* (*Suc k*))
proof −
let *?ck* = (*c* / *of-nat* (*Suc k*))
let *?r* = *fps-radical r* (*Suc k*)
have *eq0*[*simp*]: *?ck* ∗ *of-nat* (*Suc k*) = *c of-nat* (*Suc k*) ∗ *?ck* = *c*
by (*simp-all del*: *of-nat-Suc*)
have *th0*: *fps-exp ?ck* ^ (*Suc k*) = *fps-exp c* **unfolding** *fps-exp-power-mult eq0*
..
have *th*: *r* (*Suc k*) (*fps-exp c* $*0*) ^ *Suc k* = *fps-exp c* $ *0*

r (Suc k) (fps-exp c $ 0) = fps-exp ?ck $ 0 fps-exp c $ 0 ≠ 0 **using** *r* **by** *simp-all*
from *th0 radical-unique*[**where** *r=r* **and** *k=k*, *OF th*] **show** *?thesis*
by *auto*
qed

lemma *fps-exp-compose-linear* [*simp*]:
fps-exp (*d*::*'a*::*field-char-0*) *oo* (*fps-const c * fps-X*) = *fps-exp* (*c * d*)
by (*simp add*: *fps-compose-linear fps-exp-def fps-eq-iff power-mult-distrib*)

lemma *fps-fps-exp-compose-minus* [*simp*]:
fps-compose (*fps-exp c*) (−*fps-X*) = *fps-exp* (−*c* :: *'a* :: *field-char-0*)
using *fps-exp-compose-linear*[*of c −1* :: *'a*]
unfolding *fps-const-neg* [*symmetric*] *fps-const-1-eq-1* **by** *simp*

lemma *fps-exp-eq-iff* [*simp*]: *fps-exp c = fps-exp d* ⟷ *c* = (*d* :: *'a* :: *field-char-0*)
proof
assume *fps-exp c = fps-exp d*
from *arg-cong*[*of - - λF. F $ 1*, *OF this*] **show** *c = d* **by** *simp*
qed *simp-all*

lemma *fps-exp-eq-fps-const-iff* [*simp*]:
fps-exp (*c* :: *'a* :: *field-char-0*) = *fps-const c'* ⟷ *c = 0 ∧ c' = 1*
proof
assume *c = 0 ∧ c' = 1*
thus *fps-exp c = fps-const c'* **by** (*simp add*: *fps-eq-iff*)
next
assume *fps-exp c = fps-const c'*
from *arg-cong*[*of - - λF. F $ 1*, *OF this*] *arg-cong*[*of - - λF. F $ 0*, *OF this*]
show *c = 0 ∧ c' = 1* **by** *simp-all*
qed

lemma *fps-exp-neq-0* [*simp*]: ¬*fps-exp* (*c* :: *'a* :: *field-char-0*) = *0*
unfolding *fps-const-0-eq-0* [*symmetric*] *fps-exp-eq-fps-const-iff* **by** *simp*

lemma *fps-exp-eq-1-iff* [*simp*]: *fps-exp* (*c* :: *'a* :: *field-char-0*) = *1* ⟷ *c = 0*
unfolding *fps-const-1-eq-1* [*symmetric*] *fps-exp-eq-fps-const-iff* **by** *simp*

lemma *fps-exp-neq-numeral-iff* [*simp*]:
fps-exp (*c* :: *'a* :: *field-char-0*) = *numeral n* ⟷ *c = 0 ∧ n = Num.One*
unfolding *numeral-fps-const fps-exp-eq-fps-const-iff* **by** *simp*

5.19.2 Logarithmic series

lemma *Abs-fps-if-0*:
Abs-fps (*λn. if n = 0 then* (*v*::*'a*::*ring-1*) *else f n*) =
*fps-const v + fps-X * Abs-fps* (*λn. f* (*Suc n*))
by (*simp add*: *fps-eq-iff*)

definition *fps-ln* :: *'a*::*field-char-0* ⇒ *'a fps*
where *fps-ln c* = *fps-const* (*1*/*c*) ∗ *Abs-fps* (λ*n*. *if n* = *0 then 0 else* (− *1*) ^ (*n* − *1*) / *of-nat n*)

lemma *fps-ln-deriv*: *fps-deriv* (*fps-ln c*) = *fps-const* (*1*/*c*) ∗ *inverse* (*1* + *fps-X*)
unfolding *fps-inverse-fps-X-plus1*
by (*simp add*: *fps-ln-def fps-eq-iff del*: *of-nat-Suc*)

lemma *fps-ln-nth*: *fps-ln c* $ *n* = (*if n* = *0 then 0 else 1*/*c* ∗ ((− *1*) ^ (*n* − *1*) / *of-nat n*))
by (*simp add*: *fps-ln-def field-simps*)

lemma *fps-ln-0* [*simp*]: *fps-ln c* $ *0* = *0* **by** (*simp add*: *fps-ln-def*)

lemma *fps-ln-fps-exp-inv*:
fixes *a* :: *'a*::*field-char-0*
assumes *a*: *a* ≠ *0*
shows *fps-ln a* = *fps-inv* (*fps-exp a* − *1*) (**is** *?l* = *?r*)
proof −
let *?b* = *fps-exp a* − *1*
have *b0*: *?b* $ *0* = *0* **by** *simp*
have *b1*: *?b* $ *1* ≠ *0* **by** (*simp add*: *a*)
have *fps-deriv* (*fps-exp a* − *1*) *oo fps-inv* (*fps-exp a* − *1*) =
(*fps-const a* ∗ (*fps-exp a* − *1*) + *fps-const a*) *oo fps-inv* (*fps-exp a* − *1*)
by (*simp add*: *field-simps*)
also have ... = *fps-const a* ∗ (*fps-X* + *1*)
by (*simp add*: *fps-compose-add-distrib fps-inv-right*[*OF b0 b1*] *distrib-left flip*: *fps-const-mult-apply-left*)
finally have *eq*: *fps-deriv* (*fps-exp a* − *1*) *oo fps-inv* (*fps-exp a* − *1*) = *fps-const a* ∗ (*fps-X* + *1*) .
from *fps-inv-deriv*[*OF b0 b1*, *unfolded eq*]
have *fps-deriv* (*fps-inv ?b*) = *fps-const* (*inverse a*) / (*fps-X* + *1*)
using *a* **by** (*simp add*: *fps-const-inverse eq fps-divide-def fps-inverse-mult*)
then have *fps-deriv ?l* = *fps-deriv ?r*
by (*simp add*: *fps-ln-deriv add.commute fps-divide-def divide-inverse*)
then show *?thesis* **unfolding** *fps-deriv-eq-iff*
by (*simp add*: *fps-ln-nth fps-inv-def*)
qed

lemma *fps-ln-mult-add*:
assumes *c0*: *c*≠*0*
and *d0*: *d*≠*0*
shows *fps-ln c* + *fps-ln d* = *fps-const* (*c*+*d*) ∗ *fps-ln* (*c*∗*d*)
(**is** *?r* = *?l*)
proof−
from *c0 d0* **have** *eq*: *1*/*c* + *1*/*d* = (*c*+*d*)/(*c*∗*d*) **by** (*simp add*: *field-simps*)
have *fps-deriv ?r* = *fps-const* (*1*/*c* + *1*/*d*) ∗ *inverse* (*1* + *fps-X*)
by (*simp add*: *fps-ln-deriv fps-const-add*[*symmetric*] *algebra-simps del*: *fps-const-add*)
also have ... = *fps-deriv ?l*

by (*simp add*: *eq fps-ln-deriv*)
finally show *?thesis*
unfolding *fps-deriv-eq-iff* **by** *simp*
qed

lemma *fps-X-dvd-fps-ln* [*simp*]: *fps-X dvd fps-ln c*
proof −
have *fps-ln c* = *fps-X* ∗ *Abs-fps* (λ*n*. (−*1*) ^ *n* / (*of-nat* (*Suc n*) ∗ *c*))
by (*intro fps-ext*) (*simp add*: *fps-ln-def of-nat-diff*)
thus *?thesis* **by** *simp*
qed

5.19.3 Binomial series

definition *fps-binomial a* = *Abs-fps* (λ*n*. *a gchoose n*)

lemma *fps-binomial-nth*[*simp*]: *fps-binomial a* $ *n* = *a gchoose n*
by (*simp add*: *fps-binomial-def*)

lemma *fps-binomial-ODE-unique*:
fixes *c* :: *'a*::*field-char-0*
shows *fps-deriv a* = (*fps-const c* ∗ *a*) / (*1* + *fps-X*) ⟷ *a* = *fps-const* (*a*$*0*) ∗ *fps-binomial c*
(**is** *?lhs* ⟷ *?rhs*)
proof
let *?da* = *fps-deriv a*
let *?x1* = (*1* + *fps-X*):: *'a fps*
let *?l* = *?x1* ∗ *?da*
let *?r* = *fps-const c* ∗ *a*

have *eq*: *?l* = *?r* ⟷ *?lhs*
proof −
have *x10*: *?x1* $ *0* ≠ *0* **by** *simp*
have *?l* = *?r* ⟷ *inverse ?x1* ∗ *?l* = *inverse ?x1* ∗ *?r* **by** *simp*
also have ... ⟷ *?da* = (*fps-const c* ∗ *a*) / *?x1*
unfolding *fps-divide-def mult.assoc*[*symmetric*] *inverse-mult-eq-1*[*OF x10*]
by (*simp add*: *field-simps*)
finally show *?thesis* **.**
qed

show *?rhs* **if** *?lhs*
proof −
from *eq that* **have** *h*: *?l* = *?r* **..**
have *th0*: *a*$ *Suc n* = ((*c* − *of-nat n*) / *of-nat* (*Suc n*)) ∗ *a* $*n* **for** *n*
proof −
from *h* **have** *?l* $ *n* = *?r* $ *n* **by** *simp*
then show *?thesis*
by (*simp add*: *field-simps del*: *of-nat-Suc split*: *if-split-asm*)
qed

```
    have th1: a $ n = (c gchoose n) * a $ 0 for n
    proof (induct n)
      case 0
      then show ?case by simp
    next
      case (Suc m)
      have (c - of-nat m) * (c gchoose m) = (c gchoose Suc m) * of-nat (Suc m)
        by (metis gbinomial-absorb-comp gbinomial-absorption mult.commute)
      with Suc show ?case
        unfolding th0
        by (simp add: divide-simps del: of-nat-Suc)
    qed
    show ?thesis
    by (metis expand-fps-eq fps-binomial-nth fps-mult-right-const-nth mult.commute
th1)
  qed

  show ?lhs if ?rhs
  proof -
    have th00: x * (a $ 0 * y) = a $ 0 * (x * y) for x y
      by (simp add: mult.commute)
    have ?l = (1 + fps-X) * fps-deriv (fps-const (a $ 0) * fps-binomial c)
      using that by auto
    also have ... = fps-const c * (fps-const (a $ 0) * fps-binomial c)
    proof (clarsimp simp add: fps-eq-iff algebra-simps)
       show a $ 0 * (c gchoose Suc n) + (of-nat n * ((c gchoose n) * a $ 0) +
of-nat n * (a $ 0 * (c gchoose Suc n)))
         = c * ((c gchoose n) * a $ 0) for n
      unfolding mult.assoc[symmetric]
      by (simp add: field-simps gbinomial-mult-1)
  qed
    also have ... = ?r
      using that by auto
    finally have ?l = ?r .
    with eq show ?thesis ..
  qed
qed

lemma fps-binomial-ODE-unique':
  (fps-deriv a = fps-const c * a / (1 + fps-X) ∧ a $ 0 = 1) ⟷ (a = fps-binomial
c)
  by (subst fps-binomial-ODE-unique) auto

lemma fps-binomial-deriv: fps-deriv (fps-binomial c) = fps-const c * fps-binomial
c / (1 + fps-X)
proof -
  let ?a = fps-binomial c
  have th0: ?a = fps-const (?a$0) * ?a by (simp)
  from iffD2[OF fps-binomial-ODE-unique, OF th0] show ?thesis .
```

qed

lemma *fps-binomial-add-mult*: *fps-binomial* (*c*+*d*) = *fps-binomial c* ∗ *fps-binomial d* (**is** *?l* = *?r*)
proof −
  let *?P* = *?r* − *?l*
  let *?b* = *fps-binomial*
  let *?db* = λ*x*. *fps-deriv* (*?b x*)
  have *fps-deriv ?P* = *?db c* ∗ *?b d* + *?b c* ∗ *?db d* − *?db* (*c* + *d*) **by** *simp*
  also have ... = *inverse* (1 + *fps-X*) ∗
    (*fps-const c* ∗ *?b c* ∗ *?b d* + *fps-const d* ∗ *?b c* ∗ *?b d* − *fps-const* (*c*+*d*) ∗ *?b* (*c* + *d*))
    unfolding *fps-binomial-deriv*
    by (*simp add*: *fps-divide-def field-simps*)
  also have ... = (*fps-const* (*c* + *d*)/ (1 + *fps-X*)) ∗ *?P*
   by (*simp add*: *field-simps fps-divide-unit fps-const-add*[*symmetric*] *del*: *fps-const-add*)
  finally have *th0*: *fps-deriv ?P* = *fps-const* (*c*+*d*) ∗ *?P* / (1 + *fps-X*)
    by (*simp add*: *fps-divide-def*)
  have *?P* = *fps-const* (*?P*$*0*) ∗ *?b* (*c* + *d*)
    unfolding *fps-binomial-ODE-unique*[*symmetric*]
    using *th0* **by** *simp*
  then have *?P* = *0* **by** (*simp add*: *fps-mult-nth*)
  then show *?thesis* **by** *simp*
qed

lemma *fps-binomial-minus-one*: *fps-binomial* (− 1) = *inverse* (1 + *fps-X*)
  (**is** *?l* = *inverse ?r*)
proof−
  have *th*: *?r*$*0* ≠ *0* **by** *simp*
  have *th'*: *fps-deriv* (*inverse ?r*) = *fps-const* (− 1) ∗ *inverse ?r* / (1 + *fps-X*)
    by (*simp add*: *fps-inverse-deriv*[*OF th*] *fps-divide-def*
      power2-eq-square mult.commute fps-const-neg[*symmetric*] *del*: *fps-const-neg*)
  have *eq*: *inverse ?r* $ *0* = *1*
    by (*simp add*: *fps-inverse-def*)
  from *iffD1*[*OF fps-binomial-ODE-unique*[*of inverse* (1 + *fps-X*) − 1] *th'*] *eq*
  show *?thesis* **by** (*simp add*: *fps-inverse-def*)
qed

lemma *fps-binomial-of-nat*: *fps-binomial* (*of-nat n*) = (*1* + *fps-X* :: *'a* :: *field-char-0 fps*) ^ *n*
proof (*cases n* = *0*)
  case [*simp*]: *True*
  have *fps-deriv* ((*1* + *fps-X*) ^ *n* :: *'a fps*) = *0* **by** *simp*
  also have ... = *fps-const* (*of-nat n*) ∗ (*1* + *fps-X*) ^ *n* / (*1* + *fps-X*) **by** (*simp add*: *fps-binomial-def*)
 finally show *?thesis* **by** (*subst sym*, *subst fps-binomial-ODE-unique'* [*symmetric*]) *simp-all*
next
  case *False*

have *fps-deriv* ((*1* + *fps-X*) ^ *n* :: *'a fps*) = *fps-const* (*of-nat n*) * (*1* + *fps-X*) ^ (*n* − *1*)
by (*simp add*: *fps-deriv-power*)
also have (*1* + *fps-X* :: *'a fps*) $ *0* ≠ *0* **by** *simp*
hence (*1* + *fps-X* :: *'a fps*) ≠ *0* **by** (*intro notI*) (*simp only*: , *simp*)
with *False* **have** (*1* + *fps-X* :: *'a fps*) ^ (*n* − *1*) = (*1* + *fps-X*) ^ *n* / (*1* + *fps-X*)
by (*cases n*) (*simp-all*)
also have *fps-const* (*of-nat n* :: *'a*) * ((*1* + *fps-X*) ^ *n* / (*1* + *fps-X*)) =
fps-const (*of-nat n*) * (*1* + *fps-X*) ^ *n* / (*1* + *fps-X*)
by (*simp add*: *unit-div-mult-swap*)
finally show *?thesis*
by (*subst sym*, *subst fps-binomial-ODE-unique'* [*symmetric*]) (*simp-all add*: *fps-power-nth*)
qed

lemma *fps-binomial-0* [*simp*]: *fps-binomial 0* = *1*
using *fps-binomial-of-nat*[*of 0*] **by** *simp*

lemma *fps-binomial-power*: *fps-binomial a* ^ *n* = *fps-binomial* (*of-nat n* * *a*)
by (*induction n*) (*simp-all add*: *fps-binomial-add-mult ring-distribs*)

lemma *fps-binomial-1*: *fps-binomial 1* = *1* + *fps-X*
using *fps-binomial-of-nat*[*of 1*] **by** *simp*

lemma *fps-binomial-minus-of-nat*:
fps-binomial (− *of-nat n*) = *inverse* ((*1* + *fps-X* :: *'a* :: *field-char-0 fps*) ^ *n*)
by (*rule sym*, *rule fps-inverse-unique*)
(*simp add*: *fps-binomial-of-nat* [*symmetric*] *fps-binomial-add-mult* [*symmetric*])

lemma *one-minus-const-fps-X-power*:
c ≠ *0* ⟹ (*1* − *fps-const c* * *fps-X*) ^ *n* =
fps-compose (*fps-binomial* (*of-nat n*)) (−*fps-const c* * *fps-X*)
by (*subst fps-binomial-of-nat*)
(*simp add*: *fps-compose-power* [*symmetric*] *fps-compose-add-distrib fps-const-neg* [*symmetric*]
del: *fps-const-neg*)

lemma *one-minus-fps-X-const-neg-power*:
inverse ((*1* − *fps-const c* * *fps-X*) ^ *n*) =
fps-compose (*fps-binomial* (−*of-nat n*)) (−*fps-const c* * *fps-X*)
proof (*cases c* = *0*)
case *False*
thus *?thesis*
by (*subst fps-binomial-minus-of-nat*)
(*simp add*: *fps-compose-power* [*symmetric*] *fps-inverse-compose fps-compose-add-distrib*
fps-const-neg [*symmetric*] *del*: *fps-const-neg*)
qed *simp*

lemma *fps-X-plus-const-power*:
  $c \neq 0 \Longrightarrow (\mathit{fps\text{-}X} + \mathit{fps\text{-}const}\ c) \hat{}\ n =$
    fps-const $(c\hat{}n) *$ *fps-compose* (*fps-binomial* (*of-nat* n)) (*fps-const* (*inverse* c)
$*$ *fps-X*)
  by (*subst fps-binomial-of-nat*)
   (*simp add*: *fps-compose-power* [*symmetric*] *fps-binomial-of-nat fps-compose-add-distrib*
            fps-const-power [*symmetric*] *power-mult-distrib* [*symmetric*]
            algebra-simps inverse-mult-eq-1' *del*: *fps-const-power*)

lemma *fps-X-plus-const-neg-power*:
  $c \neq 0 \Longrightarrow$ *inverse* $((\mathit{fps\text{-}X} + \mathit{fps\text{-}const}\ c) \hat{}\ n) =$
     fps-const (*inverse* $c\hat{}n$) $*$ *fps-compose* (*fps-binomial* ($-$*of-nat* n)) (*fps-const*
(*inverse* c) $*$ *fps-X*)
  by (*subst fps-binomial-minus-of-nat*)
   (*simp add*: *fps-compose-power* [*symmetric*] *fps-binomial-of-nat fps-compose-add-distrib*
       fps-const-power [*symmetric*] *power-mult-distrib* [*symmetric*] *fps-inverse-compose*

        algebra-simps fps-const-inverse [*symmetric*] *fps-inverse-mult* [*symmetric*]
          fps-inverse-power [*symmetric*] *inverse-mult-eq-1'*
       del: *fps-const-power*)

lemma *one-minus-const-fps-X-neg-power'*:
  fixes $c :: 'a :: \mathit{field\text{-}char\text{-}0}$
  assumes $n > 0$
  shows *inverse* $((1 - \mathit{fps\text{-}const}\ c * \mathit{fps\text{-}X}) \hat{}\ n) = \mathit{Abs\text{-}fps}\ (\lambda k.\ \mathit{of\text{-}nat}\ ((n + k -$
$1)$ *choose* $k) * c\hat{}k)$
proof $-$
  have §: $\bigwedge j.\ \mathit{Abs\text{-}fps}\ (\lambda na.\ (-\ c) \hat{}\ na * \mathit{fps\text{-}binomial}\ (-\ \mathit{of\text{-}nat}\ n)\ \$\ na)\ \$\ j =$
       Abs-fps $(\lambda k.$ *of-nat* $(n + k - 1$ *choose* $k) * c \hat{}\ k)\ \$\ j$
    using *assms*
   by (*simp add*: *gbinomial-minus binomial-gbinomial of-nat-diff flip*: *power-mult-distrib*
mult.assoc)
  show *?thesis*
    apply (*rule fps-ext*)
    using §
     by (*metis* (*no-types*, *lifting*) *one-minus-fps-X-const-neg-power fps-const-neg*
fps-compose-linear fps-nth-Abs-fps)
qed

Vandermonde's Identity as a consequence.

lemma *gbinomial-Vandermonde*:
  sum $(\lambda k.$ (a *gchoose* $k) *$ (b *gchoose* $(n - k))) \{0..n\} = (a + b)$ *gchoose* n
proof $-$
  let *?ba* $=$ *fps-binomial* a
  let *?bb* $=$ *fps-binomial* b
  let *?bab* $=$ *fps-binomial* $(a + b)$
  from *fps-binomial-add-mult*[*of* a b] **have** *?bab* \$ $n = (?ba * ?bb)\$n$ **by** *simp*
  then show *?thesis* **by** (*simp add*: *fps-mult-nth*)

qed

lemma *binomial-Vandermonde*:
sum (λ*k*. (*a choose k*) ∗ (*b choose* (*n* − *k*))) {*0*..*n*} = (*a* + *b*) *choose n*
using *gbinomial-Vandermonde*[*of* (*of-nat a*) *of-nat b n*]
by (*simp only*: *binomial-gbinomial*[*symmetric*] *of-nat-mult*[*symmetric*]
of-nat-sum[*symmetric*] *of-nat-add*[*symmetric*] *of-nat-eq-iff*)

lemma *binomial-Vandermonde-same*: *sum* (λ*k*. (*n choose k*)^2) {*0*..*n*} = (*2* ∗ *n*) *choose n*
using *binomial-Vandermonde*[*of n n n*, *symmetric*]
unfolding *mult-2*
by (*metis atMost-atLeast0 choose-square-sum mult-2*)

lemma *Vandermonde-pochhammer-lemma*:
fixes *a* :: *'a*::*field-char-0*
assumes *b*: ⋀*j*. *j*<*n* ⟹ *b* ≠ *of-nat j*
shows *sum* (λ*k*. (*pochhammer* (− *a*) *k* ∗ *pochhammer* (− (*of-nat n*)) *k*) /
(*of-nat* (*fact k*) ∗ *pochhammer* (*b* − *of-nat n* + *1*) *k*)) {*0*..*n*} =
pochhammer (− (*a* + *b*)) *n* / *pochhammer* (− *b*) *n*
(**is** *?l* = *?r*)
proof −
let *?m1* = λ*m*. (− *1* :: *'a*) ^ *m*
let *?f* = λ*m*. *of-nat* (*fact m*)
let *?p* = λ(*x*::*'a*). *pochhammer* (− *x*)
from *b* **have** *bn0*: *?p b n* ≠ *0*
unfolding *pochhammer-eq-0-iff* **by** *simp*
have *th00*:
b gchoose (*n* − *k*) =
(*?m1 n* ∗ *?p b n* ∗ *?m1 k* ∗ *?p* (*of-nat n*) *k*) / (*?f n* ∗ *pochhammer* (*b* − *of-nat n* + *1*) *k*)
(**is** *?gchoose*)
pochhammer (*1* + *b* − *of-nat n*) *k* ≠ *0*
(**is** *?pochhammer*)
if *kn*: *k* ∈ {*0*..*n*} **for** *k*
proof −
from *kn* **have** *k* ≤ *n* **by** *simp*
have *nz*: *pochhammer* (*1* + *b* − *of-nat n*) *n* ≠ *0*
proof
assume *pochhammer* (*1* + *b* − *of-nat n*) *n* = *0*
then have *c*: *pochhammer* (*b* − *of-nat n* + *1*) *n* = *0*
by (*simp add*: *algebra-simps*)
then obtain *j* **where** *j*: *j* < *n b* − *of-nat n* + *1* = − *of-nat j*
unfolding *pochhammer-eq-0-iff* **by** *blast*
from *j* **have** *b* = *of-nat n* − *of-nat j* − *of-nat 1*
by (*simp add*: *algebra-simps*)
then show *False*
using ‹*j* < *n*› *j b*
by (*metis bn0 c mult-cancel-right2 pochhammer-minus*)

```
  qed

  from nz kn [simplified] have nz': pochhammer (1 + b − of-nat n) k ≠ 0
    by (rule pochhammer-neq-0-mono)

  consider k = 0 ∨ n = 0 | k ≠ 0 n ≠ 0
    by blast
  then have b gchoose (n − k) =
   (?m1 n * ?p b n * ?m1 k * ?p (of-nat n) k) / (?f n * pochhammer (b − of-nat
n + 1) k)
  proof cases
    case 1
    then show ?thesis
      using kn by (cases k = 0) (simp-all add: gbinomial-pochhammer)
  next
    case neq: 2
    then obtain m where m: n = Suc m
      by (cases n) auto
    from neq(1) obtain h where h: k = Suc h
      by (cases k) auto
    show ?thesis
    proof (cases k = n)
      case True
      with pochhammer-minus'[where k=k and b=b] bn0 show ?thesis
        by (simp add: pochhammer-same)
    next
      case False
      with kn have kn': k < n
        by simp
      have h ≤ m
        using ‹k ≤ n› h m by blast
      have m1nk: ?m1 n = prod (λi. − 1) {..m} ?m1 k = prod (λi. − 1) {0..h}
        by (simp-all add: m h)
      have bnz0: pochhammer (b − of-nat n + 1) k ≠ 0
        using bn0 kn
        unfolding pochhammer-eq-0-iff
        by (metis add.commute add-diff-eq nz' pochhammer-eq-0-iff)
      have eq1: prod (λk. (1::'a) + of-nat m − of-nat k) {..h} =
        prod of-nat {Suc (m − h) .. Suc m}
        using kn' h m
       by (intro prod.reindex-bij-witness[where i=λk. Suc m − k and j=λk. Suc
m − k])
          (auto simp: of-nat-diff)
       have (∏ i = 0..<k. 1 + of-nat n − of-nat k + of-nat i) = (∏ x = n −
k..<n. (1::'a) + of-nat x)
        using ‹k ≤ n›
          using prod.atLeastLessThan-shift-bounds [where ?'a = 'a, of λi. 1 +
of-nat i 0 n − k k]
        by (auto simp add: of-nat-diff field-simps)
```

```
      then have fact (n − k) ∗ pochhammer ((1::'a) + of-nat n − of-nat k) k =
fact n
        using ‹k ≤ n›
        by (auto simp add: fact-split [of k n] pochhammer-prod field-simps)
      then have th1: (?m1 k ∗ ?p (of-nat n) k) / ?f n = 1 / of-nat(fact (n − k))
        by (simp add: pochhammer-minus field-simps)
       have ?m1 n ∗ ?p b n = pochhammer (b − of-nat m) (Suc m)
        by (simp add: pochhammer-minus field-simps m)
       also have ... = (∏ i = 0..m. b − of-nat i)
      by (auto simp add: pochhammer-prod-rev of-nat-diff prod.atLeast-Suc-atMost-Suc-shift
simp del: prod.cl-ivl-Suc)
       finally have th20: ?m1 n ∗ ?p b n = prod (λi. b − of-nat i) {0..m} .
       have (∏ x = 0..h. b − of-nat m + of-nat (h − x)) = (∏ i = m − h..m. b
− of-nat i)
        using ‹h ≤ m› prod.atLeastAtMost-shift-0 [of m − h m, where ?'a = 'a]
        by (auto simp add: of-nat-diff field-simps)
       then have th21:pochhammer (b − of-nat n + 1) k = prod (λi. b − of-nat
i) {n − k .. n − 1}
      using kn by (simp add: pochhammer-prod-rev m h prod.atLeast-Suc-atMost-Suc-shift
del: prod.op-ivl-Suc del: prod.cl-ivl-Suc)
       have ?m1 n ∗ ?p b n =
         prod (λi. b − of-nat i) {0.. n − k − 1} ∗ pochhammer (b − of-nat n +
1) k
        using kn' m h unfolding th20 th21
        by (auto simp flip: prod.union-disjoint intro: prod.cong)
       then have th2: (?m1 n ∗ ?p b n)/pochhammer (b − of-nat n + 1) k =
        prod (λi. b − of-nat i) {0.. n − k − 1}
        using nz' by (simp add: field-simps)
      have (?m1 n ∗ ?p b n ∗ ?m1 k ∗ ?p (of-nat n) k) / (?f n ∗ pochhammer (b
− of-nat n + 1) k) =
        ((?m1 k ∗ ?p (of-nat n) k) / ?f n) ∗ ((?m1 n ∗ ?p b n)/pochhammer (b −
of-nat n + 1) k)
        using bnz0
        by (simp add: field-simps)
       also have ... = b gchoose (n − k)
        unfolding th1 th2
        using kn' m h
        by (auto simp: field-simps gbinomial-mult-fact intro: prod.cong)
       finally show ?thesis by simp
     qed
   qed
   then show ?gchoose and ?pochhammer
     using nz' by force+
 qed
 have ?r = ((a + b) gchoose n) ∗ (of-nat (fact n) / (?m1 n ∗ pochhammer (− b)
n))
   unfolding gbinomial-pochhammer
   using bn0 by (auto simp add: field-simps)
 also have ... = ?l
```

```
    using bn0
    unfolding gbinomial-Vandermonde[symmetric]
    apply (simp add: th00)
   by (simp add: gbinomial-pochhammer sum-distrib-right sum-distrib-left field-simps)
  finally show ?thesis by simp
qed

lemma Vandermonde-pochhammer:
  fixes a :: 'a::field-char-0
  assumes c: ∀ i ∈ {0..< n}. c ≠ − of-nat i
  shows sum (λk. (pochhammer a k * pochhammer (− (of-nat n)) k) /
   (of-nat (fact k) * pochhammer c k)) {0..n} = pochhammer (c − a) n / pochhammer c n
proof −
  let ?a = − a
  let ?b = c + of-nat n − 1
  have h: ?b ≠ of-nat j if j < n for j
  proof −
    have c ≠ − of-nat (n − j − 1)
      using c that by (auto simp: dest!: bspec [where x = n−j−1])
    with that show ?thesis
      by (auto simp add: algebra-simps of-nat-diff)
  qed
  have th0: pochhammer (− (?a + ?b)) n = (− 1)^n * pochhammer (c − a) n
    unfolding pochhammer-minus
    by (simp add: algebra-simps)
  have th1: pochhammer (− ?b) n = (− 1)^n * pochhammer c n
    unfolding pochhammer-minus
    by simp
  have nz: pochhammer c n ≠ 0 using c
    by (simp add: pochhammer-eq-0-iff)
 from Vandermonde-pochhammer-lemma[where a = ?a and b=?b and n=n, OF h, unfolded th0 th1]
  show ?thesis
    using nz by (simp add: field-simps sum-distrib-left)
qed
```

5.19.4 Trigonometric functions

```
definition fps-sin (c::'a::field-char-0) =
  Abs-fps (λn. if even n then 0 else (− 1) ^((n − 1) div 2) * c^n /(of-nat (fact n)))

definition fps-cos (c::'a::field-char-0) =
  Abs-fps (λn. if even n then (− 1) ^(n div 2) * c^n / (of-nat (fact n)) else 0)

lemma fps-sin-0 [simp]: fps-sin 0 = 0
  by (intro fps-ext) (auto simp: fps-sin-def elim!: oddE)
```

lemma *fps-cos-0* [*simp*]: *fps-cos 0 = 1*
by (*intro fps-ext*) (*simp add*: *fps-cos-def*)

lemma *fps-sin-deriv*:
fps-deriv (*fps-sin c*) = *fps-const c* ∗ *fps-cos c*
(**is** *?lhs* = *?rhs*)
proof (*rule fps-ext*)
fix *n* :: *nat*
show *?lhs* $ *n* = *?rhs* $ *n*
proof (*cases even n*)
case *True*
have *?lhs*$*n* = *of-nat* (*n*+1) ∗ (*fps-sin c* $ (*n*+1)) **by** *simp*
also have ... = *of-nat* (*n*+1) ∗ ((− 1)^(*n div 2*) ∗ *c*^*Suc n* / *of-nat* (*fact* (*Suc n*)))
using *True* **by** (*simp add*: *fps-sin-def*)
also have ... = (− 1)^(*n div 2*) ∗ *c*^*Suc n* ∗ (*of-nat* (*n*+1) / (*of-nat* (*Suc n*) ∗ *of-nat* (*fact n*)))
unfolding *fact-Suc of-nat-mult*
by (*simp add*: *field-simps del*: *of-nat-add of-nat-Suc*)
also have ... = (− 1)^(*n div 2*) ∗ *c*^*Suc n* / *of-nat* (*fact n*)
by (*simp add*: *field-simps del*: *of-nat-add of-nat-Suc*)
finally show *?thesis*
using *True* **by** (*simp add*: *fps-cos-def field-simps*)
next
case *False*
then show *?thesis*
by (*simp-all add*: *fps-deriv-def fps-sin-def fps-cos-def*)
qed
qed

lemma *fps-cos-deriv*: *fps-deriv* (*fps-cos c*) = *fps-const* (− *c*)∗ (*fps-sin c*)
(**is** *?lhs* = *?rhs*)
proof (*rule fps-ext*)
have *th0*: − ((− 1::'*a*) ^ *n*) = (− 1)^*Suc n* **for** *n*
by *simp*
show *?lhs* $ *n* = *?rhs* $ *n* **for** *n*
proof (*cases even n*)
case *False*
then have *n0*: *n* ≠ *0* **by** *presburger*
from *False* **have** *th1*: *Suc* ((*n* − *1*) *div 2*) = *Suc n div 2*
by (*cases n*) *simp-all*
have *?lhs*$*n* = *of-nat* (*n*+1) ∗ (*fps-cos c* $ (*n*+1)) **by** *simp*
also have ... = *of-nat* (*n*+1) ∗ ((− 1)^((*n* + *1*) *div 2*) ∗ *c*^*Suc n* / *of-nat* (*fact* (*Suc n*)))
using *False* **by** (*simp add*: *fps-cos-def*)
also have ... = (− 1)^((*n* + *1*) *div 2*) ∗ *c*^*Suc n* ∗ (*of-nat* (*n*+1) / (*of-nat* (*Suc n*) ∗ *of-nat* (*fact n*)))
unfolding *fact-Suc of-nat-mult*
by (*simp add*: *field-simps del*: *of-nat-add of-nat-Suc*)

```
    also have ... = (- 1)^((n + 1) div 2) * c^Suc n / of-nat (fact n)
      by (simp add: field-simps del: of-nat-add of-nat-Suc)
    also have ... = (- ((- 1)^((n - 1) div 2))) * c^Suc n / of-nat (fact n)
      unfolding th0 unfolding th1 by simp
    finally show ?thesis
      using False by (simp add: fps-sin-def field-simps)
  next
    case True
    then show ?thesis
      by (simp-all add: fps-deriv-def fps-sin-def fps-cos-def)
  qed
qed

lemma fps-sin-cos-sum-of-squares: (fps-cos c)^2 + (fps-sin c)^2 = 1
  (is ?lhs = -)
proof -
  have fps-deriv ?lhs = 0
   by (simp add: fps-deriv-power fps-sin-deriv fps-cos-deriv field-simps flip: fps-const-neg)
  then have ?lhs = fps-const (?lhs $ 0)
    unfolding fps-deriv-eq-0-iff .
  also have ... = 1
    by (simp add: fps-eq-iff numeral-2-eq-2 fps-mult-nth fps-cos-def fps-sin-def)
  finally show ?thesis .
qed

lemma fps-sin-nth-0 [simp]: fps-sin c $ 0 = 0
  unfolding fps-sin-def by simp

lemma fps-sin-nth-1 [simp]: fps-sin c $ Suc 0 = c
  unfolding fps-sin-def by simp

lemma fps-sin-nth-add-2:
    fps-sin c $ (n + 2) = - (c * c * fps-sin c $ n / (of-nat (n + 1) * of-nat (n +
2)))
proof (cases n)
  case (Suc n')
  then show ?thesis
    unfolding fps-sin-def by (simp add: field-simps)
qed (auto simp: fps-sin-def)


lemma fps-cos-nth-0 [simp]: fps-cos c $ 0 = 1
  unfolding fps-cos-def by simp

lemma fps-cos-nth-1 [simp]: fps-cos c $ Suc 0 = 0
  unfolding fps-cos-def by simp

lemma fps-cos-nth-add-2:
  fps-cos c $ (n + 2) = - (c * c * fps-cos c $ n / (of-nat (n + 1) * of-nat (n +
```

2)))
proof (*cases n*)
  case (*Suc n'*)
  then show *?thesis*
    unfolding *fps-cos-def* **by** (*simp add*: *field-simps*)
qed (*auto simp*: *fps-cos-def*)

lemma *nat-add-1-add-1*: (*n*::*nat*) + *1* + *1* = *n* + *2*
  by *simp*

lemma *eq-fps-sin*:
  assumes *a0*: *a* $ *0* = *0*
    and *a1*: *a* $ *1* = *c*
    and *a2*: *fps-deriv* (*fps-deriv a*) = − (*fps-const c* ∗ *fps-const c* ∗ *a*)
  shows *fps-sin c* = *a*
proof (*rule fps-ext*)
  fix *n*
  show *fps-sin c* $ *n* = *a* $ *n*
  proof (*induction n rule*: *nat-induct2*)
    case (*step n*)
    then have *of-nat* (*n* + *1*) ∗ (*of-nat* (*n* + *2*) ∗ *a* $ (*n* + *2*)) =
     − (*c* ∗ *c* ∗ *fps-sin c* $ *n*)
      using *a2*
    by (*metis fps-const-mult fps-deriv-nth fps-mult-left-const-nth fps-neg-nth nat-add-1-add-1*)
    with *step* **show** *?case*
     by (*metis* (*no-types*, *lifting*) *a0 add.commute add.inverse-inverse fps-sin-nth-0*
fps-sin-nth-add-2 mult-divide-mult-cancel-left-if mult-minus-right nonzero-mult-div-cancel-left
not-less-zero of-nat-eq-0-iff plus-1-eq-Suc zero-less-Suc)
  qed (*use assms* **in** *auto*)
qed

lemma *eq-fps-cos*:
  assumes *a0*: *a* $ *0* = *1*
    and *a1*: *a* $ *1* = *0*
    and *a2*: *fps-deriv* (*fps-deriv a*) = − (*fps-const c* ∗ *fps-const c* ∗ *a*)
  shows *fps-cos c* = *a*
proof (*rule fps-ext*)
  fix *n*
  show *fps-cos c* $ *n* = *a* $ *n*
  proof (*induction n rule*: *nat-induct2*)
    case (*step n*)
    then have *of-nat* (*n* + *1*) ∗ (*of-nat* (*n* + *2*) ∗ *a* $ (*n* + *2*)) =
     − (*c* ∗ *c* ∗ *fps-cos c* $ *n*)
      using *a2*
    by (*metis fps-const-mult fps-deriv-nth fps-mult-left-const-nth fps-neg-nth nat-add-1-add-1*)
    with *step* **show** *?case*
     by (*metis* (*no-types*, *lifting*) *a0 add.commute add.inverse-inverse fps-cos-nth-0*
fps-cos-nth-add-2 mult-divide-mult-cancel-left-if mult-minus-right nonzero-mult-div-cancel-left
not-less-zero of-nat-eq-0-iff plus-1-eq-Suc zero-less-Suc)

qed (*use assms* **in** *auto*)
qed

lemma *fps-sin-add*: *fps-sin* (*a* + *b*) = *fps-sin a* ∗ *fps-cos b* + *fps-cos a* ∗ *fps-sin b*
proof −
have *fps-deriv* (*fps-deriv* (*fps-sin a* ∗ *fps-cos b* + *fps-cos a* ∗ *fps-sin b*)) =
− (*fps-const* (*a* + *b*) ∗ *fps-const* (*a* + *b*) ∗ (*fps-sin a* ∗ *fps-cos b* + *fps-cos a* ∗ *fps-sin b*))
by (*simp flip*: *fps-const-neg fps-const-add fps-const-mult*
add: *fps-sin-deriv fps-cos-deriv algebra-simps*)
then show *?thesis*
by (*auto intro*: *eq-fps-sin*)
qed

lemma *fps-cos-add*: *fps-cos* (*a* + *b*) = *fps-cos a* ∗ *fps-cos b* − *fps-sin a* ∗ *fps-sin b*
proof −
have *fps-deriv*
(*fps-deriv* (*fps-cos a* ∗ *fps-cos b* − *fps-sin a* ∗ *fps-sin b*)) =
− (*fps-const* (*a* + *b*) ∗ *fps-const* (*a* + *b*) ∗
(*fps-cos a* ∗ *fps-cos b* − *fps-sin a* ∗ *fps-sin b*))
by (*simp flip*: *fps-const-neg fps-const-add fps-const-mult*
add: *fps-sin-deriv fps-cos-deriv algebra-simps*)
then show *?thesis*
by (*auto intro*: *eq-fps-cos*)
qed

lemma *fps-sin-even*: *fps-sin* (− *c*) = − *fps-sin c*
by (*simp add*: *fps-eq-iff fps-sin-def*)

lemma *fps-cos-odd*: *fps-cos* (− *c*) = *fps-cos c*
by (*simp add*: *fps-eq-iff fps-cos-def*)

definition *fps-tan c* = *fps-sin c* / *fps-cos c*

lemma *fps-tan-0* [*simp*]: *fps-tan 0* = *0*
by (*simp add*: *fps-tan-def*)

lemma *fps-tan-deriv*: *fps-deriv* (*fps-tan c*) = *fps-const c* / (*fps-cos c*)^2
proof −
have *th0*: *fps-cos c* $ *0* ≠ *0* **by** (*simp add*: *fps-cos-def*)
from *this* **have** *fps-cos c* ≠ *0* **by** (*intro notI*) *simp*
hence *fps-deriv* (*fps-tan c*) =
fps-const c ∗ (*fps-cos c*^*2* + *fps-sin c*^*2*) / (*fps-cos c*^*2*)
by (*simp add*: *fps-tan-def fps-divide-deriv power2-eq-square algebra-simps*
fps-sin-deriv fps-cos-deriv fps-const-neg[*symmetric*] *div-mult-swap*
del: *fps-const-neg*)
also note *fps-sin-cos-sum-of-squares*
finally show *?thesis* **by** *simp*

qed

Connection to *fps-exp* over the complex numbers — Euler and de Moivre.

lemma *fps-exp-ii-sin-cos*: *fps-exp* (i ∗ *c*) = *fps-cos c* + *fps-const* i ∗ *fps-sin c*
(**is** *?l* = *?r*)
proof −
have *?l* $ *n* = *?r* $ *n* **for** *n*
proof (*cases even n*)
case *True*
then obtain *m* **where** *m*: *n* = *2* ∗ *m* **..**
show *?thesis*
by (*simp add*: *m fps-sin-def fps-cos-def power-mult-distrib power-mult power-minus*
[*of c* ^ *2*])
next
case *False*
then obtain *m* **where** *m*: *n* = *2* ∗ *m* + *1* **..**
show *?thesis*
by (*simp add*: *m fps-sin-def fps-cos-def power-mult-distrib*
power-mult power-minus [*of c* ^ *2*])
qed
then show *?thesis*
by (*simp add*: *fps-eq-iff*)
qed

lemma *fps-exp-minus-ii-sin-cos*: *fps-exp* (− (i ∗ *c*)) = *fps-cos c* − *fps-const* i ∗
fps-sin c
unfolding *minus-mult-right fps-exp-ii-sin-cos* **by** (*simp add*: *fps-sin-even fps-cos-odd*)

lemma *fps-cos-fps-exp-ii*: *fps-cos c* = (*fps-exp* (i ∗ *c*) + *fps-exp* (− i ∗ *c*)) /
fps-const 2
proof −
have *th*: *fps-cos c* + *fps-cos c* = *fps-cos c* ∗ *fps-const 2*
by (*simp add*: *numeral-fps-const*)
show *?thesis*
unfolding *fps-exp-ii-sin-cos minus-mult-commute*
by (*simp add*: *fps-sin-even fps-cos-odd numeral-fps-const fps-divide-unit fps-const-inverse*
th)
qed

lemma *fps-sin-fps-exp-ii*: *fps-sin c* = (*fps-exp* (i ∗ *c*) − *fps-exp* (− i ∗ *c*)) / *fps-const*
(*2*∗i)
proof −
have *th*: *fps-const* i ∗ *fps-sin c* + *fps-const* i ∗ *fps-sin c* = *fps-sin c* ∗ *fps-const*
(*2* ∗ i)
by (*simp add*: *fps-eq-iff numeral-fps-const*)
show *?thesis*
unfolding *fps-exp-ii-sin-cos minus-mult-commute*
by (*simp add*: *fps-sin-even fps-cos-odd fps-divide-unit fps-const-inverse th*)
qed

lemma *fps-tan-fps-exp-ii*:
fps-tan c = (*fps-exp* (i ∗ *c*) − *fps-exp* (− i ∗ *c*)) /
(*fps-const* i ∗ (*fps-exp* (i ∗ *c*) + *fps-exp* (− i ∗ *c*)))
unfolding *fps-tan-def fps-sin-fps-exp-ii fps-cos-fps-exp-ii*
by (*simp add*: *fps-divide-unit fps-inverse-mult fps-const-inverse*)

lemma *fps-demoivre*:
(*fps-cos a* + *fps-const* i ∗ *fps-sin a*)^*n* =
fps-cos (*of-nat n* ∗ *a*) + *fps-const* i ∗ *fps-sin* (*of-nat n* ∗ *a*)
unfolding *fps-exp-ii-sin-cos*[*symmetric*] *fps-exp-power-mult*
by (*simp add*: *ac-simps*)

5.20 Hypergeometric series

definition *fps-hypergeo as bs* (*c*::′*a*::*field-char-0*) =
Abs-fps (λ*n*. (*foldl* (λ*r a*. *r*∗ *pochhammer a n*) *1 as* ∗ *c*^*n*) /
(*foldl* (λ*r b*. *r* ∗ *pochhammer b n*) *1 bs* ∗ *of-nat* (*fact n*)))

lemma *fps-hypergeo-nth*[*simp*]: *fps-hypergeo as bs c* $ *n* =
(*foldl* (λ*r a*. *r*∗ *pochhammer a n*) *1 as* ∗ *c*^*n*) /
(*foldl* (λ*r b*. *r* ∗ *pochhammer b n*) *1 bs* ∗ *of-nat* (*fact n*))
by (*simp add*: *fps-hypergeo-def*)

lemma *foldl-mult-start*:
fixes *v* :: ′*a*::*comm-ring-1*
shows *foldl* (λ*r x*. *r* ∗ *f x*) *v as* ∗ *x* = *foldl* (λ*r x*. *r* ∗ *f x*) (*v* ∗ *x*) *as*
by (*induct as arbitrary*: *x v*) (*auto simp add*: *algebra-simps*)

lemma *foldr-mult-foldl*:
fixes *v* :: ′*a*::*comm-ring-1*
shows *foldr* (λ*x r*. *r* ∗ *f x*) *as v* = *foldl* (λ*r x*. *r* ∗ *f x*) *v as*
by (*induct as arbitrary*: *v*) (*simp-all add*: *foldl-mult-start*)

lemma *fps-hypergeo-nth-alt*:
fps-hypergeo as bs c $ *n* = *foldr* (λ*a r*. *r* ∗ *pochhammer a n*) *as* (*c* ^ *n*) /
foldr (λ*b r*. *r* ∗ *pochhammer b n*) *bs* (*of-nat* (*fact n*))
by (*simp add*: *foldl-mult-start foldr-mult-foldl*)

lemma *fps-hypergeo-fps-exp*[*simp*]: *fps-hypergeo* [] [] *c* = *fps-exp c*
by (*simp add*: *fps-eq-iff*)

lemma *fps-hypergeo-1-0*[*simp*]: *fps-hypergeo* [*1*] [] *c* = *1*/(*1* − *fps-const c* ∗ *fps-X*)
proof −
let *?a* = (*Abs-fps* (λ*n*. *1*)) *oo* (*fps-const c* ∗ *fps-X*)
have *th0*: (*fps-const c* ∗ *fps-X*) $ *0* = *0* **by** *simp*
show *?thesis* **unfolding** *gp*[*OF th0*, *symmetric*]
by (*simp add*: *fps-eq-iff pochhammer-fact*[*symmetric*]
fps-compose-nth power-mult-distrib if-distrib cong del: *if-weak-cong*)

qed

lemma *fps-hypergeo-B*[*simp*]: *fps-hypergeo* [−*a*] [] (− *1*) = *fps-binomial a*
by (*simp add*: *fps-eq-iff gbinomial-pochhammer algebra-simps*)

lemma *fps-hypergeo-0*[*simp*]: *fps-hypergeo as bs c* $ *0* = *1*
proof −
have *foldl* (λ(*r*::′*a*) (*a*::′*a*). *r*) *1 as* = *1* **for** *as*
by (*induction as*) *auto*
then show *?thesis*
by *auto*
qed

lemma *foldl-prod-prod*:
foldl (λ(*r*::′*b*::*comm-ring-1*) (*x*::′*a*::*comm-ring-1*). *r* ∗ *f x*) *v as* ∗ *foldl* (λ*r x*. *r* ∗ *g x*) *w as* =
foldl (λ*r x*. *r* ∗ *f x* ∗ *g x*) (*v* ∗ *w*) *as*
by (*induct as arbitrary*: *v w*) (*simp-all add*: *algebra-simps*)

lemma *fps-hypergeo-rec*:
fps-hypergeo as bs c $ *Suc n* = ((*foldl* (λ*r a*. *r*∗ (*a* + *of-nat n*)) *c as*) /
(*foldl* (λ*r b*. *r* ∗ (*b* + *of-nat n*)) (*of-nat* (*Suc n*)) *bs*)) ∗ *fps-hypergeo as bs c* $ *n*
apply (*simp add*: *foldl-mult-start del*: *of-nat-Suc of-nat-add fact-Suc*)
unfolding *foldl-prod-prod*[*unfolded foldl-mult-start*] *pochhammer-Suc*
by (*simp add*: *algebra-simps*)

lemma *fps-XD-nth*[*simp*]: *fps-XD a* $ *n* = *of-nat n* ∗ *a*$*n*
by (*simp add*: *fps-XD-def*)

lemma *fps-XD-0th*[*simp*]: *fps-XD a* $ *0* = *0*
by *simp*
lemma *fps-XD-Suc*[*simp*]: *fps-XD a* $ *Suc n* = *of-nat* (*Suc n*) ∗ *a* $ *Suc n*
by *simp*

definition *fps-XDp c a* = *fps-XD a* + *fps-const c* ∗ *a*

lemma *fps-XDp-nth*[*simp*]: *fps-XDp c a* $ *n* = (*c* + *of-nat n*) ∗ *a*$*n*
by (*simp add*: *fps-XDp-def algebra-simps*)

lemma *fps-XDp-commute*: *fps-XDp b* ∘ *fps-XDp* (*c*::′*a*::*comm-ring-1*) = *fps-XDp c* ∘ *fps-XDp b*
by (*simp add*: *fps-XDp-def fun-eq-iff fps-eq-iff algebra-simps*)

lemma *fps-XDp0* [*simp*]: *fps-XDp 0* = *fps-XD*
by (*simp add*: *fun-eq-iff fps-eq-iff*)

lemma *fps-XDp-fps-integral* [*simp*]:

fixes $a :: 'a::\{division\text{-}ring,ring\text{-}char\text{-}0\}$ *fps*
shows *fps-XDp 0* (*fps-integral a c*) = *fps-X* ∗ *a*
using *fps-deriv-fps-integral*[*of a c*]
by (*simp add*: *fps-XD-def*)

lemma *fps-hypergeo-minus-nat*:
fps-hypergeo [− *of-nat n*] [− *of-nat* (*n* + *m*)] (*c*::*'a*::*field-char-0*) $ *k* =
(*if k* ≤ *n then*
pochhammer (− *of-nat n*) *k* ∗ *c* ^ *k* / (*pochhammer* (− *of-nat* (*n* + *m*)) *k* ∗ *of-nat* (*fact k*))
else 0)
fps-hypergeo [− *of-nat m*] [− *of-nat* (*m* + *n*)] (*c*::*'a*::*field-char-0*) $ *k* =
(*if k* ≤ *m then*
pochhammer (− *of-nat m*) *k* ∗ *c* ^ *k* / (*pochhammer* (− *of-nat* (*m* + *n*)) *k* ∗ *of-nat* (*fact k*))
else 0)
by (*simp-all add*: *pochhammer-eq-0-iff*)

lemma *pochhammer-rec-if*: *pochhammer a n* = (*if n* = *0 then 1 else a* ∗ *pochhammer* (*a* + *1*) (*n* − *1*))
by (*cases n*) (*simp-all add*: *pochhammer-rec*)

lemma *fps-XDp-foldr-nth* [*simp*]: *foldr* (λ*c r*. *fps-XDp c* ∘ *r*) *cs* (λ*c*. *fps-XDp c a*) *c0* $ *n* =
foldr (λ*c r*. (*c* + *of-nat n*) ∗ *r*) *cs* (*c0* + *of-nat n*) ∗ *a*$*n*
by (*induct cs arbitrary*: *c0*) (*simp-all add*: *algebra-simps*)

lemma *genric-fps-XDp-foldr-nth*:
assumes *f*: ∀ *n c a*. *f c a* $ *n* = (*of-nat n* + *k c*) ∗ *a*$*n*
shows *foldr* (λ*c r*. *f c* ∘ *r*) *cs* (λ*c*. *g c a*) *c0* $ *n* =
foldr (λ*c r*. (*k c* + *of-nat n*) ∗ *r*) *cs* (*g c0 a* $ *n*)
by (*induct cs arbitrary*: *c0*) (*simp-all add*: *algebra-simps f*)

lemma *dist-less-imp-nth-equal*:
assumes *dist f g* < *inverse* (*2* ^ *i*)
and*j* ≤ *i*
shows *f* $ *j* = *g* $ *j*
proof (*rule ccontr*)
assume *f* $ *j* ≠ *g* $ *j*
hence *f* ≠ *g* **by** *auto*
with *assms* **have** *i* < *subdegree* (*f* − *g*)
by (*simp add*: *if-split-asm dist-fps-def*)
also have ... ≤ *j*
using ‹*f* $ *j* ≠ *g* $ *j*› **by** (*intro subdegree-leI*) *simp-all*
finally show *False* **using** ‹*j* ≤ *i*› **by** *simp*
qed

lemma *nth-equal-imp-dist-less*:
assumes ⋀*j*. *j* ≤ *i* ⟹ *f* $ *j* = *g* $ *j*

```
  shows dist f g < inverse (2 ^ i)
proof (cases f = g)
  case True
  then show ?thesis by simp
next
  case False
  with assms have dist f g = inverse (2 ^ subdegree (f - g))
    by (simp add: if-split-asm dist-fps-def)
  moreover
  from assms and False have i < subdegree (f - g)
    by (intro subdegree-greaterI) simp-all
  ultimately show ?thesis by simp
qed

lemma dist-less-eq-nth-equal: dist f g < inverse (2 ^ i) ⟷ (∀j ≤ i. f $ j = g $
j)
  using dist-less-imp-nth-equal nth-equal-imp-dist-less by blast

instance fps :: (comm-ring-1) complete-space
proof
  fix fps-X :: nat ⇒ 'a fps
  assume Cauchy fps-X
  obtain M where M: ∀i. ∀m ≥ M i. ∀j ≤ i. fps-X (M i) $ j = fps-X m $ j
  proof -
    have ∃M. ∀m ≥ M. ∀j≤i. fps-X M $ j = fps-X m $ j for i
    proof -
      have 0 < inverse ((2::real)^i) by simp
      from metric-CauchyD[OF ‹Cauchy fps-X› this] dist-less-imp-nth-equal
      show ?thesis by blast
    qed
    then show ?thesis using that by metis
  qed

  show convergent fps-X
  proof (rule convergentI)
    show fps-X ⟶ Abs-fps (λi. fps-X (M i) $ i)
      unfolding tendsto-iff
    proof safe
      fix e::real assume e: 0 < e
    have (λn. inverse (2 ^ n) :: real) ⟶ 0 by (rule LIMSEQ-inverse-realpow-zero)
simp-all
      from this and e have eventually (λi. inverse (2 ^ i) < e) sequentially
        by (rule order-tendstoD)
      then obtain i where inverse (2 ^ i) < e
        by (auto simp: eventually-sequentially)
      have eventually (λx. M i ≤ x) sequentially
        by (auto simp: eventually-sequentially)
      then show eventually (λx. dist (fps-X x) (Abs-fps (λi. fps-X (M i) $ i)) <
e) sequentially
```

```
      proof eventually-elim
        fix x
        assume x: M i ≤ x
        have fps-X (M i) $ j = fps-X (M j) $ j if j ≤ i for j
          using M that by (metis nat-le-linear)
        with x have dist (fps-X x) (Abs-fps (λj. fps-X (M j) $ j)) < inverse (2 ^ i)
          using M by (force simp: dist-less-eq-nth-equal)
        also note ‹inverse (2 ^ i) < e›
        finally show dist (fps-X x) (Abs-fps (λj. fps-X (M j) $ j)) < e .
      qed
    qed
  qed
qed


bundle fps-syntax
begin
notation fps-nth (infixl ‹$› 75)
end

unbundle no fps-syntax

end
```

6 Converting polynomials to formal power series

```
theory Polynomial-FPS
  imports Polynomial Formal-Power-Series
begin

context
  includes fps-syntax
begin

definition fps-of-poly where
  fps-of-poly p = Abs-fps (coeff p)

lemma fps-of-poly-eq-iff: fps-of-poly p = fps-of-poly q ⟷ p = q
  by (simp add: fps-of-poly-def poly-eq-iff fps-eq-iff)

lemma fps-of-poly-nth [simp]: fps-of-poly p $ n = coeff p n
  by (simp add: fps-of-poly-def)

lemma fps-of-poly-const: fps-of-poly [:c:] = fps-const c
proof (subst fps-eq-iff, clarify)
  fix n :: nat show fps-of-poly [:c:] $ n = fps-const c $ n
    by (cases n) (auto simp: fps-of-poly-def)
qed
```

lemma *fps-of-poly-0* [*simp*]: *fps-of-poly 0 = 0*
by (*subst fps-const-0-eq-0* [*symmetric*], *subst fps-of-poly-const* [*symmetric*]) *simp*

lemma *fps-of-poly-1* [*simp*]: *fps-of-poly 1 = 1*
by (*simp add*: *fps-eq-iff*)

lemma *fps-of-poly-1'* [*simp*]: *fps-of-poly* [:*1*:] = *1*
by (*subst fps-const-1-eq-1* [*symmetric*], *subst fps-of-poly-const* [*symmetric*])
(*simp add*: *one-poly-def*)

lemma *fps-of-poly-numeral* [*simp*]: *fps-of-poly* (*numeral n*) = *numeral n*
by (*simp add*: *numeral-fps-const fps-of-poly-const* [*symmetric*] *numeral-poly*)

lemma *fps-of-poly-numeral'* [*simp*]: *fps-of-poly* [:*numeral n*:] = *numeral n*
by (*simp add*: *numeral-fps-const fps-of-poly-const* [*symmetric*] *numeral-poly*)

lemma *fps-of-poly-fps-X* [*simp*]: *fps-of-poly* [:*0*, *1*:] = *fps-X*
by (*auto simp add*: *fps-of-poly-def fps-eq-iff coeff-pCons split*: *nat.split*)

lemma *fps-of-poly-add*: *fps-of-poly* (*p* + *q*) = *fps-of-poly p* + *fps-of-poly q*
by (*simp add*: *fps-of-poly-def plus-poly.rep-eq fps-plus-def*)

lemma *fps-of-poly-diff*: *fps-of-poly* (*p* − *q*) = *fps-of-poly p* − *fps-of-poly q*
by (*simp add*: *fps-of-poly-def minus-poly.rep-eq fps-minus-def*)

lemma *fps-of-poly-uminus*: *fps-of-poly* (−*p*) = −*fps-of-poly p*
by (*simp add*: *fps-of-poly-def uminus-poly.rep-eq fps-uminus-def*)

lemma *fps-of-poly-mult*: *fps-of-poly* (*p* ∗ *q*) = *fps-of-poly p* ∗ *fps-of-poly q*
by (*simp add*: *fps-of-poly-def fps-times-def fps-eq-iff coeff-mult atLeast0AtMost*)

lemma *fps-of-poly-smult*:
fps-of-poly (*smult c p*) = *fps-const c* ∗ *fps-of-poly p*
using *fps-of-poly-mult*[*of* [:*c*:] *p*] **by** (*simp add*: *fps-of-poly-mult fps-of-poly-const*)

lemma *fps-of-poly-sum*: *fps-of-poly* (*sum f A*) = *sum* (λ*x*. *fps-of-poly* (*f x*)) *A*
by (*cases finite A*, *induction rule*: *finite-induct*) (*simp-all add*: *fps-of-poly-add*)

lemma *fps-of-poly-sum-list*: *fps-of-poly* (*sum-list xs*) = *sum-list* (*map fps-of-poly xs*)
by (*induction xs*) (*simp-all add*: *fps-of-poly-add*)

lemma *fps-of-poly-prod*: *fps-of-poly* (*prod f A*) = *prod* (λ*x*. *fps-of-poly* (*f x*)) *A*
by (*cases finite A*, *induction rule*: *finite-induct*) (*simp-all add*: *fps-of-poly-mult*)

lemma *fps-of-poly-prod-list*: *fps-of-poly* (*prod-list xs*) = *prod-list* (*map fps-of-poly xs*)
by (*induction xs*) (*simp-all add*: *fps-of-poly-mult*)

lemma *fps-of-poly-pCons*:
fps-of-poly (*pCons* (*c* :: *'a* :: *semiring-1*) *p*) = *fps-const c* + *fps-of-poly p* ∗ *fps-X*
by (*subst fps-mult-fps-X-commute* [*symmetric*], *intro fps-ext*)
(*auto simp*: *fps-of-poly-def coeff-pCons split*: *nat.split*)

lemma *fps-of-poly-pderiv*: *fps-of-poly* (*pderiv p*) = *fps-deriv* (*fps-of-poly p*)
by (*intro fps-ext*) (*simp add*: *fps-of-poly-nth coeff-pderiv*)

lemma *fps-of-poly-power*: *fps-of-poly* (*p* ^ *n*) = *fps-of-poly p* ^ *n*
by (*induction n*) (*simp-all add*: *fps-of-poly-mult*)

lemma *fps-of-poly-monom*: *fps-of-poly* (*monom* (*c* :: *'a* :: *comm-ring-1*) *n*) = *fps-const c* ∗ *fps-X* ^ *n*
by (*intro fps-ext*) *simp-all*

lemma *fps-of-poly-monom'*: *fps-of-poly* (*monom* (*1* :: *'a* :: *comm-ring-1*) *n*) = *fps-X* ^ *n*
by (*simp add*: *fps-of-poly-monom*)

lemma *fps-of-poly-div*:
assumes (*q* :: *'a* :: *field poly*) *dvd p*
shows *fps-of-poly* (*p div q*) = *fps-of-poly p* / *fps-of-poly q*
proof (*cases q* = *0*)
case *False*
from *False fps-of-poly-eq-iff*[*of q 0*] **have** *nz*: *fps-of-poly q* ≠ *0* **by** *simp*
from *assms* **have** *p* = (*p div q*) ∗ *q* **by** *simp*
also have *fps-of-poly* ... = *fps-of-poly* (*p div q*) ∗ *fps-of-poly q*
by (*simp add*: *fps-of-poly-mult*)
also from *nz* **have** ... / *fps-of-poly q* = *fps-of-poly* (*p div q*)
by (*intro nonzero-mult-div-cancel-right*) (*auto simp*: *fps-of-poly-0*)
finally show *?thesis* **..**
qed *simp*

lemma *fps-of-poly-divide-numeral*:
fps-of-poly (*smult* (*inverse* (*numeral c* :: *'a* :: *field*)) *p*) = *fps-of-poly p* / *numeral c*
proof −
have *smult* (*inverse* (*numeral c*)) *p* = [:*inverse* (*numeral c*):] ∗ *p* **by** *simp*
also have *fps-of-poly* ... = *fps-of-poly p* / *numeral c*
by (*subst fps-of-poly-mult*) (*simp add*: *numeral-fps-const fps-of-poly-pCons*)
finally show *?thesis* **by** *simp*
qed

lemma *subdegree-fps-of-poly*:
assumes *p* ≠ *0*
defines *n* ≡ *Polynomial.order 0 p*
shows *subdegree* (*fps-of-poly p*) = *n*
proof (*rule subdegreeI*)

from *assms* **have** *monom 1 n dvd p* **by** (*simp add*: *monom-1-dvd-iff*)
thus *zero*: *fps-of-poly p $ i = 0* **if** *i < n* **for** *i*
using *that* **by** (*simp add*: *monom-1-dvd-iff'*)

from *assms* **have** ¬*monom 1 (Suc n) dvd p*
by (*auto simp*: *monom-1-dvd-iff simp del*: *power-Suc*)
then obtain *k* **where** *k*: *k ≤ n fps-of-poly p $ k ≠ 0*
by (*auto simp*: *monom-1-dvd-iff' less-Suc-eq-le*)
with *zero*[*of k*] **have** *k = n* **by** *linarith*
with *k* **show** *fps-of-poly p $ n ≠ 0* **by** *simp*
qed

lemma *fps-of-poly-dvd*:
assumes *p dvd q*
shows *fps-of-poly (p :: 'a :: field poly) dvd fps-of-poly q*
proof (*cases p = 0 ∨ q = 0*)
case *False*
with *assms fps-of-poly-eq-iff*[*of p 0*] *fps-of-poly-eq-iff*[*of q 0*] **show** *?thesis*
by (*auto simp*: *fps-dvd-iff subdegree-fps-of-poly dvd-imp-order-le*)
qed (*insert assms*, *auto*)

lemmas *fps-of-poly-simps* =
fps-of-poly-0 fps-of-poly-1 fps-of-poly-numeral fps-of-poly-const fps-of-poly-fps-X
fps-of-poly-add fps-of-poly-diff fps-of-poly-uminus fps-of-poly-mult fps-of-poly-smult
fps-of-poly-sum fps-of-poly-sum-list fps-of-poly-prod fps-of-poly-prod-list
fps-of-poly-pCons fps-of-poly-pderiv fps-of-poly-power fps-of-poly-monom
fps-of-poly-divide-numeral

lemma *fps-of-poly-pcompose*:
assumes *coeff q 0 = (0 :: 'a :: idom)*
shows *fps-of-poly (pcompose p q) = fps-compose (fps-of-poly p) (fps-of-poly q)*
using *assms* **by** (*induction p rule*: *pCons-induct*)
(*auto simp*: *pcompose-pCons fps-of-poly-simps fps-of-poly-pCons*
fps-compose-add-distrib fps-compose-mult-distrib)

lemmas *reify-fps-atom* =
fps-of-poly-0 fps-of-poly-1' fps-of-poly-numeral' fps-of-poly-const fps-of-poly-fps-X

The following simproc can reduce the equality of two polynomial FPSs two equality of the respective polynomials. A polynomial FPS is one that only has finitely many non-zero coefficients and can therefore be written as *fps-of-poly p* for some polynomial *p*.

This may sound trivial, but it covers a number of annoying side conditions like *1 + fps-X ≠ 0* that would otherwise not be solved automatically.

ML ‹

(∗ *TODO*: *Support for division* ∗)

```
signature POLY-FPS = sig

val reify-conv : conv
val eq-conv : conv
val eq-simproc : cterm -> thm option

end


structure Poly-Fps = struct

fun const-binop-conv s conv ct =
  case Thm.term-of ct of
    (Const (s', -) $ - $ -) =>
      if s = s' then
        Conv.binop-conv conv ct
      else
        raise CTERM (const-binop-conv, [ct])
  | - => raise CTERM (const-binop-conv, [ct])

fun reify-conv ct =
  let
    val rewr = Conv.rewrs-conv o map (fn thm => thm RS @{thm eq-reflection})
    val un = Conv.arg-conv reify-conv
    val bin = Conv.binop-conv reify-conv
  in
    case Thm.term-of ct of
      (Const (const-name ‹fps-of-poly›, -) $ -) => ct |> Conv.all-conv
    | (Const (const-name ‹Groups.plus›, -) $ - $ -) => ct |> (
        bin then-conv rewr @{thms fps-of-poly-add [symmetric]})
    | (Const (const-name ‹Groups.uminus›, -) $ -) => ct |> (
        un then-conv rewr @{thms fps-of-poly-uminus [symmetric]})
    | (Const (const-name ‹Groups.minus›, -) $ - $ -) => ct |> (
        bin then-conv rewr @{thms fps-of-poly-diff [symmetric]})
    | (Const (const-name ‹Groups.times›, -) $ - $ -) => ct |> (
        bin then-conv rewr @{thms fps-of-poly-mult [symmetric]})
   | (Const (const-name ‹Rings.divide›, -) $ - $ (Const (const-name ‹Num.numeral›,
-) $ -))
        => ct |> (Conv.fun-conv (Conv.arg-conv reify-conv)
             then-conv rewr @{thms fps-of-poly-divide-numeral [symmetric]})
    | (Const (const-name ‹Power.power›, -) $ Const (const-name ‹fps-X›,-) $ -)
=> ct |> (
        rewr @{thms fps-of-poly-monom' [symmetric]})
    | (Const (const-name ‹Power.power›, -) $ - $ -) => ct |> (
        Conv.fun-conv (Conv.arg-conv reify-conv)
        then-conv rewr @{thms fps-of-poly-power [symmetric]})
    | - => ct |> (
        rewr @{thms reify-fps-atom [symmetric]})
  end
```

```
fun eq-conv ct =
  case Thm.term-of ct of
    (Const (const-name ‹HOL.eq›, -) $ - $ -) => ct |> (
      Conv.binop-conv reify-conv
      then-conv Conv.rewr-conv @{thm fps-of-poly-eq-iff[THEN eq-reflection]})
  | - => raise CTERM (poly-fps-eq-conv, [ct])

val eq-simproc = try eq-conv

end
›

simproc-setup poly-fps-eq ((f :: 'a fps) = g) = ‹K (K Poly-Fps.eq-simproc)›

lemma fps-of-poly-linear: fps-of-poly [:a,1 :: 'a :: field:] = fps-X + fps-const a
  by simp

lemma fps-of-poly-linear': fps-of-poly [:1,a :: 'a :: field:] = 1 + fps-const a * fps-X
  by simp

lemma fps-of-poly-cutoff [simp]:
  fps-of-poly (poly-cutoff n p) = fps-cutoff n (fps-of-poly p)
  by (simp add: fps-eq-iff coeff-poly-cutoff)

lemma fps-of-poly-shift [simp]: fps-of-poly (poly-shift n p) = fps-shift n (fps-of-poly
p)
  by (simp add: fps-eq-iff coeff-poly-shift)


definition poly-subdegree :: 'a::zero poly ⇒ nat where
  poly-subdegree p = subdegree (fps-of-poly p)

lemma coeff-less-poly-subdegree:
  k < poly-subdegree p ⟹ coeff p k = 0
  unfolding poly-subdegree-def using nth-less-subdegree-zero[of k fps-of-poly p] by
simp


definition prefix-length :: ('a ⇒ bool) ⇒ 'a list ⇒ nat where
  prefix-length P xs = length (takeWhile P xs)

primrec prefix-length-aux :: ('a ⇒ bool) ⇒ nat ⇒ 'a list ⇒ nat where
  prefix-length-aux P acc [] = acc
| prefix-length-aux P acc (x#xs) = (if P x then prefix-length-aux P (Suc acc) xs
else acc)

lemma prefix-length-aux-correct: prefix-length-aux P acc xs = prefix-length P xs +
```

acc
by (*induction xs arbitrary*: *acc*) (*simp-all add*: *prefix-length-def*)

lemma *prefix-length-code* [*code*]: *prefix-length P xs* = *prefix-length-aux P 0 xs*
by (*simp add*: *prefix-length-aux-correct*)

lemma *prefix-length-le-length*: *prefix-length P xs* ≤ *length xs*
by (*induction xs*) (*simp-all add*: *prefix-length-def*)

lemma *prefix-length-less-length*: (∃ *x*∈*set xs*. ¬*P x*) ⟹ *prefix-length P xs* < *length xs*
by (*induction xs*) (*simp-all add*: *prefix-length-def*)

lemma *nth-prefix-length*:
(∃ *x*∈*set xs*. ¬*P x*) ⟹ ¬*P* (*xs* ! *prefix-length P xs*)
by (*induction xs*) (*simp-all add*: *prefix-length-def*)

lemma *nth-less-prefix-length*:
n < *prefix-length P xs* ⟹ *P* (*xs* ! *n*)
by (*induction xs arbitrary*: *n*)
(*auto simp*: *prefix-length-def nth-Cons split*: *if-splits nat.splits*)

lemma *poly-subdegree-code* [*code*]: *poly-subdegree p* = *prefix-length* ((=) *0*) (*coeffs p*)
proof (*cases p* = *0*)
case *False*
note [*simp*] = *this*
define *n* **where** *n* = *prefix-length* ((=) *0*) (*coeffs p*)
from *False* **have** ∃ *k*. *coeff p k* ≠ *0* **by** (*auto simp*: *poly-eq-iff*)
hence *ex*: ∃ *x*∈*set* (*coeffs p*). *x* ≠ *0* **by** (*auto simp*: *coeffs-def*)
hence *n-less*: *n* < *length* (*coeffs p*) **and** *nonzero*: *coeffs p* ! *n* ≠ *0*
unfolding *n-def* **by** (*auto intro*!: *prefix-length-less-length nth-prefix-length*)
show *?thesis* **unfolding** *poly-subdegree-def*
proof (*intro subdegreeI*)
from *n-less* **have** *fps-of-poly p* $ *n* = *coeffs p* ! *n*
by (*subst coeffs-nth*) (*simp-all add*: *degree-eq-length-coeffs*)
with *nonzero* **show** *fps-of-poly p* $ *prefix-length* ((=) *0*) (*coeffs p*) ≠ *0*
unfolding *n-def* **by** *simp*
next
fix *k* **assume** *A*: *k* < *prefix-length* ((=) *0*) (*coeffs p*)
also have ... ≤ *length* (*coeffs p*) **by** (*rule prefix-length-le-length*)
finally show *fps-of-poly p* $ *k* = *0*
using *nth-less-prefix-length*[*OF A*]
by (*simp add*: *coeffs-nth degree-eq-length-coeffs*)
qed
qed (*simp-all add*: *poly-subdegree-def prefix-length-def*)

end

Truncation of formal power series: all monomials cx^k with $k \geq n$ are removed; the remainder is a polynomial of degree at most $n - 1$.

lift-definition *truncate-fps* :: *nat* ⇒ *'a fps* ⇒ *'a* :: *zero poly* **is**
  λn F k. if k ≥ n then 0 else fps-nth F k
proof *goal-cases*
  case (*1 n F*)
  have *eventually* (*λk.* (*if n ≤ k then 0 else fps-nth F k*) = *0*) *at-top*
    using *eventually-ge-at-top*[*of n*] **by** *eventually-elim auto*
  thus *?case*
    by (*simp add*: *cofinite-eq-sequentially*)
qed

lemma *coeff-truncate-fps'* [*simp*]:
  k ≥ n ⟹ coeff (*truncate-fps n F*) *k* = *0*
  k < n ⟹ coeff (*truncate-fps n F*) *k* = *fps-nth F k*
  by (*transfer*; *simp*; *fail*)+

lemma *coeff-truncate-fps*: *coeff* (*truncate-fps n F*) *k* = (*if k < n then fps-nth F k else 0*)
  by *auto*

lemma *truncate-0-fps* [*simp*]: *truncate-fps 0 F* = *0*
  by (*rule poly-eqI*) *auto*

lemma *degree-truncate-fps*: *n > 0 ⟹ degree* (*truncate-fps n F*) < *n*
  by (*rule degree-lessI*) *auto*

lemma *truncate-fps-0* [*simp*]: *truncate-fps n 0* = *0*
  by (*rule poly-eqI*) (*auto simp*: *coeff-truncate-fps*)

lemma *truncate-fps-add*: *truncate-fps n* (*f* + *g*) = *truncate-fps n f* + *truncate-fps n g*
  by (*rule poly-eqI*) (*auto simp*: *coeff-truncate-fps*)

lemma *truncate-fps-diff*: *truncate-fps n* (*f* − *g*) = *truncate-fps n f* − *truncate-fps n g*
  by (*rule poly-eqI*) (*auto simp*: *coeff-truncate-fps*)

lemma *truncate-fps-uminus*: *truncate-fps n* (−*f*) = −*truncate-fps n f*
  by (*rule poly-eqI*) (*auto simp*: *coeff-truncate-fps*)

lemma *fps-of-poly-truncate* [*simp*]: *fps-of-poly* (*truncate-fps n f*) = *fps-cutoff n f*
  by (*rule fps-ext*) *auto*

end

7 A formalization of formal Laurent series

theory *Formal-Laurent-Series*
imports
  Polynomial-FPS
begin

7.1 The type of formal Laurent series

7.1.1 Type definition

typedef (**overloaded**) *'a fls* = $\{f::int \Rightarrow 'a::zero.\ \forall_\infty\ n::nat.\ f\ (-\ int\ n) = 0\}$
  morphisms *fls-nth Abs-fls*
proof
  show $(\lambda x.\ 0) \in \{f::int \Rightarrow 'a::zero.\ \forall_\infty\ n::nat.\ f\ (-\ int\ n) = 0\}$
    by *simp*
qed

setup-lifting *type-definition-fls*

unbundle *fps-syntax*
notation *fls-nth* (**infixl** ‹$$› *75*)

lemmas *fls-eqI* = *iffD1*[*OF fls-nth-inject*, *OF iffD2*, *OF fun-eq-iff*, *OF allI*]

lemma *fls-eq-iff*: $f = g \longleftrightarrow (\forall\, n.\ f\ \$\$\ n = g\ \$\$\ n)$
  by (*simp add*: *fls-nth-inject*[*symmetric*] *fun-eq-iff*)

lemma *nth-Abs-fls* [*simp*]: $\forall_\infty n.\ f\ (-\ int\ n) = 0 \Longrightarrow$ *Abs-fls f* $\$\$\ n = f\ n$
 by (*simp add*: *Abs-fls-inverse*[*OF CollectI*])

lemmas *nth-Abs-fls-finite-nonzero-neg-nth* = *nth-Abs-fls*[*OF iffD2*, *OF eventually-cofinite*]
lemmas *nth-Abs-fls-ex-nat-lower-bound* = *nth-Abs-fls*[*OF iffD2*, *OF MOST-nat*]
lemmas *nth-Abs-fls-nat-lower-bound* = *nth-Abs-fls-ex-nat-lower-bound*[*OF exI*]

lemma *nth-Abs-fls-ex-lower-bound*:
  assumes $\exists\, N.\ \forall\, n{<}N.\ f\ n = 0$
  shows *Abs-fls f* $\$\$\ n = f\ n$
proof (*intro nth-Abs-fls-ex-nat-lower-bound*)
  from *assms* **obtain** *N::int* **where** $\forall\, n{<}N.\ f\ n = 0$ **by** *fast*
  hence $\forall\, n > (if\ N < 0\ then\ nat\ (-N)\ else\ 0).\ f\ (-int\ n) = 0$ **by** *auto*
  thus $\exists\, M.\ \forall\, n{>}M.\ f\ (-\ int\ n) = 0$ **by** *fast*
qed

lemmas *nth-Abs-fls-lower-bound* = *nth-Abs-fls-ex-lower-bound*[*OF exI*]

lemmas *MOST-fls-neg-nth-eq-0* [*simp*] = *CollectD*[*OF fls-nth*]
lemmas *fls-finite-nonzero-neg-nth* = *iffD1*[*OF eventually-cofinite MOST-fls-neg-nth-eq-0*]

lemma *fls-nth-vanishes-below-natE*:
fixes $f :: 'a::zero\ fls$
obtains $N :: nat$
where $\forall n{>}N.\ f\$\$(-int\ n) = 0$
using *iffD1*[*OF MOST-nat MOST-fls-neg-nth-eq-0*]
by *blast*

lemma *fls-nth-vanishes-belowE*:
fixes $f :: 'a::zero\ fls$
obtains $N :: int$
where $\forall n{<}N.\ f\$\$n = 0$
proof−
obtain $K :: nat$ **where** K: $\forall n{>}K.\ f\$\$(-int\ n) = 0$ **by** (*elim fls-nth-vanishes-below-natE*)
have $\forall n < -int\ K.\ f\$\$n = 0$
proof *clarify*
fix n **assume** n: $n < -int\ K$
define m **where** $m \equiv nat\ (-n)$
with n **have** $m > K$ **by** *simp*
moreover from n *m-def* **have** $f\$\$n = f\ \$\$\ (-int\ m)$ **by** *simp*
ultimately show $f\ \$\$\ n = 0$ **using** K **by** *simp*
qed
thus $(\bigwedge N.\ \forall n{<}N.\ f\ \$\$\ n = 0 \Longrightarrow thesis) \Longrightarrow thesis$ **by** *fast*
qed

7.1.2 Definition of basic Laurent series

instantiation *fls* :: (*zero*) *zero*
begin
lift-definition *zero-fls* :: $'a\ fls$ **is** λ-. *0* **by** *simp*
instance ..
end

lemma *fls-zero-nth* [*simp*]: $0\ \$\$\ n = 0$
by (*simp add*: *zero-fls-def*)

lemma *fls-zero-eqI*: $(\bigwedge n.\ f\$\$n = 0) \Longrightarrow f = 0$
by (*fastforce intro*: *fls-eqI*)

lemma *fls-nonzeroI*: $f\$\$n \neq 0 \Longrightarrow f \neq 0$
by *auto*

lemma *fls-nonzero-nth*: $f \neq 0 \longleftrightarrow (\exists\ n.\ f\ \$\$\ n \neq 0)$
using *fls-zero-eqI* **by** *fastforce*

lemma *fls-trivial-delta-eq-zero* [*simp*]: $b = 0 \Longrightarrow Abs\text{-}fls\ (\lambda n.\ if\ n{=}a\ then\ b\ else\ 0) = 0$
by (*intro fls-zero-eqI*) *simp*

lemma *fls-delta-nth* [*simp*]:

Abs-fls (λ*n*. *if n=a then b else 0*) $$ *n* = (*if n=a then b else 0*)
using *nth-Abs-fls-lower-bound*[*of a* λ*n*. *if n=a then b else 0*] **by** *simp*

instantiation *fls* :: ({*zero*,*one*}) *one*
begin
lift-definition *one-fls* :: *'a fls* **is** λ*k*. *if k = 0 then 1 else 0*
by (*simp add*: *eventually-cofinite*)
instance ..
end

lemma *fls-one-nth* [*simp*]:
1 $$ *n* = (*if n = 0 then 1 else 0*)
by (*simp add*: *one-fls-def eventually-cofinite*)

instance *fls* :: (*zero-neq-one*) *zero-neq-one*
proof (*standard*, *standard*)
assume (*0*::*'a fls*) = (*1*::*'a fls*)
hence (*0*::*'a fls*) $$ *0* = (*1*::*'a fls*) $$ *0* **by** *simp*
thus *False* **by** *simp*
qed

definition *fls-const* :: *'a*::*zero* ⇒ *'a fls*
where *fls-const c* ≡ *Abs-fls* (λ*n*. *if n = 0 then c else 0*)

lemma *fls-const-nth* [*simp*]: *fls-const c* $$ *n* = (*if n = 0 then c else 0*)
by (*simp add*: *fls-const-def eventually-cofinite*)

lemma *fls-const-0* [*simp*]: *fls-const 0* = *0*
unfolding *fls-const-def* **using** *fls-trivial-delta-eq-zero* **by** *fast*

lemma *fls-const-nonzero*: *c* ≠ *0* ⟹ *fls-const c* ≠ *0*
using *fls-nonzeroI*[*of fls-const c 0*] **by** *simp*

lemma *fls-const-eq-0-iff* [*simp*]: *fls-const c* = *0* ⟷ *c* = *0*
by (*auto simp*: *fls-eq-iff*)

lemma *fls-const-1* [*simp*]: *fls-const 1* = *1*
unfolding *fls-const-def one-fls-def* **..**

lemma *fls-const-eq-1-iff* [*simp*]: *fls-const c* = *1* ⟷ *c* = *1*
by (*auto simp*: *fls-eq-iff*)

lift-definition *fls-X* :: *'a*::{*zero*,*one*} *fls*
is λ*n*. *if n = 1 then 1 else 0*
by *simp*

lemma *fls-X-nth* [*simp*]:
fls-X $$ *n* = (*if n = 1 then 1 else 0*)
by (*simp add*: *fls-X-def*)

lemma *fls-X-nonzero* [*simp*]: (*fls-X* :: *'a* :: *zero-neq-one fls*) ≠ *0*
by (*intro fls-nonzeroI*) *simp*

lift-definition *fls-X-inv* :: *'a*::{*zero*,*one*} *fls*
is λ*n*. *if n* = −*1 then 1 else 0*
by (*simp add*: *eventually-cofinite*)

lemma *fls-X-inv-nth* [*simp*]:
fls-X-inv $$ *n* = (*if n* = −*1 then 1 else 0*)
by (*simp add*: *fls-X-inv-def eventually-cofinite*)

lemma *fls-X-inv-nonzero* [*simp*]: (*fls-X-inv* :: *'a* :: *zero-neq-one fls*) ≠ *0*
by (*intro fls-nonzeroI*) *simp*

7.2 Subdegrees

lemma *unique-fls-subdegree*:
assumes *f* ≠ *0*
shows ∃!*n*. *f*$$*n* ≠ *0* ∧ (∀ *m*. *f*$$*m* ≠ *0* ⟶ *n* ≤ *m*)
proof−
obtain *N*::*nat* **where** *N*: ∀ *n*>*N*. *f*$$(−*int n*) = *0* **by** (*elim fls-nth-vanishes-below-natE*)
define *M* **where** *M* ≡ −*int N*
have *M*: ⋀*m*. *f*$$*m* ≠ *0* ⟹ *M* ≤ *m*
proof−
fix *m* **assume** *m*: *f*$$*m* ≠ *0*
show *M* ≤ *m*
proof (*cases m*<*0*)
case *True* **with** *m N M-def* **show** *?thesis*
using *allE*[*OF N*, *of nat* (−*m*) *False*] **by** *force*
qed (*simp add*: *M-def*)
qed
have ¬ (∀ *k*::*nat*. *f*$$(*M* + *int k*) = *0*)
proof
assume *above0*: ∀ *k*::*nat*. *f*$$(*M* + *int k*) = *0*
have *f*=*0*
proof (*rule fls-zero-eqI*)
fix *n* **show** *f*$$*n* = *0*
proof (*cases M* ≤ *n*)
case *True*
define *k* **where** *k* = *nat* (*n* − *M*)
from *True* **have** *n* = *M* + *int k* **by** (*simp add*: *k-def*)
with *above0* **show** *?thesis* **by** *simp*
next
case *False* **with** *M* **show** *?thesis* **by** *auto*
qed
qed
with *assms* **show** *False* **by** *fast*
qed

hence *ex-k*: $\exists\, k{::}nat.\ f\$\$(M + int\ k) \neq 0$ **by** *fast*
define k **where** $k \equiv (LEAST\ k{::}nat.\ f\$\$(M + int\ k) \neq 0)$
define n **where** $n \equiv M + int\ k$
from *k-def n-def* **have** *fn*: $f\$\$n \neq 0$ **using** *LeastI-ex*[*OF ex-k*] **by** *simp*
moreover have $\forall\, m.\ f\$\$m \neq 0 \longrightarrow n \leq m$
proof (*clarify*)
fix m **assume** m: $f\$\$m \neq 0$
with M **have** $M \leq m$ **by** *fast*
define l **where** $l = nat\ (m - M)$
from ‹$M \leq m$› **have** l: $m = M + int\ l$ **by** (*simp add*: *l-def*)
with *n-def m k-def l* **show** $n \leq m$
using *Least-le*[*of* $\lambda k.\ f\$\$(M + int\ k) \neq 0\ l$] **by** *auto*
qed
moreover have $\bigwedge n'.\ f\$\$n' \neq 0 \Longrightarrow (\forall\, m.\ f\$\$m \neq 0 \longrightarrow n' \leq m) \Longrightarrow n' = n$
proof−
fix n' :: *int*
assume n': $f\$\$n' \neq 0\ \forall\, m.\ f\$\$m \neq 0 \longrightarrow n' \leq m$
from $n'(1)$ M **have** $M \leq n'$ **by** *fast*
define l **where** $l = nat\ (n' - M)$
from ‹$M \leq n'$› **have** l: $n' = M + int\ l$ **by** (*simp add*: *l-def*)
with *n-def k-def n' fn* **show** $n' = n$
using *Least-le*[*of* $\lambda k.\ f\$\$(M + int\ k) \neq 0\ l$] **by** *force*
qed
ultimately show *?thesis*
using *ex1I*[*of* $\lambda n.\ f\$\$n \neq 0 \wedge (\forall\, m.\ f\$\$m \neq 0 \longrightarrow n \leq m)\ n$] **by** *blast*
qed

definition *fls-subdegree* :: ('*a*::*zero*) *fls* ⇒ *int*
where *fls-subdegree f* ≡ (*if f* = *0 then 0 else LEAST n*::*int*. *f*$$*n* ≠ *0*)

lemma *fls-zero-subdegree* [*simp*]: *fls-subdegree 0* = *0*
by (*simp add*: *fls-subdegree-def*)

lemma *nth-fls-subdegree-nonzero* [*simp*]: *f* ≠ *0* ⟹ *f* $$ *fls-subdegree f* ≠ *0*
using *Least1I*[*OF unique-fls-subdegree*] **by** (*simp add*: *fls-subdegree-def*)

lemma *nth-fls-subdegree-zero-iff*: (*f* $$ *fls-subdegree f* = *0*) ⟷ (*f* = *0*)
using *nth-fls-subdegree-nonzero* **by** *auto*

lemma *fls-subdegree-leI*: *f* $$ *n* ≠ *0* ⟹ *fls-subdegree f* ≤ *n*
using *Least1-le*[*OF unique-fls-subdegree*]
by (*auto simp*: *fls-subdegree-def*)

lemma *fls-subdegree-leI'*: *f* $$ *n* ≠ *0* ⟹ *n* ≤ *m* ⟹ *fls-subdegree f* ≤ *m*
using *fls-subdegree-leI* **by** *fastforce*

lemma *fls-eq0-below-subdegree* [*simp*]: *n* < *fls-subdegree f* ⟹ *f* $$ *n* = *0*
using *fls-subdegree-leI* **by** *fastforce*

lemma *fls-subdegree-geI*: $f \neq 0 \Longrightarrow (\bigwedge k.\ k < n \Longrightarrow f\ \$\$\ k = 0) \Longrightarrow n \leq$ *fls-subdegree f*
using *nth-fls-subdegree-nonzero* **by** *force*

lemma *fls-subdegree-ge0I*: $(\bigwedge k.\ k < 0 \Longrightarrow f\ \$\$\ k = 0) \Longrightarrow 0 \leq$ *fls-subdegree f*
using *fls-subdegree-geI*[*of f 0*] **by** (*cases f=0*) *auto*

lemma *fls-subdegree-greaterI*:
assumes $f \neq 0$ $\bigwedge k.\ k \leq n \Longrightarrow f\ \$\$\ k = 0$
shows $n <$ *fls-subdegree f*
using *assms*(*1*) *assms*(*2*)[*of fls-subdegree f*] *nth-fls-subdegree-nonzero*[*of f*]
by *force*

lemma *fls-subdegree-eqI*: $f\ \$\$\ n \neq 0 \Longrightarrow (\bigwedge k.\ k < n \Longrightarrow f\ \$\$\ k = 0) \Longrightarrow$ *fls-subdegree f* $= n$
using *fls-subdegree-leI fls-subdegree-geI*[*of f*]
by *fastforce*

lemma *fls-delta-subdegree* [*simp*]:
$b \neq 0 \Longrightarrow$ *fls-subdegree* (*Abs-fls* ($\lambda n.$ *if n=a then b else 0*)) $= a$
by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-delta0-subdegree*: *fls-subdegree* (*Abs-fls* ($\lambda n.$ *if n=0 then a else 0*)) $= 0$
by (*cases a=0*) *simp-all*

lemma *fls-one-subdegree* [*simp*]: *fls-subdegree 1* $= 0$
by (*auto intro*: *fls-delta0-subdegree simp*: *one-fls-def*)

lemma *fls-const-subdegree* [*simp*]: *fls-subdegree* (*fls-const c*) $= 0$
by (*cases c=0*) (*auto intro*: *fls-subdegree-eqI*)

lemma *fls-X-subdegree* [*simp*]: *fls-subdegree* (*fls-X*::*'a*::{*zero-neq-one*} *fls*) $= 1$
by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-X-inv-subdegree* [*simp*]: *fls-subdegree* (*fls-X-inv*::*'a*::{*zero-neq-one*} *fls*) $= -1$
by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-eq-above-subdegreeI*:
assumes $N \leq$ *fls-subdegree f* $N \leq$ *fls-subdegree g* $\forall k{\geq}N.\ f\ \$\$\ k = g\ \$\$\ k$
shows $f = g$
proof (*rule fls-eqI*)
fix *n* **from** *assms* **show** $f\ \$\$\ n = g\ \$\$\ n$ **by** (*cases* $n < N$) *auto*
qed

7.3 Shifting

7.3.1 Shift definition

definition *fls-shift* :: *int* $\Rightarrow$ (*'a*::*zero*) *fls* $\Rightarrow$ *'a fls*

where *fls-shift n f* ≡ *Abs-fls* (λ*k*. *f* $$ (*k*+*n*))
— Since the index set is unbounded in both directions, we can shift in either direction.

lemma *fls-shift-nth* [*simp*]: *fls-shift m f* $$ *n* = *f* $$ (*n*+*m*)
unfolding *fls-shift-def*
proof (*rule nth-Abs-fls-ex-lower-bound*)
obtain *K*::*int* **where** *K*: ∀ *n*<*K*. *f*$$*n* = *0* **by** (*elim fls-nth-vanishes-belowE*)
hence ∀ *n*<*K*−*m*. *f*$$(*n*+*m*) = *0* **by** *auto*
thus ∃ *N*. ∀ *n*<*N*. *f* $$ (*n* + *m*) = *0* **by** *fast*
qed

lemma *fls-shift-eq-iff*: (*fls-shift m f* = *fls-shift m g*) ⟷ (*f* = *g*)
proof (*rule iffI*, *rule fls-eqI*)
fix *k*
assume *1*: *fls-shift m f* = *fls-shift m g*
have *f* $$ *k* = *fls-shift m g* $$ (*k* − *m*) **by** (*simp add*: *1*[*symmetric*])
thus *f* $$ *k* = *g* $$ *k* **by** *simp*
qed (*intro fls-eqI*, *simp*)

lemma *fls-shift-0* [*simp*]: *fls-shift 0 f* = *f*
by (*intro fls-eqI*) *simp*

lemma *fls-shift-subdegree* [*simp*]:
f ≠ *0* ⟹ *fls-subdegree* (*fls-shift n f*) = *fls-subdegree f* − *n*
by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-shift-fls-shift* [*simp*]: *fls-shift m* (*fls-shift k f*) = *fls-shift* (*k*+*m*) *f*
by (*intro fls-eqI*) (*simp add*: *algebra-simps*)

lemma *fls-shift-fls-shift-reorder*:
fls-shift m (*fls-shift k f*) = *fls-shift k* (*fls-shift m f*)
using *fls-shift-fls-shift*[*of m k f*] *fls-shift-fls-shift*[*of k m f*] **by** (*simp add*: *add.commute*)

lemma *fls-shift-zero* [*simp*]: *fls-shift m 0* = *0*
by (*intro fls-zero-eqI*) *simp*

lemma *fls-shift-eq0-iff*: *fls-shift m f* = *0* ⟷ *f* = *0*
using *fls-shift-eq-iff*[*of m f 0*] **by** *simp*

lemma *fls-shift-eq-1-iff*: *fls-shift n f* = *1* ⟷ *f* = *fls-shift* (−*n*) *1*
by (*metis add-minus-cancel fls-shift-eq-iff fls-shift-fls-shift*)

lemma *fls-shift-nonneg-subdegree*: *m* ≤ *fls-subdegree f* ⟹ *fls-subdegree* (*fls-shift m f*) ≥ *0*
by (*cases f*=*0*) (*auto intro*: *fls-subdegree-geI*)

lemma *fls-shift-delta*:
fls-shift m (*Abs-fls* (λ*n*. *if n*=*a then b else 0*)) = *Abs-fls* (λ*n*. *if n*=*a*−*m then b*

else 0)
by (*intro fls-eqI*) *simp*

lemma *fls-shift-const*:
fls-shift m (*fls-const c*) = *Abs-fls* (λ*n*. *if n*=−*m then c else 0*)
by (*intro fls-eqI*) *simp*

lemma *fls-shift-const-nth*:
fls-shift m (*fls-const c*) $$ *n* = (*if n*=−*m then c else 0*)
by (*simp add*: *fls-shift-const*)

lemma *fls-X-conv-shift-1*: *fls-X* = *fls-shift* (−*1*) *1*
by (*intro fls-eqI*) *simp*

lemma *fls-X-shift-to-one* [*simp*]: *fls-shift 1 fls-X* = *1*
using *fls-shift-fls-shift*[*of* −*1 1 1*] **by** (*simp add*: *fls-X-conv-shift-1*)

lemma *fls-X-inv-conv-shift-1*: *fls-X-inv* = *fls-shift 1 1*
by (*intro fls-eqI*) *simp*

lemma *fls-X-inv-shift-to-one* [*simp*]: *fls-shift* (−*1*) *fls-X-inv* = *1*
using *fls-shift-fls-shift*[*of 1* −*1 1*] **by** (*simp add*: *fls-X-inv-conv-shift-1*)

lemma *fls-X-fls-X-inv-conv*:
fls-X = *fls-shift* (−*2*) *fls-X-inv fls-X-inv* = *fls-shift 2 fls-X*
by (*simp-all add*: *fls-X-conv-shift-1 fls-X-inv-conv-shift-1*)

7.3.2 Base factor

Similarly to the *unit-factor* for formal power series, we can decompose a formal Laurent series as a power of the implied variable times a series of subdegree 0. (See lemma *fls-base-factor-X-power-decompose*.) But we will call this something other *unit-factor* because it will not satisfy assumption *is-unit-unit-factor* of *semidom-divide-unit-factor*.

definition *fls-base-factor* :: ('*a*::*zero*) *fls* ⇒ '*a fls*
where *fls-base-factor-def*[*simp*]: *fls-base-factor f* = *fls-shift* (*fls-subdegree f*) *f*

lemma *fls-base-factor-nth*: *fls-base-factor f* $$ *n* = *f* $$ (*n* + *fls-subdegree f*)
by *simp*

lemma *fls-base-factor-nonzero* [*simp*]: *f* ≠ *0* ⟹ *fls-base-factor f* ≠ *0*
using *fls-nonzeroI*[*of fls-base-factor f 0*] **by** *simp*

lemma *fls-base-factor-subdegree* [*simp*]: *fls-subdegree* (*fls-base-factor f*) = *0*
by (*cases f*=*0*) *auto*

lemma *fls-base-factor-base* [*simp*]:
fls-base-factor f $$ *fls-subdegree* (*fls-base-factor f*) = *f* $$ *fls-subdegree f*

using *fls-base-factor-subdegree*[*of f*] **by** *simp*

lemma *fls-conv-base-factor-shift-subdegree*:
f = *fls-shift* (−*fls-subdegree f*) (*fls-base-factor f*)
by *simp*

lemma *fls-base-factor-idem*:
fls-base-factor (*fls-base-factor* (*f*::′*a*::*zero fls*)) = *fls-base-factor f*
using *fls-base-factor-subdegree*[*of f*] **by** *simp*

lemma *fls-base-factor-zero*: *fls-base-factor* (*0*::′*a*::*zero fls*) = *0*
by *simp*

lemma *fls-base-factor-zero-iff*: *fls-base-factor* (*f*::′*a*::*zero fls*) = *0* ⟷ *f* = *0*
proof
have *fls-shift* (−*fls-subdegree f*) (*fls-shift* (*fls-subdegree f*) *f*) = *f* **by** *simp*
thus *fls-base-factor f* = *0* ⟹ *f*=*0* **by** *simp*
qed *simp*

lemma *fls-base-factor-nth-0*: *f* ≠ *0* ⟹ *fls-base-factor f* $$ *0* ≠ *0*
by *simp*

lemma *fls-base-factor-one*: *fls-base-factor* (*1*::′*a*::{*zero*,*one*} *fls*) = *1*
by *simp*

lemma *fls-base-factor-const*: *fls-base-factor* (*fls-const c*) = *fls-const c*
by *simp*

lemma *fls-base-factor-delta*:
fls-base-factor (*Abs-fls* (λ*n*. *if n*=*a then c else 0*)) = *fls-const c*
by (*cases c*=*0*) (*auto intro*: *fls-eqI*)

lemma *fls-base-factor-X*: *fls-base-factor* (*fls-X*::′*a*::{*zero-neq-one*} *fls*) = *1*
by *simp*

lemma *fls-base-factor-X-inv*: *fls-base-factor* (*fls-X-inv*::′*a*::{*zero-neq-one*} *fls*) = *1*
by *simp*

lemma *fls-base-factor-shift* [*simp*]: *fls-base-factor* (*fls-shift n f*) = *fls-base-factor f*
by (*cases f*=*0*) *simp-all*

7.4 Conversion between formal power and Laurent series

7.4.1 Converting Laurent to power series

We can truncate a Laurent series at index 0 to create a power series, called the regular part.

lift-definition *fls-regpart* :: (′*a*::*zero*) *fls* ⇒ ′*a fps*
is λ*f*. *Abs-fps* (λ*n*. *f* (*int n*))

.

lemma *fls-regpart-nth* [*simp*]: *fls-regpart f $ n = f $$ (int n)*
by (*simp add*: *fls-regpart-def*)

lemma *fls-regpart-zero* [*simp*]: *fls-regpart 0 = 0*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-one* [*simp*]: *fls-regpart 1 = 1*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-Abs-fls*:
$\forall_\infty n$. *F* (− *int n*) = *0* ⟹ *fls-regpart* (*Abs-fls F*) = *Abs-fps* (λ*n*. *F* (*int n*))
by (*intro fps-ext*) *auto*

lemma *fls-regpart-delta*:
fls-regpart (*Abs-fls* (λ*n*. *if n=a then b else 0*)) =
(*if a < 0 then 0 else Abs-fps* (λ*n*. *if n=nat a then b else 0*))
by (*rule fps-ext*, *auto*)

lemma *fls-regpart-const* [*simp*]: *fls-regpart* (*fls-const c*) = *fps-const c*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-fls-X* [*simp*]: *fls-regpart fls-X = fps-X*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-fls-X-inv* [*simp*]: *fls-regpart fls-X-inv = 0*
by (*intro fps-ext*) *simp*

lemma *fls-regpart-eq0-imp-nonpos-subdegree*:
assumes *fls-regpart f = 0*
shows *fls-subdegree f* ≤ *0*
proof (*cases f=0*)
case *False*
have *fls-subdegree f* ≥ *0* ⟹ *f $$ fls-subdegree f = 0*
proof−
assume *fls-subdegree f* ≥ *0*
hence *f $$* (*fls-subdegree f*) = (*fls-regpart f*) $ (*nat* (*fls-subdegree f*)) **by** *simp*
with *assms* **show** *f $$* (*fls-subdegree f*) = *0* **by** *simp*
qed
with *False* **show** *?thesis* **by** *fastforce*
qed *simp*

lemma *fls-subdegree-lt-fls-regpart-subdegree*:
fls-subdegree f ≤ *int* (*subdegree* (*fls-regpart f*))
using *fls-subdegree-leI nth-subdegree-nonzero*[*of fls-regpart f*]
by (*cases* (*fls-regpart f*) = *0*)
(*simp-all add*: *fls-regpart-eq0-imp-nonpos-subdegree*)

lemma *fls-regpart-subdegree-conv*:
assumes *fls-subdegree f ≥ 0*
shows *subdegree (fls-regpart f) = nat (fls-subdegree f)*
— This is the best we can do since if the subdegree is negative, we might still have the bad luck that the term at index 0 is equal to 0.
proof (*cases f=0*)
case *False* **with** *assms* **show** *?thesis* **by** (*intro subdegreeI*) *simp-all*
qed *simp*

lemma *fls-eq-conv-fps-eqI*:
assumes *0 ≤ fls-subdegree f 0 ≤ fls-subdegree g fls-regpart f = fls-regpart g*
shows *f = g*
proof (*rule fls-eq-above-subdegreeI*, *rule assms(1)*, *rule assms(2)*, *clarify*)
fix *k*::*int* **assume** *0 ≤ k*
with *assms(3)* **show** *f $$ k = g $$ k*
using *fls-regpart-nth*[*of f nat k*] *fls-regpart-nth*[*of g*] **by** *simp*
qed

lemma *fls-regpart-shift-conv-fps-shift*:
m ≥ 0 ⟹ fls-regpart (fls-shift m f) = fps-shift (nat m) (fls-regpart f)
by (*intro fps-ext*) *simp-all*

lemma *fps-shift-fls-regpart-conv-fls-shift*:
fps-shift m (fls-regpart f) = fls-regpart (fls-shift m f)
by (*intro fps-ext*) *simp-all*

lemma *fps-unit-factor-fls-regpart*:
fls-subdegree f ≥ 0 ⟹ unit-factor (fls-regpart f) = fls-regpart (fls-base-factor f)
by (*auto intro*: *fps-ext simp*: *fls-regpart-subdegree-conv*)

The terms below the zeroth form a polynomial in the inverse of the implied variable, called the principle part.

lift-definition *fls-prpart* :: (*'a*::*zero*) *fls* ⇒ *'a poly*
is *λf. Abs-poly (λn. if n = 0 then 0 else f (− int n))*
.

lemma *fls-prpart-coeff* [*simp*]: *coeff (fls-prpart f) n = (if n = 0 then 0 else f $$ (− int n))*
proof−
have *{x. (if x = 0 then 0 else f $$ − int x) ≠ 0} ⊆ {x. f $$ − int x ≠ 0}*
by *auto*
hence *finite {x. (if x = 0 then 0 else f $$ − int x) ≠ 0}*
using *fls-finite-nonzero-neg-nth*[*of f*] **by** (*simp add*: *rev-finite-subset*)
hence *coeff (fls-prpart f) = (λn. if n = 0 then 0 else f $$ (− int n))*
using *Abs-poly-inverse*[*OF CollectI*, *OF iffD2*, *OF eventually-cofinite*]
by (*simp add*: *fls-prpart-def*)
thus *?thesis* **by** *simp*
qed

lemma *fls-prpart-eq0-iff*: (*fls-prpart f = 0*) ⟷ (*fls-subdegree f ≥ 0*)
proof
  assume *1*: *fls-prpart f = 0*
  show *fls-subdegree f ≥ 0*
  proof (*intro fls-subdegree-ge0I*)
    fix *k*::*int* **assume** *k < 0*
    with *1* **show** *f $$ k = 0* **using** *fls-prpart-coeff*[*of f nat* (−*k*)] **by** *simp*
  qed
qed (*intro poly-eqI*, *simp*)

lemma *fls-prpart0* [*simp*]: *fls-prpart 0 = 0*
  by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-prpart-one* [*simp*]: *fls-prpart 1 = 0*
  by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-prpart-delta*:
  fls-prpart (*Abs-fls* (λ*n*. *if n=a then b else 0*)) =
    (*if a<0 then Poly* (*replicate* (*nat* (−*a*)) *0* @ [*b*]) *else 0*)
  by (*intro poly-eqI*) (*auto simp*: *nth-default-def nth-append*)

lemma *fls-prpart-const* [*simp*]: *fls-prpart* (*fls-const c*) = *0*
  by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-prpart-X* [*simp*]: *fls-prpart fls-X = 0*
  by (*intro poly-eqI*) *simp*

lemma *fls-prpart-X-inv*: *fls-prpart fls-X-inv* = [:*0*,*1*:]
proof (*intro poly-eqI*)
  fix *n* **show** *coeff* (*fls-prpart fls-X-inv*) *n* = *coeff* [:*0*,*1*:] *n*
  proof (*cases n*)
    case (*Suc i*) **thus** *?thesis* **by** (*cases i*) *simp-all*
  qed *simp*
qed

lemma *degree-fls-prpart* [*simp*]:
  degree (*fls-prpart f*) = *nat* (−*fls-subdegree f*)
proof (*cases f=0*)
  case *False* **show** *?thesis* **unfolding** *degree-def*
  proof (*intro Least-equality*)
    fix *N* **assume** *N*: ∀ *i*>*N*. *coeff* (*fls-prpart f*) *i* = *0*
    have ∀ *i* < −*int N*. *f $$ i* = *0*
    proof *clarify*
      fix *i* **assume** *i*: *i* < −*int N*
      hence *nat* (−*i*) > *N* **by** *simp*
      with *N i* **show** *f $$ i* = *0* **using** *fls-prpart-coeff*[*of f nat* (−*i*)] **by** *auto*
    qed
    with *False* **have** *fls-subdegree f* ≥ −*int N* **using** *fls-subdegree-geI* **by** *auto*
    thus *nat* (− *fls-subdegree f*) ≤ *N* **by** *simp*

qed *auto*
qed *simp*

lemma *fls-prpart-shift*:
assumes $m \leq 0$
shows *fls-prpart* (*fls-shift* m f) = *pCons* 0 (*poly-shift* (*Suc* (*nat* $(-m)$)) (*fls-prpart* f))
proof (*intro poly-eqI*)
fix n
define *LHS RHS*
where *LHS* ≡ *fls-prpart* (*fls-shift* m f)
and *RHS* ≡ *pCons* 0 (*poly-shift* (*Suc* (*nat* $(-m)$)) (*fls-prpart* f))
show *coeff LHS* n = *coeff RHS* n
proof (*cases* n)
case (*Suc* k)
from *assms* **have** *1*: $-$*int* (*Suc* k + *nat* $(-m)$) = $-$*int* (*Suc* k) + m **by** *simp*
have *coeff RHS* n = f $$ ($-$*int* (*Suc* k) + m)
using *arg-cong*[*OF 1*, *of* ($$) f] **by** (*simp add*: *Suc RHS-def coeff-poly-shift*)
with *Suc* **show** *?thesis* **by** (*simp add*: *LHS-def*)
qed (*simp add*: *LHS-def RHS-def*)
qed

lemma *fls-prpart-base-factor*: *fls-prpart* (*fls-base-factor* f) = *0*
using *fls-base-factor-subdegree*[*of f*] **by** (*simp add*: *fls-prpart-eq0-iff*)

The essential data of a formal Laurant series resides from the subdegree up.

abbreviation *fls-base-factor-to-fps* :: ('*a*::*zero*) *fls* ⇒ '*a fps*
where *fls-base-factor-to-fps* f ≡ *fls-regpart* (*fls-base-factor* f)

lemma *fls-base-factor-to-fps-conv-fps-shift*:
assumes *fls-subdegree* $f \geq 0$
shows *fls-base-factor-to-fps* f = *fps-shift* (*nat* (*fls-subdegree* f)) (*fls-regpart* f)
by (*simp add*: *assms fls-regpart-shift-conv-fps-shift*)

lemma *fls-base-factor-to-fps-nth*:
fls-base-factor-to-fps f $ n = f $$ (*fls-subdegree* f + *int* n)
by (*simp add*: *algebra-simps*)

lemma *fls-base-factor-to-fps-base*: $f \neq 0 \Longrightarrow$ *fls-base-factor-to-fps* f $ *0* ≠ *0*
by *simp*

lemma *fls-base-factor-to-fps-nonzero*: $f \neq 0 \Longrightarrow$ *fls-base-factor-to-fps* $f \neq 0$
using *fps-nonzeroI*[*of fls-base-factor-to-fps f 0*] *fls-base-factor-to-fps-base* **by** *simp*

lemma *fls-base-factor-to-fps-subdegree* [*simp*]: *subdegree* (*fls-base-factor-to-fps* f) = *0*
by (*cases f=0*) *auto*

lemma *fls-base-factor-to-fps-trivial*:

fls-subdegree f = 0 ⟹ fls-base-factor-to-fps f = fls-regpart f
by *simp*

lemma *fls-base-factor-to-fps-zero*: *fls-base-factor-to-fps 0 = 0*
by *simp*

lemma *fls-base-factor-to-fps-one*: *fls-base-factor-to-fps 1 = 1*
by *simp*

lemma *fls-base-factor-to-fps-delta*:
fls-base-factor-to-fps (Abs-fls (λn. if n=a then c else 0)) = fps-const c
using *fls-base-factor-delta*[*of a c*] **by** *simp*

lemma *fls-base-factor-to-fps-const*:
fls-base-factor-to-fps (fls-const c) = fps-const c
by *simp*

lemma *fls-base-factor-to-fps-X*:
fls-base-factor-to-fps (fls-X::'a::{zero-neq-one} fls) = 1
by *simp*

lemma *fls-base-factor-to-fps-X-inv*:
fls-base-factor-to-fps (fls-X-inv::'a::{zero-neq-one} fls) = 1
by *simp*

lemma *fls-base-factor-to-fps-shift*:
fls-base-factor-to-fps (fls-shift m f) = fls-base-factor-to-fps f
using *fls-base-factor-shift*[*of m f*] **by** *simp*

lemma *fls-base-factor-to-fps-base-factor*:
fls-base-factor-to-fps (fls-base-factor f) = fls-base-factor-to-fps f
using *fls-base-factor-to-fps-shift* **by** *simp*

lemma *fps-unit-factor-fls-base-factor*:
unit-factor (fls-base-factor-to-fps f) = fls-base-factor-to-fps f
using *fls-base-factor-to-fps-subdegree*[*of f*] **by** *simp*

7.4.2 Converting power to Laurent series

We can extend a power series by 0s below to create a Laurent series.

definition *fps-to-fls* :: ('*a*::*zero*) *fps* ⇒ '*a fls*
where *fps-to-fls f* ≡ *Abs-fls* (λ*k*::*int*. *if k<0 then 0 else f* $ (*nat k*))

lemma *fps-to-fls-nth* [*simp*]:
(*fps-to-fls f*) $$ *n* = (*if n* < *0 then 0 else f*$(*nat n*))
using *nth-Abs-fls-lower-bound*[*of 0* (λ*k*::*int*. *if k<0 then 0 else f* $ (*nat k*))]
unfolding *fps-to-fls-def*
by *simp*

lemma *fps-to-fls-eq-imp-fps-eq*:
assumes *fps-to-fls f = fps-to-fls g*
shows *f = g*
proof (*intro fps-ext*)
fix *n*
have *f $ n = fps-to-fls g $$ int n* **by** (*simp add*: *assms*[*symmetric*])
thus *f $ n = g $ n* **by** *simp*
qed

lemma *fps-to-fls-eq-iff* [*simp*]: *fps-to-fls f = fps-to-fls g ⟷ f = g*
using *fps-to-fls-eq-imp-fps-eq* **by** *blast*

lemma *fps-zero-to-fls* [*simp*]: *fps-to-fls 0 = 0*
by (*intro fls-zero-eqI*) *simp*

lemma *fps-to-fls-nonzeroI*: *f ≠ 0 ⟹ fps-to-fls f ≠ 0*
using *fps-to-fls-eq-imp-fps-eq*[*of f 0*] **by** *auto*

lemma *fps-one-to-fls* [*simp*]: *fps-to-fls 1 = 1*
by (*intro fls-eqI*) *simp*

lemma *fps-to-fls-Abs-fps*:
fps-to-fls (*Abs-fps F*) = *Abs-fls* (*λn. if n<0 then 0 else F* (*nat n*))
using *nth-Abs-fls-lower-bound*[*of 0* (*λn::int. if n<0 then 0 else F* (*nat n*))]
by (*intro fls-eqI*) *simp*

lemma *fps-delta-to-fls*:
fps-to-fls (*Abs-fps* (*λn. if n=a then b else 0*)) = *Abs-fls* (*λn. if n=int a then b else 0*)
using *fls-eqI*[*of - Abs-fls* (*λn. if n=int a then b else 0*)] **by** *force*

lemma *fps-const-to-fls* [*simp*]: *fps-to-fls* (*fps-const c*) = *fls-const c*
by (*intro fls-eqI*) *simp*

lemma *fps-X-to-fls* [*simp*]: *fps-to-fls fps-X = fls-X*
by (*fastforce intro*: *fls-eqI*)

lemma *fps-to-fls-eq-0-iff* [*simp*]: (*fps-to-fls f = 0*) ⟷ (*f=0*)
using *fps-to-fls-nonzeroI* **by** *auto*

lemma *fps-to-fls-eq-1-iff* [*simp*]: *fps-to-fls f = 1 ⟷ f = 1*
using *fps-to-fls-eq-iff* **by** *fastforce*

lemma *fls-subdegree-fls-to-fps-gt0*: *fls-subdegree* (*fps-to-fls f*) ≥ *0*
proof (*cases f=0*)
case *False* **show** *?thesis*
proof (*rule fls-subdegree-geI*, *rule fls-nonzeroI*)
from *False* **show** *fps-to-fls f $$ int* (*subdegree f*) ≠ *0*
by *simp*

qed *simp*
qed *simp*

lemma *fls-subdegree-fls-to-fps*: *fls-subdegree* (*fps-to-fls f*) = *int* (*subdegree f*)
proof (*cases f=0*)
case *False*
have *subdegree f* = *nat* (*fls-subdegree* (*fps-to-fls f*))
proof (*rule subdegreeI*)
from *False* **show** *f* $ (*nat* (*fls-subdegree* (*fps-to-fls f*))) ≠ *0*
using *fls-subdegree-fls-to-fps-gt0*[*of f*] *nth-fls-subdegree-nonzero*[*of fps-to-fls f*]
fps-to-fls-nonzeroI[*of f*]
by *simp*
next
fix *k* **assume** *k*: *k* < *nat* (*fls-subdegree* (*fps-to-fls f*))
thus *f* $ *k* = *0*
using *fls-eq0-below-subdegree*[*of int k fps-to-fls f*] **by** *simp*
qed
thus *?thesis* **by** (*simp add*: *fls-subdegree-fls-to-fps-gt0*)
qed *simp*

lemma *fps-shift-to-fls* [*simp*]:
n ≤ *subdegree f* ⟹ *fps-to-fls* (*fps-shift n f*) = *fls-shift* (*int n*) (*fps-to-fls f*)
by (*auto intro*: *fls-eqI simp*: *nat-add-distrib nth-less-subdegree-zero*)

lemma *fls-base-factor-fps-to-fls*: *fls-base-factor* (*fps-to-fls f*) = *fps-to-fls* (*unit-factor f*)
using *nth-less-subdegree-zero*[*of - f*]
by (*auto intro*: *fls-eqI simp*: *fls-subdegree-fls-to-fps nat-add-distrib*)

lemma *fls-regpart-to-fls-trivial* [*simp*]:
fls-subdegree f ≥ *0* ⟹ *fps-to-fls* (*fls-regpart f*) = *f*
by (*intro fls-eqI*) *simp*

lemma *fls-regpart-fps-trivial* [*simp*]: *fls-regpart* (*fps-to-fls f*) = *f*
by (*intro fps-ext*) *simp*

lemma *fps-to-fls-base-factor-to-fps*:
fps-to-fls (*fls-base-factor-to-fps f*) = *fls-base-factor f*
by (*intro fls-eqI*) *simp*

lemma *fls-conv-base-factor-to-fps-shift-subdegree*:
f = *fls-shift* (−*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*))
using *fps-to-fls-base-factor-to-fps*[*of f*] *fps-to-fls-base-factor-to-fps*[*of f*] **by** *simp*

lemma *fls-base-factor-to-fps-to-fls*:
fls-base-factor-to-fps (*fps-to-fls f*) = *unit-factor f*
using *fls-base-factor-fps-to-fls*[*of f*] *fls-regpart-fps-trivial*[*of unit-factor f*]
by *simp*

lemma *fls-as-fps*:
fixes $f :: 'a :: zero\ fls$ **and** $n :: int$
assumes n: $n \geq -fls\text{-}subdegree\ f$
obtains f' **where** $f = fls\text{-}shift\ n\ (fps\text{-}to\text{-}fls\ f')$
proof −
have $fls\text{-}subdegree\ (fls\text{-}shift\ (-\ n)\ f) \geq 0$
by (*rule fls-shift-nonneg-subdegree*) (*use n* **in** *simp*)
hence $f = fls\text{-}shift\ n\ (fps\text{-}to\text{-}fls\ (fls\text{-}regpart\ (fls\text{-}shift\ (-n)\ f)))$
by (*subst fls-regpart-to-fls-trivial*) *simp-all*
thus *?thesis*
by (*rule that*)
qed

lemma *fls-as-fps'*:
fixes $f :: 'a :: zero\ fls$ **and** $n :: int$
assumes n: $n \geq -fls\text{-}subdegree\ f$
shows $\exists f'.\ f = fls\text{-}shift\ n\ (fps\text{-}to\text{-}fls\ f')$
using *fls-as-fps*[*OF assms*] **by** *metis*

abbreviation
$fls\text{-}regpart\text{-}as\text{-}fls\ f \equiv fps\text{-}to\text{-}fls\ (fls\text{-}regpart\ f)$
abbreviation
$fls\text{-}prpart\text{-}as\text{-}fls\ f \equiv$
$fls\text{-}shift\ (-fls\text{-}subdegree\ f)\ (fps\text{-}to\text{-}fls\ (fps\text{-}of\text{-}poly\ (reflect\text{-}poly\ (fls\text{-}prpart\ f))))$

lemma *fls-regpart-as-fls-nth*:
$fls\text{-}regpart\text{-}as\text{-}fls\ f\ \$\$\ n = (if\ n < 0\ then\ 0\ else\ f\ \$\$\ n)$
by *simp*

lemma *fls-regpart-idem*:
$fls\text{-}regpart\ (fls\text{-}regpart\text{-}as\text{-}fls\ f) = fls\text{-}regpart\ f$
by *simp*

lemma *fls-prpart-as-fls-nth*:
$fls\text{-}prpart\text{-}as\text{-}fls\ f\ \$\$\ n = (if\ n < 0\ then\ f\ \$\$\ n\ else\ 0)$
proof (*cases* $n < fls\text{-}subdegree\ f\ n < 0$ *rule*: *case-split*[*case-product case-split*])
case *False-True*
hence $nat\ (-fls\text{-}subdegree\ f) - nat\ (n - fls\text{-}subdegree\ f) = nat\ (-n)$ **by** *auto*
with *False-True* **show** *?thesis*
using *coeff-reflect-poly*[*of fls-prpart f nat* $(n - fls\text{-}subdegree\ f)$] **by** *auto*
next
case *False-False* **thus** *?thesis*
using *coeff-reflect-poly*[*of fls-prpart f nat* $(n - fls\text{-}subdegree\ f)$] **by** *auto*
qed *simp-all*

lemma *fls-prpart-idem* [*simp*]: $fls\text{-}prpart\ (fls\text{-}prpart\text{-}as\text{-}fls\ f) = fls\text{-}prpart\ f$
using *fls-prpart-as-fls-nth*[*of f*] **by** (*intro poly-eqI*) *simp*

lemma *fls-regpart-prpart*: $fls\text{-}regpart\ (fls\text{-}prpart\text{-}as\text{-}fls\ f) = 0$

using *fls-prpart-as-fls-nth*[*of f*] **by** (*intro fps-ext*) *simp*

lemma *fls-prpart-regpart*: *fls-prpart* (*fls-regpart-as-fls f*) = *0*
by (*intro poly-eqI*) *simp*

7.5 Algebraic structures

7.5.1 Addition

instantiation *fls* :: (*monoid-add*) *plus*
begin
lift-definition *plus-fls* :: *'a fls* ⇒ *'a fls* ⇒ *'a fls* **is** λ*f g n*. *f n* + *g n*
proof−
fix *f f'* :: *int* ⇒ *'a*
assume ∀$_\infty$*n*. *f* (− *int n*) = *0* ∀$_\infty$*n*. *f'* (− *int n*) = *0*
from *this* **obtain** *N N'* **where** ∀ *n*>*N*. *f* (−*int n*) = *0* ∀ *n*>*N'*. *f'* (−*int n*) = *0*
by (*auto simp*: *MOST-nat*)
hence ∀ *n* > *max N N'*. *f* (−*int n*) + *f'* (−*int n*) = *0* **by** *auto*
hence ∃ *K*. ∀ *n*>*K*. *f* (−*int n*) + *f'* (−*int n*) = *0* **by** *fast*
thus ∀$_\infty$*n*. *f* (− *int n*) + *f'* (−*int n*) = *0* **by** (*simp add*: *MOST-nat*)
qed
instance ..
end

lemma *fls-plus-nth* [*simp*]: (*f* + *g*) $$ *n* = *f* $$ *n* + *g* $$ *n*
by *transfer simp*

lemma *fls-plus-const*: *fls-const x* + *fls-const y* = *fls-const* (*x*+*y*)
by (*intro fls-eqI*) *simp*

lemma *fls-plus-subdegree*:
f + *g* ≠ *0* ⟹ *fls-subdegree* (*f* + *g*) ≥ *min* (*fls-subdegree f*) (*fls-subdegree g*)
by (*auto intro*: *fls-subdegree-geI*)

lemma *fls-shift-plus* [*simp*]:
fls-shift m (*f* + *g*) = (*fls-shift m f*) + (*fls-shift m g*)
by (*intro fls-eqI*) *simp*

lemma *fls-regpart-plus* [*simp*]: *fls-regpart* (*f* + *g*) = *fls-regpart f* + *fls-regpart g*
by (*intro fps-ext*) *simp*

lemma *fls-prpart-plus* [*simp*] : *fls-prpart* (*f* + *g*) = *fls-prpart f* + *fls-prpart g*
by (*intro poly-eqI*) *simp*

lemma *fls-decompose-reg-pr-parts*:
fixes *f* :: *'a* :: *monoid-add fls*
defines *R* ≡ *fls-regpart-as-fls f*
and *P* ≡ *fls-prpart-as-fls f*
shows *f* = *P* + *R*

and $f = R + P$
using *fls-prpart-as-fls-nth*[*of f*]
by (*auto intro*: *fls-eqI simp add*: *assms*)

lemma *fps-to-fls-plus* [*simp*]: *fps-to-fls* $(f + g) =$ *fps-to-fls* $f +$ *fps-to-fls* g
by (*intro fls-eqI*) *simp*

instance *fls* :: (*monoid-add*) *monoid-add*
proof
fix $a\ b\ c :: 'a$ *fls*
show $a + b + c = a + (b + c)$ **by** *transfer* (*simp add*: *add.assoc*)
show $0 + a = a$ **by** *transfer simp*
show $a + 0 = a$ **by** *transfer simp*
qed

instance *fls* :: (*comm-monoid-add*) *comm-monoid-add*
by (*standard*, *transfer*, *auto simp*: *add.commute*)

lemma *fls-nth-sum*: *fls-nth* $(\sum x{\in}A.\ f\ x)\ n = (\sum x{\in}A.$ *fls-nth* $(f\ x)\ n)$
by (*induction A rule*: *infinite-finite-induct*) *auto*

7.5.2 Subtraction and negatives

instantiation *fls* :: (*group-add*) *minus*
begin
lift-definition *minus-fls* :: $'a$ *fls* $\Rightarrow$ $'a$ *fls* $\Rightarrow$ $'a$ *fls* **is** $\lambda f\ g\ n.\ f\ n - g\ n$
proof−
fix $f\ f' ::$ *int* $\Rightarrow$ $'a$
assume $\forall_\infty n.\ f\ (-\ int\ n) = 0\ \forall_\infty n.\ f'\ (-\ int\ n) = 0$
from *this* **obtain** $N\ N'$ **where** $\forall\, n{>}N.\ f\ (-int\ n) = 0\ \forall\, n{>}N'.\ f'\ (-int\ n) = 0$
by (*auto simp*: *MOST-nat*)
hence $\forall\, n > max\ N\ N'.\ f\ (-int\ n) - f'\ (-int\ n) = 0$ **by** *auto*
hence $\exists\, K.\ \forall\, n{>}K.\ f\ (-int\ n) - f'\ (-int\ n) = 0$ **by** *fast*
thus $\forall_\infty n.\ f\ (-\ int\ n) - f'\ (-int\ n) = 0$ **by** (*simp add*: *MOST-nat*)
qed
instance ..
end

lemma *fls-minus-nth* [*simp*]: $(f - g)$ \$\$ $n = f$ \$\$ $n - g$ \$\$ n
by *transfer simp*

lemma *fls-minus-const*: *fls-const* $x -$ *fls-const* $y =$ *fls-const* $(x{-}y)$
by (*intro fls-eqI*) *simp*

lemma *fls-subdegree-minus*:
$f - g \neq 0 \Longrightarrow$ *fls-subdegree* $(f - g) \geq min$ (*fls-subdegree* f) (*fls-subdegree* g)
by (*intro fls-subdegree-geI*) *simp-all*

lemma *fls-shift-minus* [*simp*]: *fls-shift m* (*f* − *g*) = (*fls-shift m f*) − (*fls-shift m g*)
by (*auto intro*: *fls-eqI*)

lemma *fls-regpart-minus* [*simp*]: *fls-regpart* (*f* − *g*) = *fls-regpart f* − *fls-regpart g*
by (*intro fps-ext*) *simp*

lemma *fls-prpart-minus* [*simp*] : *fls-prpart* (*f* − *g*) = *fls-prpart f* − *fls-prpart g*
by (*intro poly-eqI*) *simp*

lemma *fps-to-fls-minus* [*simp*]: *fps-to-fls* (*f* − *g*) = *fps-to-fls f* − *fps-to-fls g*
by (*intro fls-eqI*) *simp*

instantiation *fls* :: (*group-add*) *uminus*
begin
lift-definition *uminus-fls* :: *'a fls* ⇒ *'a fls* **is** λ*f n*. − *f n*
proof−
fix *f* :: *int* ⇒ *'a* **assume** ∀ ∞*n*. *f* (− *int n*) = *0*
from *this* **obtain** *N* **where** ∀ *n*>*N*. *f* (−*int n*) = *0*
by (*auto simp*: *MOST-nat*)
hence ∀ *n*>*N*. − *f* (−*int n*) = *0* **by** *auto*
hence ∃ *K*. ∀ *n*>*K*. − *f* (−*int n*) = *0* **by** *fast*
thus ∀ ∞*n*. − *f* (− *int n*) = *0* **by** (*simp add*: *MOST-nat*)
qed
instance ..
end

lemma *fls-uminus-nth* [*simp*]: (−*f*) $$ *n* = − (*f* $$ *n*)
by *transfer simp*

lemma *fls-const-uminus*[*simp*]: *fls-const* (−*x*) = −*fls-const x*
by (*intro fls-eqI*) *simp*

lemma *fls-shift-uminus* [*simp*]: *fls-shift m* (− *f*) = − (*fls-shift m f*)
by (*auto intro*: *fls-eqI*)

lemma *fls-regpart-uminus* [*simp*]: *fls-regpart* (− *f*) = − *fls-regpart f*
by (*intro fps-ext*) *simp*

lemma *fls-prpart-uminus* [*simp*] : *fls-prpart* (− *f*) = − *fls-prpart f*
by (*intro poly-eqI*) *simp*

lemma *fps-to-fls-uminus* [*simp*]: *fps-to-fls* (− *f*) = − *fps-to-fls f*
by (*intro fls-eqI*) *simp*

instance *fls* :: (*group-add*) *group-add*
proof
fix *a b* :: *'a fls*
show − *a* + *a* = *0* **by** *transfer simp*

show $a + - b = a - b$ **by** *transfer simp*
qed

instance *fls* :: (*ab-group-add*) *ab-group-add*
proof
fix $a\ b$ :: $'a$ *fls*
show $- a + a = 0$ **by** *transfer simp*
show $a - b = a + - b$ **by** *transfer simp*
qed

lemma *fls-uminus-subdegree* [*simp*]: *fls-subdegree* $(-f)$ = *fls-subdegree f*
by (*cases f=0*) (*auto intro*: *fls-subdegree-eqI*)

lemma *fls-subdegree-minus-sym*: *fls-subdegree* $(g - f)$ = *fls-subdegree* $(f - g)$
using *fls-uminus-subdegree*[*of g−f*] **by** (*simp add*: *algebra-simps*)

lemma *fls-regpart-sub-prpart*: *fls-regpart* $(f$ − *fls-prpart-as-fls f*$)$ = *fls-regpart f*
using *fls-decompose-reg-pr-parts*(*2*)[*of f*]
add-diff-cancel[*of fls-regpart-as-fls f fls-prpart-as-fls f*]
by *simp*

lemma *fls-prpart-sub-regpart*: *fls-prpart* $(f$ − *fls-regpart-as-fls f*$)$ = *fls-prpart f*
using *fls-decompose-reg-pr-parts*(*1*)[*of f*]
add-diff-cancel[*of fls-prpart-as-fls f fls-regpart-as-fls f*]
by *simp*

7.5.3 Multiplication

instantiation *fls* :: ({*comm-monoid-add*, *times*}) *times*
begin
definition *fls-times-def*:
$(*) = (\lambda f\ g.$
fls-shift
(− (*fls-subdegree f* + *fls-subdegree g*))
(*fps-to-fls* (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*))
)
instance ..
end

lemma *fls-times-nth-eq0*: $n <$ *fls-subdegree f* + *fls-subdegree g* $\Longrightarrow$ $(f * g)$ $$ n = *0*
by (*simp add*: *fls-times-def*)

lemma *fls-times-nth*:
fixes *f df g dg*
defines *df* ≡ *fls-subdegree f* **and** *dg* ≡ *fls-subdegree g*
shows $(f * g)$ $$ $n = (\sum i{=}df + dg..n.\ f$ $$ $(i - dg) * g$ $$ $(dg + n - i))$
and $(f * g)$ $$ $n = (\sum i{=}df..n - dg.\ f$ $$ $i * g$ $$ $(n - i))$
and $(f * g)$ $$ $n = (\sum i{=}dg..n - df.\ f$ $$ $(df + i - dg) * g$ $$ $(dg + n -$

```
df − i))
  and     (f ∗ g) $$ n = (∑ i=0..n − (df + dg). f $$ (df + i) ∗ g $$ (n − df −
i))
proof−

  define dfg where dfg ≡ df + dg

  show 4: (f ∗ g) $$ n = (∑ i=0..n − dfg. f $$ (df + i) ∗ g $$ (n − df − i))
  proof (cases n < dfg)
    case False
    from False assms have
      (f ∗ g) $$ n =
        (∑ i = 0..nat (n − dfg). f $$ (df + int i) ∗ g $$ (dg + int (nat (n − dfg)
− i)))
      using fps-mult-nth[of fls-base-factor-to-fps f fls-base-factor-to-fps g]
            fls-base-factor-to-fps-nth[of f]
            fls-base-factor-to-fps-nth[of g]
      by    (simp add: dfg-def fls-times-def algebra-simps)
    moreover from False have index:
      ⋀i. i ∈ {0..nat (n − dfg)} ⟹ dg + int (nat (n − dfg) − i) = n − df − int
i
      by (auto simp: dfg-def)
    ultimately have
      (f ∗ g) $$ n = (∑ i=0..nat (n − dfg). f $$ (df + int i) ∗ g $$ (n − df − int
i))
      by (simp del: of-nat-diff)
    moreover have
      (∑ i=0..nat (n − dfg). f $$ (df + int i) ∗  g $$ (n − df − int i)) =
        (∑ i=0..n − dfg. f $$ (df + i) ∗  g $$ (n − df − i))
    proof (intro sum.reindex-cong)
      show inj-on nat {0..n − dfg} by standard auto
      show {0..nat (n − dfg)} = nat ‘ {0..n − dfg}
      proof
        show {0..nat (n − dfg)} ⊆ nat ‘ {0..n − dfg}
        proof
          fix i assume i ∈ {0..nat (n − dfg)}
          hence i: i ≥ 0 i ≤ nat (n − dfg) by auto
          with False have int i ≥ 0 int i ≤ n − dfg by auto
          hence int i ∈ {0..n − dfg} by simp
          moreover from i(1) have i = nat (int i) by simp
          ultimately show i ∈ nat ‘ {0..n − dfg} by fast
        qed
      qed (auto simp: False)
    qed (simp add: False)
    ultimately show (f ∗ g) $$ n = (∑ i=0..n − dfg. f $$ (df + i) ∗  g $$ (n −
df − i))
      by simp
  qed (simp add: fls-times-nth-eq0 assms dfg-def)
```

have
(∑ i=dfg..n. f $$ (i − dg) ∗ g $$ (dg + n − i)) =
(∑ i=0..n − dfg. f $$ (df + i) ∗ g $$ (n − df − i))
proof (*intro sum.reindex-cong*)
define T **where** T ≡ λi. i + dfg
show *inj-on T* {0..n − dfg} **by** *standard* (*simp add*: *T-def*)
qed (*simp-all add*: *dfg-def algebra-simps*)
with *4* **show** *1*: (f ∗ g) $$ n = (∑ i=dfg..n. f $$ (i − dg) ∗ g $$ (dg + n − i))
by *simp*

have
(∑ i=dfg..n. f $$ (i − dg) ∗ g $$ (dg + n − i)) = (∑ i=df..n − dg. f $$ i ∗ g $$ (n − i))
proof (*intro sum.reindex-cong*)
define T **where** T ≡ λi. i + dg
show *inj-on T* {df..n − dg} **by** *standard* (*simp add*: *T-def*)
qed (*auto simp*: *dfg-def*)
with *1* **show** (f ∗ g) $$ n = (∑ i=df..n − dg. f $$ i ∗ g $$ (n − i))
by *simp*

have
(∑ i=dfg..n. f $$ (i − dg) ∗ g $$ (dg + n − i)) =
(∑ i=dg..n − df. f $$ (df + i − dg) ∗ g $$ (dg + n − df − i))
proof (*intro sum.reindex-cong*)
define T **where** T ≡ λi. i + df
show *inj-on T* {dg..n − df} **by** *standard* (*simp add*: *T-def*)
qed (*simp-all add*: *dfg-def algebra-simps*)
with *1* **show** (f ∗ g) $$ n = (∑ i=dg..n − df. f $$ (df + i − dg) ∗ g $$ (dg + n − df − i))
by *simp*

qed

lemma *fls-times-base* [*simp*]:
(f ∗ g) $$ (*fls-subdegree* f + *fls-subdegree* g) =
(f $$ *fls-subdegree* f) ∗ (g $$ *fls-subdegree* g)
by (*simp add*: *fls-times-nth*(*1*))

instance *fls* :: ({*comm-monoid-add*, *mult-zero*}) *mult-zero*
proof
fix a :: 'a *fls*
have
(0::'a *fls*) ∗ a =
fls-shift (*fls-subdegree* a) (*fps-to-fls* ((0::'a *fps*)∗(*fls-base-factor-to-fps* a)))
by (*simp add*: *fls-times-def*)
moreover have
a ∗ (0::'a *fls*) =
fls-shift (*fls-subdegree* a) (*fps-to-fls* ((*fls-base-factor-to-fps* a)∗(0::'a *fps*)))

```
    by (simp add: fls-times-def)
  ultimately show 0 * a = (0::'a fls) a * 0 = (0::'a fls)
    by auto
qed

lemma fls-mult-one:
  fixes f :: 'a::{comm-monoid-add, mult-zero, monoid-mult} fls
  shows 1 * f = f
  and   f * 1 = f
  using fls-conv-base-factor-to-fps-shift-subdegree[of f]
  by    (simp-all add: fls-times-def fps-one-mult)

lemma fls-mult-const-nth [simp]:
  fixes f :: 'a::{comm-monoid-add, mult-zero} fls
  shows (fls-const x * f) $$ n = x * f$$n
  and   (f * fls-const x ) $$ n = f$$n * x
proof-
  show (fls-const x * f) $$ n = x * f$$n
  proof (cases n<fls-subdegree f)
    case False
    hence {fls-subdegree f..n} = insert (fls-subdegree f) {fls-subdegree f+1..n} by
auto
    thus ?thesis by (simp add: fls-times-nth(1))
  qed (simp add: fls-times-nth-eq0)
  show (f * fls-const x ) $$ n = f$$n * x
  proof (cases n<fls-subdegree f)
    case False
    hence {fls-subdegree f..n} = insert n {fls-subdegree f..n-1} by auto
    thus ?thesis by (simp add: fls-times-nth(1))
  qed (simp add: fls-times-nth-eq0)
qed

lemma fls-const-mult-const[simp]:
  fixes x y :: 'a::{comm-monoid-add, mult-zero}
  shows fls-const x * fls-const y = fls-const (x*y)
  by    (intro fls-eqI) simp

lemma fls-subdegree-add-eq1:
  assumes f ≠ 0 fls-subdegree f < fls-subdegree g
  shows   fls-subdegree (f + g) = fls-subdegree f
proof (intro antisym)
  from assms have *: fls-nth (f + g) (fls-subdegree f) ≠ 0
    by auto
  from * show fls-subdegree (f + g) ≤ fls-subdegree f
    by (rule fls-subdegree-leI)
  from * have f + g ≠ 0
    using fls-nonzeroI by blast
  thus fls-subdegree f ≤ fls-subdegree (f + g)
    using assms(2) fls-plus-subdegree by force
```

qed

lemma *fls-subdegree-add-eq2*:
assumes $g \neq 0$ *fls-subdegree* $g <$ *fls-subdegree* f
shows *fls-subdegree* $(f + g) =$ *fls-subdegree* g
proof (*intro antisym*)
from *assms* **have** $*$: *fls-nth* $(f + g)$ (*fls-subdegree* g) $\neq 0$
by *auto*
from $*$ **show** *fls-subdegree* $(f + g) \leq$ *fls-subdegree* g
by (*rule fls-subdegree-leI*)
from $*$ **have** $f + g \neq 0$
using *fls-nonzeroI* **by** *blast*
thus *fls-subdegree* $g \leq$ *fls-subdegree* $(f + g)$
using *assms*(2) *fls-plus-subdegree* **by** *force*
qed

lemma *fls-subdegree-diff-eq1*:
assumes $f \neq 0$ *fls-subdegree* $f <$ *fls-subdegree* g
shows *fls-subdegree* $(f - g) =$ *fls-subdegree* f
using *fls-subdegree-add-eq1*[*of* f $-g$] *assms* **by** *simp*

lemma *fls-subdegree-diff-eq2*:
assumes $g \neq 0$ *fls-subdegree* $g <$ *fls-subdegree* f
shows *fls-subdegree* $(f - g) =$ *fls-subdegree* g
using *fls-subdegree-add-eq2*[*of* $-g$ f] *assms* **by** *simp*

lemma *nat-minus-fls-subdegree-plus-const-eq*:
nat $(-$*fls-subdegree* $(F +$ *fls-const* $c))$ = *nat* $(-$*fls-subdegree* $F)$
proof (*cases fls-subdegree* $F < 0$)
case *True*
hence *fls-subdegree* $(F +$ *fls-const* $c) =$ *fls-subdegree* F
by (*intro fls-subdegree-add-eq1*) *auto*
thus *?thesis*
by *simp*
next
case *False*
thus *?thesis*
by (*auto simp*: *fls-subdegree-ge0I*)
qed

lemma *fls-mult-subdegree-ge*:
fixes f g :: $'a$::{*comm-monoid-add*,*mult-zero*} *fls*
assumes $f*g \neq 0$
shows *fls-subdegree* $(f*g) \geq$ *fls-subdegree* $f +$ *fls-subdegree* g
by (*auto intro*: *fls-subdegree-geI simp*: *assms fls-times-nth-eq0*)

lemma *fls-mult-subdegree-ge-0*:
fixes f g :: $'a$::{*comm-monoid-add*,*mult-zero*} *fls*
assumes *fls-subdegree* $f \geq 0$ *fls-subdegree* $g \geq 0$

shows *fls-subdegree* $(f*g) \geq 0$
using *assms fls-mult-subdegree-ge*[*of f g*]
by *fastforce*

lemma *fls-mult-nonzero-base-subdegree-eq*:
fixes $f\ g$ :: $'a$::{*comm-monoid-add*,*mult-zero*} *fls*
assumes f $$ (*fls-subdegree f*) $*$ g $$ (*fls-subdegree g*) $\neq 0$
shows *fls-subdegree* $(f*g)$ = *fls-subdegree f* + *fls-subdegree g*
proof−
from *assms* **have** *fls-subdegree* $(f*g) \geq$ *fls-subdegree f* + *fls-subdegree g*
using *fls-nonzeroI*[*of f*∗*g fls-subdegree f* + *fls-subdegree g*]
fls-mult-subdegree-ge[*of f g*]
by *simp*
moreover from *assms* **have** *fls-subdegree* $(f*g) \leq$ *fls-subdegree f* + *fls-subdegree g*
by (*intro fls-subdegree-leI*) *simp*
ultimately show *?thesis* **by** *simp*
qed

lemma *fls-subdegree-mult* [*simp*]:
fixes $f\ g$ :: $'a$::*semiring-no-zero-divisors fls*
assumes $f \neq 0$ $g \neq 0$
shows *fls-subdegree* $(f * g)$ = *fls-subdegree f* + *fls-subdegree g*
using *assms*
by (*auto intro*: *fls-subdegree-eqI simp*: *fls-times-nth-eq0*)

lemma *fls-shifted-times-simps*:
fixes $f\ g$ :: $'a$::{*comm-monoid-add*, *mult-zero*} *fls*
shows f $*$ (*fls-shift n g*) = *fls-shift n* $(f*g)$ (*fls-shift n f*) $*$ g = *fls-shift n* $(f*g)$
proof−

show f $*$ (*fls-shift n g*) = *fls-shift n* $(f*g)$
proof (*cases g=0*)
case *False*
hence
f $*$ (*fls-shift n g*) =
fls-shift (− (*fls-subdegree f* + (*fls-subdegree g* − *n*)))
(*fps-to-fls* (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*))
unfolding *fls-times-def* **by** (*simp add*: *fls-base-factor-to-fps-shift*)
thus f $*$ (*fls-shift n g*) = *fls-shift n* $(f*g)$
by (*simp add*: *algebra-simps fls-times-def*)
qed *auto*

show (*fls-shift n f*)$*g$ = *fls-shift n* $(f*g)$
proof (*cases f=0*)
case *False*
hence
(*fls-shift n f*)$*g$ =
fls-shift (− ((*fls-subdegree f* − *n*) + *fls-subdegree g*))

```
      (fps-to-fls (fls-base-factor-to-fps f * fls-base-factor-to-fps g))
    unfolding fls-times-def by (simp add: fls-base-factor-to-fps-shift)
  thus (fls-shift n f) * g = fls-shift n (f*g)
    by (simp add: algebra-simps fls-times-def)
 qed auto

qed

lemma fls-shifted-times-transfer:
 fixes f g :: 'a::{comm-monoid-add, mult-zero} fls
 shows fls-shift n f * g = f * fls-shift n g
 using fls-shifted-times-simps(1)[of f n g] fls-shifted-times-simps(2)[of n f g]
 by   simp

lemma fls-times-both-shifted-simp:
 fixes f g :: 'a::{comm-monoid-add, mult-zero} fls
 shows (fls-shift m f) * (fls-shift n g) = fls-shift (m+n) (f*g)
 by   (simp add: fls-shifted-times-simps)

lemma fls-base-factor-mult-base-factor:
 fixes f g :: 'a::{comm-monoid-add, mult-zero} fls
 shows fls-base-factor (f * fls-base-factor g) = fls-base-factor (f * g)
 and   fls-base-factor (fls-base-factor f * g) = fls-base-factor (f * g)
 using fls-base-factor-shift[of fls-subdegree g f*g]
       fls-base-factor-shift[of fls-subdegree f f*g]
 by   (simp-all add: fls-shifted-times-simps)

lemma fls-base-factor-mult-both-base-factor:
 fixes f g :: 'a::{comm-monoid-add,mult-zero} fls
 shows fls-base-factor (fls-base-factor f * fls-base-factor g) = fls-base-factor (f *
g)
 using fls-base-factor-mult-base-factor(1)[of fls-base-factor f g]
       fls-base-factor-mult-base-factor(2)[of f g]
 by   simp

lemma fls-base-factor-mult:
 fixes f g :: 'a::semiring-no-zero-divisors fls
 shows fls-base-factor (f * g) = fls-base-factor f * fls-base-factor g
 by   (cases f≠0 ∧ g≠0)
      (auto simp: fls-times-both-shifted-simp)

lemma fls-times-conv-base-factor-times:
 fixes f g :: 'a::{comm-monoid-add, mult-zero} fls
 shows
   f * g =
    fls-shift (−(fls-subdegree f + fls-subdegree g)) (fls-base-factor f * fls-base-factor
g)
 by (simp add: fls-times-both-shifted-simp)
```

lemma *fls-times-base-factor-conv-shifted-times*:
— Convenience form of lemma *fls-times-both-shifted-simp*.
fixes *f g* :: *'a*::{*comm-monoid-add*, *mult-zero*} *fls*
shows
fls-base-factor f ∗ *fls-base-factor g* = *fls-shift* (*fls-subdegree f* + *fls-subdegree g*)
(*f* ∗ *g*)
by (*simp add*: *fls-times-both-shifted-simp*)

lemma *fls-times-conv-regpart*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *fls-regpart* (*f* ∗ *g*) = *fls-regpart f* ∗ *fls-regpart g*
proof−
from *assms* **have** *1*:
f ∗ *g* =
fls-shift (− (*fls-subdegree f* + *fls-subdegree g*)) (
fps-to-fls (
fps-shift (*nat* (*fls-subdegree f*) + *nat* (*fls-subdegree g*)) (
fls-regpart f ∗ *fls-regpart g*
)
)
)
by (*simp add*:
fls-times-def fls-base-factor-to-fps-conv-fps-shift[*symmetric*]
fls-regpart-subdegree-conv fps-shift-mult-both[*symmetric*]
)
show *?thesis*
proof (*cases fls-regpart f* ∗ *fls-regpart g* = *0*)
case *False*
with *assms* **have**
subdegree (*fls-regpart f* ∗ *fls-regpart g*) ≥
nat (*fls-subdegree f*) + *nat* (*fls-subdegree g*)
by (*simp add*: *fps-mult-subdegree-ge fls-regpart-subdegree-conv*[*symmetric*])
with *1 assms* **show** *?thesis* **by** *simp*
qed (*simp add*: *1*)
qed

lemma *fls-base-factor-to-fps-mult-conv-unit-factor*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
shows
fls-base-factor-to-fps (*f* ∗ *g*) =
unit-factor (*fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*)
using *fls-base-factor-mult-both-base-factor*[*of f g*]
fps-unit-factor-fls-regpart[*of fls-base-factor f* ∗ *fls-base-factor g*]
fls-base-factor-subdegree[*of f*] *fls-base-factor-subdegree*[*of g*]
fls-mult-subdegree-ge-0[*of fls-base-factor f fls-base-factor g*]
fls-times-conv-regpart[*of fls-base-factor f fls-base-factor g*]
by *simp*

lemma *fls-base-factor-to-fps-mult'*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes (*f* $$ *fls-subdegree f*) ∗ (*g* $$ *fls-subdegree g*) ≠ *0*
shows *fls-base-factor-to-fps* (*f* ∗ *g*) = *fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*
using *assms fls-mult-nonzero-base-subdegree-eq*[*of f g*]
fls-times-base-factor-conv-shifted-times[*of f g*]
fls-times-conv-regpart[*of fls-base-factor f fls-base-factor g*]
fls-base-factor-subdegree[*of f*] *fls-base-factor-subdegree*[*of g*]
by *fastforce*

lemma *fls-base-factor-to-fps-mult*:
fixes *f g* :: *'a*::*semiring-no-zero-divisors fls*
shows *fls-base-factor-to-fps* (*f* ∗ *g*) = *fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g*
using *fls-base-factor-to-fps-mult'*[*of f g*]
by (*cases f*=*0* ∨ *g*=*0*) *auto*

lemma *fls-times-conv-fps-times*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *f* ∗ *g* = *fps-to-fls* (*fls-regpart f* ∗ *fls-regpart g*)
using *assms fls-mult-subdegree-ge*[*of f g*]
by (*cases f* ∗ *g* = *0*) (*simp all add*: *fls-times-conv-regpart*[*symmetric*])

lemma *fps-times-conv-fls-times*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
shows *f* ∗ *g* = *fls-regpart* (*fps-to-fls f* ∗ *fps-to-fls g*)
using *fls-subdegree-fls-to-fps-gt0 fls-times-conv-regpart*[*symmetric*]
by *fastforce*

lemma *fls-times-fps-to-fls*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*mult-zero*} *fps*
shows *fps-to-fls* (*f* ∗ *g*) = *fps-to-fls f* ∗ *fps-to-fls g*
proof (*intro fls-eq-conv-fps-eqI*, *rule fls-subdegree-fls-to-fps-gt0*)
show *fls-subdegree* (*fps-to-fls f* ∗ *fps-to-fls g*) ≥ *0*
proof (*cases fps-to-fls f* ∗ *fps-to-fls g* = *0*)
case *False* **thus** *?thesis*
using *fls-mult-subdegree-ge fls-subdegree-fls-to-fps-gt0*[*of f*]
fls-subdegree-fls-to-fps-gt0[*of g*]
by *fastforce*
qed *simp*
qed (*simp add*: *fps-times-conv-fls-times*)

lemma *fls-X-times-conv-shift*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
shows *fls-X* ∗ *f* = *fls-shift* (−*1*) *f f* ∗ *fls-X* = *fls-shift* (−*1*) *f*
by (*simp all add*: *fls-X-conv-shift-1 fls-mult-one fls-shifted-times-simps*)

lemmas *fls-X-times-comm* = *trans-sym*[*OF fls-X-times-conv-shift*]

lemma *fls-subdegree-mult-fls-X*:
fixes f :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
assumes $f \neq 0$
shows *fls-subdegree* (*fls-X* $*$ f) = *fls-subdegree* f + 1
and *fls-subdegree* (f $*$ *fls-X*) = *fls-subdegree* f + 1
by (*auto simp*: *fls-X-times-conv-shift assms*)

lemma *fls-mult-fls-X-nonzero*:
fixes f :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
assumes $f \neq 0$
shows *fls-X* $*$ $f \neq 0$
and f $*$ *fls-X* $\neq 0$
by (*auto simp*: *fls-X-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-base-factor-mult-fls-X*:
fixes f :: *'a*::{*comm-monoid-add*,*monoid-mult*,*mult-zero*} *fls*
shows *fls-base-factor* (*fls-X* $*$ f) = *fls-base-factor* f
and *fls-base-factor* (f $*$ *fls-X*) = *fls-base-factor* f
using *fls-base-factor-shift*[*of* −1 f]
by (*auto simp*: *fls-X-times-conv-shift*)

lemma *fls-X-inv-times-conv-shift*:
fixes f :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
shows *fls-X-inv* $*$ f = *fls-shift* 1 f f $*$ *fls-X-inv* = *fls-shift* 1 f
by (*simp-all add*: *fls-X-inv-conv-shift-1 fls-mult-one fls-shifted-times-simps*)

lemmas *fls-X-inv-times-comm* = *trans-sym*[*OF fls-X-inv-times-conv-shift*]

lemma *fls-subdegree-mult-fls-X-inv*:
fixes f :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
assumes $f \neq 0$
shows *fls-subdegree* (*fls-X-inv* $*$ f) = *fls-subdegree* f − 1
and *fls-subdegree* (f $*$ *fls-X-inv*) = *fls-subdegree* f − 1
by (*auto simp*: *fls-X-inv-times-conv-shift assms*)

lemma *fls-mult-fls-X-inv-nonzero*:
fixes f :: *'a*::{*comm-monoid-add*,*mult-zero*,*monoid-mult*} *fls*
assumes $f \neq 0$
shows *fls-X-inv* $*$ $f \neq 0$
and f $*$ *fls-X-inv* $\neq 0$
by (*auto simp*: *fls-X-inv-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-base-factor-mult-fls-X-inv*:
fixes f :: *'a*::{*comm-monoid-add*,*monoid-mult*,*mult-zero*} *fls*
shows *fls-base-factor* (*fls-X-inv* $*$ f) = *fls-base-factor* f
and *fls-base-factor* (f $*$ *fls-X-inv*) = *fls-base-factor* f
using *fls-base-factor-shift*[*of* 1 f]

by (*auto simp*: *fls-X-inv-times-conv-shift*)

lemma *fls-mult-assoc-subdegree-ge-0*:
fixes *f g h* :: *'a*::*semiring-0 fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0 fls-subdegree h* ≥ *0*
shows *f* ∗ *g* ∗ *h* = *f* ∗ (*g* ∗ *h*)
using *assms*
by (*simp add*: *fls-times-conv-fps-times fls-subdegree-fls-to-fps-gt0 mult.assoc*)

lemma *fls-mult-assoc-base-factor*:
fixes *a b c* :: *'a*::*semiring-0 fls*
shows
fls-base-factor a ∗ *fls-base-factor b* ∗ *fls-base-factor c* =
fls-base-factor a ∗ (*fls-base-factor b* ∗ *fls-base-factor c*)
by (*simp add*: *fls-mult-assoc-subdegree-ge-0 del*: *fls-base-factor-def*)

lemma *fls-mult-distrib-subdegree-ge-0*:
fixes *f g h* :: *'a*::*semiring-0 fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0 fls-subdegree h* ≥ *0*
shows (*f* + *g*) ∗ *h* = *f* ∗ *h* + *g* ∗ *h*
and *h* ∗ (*f* + *g*) = *h* ∗ *f* + *h* ∗ *g*
proof−
have *fls-subdegree* (*f*+*g*) ≥ *0*
proof (*cases f*+*g* = *0*)
case *False*
with *assms*(*1*,*2*) **show** *?thesis*
using *fls-plus-subdegree* **by** *fastforce*
qed *simp*
with *assms* **show** (*f* + *g*) ∗ *h* = *f* ∗ *h* + *g* ∗ *h h* ∗ (*f* + *g*) = *h* ∗ *f* + *h* ∗ *g*
using *distrib-right*[*of fls-regpart f*] *distrib-left*[*of fls-regpart h*]
by (*simp-all add*: *fls-times-conv-fps-times*)
qed

lemma *fls-mult-distrib-base-factor*:
fixes *a b c* :: *'a*::*semiring-0 fls*
shows
fls-base-factor a ∗ (*fls-base-factor b* + *fls-base-factor c*) =
fls-base-factor a ∗ *fls-base-factor b* + *fls-base-factor a* ∗ *fls-base-factor c*
by (*simp add*: *fls-mult-distrib-subdegree-ge-0 del*: *fls-base-factor-def*)

instance *fls* :: (*semiring-0*) *semiring-0*
proof

fix *a b c* :: *'a fls*
have
a ∗ *b* ∗ *c* =
fls-shift (− (*fls-subdegree a* + *fls-subdegree b* + *fls-subdegree c*))
(*fls-base-factor a* ∗ *fls-base-factor b* ∗ *fls-base-factor c*)
by (*simp add*: *fls-times-both-shifted-simp*)

```
moreover have
  a * (b * c) =
    fls-shift (- (fls-subdegree a + fls-subdegree b + fls-subdegree c))
      (fls-base-factor a * fls-base-factor b * fls-base-factor c)
 using fls-mult-assoc-base-factor[of a b c] by (simp add: fls-times-both-shifted-simp)
ultimately show a * b * c = a * (b * c) by simp

have ab:
  fls-subdegree (fls-shift (min (fls-subdegree a) (fls-subdegree b)) a) ≥ 0
  fls-subdegree (fls-shift (min (fls-subdegree a) (fls-subdegree b)) b) ≥ 0
  by (simp-all add: fls-shift-nonneg-subdegree)
have
  (a + b) * c =
    fls-shift (- (min (fls-subdegree a) (fls-subdegree b) + fls-subdegree c)) (
      (
        fls-shift (min (fls-subdegree a) (fls-subdegree b)) a +
        fls-shift (min (fls-subdegree a) (fls-subdegree b)) b
      ) * fls-base-factor c)
  using fls-times-both-shifted-simp[of
          -min (fls-subdegree a) (fls-subdegree b)
          fls-shift (min (fls-subdegree a) (fls-subdegree b)) a +
          fls-shift (min (fls-subdegree a) (fls-subdegree b)) b
          -fls-subdegree c fls-base-factor c
        ]
  by    simp
also have
  ... =
    fls-shift (-(min (fls-subdegree a) (fls-subdegree b) + fls-subdegree c))
      (fls-shift (min (fls-subdegree a) (fls-subdegree b)) a * fls-base-factor c)
    +
    fls-shift (-(min (fls-subdegree a) (fls-subdegree b) + fls-subdegree c))
      (fls-shift (min (fls-subdegree a) (fls-subdegree b)) b * fls-base-factor c)
  using ab
  by    (simp add: fls-mult-distrib-subdegree-ge-0(1) del: fls-base-factor-def)
finally show (a + b) * c = a * c + b * c by (simp add: fls-times-both-shifted-simp)

have bc:
  fls-subdegree (fls-shift (min (fls-subdegree b) (fls-subdegree c)) b) ≥ 0
  fls-subdegree (fls-shift (min (fls-subdegree b) (fls-subdegree c)) c) ≥ 0
  by (simp-all add: fls-shift-nonneg-subdegree)
have
  a * (b + c) =
    fls-shift (- (fls-subdegree a + min (fls-subdegree b) (fls-subdegree c))) (
      fls-base-factor a * (
        fls-shift (min (fls-subdegree b) (fls-subdegree c)) b +
        fls-shift (min (fls-subdegree b) (fls-subdegree c)) c
      )
    )
```

```
    using fls-times-both-shifted-simp[of
          −fls-subdegree a fls-base-factor a
          −min (fls-subdegree b) (fls-subdegree c)
          fls-shift (min (fls-subdegree b) (fls-subdegree c)) b +
          fls-shift (min (fls-subdegree b) (fls-subdegree c)) c
        ]
    by    simp
  also have
    ... =
      fls-shift (−(fls-subdegree a + min (fls-subdegree b) (fls-subdegree c)))
        (fls-base-factor a * fls-shift (min (fls-subdegree b) (fls-subdegree c)) b)
      +
      fls-shift (−(fls-subdegree a + min (fls-subdegree b) (fls-subdegree c)))
        (fls-base-factor a * fls-shift (min (fls-subdegree b) (fls-subdegree c)) c)

    using bc
    by    (simp add: fls-mult-distrib-subdegree-ge-0(2) del: fls-base-factor-def)
  finally show a * (b + c) = a * b + a * c by (simp add: fls-times-both-shifted-simp)

qed

lemma fls-mult-commute-subdegree-ge-0:
  fixes   f g :: 'a::comm-semiring-0 fls
  assumes fls-subdegree f ≥ 0 fls-subdegree g ≥ 0
  shows   f * g = g * f
  using   assms
  by      (simp add: fls-times-conv-fps-times mult.commute)

lemma fls-mult-commute-base-factor:
  fixes a b c :: 'a::comm-semiring-0 fls
  shows fls-base-factor a * fls-base-factor b = fls-base-factor b * fls-base-factor a
  by    (simp add: fls-mult-commute-subdegree-ge-0 del: fls-base-factor-def)

instance fls :: (comm-semiring-0) comm-semiring-0
proof
  fix a b c :: 'a fls
  show a * b = b * a
    using fls-times-conv-base-factor-times[of a b] fls-times-conv-base-factor-times[of
b a]
          fls-mult-commute-base-factor[of a b]
    by    (simp add: add.commute)
qed (simp add: distrib-right)

instance fls :: (semiring-1) semiring-1
  by (standard, simp-all add: fls-mult-one)

lemma fls-of-nat: (of-nat n :: 'a::semiring-1 fls) = fls-const (of-nat n)
  by (induct n) (auto intro: fls-eqI)
```

lemma *fls-of-nat-nth*: *of-nat n $$ k = (if k=0 then of-nat n else 0)*
by (*simp add*: *fls-of-nat*)

lemma *fls-mult-of-nat-nth* [*simp*]:
shows (*of-nat k * f*) *$$ n = of-nat k * f$$n*
and (*f * of-nat k*) *$$ n = f$$n * of-nat k*
by (*simp-all add*: *fls-of-nat*)

lemma *fls-subdegree-of-nat* [*simp*]: *fls-subdegree* (*of-nat n*) = *0*
by (*simp add*: *fls-of-nat*)

lemma *fls-shift-of-nat-nth*:
fls-shift k (*of-nat a*) *$$ n = (if n=−k then of-nat a else 0)*
by (*simp add*: *fls-of-nat fls-shift-const-nth*)

lemma *fls-base-factor-of-nat* [*simp*]:
fls-base-factor (*of-nat n* :: *'a*::*semiring-1 fls*) = (*of-nat n* :: *'a fls*)
by (*simp add*: *fls-of-nat*)

lemma *fls-regpart-of-nat* [*simp*]: *fls-regpart* (*of-nat n*) = (*of-nat n* :: *'a*::*semiring-1 fps*)
by (*simp add*: *fls-of-nat fps-of-nat*)

lemma *fls-prpart-of-nat* [*simp*]: *fls-prpart* (*of-nat n*) = *0*
by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-base-factor-to-fps-of-nat*:
fls-base-factor-to-fps (*of-nat n*) = (*of-nat n* :: *'a*::*semiring-1 fps*)
by *simp*

lemma *fps-to-fls-of-nat*:
fps-to-fls (*of-nat n*) = (*of-nat n* :: *'a*::*semiring-1 fls*)
proof −
have *fps-to-fls* (*of-nat n*) = *fps-to-fls* (*fps-const* (*of-nat n*))
by (*simp add*: *fps-of-nat*)
thus *?thesis* **by** (*simp add*: *fls-of-nat*)
qed

lemma *fps-to-fls-numeral* [*simp*]: *fps-to-fls* (*numeral n*) = *numeral n*
by (*metis fps-to-fls-of-nat of-nat-numeral*)

lemma *fls-const-power*: *fls-const* (*a ^ b*) = *fls-const a ^ b*
by (*induction b*) (*auto simp flip*: *fls-const-mult-const*)

lemma *fls-const-numeral* [*simp*]: *fls-const* (*numeral n*) = *numeral n*
by (*metis fls-of-nat of-nat-numeral*)

lemma *fls-mult-of-numeral-nth* [*simp*]:
shows (*numeral k * f*) *$$ n = numeral k * f $$ n*

and $(f * numeral\ k)\ \$\$\ n = f\ \$\$\ n * numeral\ k$
by (*metis fls-const-numeral fls-mult-const-nth*)+

lemma *fls-nth-numeral'* [*simp*]:
$numeral\ n\ \$\$\ 0 = numeral\ n\ k \neq 0 \Longrightarrow numeral\ n\ \$\$\ k = 0$
by (*metis fls-const-nth fls-const-numeral*)+

instance *fls* :: (*comm-semiring-1*) *comm-semiring-1*
by *standard simp*

instance *fls* :: (*ring*) *ring* **..**

instance *fls* :: (*comm-ring*) *comm-ring* **..**

instance *fls* :: (*ring-1*) *ring-1* **..**

lemma *fls-of-int-nonneg*: (*of-int* (*int n*) :: *'a*::*ring-1 fls*) = *fls-const* (*of-int* (*int n*))
by (*induct n*) (*auto intro*: *fls-eqI*)

lemma *fls-of-int*: (*of-int i* :: *'a*::*ring-1 fls*) = *fls-const* (*of-int i*)
proof (*induct i*)
case (*neg i*)
have *of-int* (*int* (*Suc i*)) = *fls-const* (*of-int* (*int* (*Suc i*)) :: *'a*)
using *fls-of-int-nonneg*[*of Suc i*] **by** *simp*
hence $-$ *of-int* (*int* (*Suc i*)) = $-$ *fls-const* (*of-int* (*int* (*Suc i*)) :: *'a*)
by *simp*
thus *?case* **by** (*simp add*: *fls-const-uminus*[*symmetric*])
qed (*rule fls-of-int-nonneg*)

lemma *fls-of-int-nth*: *of-int n* $\$\$$ *k* = (*if k=0 then of-int n else 0*)
by (*simp add*: *fls-of-int*)

lemma *fls-mult-of-int-nth* [*simp*]:
shows (*of-int k* $*$ *f*) $\$\$$ *n* = *of-int k* $*$ *f*$\$\$*n*
and (*f* $*$ *of-int k*) $\$\$$ *n* = *f*$\$\$*n* $*$ *of-int k*
by (*simp-all add*: *fls-of-int*)

lemma *fls-subdegree-of-int* [*simp*]: *fls-subdegree* (*of-int i*) = *0*
by (*simp add*: *fls-of-int*)

lemma *fls-shift-of-int-nth*:
fls-shift k (*of-int i*) $\$\$$ *n* = (*if n=−k then of-int i else 0*)
by (*simp add*: *fls-of-int-nth*)

lemma *fls-base-factor-of-int* [*simp*]:
fls-base-factor (*of-int i* :: *'a*::*ring-1 fls*) = (*of-int i* :: *'a fls*)
by (*simp add*: *fls-of-int*)

lemma *fls-regpart-of-int* [*simp*]:
fls-regpart (*of-int i*) = (*of-int i* :: *'a*::*ring-1 fps*)
by (*simp add*: *fls-of-int fps-of-int*)

lemma *fls-prpart-of-int* [*simp*]: *fls-prpart* (*of-int n*) = *0*
by (*simp add*: *fls-prpart-eq0-iff*)

lemma *fls-base-factor-to-fps-of-int*:
fls-base-factor-to-fps (*of-int i*) = (*of-int i* :: *'a*::*ring-1 fps*)
by *simp*

lemma *fps-to-fls-of-int*:
fps-to-fls (*of-int i*) = (*of-int i* :: *'a*::*ring-1 fls*)
proof −
have *fps-to-fls* (*of-int i*) = *fps-to-fls* (*fps-const* (*of-int i*))
by (*simp add*: *fps-of-int*)
thus *?thesis* **by** (*simp add*: *fls-of-int*)
qed

instance *fls* :: (*comm-ring-1*) *comm-ring-1* **..**

instance *fls* :: (*semiring-no-zero-divisors*) *semiring-no-zero-divisors*
proof
fix *a b* :: *'a fls*
assume *a* ≠ *0* **and** *b* ≠ *0*
hence (*a* ∗ *b*) $$ (*fls-subdegree a* + *fls-subdegree b*) ≠ *0* **by** *simp*
thus *a* ∗ *b* ≠ *0* **using** *fls-nonzeroI* **by** *fast*
qed

instance *fls* :: (*semiring-1-no-zero-divisors*) *semiring-1-no-zero-divisors* **..**

instance *fls* :: (*ring-no-zero-divisors*) *ring-no-zero-divisors* **..**

instance *fls* :: (*ring-1-no-zero-divisors*) *ring-1-no-zero-divisors* **..**

instance *fls* :: (*idom*) *idom* **..**

lemma *semiring-char-fls* [*simp*]: *CHAR*(*'a* :: *comm-semiring-1 fls*) = *CHAR*(*'a*)
by (*rule CHAR-eqI*) (*auto simp*: *fls-of-nat of-nat-eq-0-iff-char-dvd fls-const-nonzero*)

instance *fls* :: ({*semiring-prime-char*,*comm-semiring-1*}) *semiring-prime-char*
by (*rule semiring-prime-charI*) *auto*
instance *fls* :: ({*comm-semiring-prime-char*,*comm-semiring-1*}) *comm-semiring-prime-char*
by *standard*
instance *fls* :: ({*comm-ring-prime-char*,*comm-semiring-1*}) *comm-ring-prime-char*
by *standard*
instance *fls* :: ({*idom-prime-char*,*comm-semiring-1*}) *idom-prime-char*
by *standard*

lemma *fls-subdegree-numeral* [*simp*]: *fls-subdegree* (*numeral n*) = *0*
by (*metis fls-subdegree-of-nat of-nat-numeral*)

lemma *fls-regpart-numeral* [*simp*]: *fls-regpart* (*numeral n*) = *numeral n*
by (*metis fls-regpart-of-nat of-nat-numeral*)

7.5.4 Powers

lemma *fls-subdegree-prod*:
fixes *F* :: *'a* ⇒ *'b* :: *field-char-0 fls*
assumes ⋀*x*. *x* ∈ *I* ⟹ *F x* ≠ *0*
shows *fls-subdegree* (∏ *x*∈*I*. *F x*) = (∑ *x*∈*I*. *fls-subdegree* (*F x*))
using *assms* **by** (*induction I rule*: *infinite-finite-induct*) *auto*

lemma *fls-subdegree-prod'*:
fixes *F* :: *'a* ⇒ *'b* :: *field-char-0 fls*
assumes ⋀*x*. *x* ∈ *I* ⟹ *fls-subdegree* (*F x*) ≠ *0*
shows *fls-subdegree* (∏ *x*∈*I*. *F x*) = (∑ *x*∈*I*. *fls-subdegree* (*F x*))
proof (*intro fls-subdegree-prod*)
show *F x* ≠ *0* **if** *x* ∈ *I* **for** *x*
using *assms*[*OF that*] **by** *auto*
qed

lemma *fls-pow-subdegree-ge*:
f^*n* ≠ *0* ⟹ *fls-subdegree* (*f*^*n*) ≥ *n* ∗ *fls-subdegree f*
proof (*induct n*)
case (*Suc n*) **thus** *?case*
using *fls-mult-subdegree-ge*[*of f f*^*n*] **by** (*fastforce simp*: *algebra-simps*)
qed *simp*

lemma *fls-pow-nth-below-subdegree*:
k < *n* ∗ *fls-subdegree f* ⟹ (*f*^*n*) $$ *k* = *0*
using *fls-pow-subdegree-ge*[*of f n*] **by** (*cases f*^*n* = *0*) *auto*

lemma *fls-pow-base* [*simp*]:
(*f* ^ *n*) $$ (*n* ∗ *fls-subdegree f*) = (*f* $$ *fls-subdegree f*) ^ *n*
proof (*induct n*)
case (*Suc n*)
show *?case*
proof (*cases Suc n* ∗ *fls-subdegree f* < *fls-subdegree f* + *fls-subdegree* (*f*^*n*))
case *True* **with** *Suc* **show** *?thesis*
by (*simp-all add*: *fls-times-nth-eq0 distrib-right*)
next
case *False*
from *False* **have**
{*0*..*int n* ∗ *fls-subdegree f* − *fls-subdegree* (*f* ^ *n*)} =
insert 0 {*1*..*int n* ∗ *fls-subdegree f* − *fls-subdegree* (*f* ^ *n*)}
by (*auto simp*: *algebra-simps*)

with *False Suc* **show** *?thesis*
by (*simp add*: *algebra-simps fls-times-nth*(*4*) *fls-pow-nth-below-subdegree*)
qed
qed *simp*

lemma *fls-pow-subdegree-eqI*:
(*f* $$ *fls-subdegree f*) ^ *n* ≠ *0* ⟹ *fls-subdegree* (*f*^*n*) = *n* * *fls-subdegree f*
using *fls-pow-nth-below-subdegree* **by** (*fastforce intro*: *fls-subdegree-eqI*)

lemma *fls-unit-base-subdegree-power*:
x * *f* $$ *fls-subdegree f* = *1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* * *fls-subdegree f*
f $$ *fls-subdegree f* * *y* = *1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* * *fls-subdegree f*
proof−
show *x* * *f* $$ *fls-subdegree f* = *1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* * *fls-subdegree f*
using *left-right-inverse-power*[*of x f* $$ *fls-subdegree f n*]
by (*auto intro*: *fls-pow-subdegree-eqI*)
show *f* $$ *fls-subdegree f* * *y* = *1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* * *fls-subdegree f*
using *left-right-inverse-power*[*of f* $$ *fls-subdegree f y n*]
by (*auto intro*: *fls-pow-subdegree-eqI*)
qed

lemma *fls-base-dvd1-subdegree-power*:
f $$ *fls-subdegree f dvd 1* ⟹ *fls-subdegree* (*f* ^ *n*) = *n* * *fls-subdegree f*
using *fls-unit-base-subdegree-power* **unfolding** *dvd-def* **by** *auto*

lemma *fls-pow-subdegree-ge0*:
assumes *fls-subdegree f* ≥ *0*
shows *fls-subdegree* (*f*^*n*) ≥ *0*
proof (*cases f*^*n* = *0*)
case *False*
moreover from *assms* **have** *int n* * *fls-subdegree f* ≥ *0* **by** *simp*
ultimately show *?thesis* **using** *fls-pow-subdegree-ge* **by** *fastforce*
qed *simp*

lemma *fls-subdegree-pow*:
fixes *f* :: *'a*::*semiring-1-no-zero-divisors fls*
shows *fls-subdegree* (*f* ^ *n*) = *n* * *fls-subdegree f*
proof (*cases f*=*0*)
case *False* **thus** *?thesis* **by** (*induct n*) (*simp-all add*: *algebra-simps*)
qed (*cases n*=*0*, *auto simp*: *zero-power*)

lemma *fls-shifted-pow*:
(*fls-shift m f*) ^ *n* = *fls-shift* (*n*∗*m*) (*f* ^ *n*)
by (*induct n*) (*simp-all add*: *fls-times-both-shifted-simp algebra-simps*)

lemma *fls-pow-conv-fps-pow*:
assumes *fls-subdegree f* ≥ *0*
shows *f* ^ *n* = *fps-to-fls* ((*fls-regpart f*) ^ *n*)
proof (*induct n*)

case (*Suc n*) **with** *assms* **show** *?case*
using *fls-pow-subdegree-ge0*[*of f n*]
by (*simp add*: *fls-times-conv-fps-times*)
qed *simp*

lemma *fps-to-fls-power*: *fps-to-fls* (*f* ^ *n*) = *fps-to-fls f* ^ *n*
by (*simp add*: *fls-pow-conv-fps-pow fls-subdegree-fls-to-fps-gt0*)

lemma *fls-pow-conv-regpart*:
fls-subdegree f ≥ *0* ⟹ *fls-regpart* (*f* ^ *n*) = (*fls-regpart f*) ^ *n*
by (*simp add*: *fls-pow-conv-fps-pow*)

These two lemmas show that shifting 1 is equivalent to powers of the implied variable.

lemma *fls-X-power-conv-shift-1*: *fls-X* ^ *n* = *fls-shift* (−*n*) *1*
by (*simp add*: *fls-X-conv-shift-1 fls-shifted-pow*)

lemma *fls-X-inv-power-conv-shift-1*: *fls-X-inv* ^ *n* = *fls-shift n 1*
by (*simp add*: *fls-X-inv-conv-shift-1 fls-shifted-pow*)

abbreviation *fls-X-intpow* ≡ (λ*i*. *fls-shift* (−*i*) *1*)
— Unifies *fls-X* and *fls-X-inv* so that *fls-X-intpow* returns the equivalent of the implied variable raised to the supplied integer argument of *fls-X-intpow*, whether positive or negative.

lemma *fls-X-intpow-nonzero*[*simp*]: (*fls-X-intpow i* :: *'a*::*zero-neq-one fls*) ≠ *0*
by (*simp add*: *fls-shift-eq0-iff*)

lemma *fls-X-intpow-power*: (*fls-X-intpow i*) ^ *n* = *fls-X-intpow* (*n* ∗ *i*)
by (*simp add*: *fls-shifted-pow*)

lemma *fls-X-power-nth* [*simp*]: *fls-X* ^ *n* $$ *k* = (*if k*=*n then 1 else 0*)
by (*simp add*: *fls-X-power-conv-shift-1*)

lemma *fls-X-inv-power-nth* [*simp*]: *fls-X-inv* ^ *n* $$ *k* = (*if k*=−*n then 1 else 0*)
by (*simp add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-X-pow-nonzero*[*simp*]: (*fls-X* ^ *n* :: *'a* :: *semiring-1 fls*) ≠ *0*
proof
assume (*fls-X* ^ *n* :: *'a fls*) = *0*
hence (*fls-X* ^ *n* :: *'a fls*) $$ *n* = *0* **by** *simp*
thus *False* **by** *simp*
qed

lemma *fls-X-inv-pow-nonzero*[*simp*]: (*fls-X-inv* ^ *n* :: *'a* :: *semiring-1 fls*) ≠ *0*
proof
assume (*fls-X-inv* ^ *n* :: *'a fls*) = *0*
hence (*fls-X-inv* ^ *n* :: *'a fls*) $$ −*n* = *0* **by** *simp*
thus *False* **by** *simp*

qed

lemma *fls-subdegree-fls-X-pow* [*simp*]: *fls-subdegree* (*fls-X* ^ *n*) = *n*
by (*intro fls-subdegree-eqI*) (*simp-all add*: *fls-X-power-conv-shift-1*)

lemma *fls-subdegree-fls-X-inv-pow* [*simp*]: *fls-subdegree* (*fls-X-inv* ^ *n*) = −*n*
by (*intro fls-subdegree-eqI*) (*simp-all add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-subdegree-fls-X-intpow* [*simp*]:
fls-subdegree ((*fls-X-intpow i*) :: 'a::*zero-neq-one fls*) = *i*
by *simp*

lemma *fls-X-pow-conv-fps-X-pow*: *fls-regpart* (*fls-X* ^ *n*) = *fps-X* ^ *n*
by (*simp add*: *fls-pow-conv-regpart*)

lemma *fls-X-inv-pow-regpart*: *n* > *0* ⟹ *fls-regpart* (*fls-X-inv* ^ *n*) = *0*
by (*auto intro*: *fps-ext simp*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-X-intpow-regpart*:
fls-regpart (*fls-X-intpow i*) = (*if i*≥*0 then fps-X* ^ *nat i else 0*)
using *fls-X-pow-conv-fps-X-pow*[*of nat i*]
fls-regpart-shift-conv-fps-shift[*of* −*i 1*]
by (*auto simp*: *fls-X-power-conv-shift-1 fps-shift-one*)

lemma *fls-X-power-times-conv-shift*:
fls-X ^ *n* ∗ *f* = *fls-shift* (−*int n*) *f f* ∗ *fls-X* ^ *n* = *fls-shift* (−*int n*) *f*
using *fls-times-both-shifted-simp*[*of* −*int n 1 0 f*]
fls-times-both-shifted-simp[*of 0 f* −*int n 1*]
by (*simp-all add*: *fls-X-power-conv-shift-1*)

lemma *fls-X-inv-power-times-conv-shift*:
fls-X-inv ^ *n* ∗ *f* = *fls-shift* (*int n*) *f f* ∗ *fls-X-inv* ^ *n* = *fls-shift* (*int n*) *f*
using *fls-times-both-shifted-simp*[*of int n 1 0 f*]
fls-times-both-shifted-simp[*of 0 f int n 1*]
by (*simp-all add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-X-intpow-times-conv-shift*:
fixes *f* :: 'a::*semiring-1 fls*
shows *fls-X-intpow i* ∗ *f* = *fls-shift* (−*i*) *f f* ∗ *fls-X-intpow i* = *fls-shift* (−*i*) *f*
by (*simp-all add*: *fls-shifted-times-simps*)

lemmas *fls-X-power-times-comm* = *trans-sym*[*OF fls-X-power-times-conv-shift*]
lemmas *fls-X-inv-power-times-comm* = *trans-sym*[*OF fls-X-inv-power-times-conv-shift*]

lemma *fls-X-intpow-times-comm*:
fixes *f* :: 'a::*semiring-1 fls*
shows *fls-X-intpow i* ∗ *f* = *f* ∗ *fls-X-intpow i*
by (*simp add*: *fls-X-intpow-times-conv-shift*)

lemma *fls-X-intpow-times-fls-X-intpow*:
(*fls-X-intpow i* :: *'a*::*semiring-1 fls*) ∗ *fls-X-intpow j* = *fls-X-intpow* (*i*+*j*)
by (*simp add*: *fls-times-both-shifted-simp*)

lemma *fls-X-intpow-diff-conv-times*:
fls-X-intpow (*i*−*j*) = (*fls-X-intpow i* :: *'a*::*semiring-1 fls*) ∗ *fls-X-intpow* (−*j*)
using *fls-X-intpow-times-fls-X-intpow*[*of i* −*j*,*symmetric*] **by** *simp*

lemma *fls-mult-fls-X-power-nonzero*:
assumes *f* ≠ *0*
shows *fls-X* ^ *n* ∗ *f* ≠ *0 f* ∗ *fls-X* ^ *n* ≠ *0*
by (*auto simp*: *fls-X-power-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-mult-fls-X-inv-power-nonzero*:
assumes *f* ≠ *0*
shows *fls-X-inv* ^ *n* ∗ *f* ≠ *0 f* ∗ *fls-X-inv* ^ *n* ≠ *0*
by (*auto simp*: *fls-X-inv-power-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-mult-fls-X-intpow-nonzero*:
fixes *f* :: *'a*::*semiring-1 fls*
assumes *f* ≠ *0*
shows *fls-X-intpow i* ∗ *f* ≠ *0 f* ∗ *fls-X-intpow i* ≠ *0*
by (*auto simp*: *fls-X-intpow-times-conv-shift fls-shift-eq0-iff assms*)

lemma *fls-subdegree-mult-fls-X-power*:
assumes *f* ≠ *0*
shows *fls-subdegree* (*fls-X* ^ *n* ∗ *f*) = *fls-subdegree f* + *n*
and *fls-subdegree* (*f* ∗ *fls-X* ^ *n*) = *fls-subdegree f* + *n*
by (*auto simp*: *fls-X-power-times-conv-shift assms*)

lemma *fls-subdegree-mult-fls-X-inv-power*:
assumes *f* ≠ *0*
shows *fls-subdegree* (*fls-X-inv* ^ *n* ∗ *f*) = *fls-subdegree f* − *n*
and *fls-subdegree* (*f* ∗ *fls-X-inv* ^ *n*) = *fls-subdegree f* − *n*
by (*auto simp*: *fls-X-inv-power-times-conv-shift assms*)

lemma *fls-subdegree-mult-fls-X-intpow*:
fixes *f* :: *'a*::*semiring-1 fls*
assumes *f* ≠ *0*
shows *fls-subdegree* (*fls-X-intpow i* ∗ *f*) = *fls-subdegree f* + *i*
and *fls-subdegree* (*f* ∗ *fls-X-intpow i*) = *fls-subdegree f* + *i*
by (*auto simp*: *fls-X-intpow-times-conv-shift assms*)

lemma *fls-X-shift*:
fls-shift (−*int n*) *fls-X* = *fls-X* ^ *Suc n*
fls-shift (*int* (*Suc n*)) *fls-X* = *fls-X-inv* ^ *n*
using *fls-X-power-conv-shift-1*[*of Suc n*, *symmetric*]
by (*simp-all add*: *fls-X-conv-shift-1 fls-X-inv-power-conv-shift-1*)

lemma *fls-X-inv-shift*:
fls-shift (*int n*) *fls-X-inv* = *fls-X-inv* ^ *Suc n*
fls-shift (− *int* (*Suc n*)) *fls-X-inv* = *fls-X* ^ *n*
using *fls-X-inv-power-conv-shift-1*[*of Suc n*, *symmetric*]
by (*simp-all add*: *fls-X-inv-conv-shift-1 fls-X-power-conv-shift-1*)

lemma *fls-X-power-base-factor*: *fls-base-factor* (*fls-X* ^ *n*) = *1*
by (*simp add*: *fls-X-power-conv-shift-1*)

lemma *fls-X-inv-power-base-factor*: *fls-base-factor* (*fls-X-inv* ^ *n*) = *1*
by (*simp add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-X-intpow-base-factor*: *fls-base-factor* (*fls-X-intpow i*) = *1*
using *fls-base-factor-shift*[*of* −*i 1*] **by** *simp*

lemma *fls-base-factor-mult-fls-X-power*:
shows *fls-base-factor* (*fls-X* ^ *n* ∗ *f*) = *fls-base-factor f*
and *fls-base-factor* (*f* ∗ *fls-X* ^ *n*) = *fls-base-factor f*
using *fls-base-factor-shift*[*of* −*int n f*]
by (*auto simp*: *fls-X-power-times-conv-shift*)

lemma *fls-base-factor-mult-fls-X-inv-power*:
shows *fls-base-factor* (*fls-X-inv* ^ *n* ∗ *f*) = *fls-base-factor f*
and *fls-base-factor* (*f* ∗ *fls-X-inv* ^ *n*) = *fls-base-factor f*
using *fls-base-factor-shift*[*of int n f*]
by (*auto simp*: *fls-X-inv-power-times-conv-shift*)

lemma *fls-base-factor-mult-fls-X-intpow*:
fixes *f* :: *'a*::*semiring-1 fls*
shows *fls-base-factor* (*fls-X-intpow i* ∗ *f*) = *fls-base-factor f*
and *fls-base-factor* (*f* ∗ *fls-X-intpow i*) = *fls-base-factor f*
using *fls-base-factor-shift*[*of* −*i f*]
by (*auto simp*: *fls-X-intpow-times-conv-shift*)

lemma *fls-X-power-base-factor-to-fps*: *fls-base-factor-to-fps* (*fls-X* ^ *n*) = *1*
proof−
define *X* **where** *X* ≡ *fls-X* :: *'a*::*semiring-1 fls*
hence *fls-base-factor* (*X* ^ *n*) = *1* **using** *fls-X-power-base-factor* **by** *simp*
thus *fls-base-factor-to-fps* (*X*^*n*) = *1* **by** *simp*
qed

lemma *fls-X-inv-power-base-factor-to-fps*: *fls-base-factor-to-fps* (*fls-X-inv* ^ *n*) = *1*
proof−
define *iX* **where** *iX* ≡ *fls-X-inv* :: *'a*::*semiring-1 fls*
hence *fls-base-factor* (*iX* ^ *n*) = *1* **using** *fls-X-inv-power-base-factor* **by** *simp*
thus *fls-base-factor-to-fps* (*iX*^*n*) = *1* **by** *simp*
qed

lemma *fls-X-intpow-base-factor-to-fps*: *fls-base-factor-to-fps* (*fls-X-intpow i*) = *1*
proof−
define *f* :: *'a fls* **where** *f* ≡ *fls-X-intpow i*
moreover have *fls-base-factor* (*fls-X-intpow i*) = *1* **by** (*rule fls-X-intpow-base-factor*)
ultimately have *fls-base-factor f* = *1* **by** *simp*
thus *fls-base-factor-to-fps f* = *1* **by** *simp*
qed

lemma *fls-base-factor-X-power-decompose*:
fixes *f* :: *'a*::*semiring-1 fls*
shows *f* = *fls-base-factor f* ∗ *fls-X-intpow* (*fls-subdegree f*)
and *f* = *fls-X-intpow* (*fls-subdegree f*) ∗ *fls-base-factor f*
by (*simp-all add*: *fls-times-both-shifted-simp*)

lemma *fls-normalized-product-of-inverses*:
assumes *f* ∗ *g* = *1*
shows *fls-base-factor f* ∗ *fls-base-factor g* =
fls-X ^ (*nat* (−(*fls-subdegree f*+*fls-subdegree g*)))
and *fls-base-factor f* ∗ *fls-base-factor g* =
fls-X-intpow (−(*fls-subdegree f*+*fls-subdegree g*))
using *fls-mult-subdegree-ge*[*of f g*]
fls-times-base-factor-conv-shifted-times[*of f g*]
by (*simp-all add*: *assms fls-X-power-conv-shift-1 algebra-simps*)

lemma *fls-fps-normalized-product-of-inverses*:
assumes *f* ∗ *g* = *1*
shows *fls-base-factor-to-fps f* ∗ *fls-base-factor-to-fps g* =
fps-X ^ (*nat* (−(*fls-subdegree f*+*fls-subdegree g*)))
using *fls-times-conv-regpart*[*of fls-base-factor f fls-base-factor g*]
fls-base-factor-subdegree[*of f*] *fls-base-factor-subdegree*[*of g*]
fls-normalized-product-of-inverses(*1*)[*OF assms*]
by (*force simp*: *fls-X-pow-conv-fps-X-pow*)

7.5.5 Inverses

abbreviation *fls-left-inverse* ::
'a::{*comm-monoid-add*,*uminus*,*times*} *fls* ⇒ *'a* ⇒ *'a fls*
where
fls-left-inverse f x ≡
fls-shift (*fls-subdegree f*) (*fps-to-fls* (*fps-left-inverse* (*fls-base-factor-to-fps f*) *x*))

abbreviation *fls-right-inverse* ::
'a::{*comm-monoid-add*,*uminus*,*times*} *fls* ⇒ *'a* ⇒ *'a fls*
where
fls-right-inverse f y ≡
fls-shift (*fls-subdegree f*) (*fps-to-fls* (*fps-right-inverse* (*fls-base-factor-to-fps f*)
y))

instantiation *fls* :: ({*comm-monoid-add*,*uminus*,*times*,*inverse*}) *inverse*

begin
definition *fls-divide-def*:
f div g =
fls-shift (*fls-subdegree g* − *fls-subdegree f*) (
fps-to-fls ((*fls-base-factor-to-fps f*) *div* (*fls-base-factor-to-fps g*))
)

definition *fls-inverse-def*:
inverse f = *fls-shift* (*fls-subdegree f*) (*fps-to-fls* (*inverse* (*fls-base-factor-to-fps f*)))
instance ..
end

lemma *fls-inverse-def'*:
inverse f = *fls-right-inverse f* (*inverse* (*f* $$ *fls-subdegree f*))
by (*simp add*: *fls-inverse-def fps-inverse-def*)

lemma *fls-lr-inverse-base*:
fls-left-inverse f x $$ (−*fls-subdegree f*) = *x*
fls-right-inverse f y $$ (−*fls-subdegree f*) = *y*
by *auto*

lemma *fls-inverse-base*:
f ≠ *0* ⟹ *inverse f* $$ (−*fls-subdegree f*) = *inverse* (*f* $$ *fls-subdegree f*)
by (*simp add*: *fls-inverse-def'*)

lemma *fls-lr-inverse-starting0*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*} *fls*
and *g* :: *'b*::{*ab-group-add*,*mult-zero*} *fls*
shows *fls-left-inverse f 0* = *0*
and *fls-right-inverse g 0* = *0*
by (*simp-all add*: *fps-lr-inverse-starting0*)

lemma *fls-lr-inverse-eq0-imp-starting0*:
fls-left-inverse f x = *0* ⟹ *x* = *0*
fls-right-inverse f x = *0* ⟹ *x* = *0*
by (*metis fls-lr-inverse-base fls-nonzeroI*)+

lemma *fls-lr-inverse-eq-0-iff*:
fixes *x* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
and *y* :: *'b*::{*ab-group-add*,*mult-zero*}
shows *fls-left-inverse f x* = *0* ⟷ *x* = *0*
and *fls-right-inverse g y* = *0* ⟷ *y* = *0*
using *fls-lr-inverse-starting0 fls-lr-inverse-eq0-imp-starting0*
by *auto*

lemma *fls-inverse-eq-0-iff'*:
fixes *f* :: *'a*::{*ab-group-add*,*inverse*,*mult-zero*} *fls*
shows *inverse f* = *0* ⟷ (*inverse* (*f* $$ *fls-subdegree f*) = *0*)

using *fls-lr-inverse-eq-0-iff*(*2*)[*of f inverse* (*f* $$ *fls-subdegree f*)]
by (*simp add*: *fls-inverse-def'*)

lemma *fls-inverse-eq-0-iff*[*simp*]:
inverse f = (*0*:: ('*a*::*division-ring*) *fls*) ⟷ *f* $$ *fls-subdegree f* = *0*
using *fls-inverse-eq-0-iff'*[*of f*] **by** (*cases f*=*0*) *auto*

lemmas *fls-inverse-eq-0'* = *iffD2*[*OF fls-inverse-eq-0-iff'*]
lemmas *fls-inverse-eq-0* = *iffD2*[*OF fls-inverse-eq-0-iff*]

lemma *fls-lr-inverse-const*:
fixes *a* :: '*a*::{*ab-group-add*,*mult-zero*}
and *b* :: '*b*::{*comm-monoid-add*,*mult-zero*,*uminus*}
shows *fls-left-inverse* (*fls-const a*) *x* = *fls-const x*
and *fls-right-inverse* (*fls-const b*) *y* = *fls-const y*
by (*simp-all add*: *fps-const-lr-inverse*)

lemma *fls-inverse-const*:
fixes *a* :: '*a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}
shows *inverse* (*fls-const a*) = *fls-const* (*inverse a*)
using *fls-lr-inverse-const*(*2*)
by (*auto simp*: *fls-inverse-def'*)

lemma *fls-lr-inverse-of-nat*:
fixes *x* :: '*a*::{*ring-1*,*mult-zero*}
and *y* :: '*b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*of-nat n*) *x* = *fls-const x*
and *fls-right-inverse* (*of-nat n*) *y* = *fls-const y*
using *fls-lr-inverse-const*
by (*auto simp*: *fls-of-nat*)

lemma *fls-inverse-of-nat*:
inverse (*of-nat n* :: '*a* :: {*semiring-1*,*inverse*,*uminus*} *fls*) = *fls-const* (*inverse* (*of-nat n*))
by (*simp add*: *fls-inverse-const fls-of-nat*)

lemma *fls-lr-inverse-of-int*:
fixes *x* :: '*a*::{*ring-1*,*mult-zero*}
shows *fls-left-inverse* (*of-int n*) *x* = *fls-const x*
and *fls-right-inverse* (*of-int n*) *x* = *fls-const x*
using *fls-lr-inverse-const*
by (*auto simp*: *fls-of-int*)

lemma *fls-inverse-of-int*:
inverse (*of-int n* :: '*a* :: {*ring-1*,*inverse*,*uminus*} *fls*) = *fls-const* (*inverse* (*of-int n*))
by (*simp add*: *fls-inverse-const fls-of-int*)

lemma *fls-lr-inverse-zero*:

fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus\}$
shows *fls-left-inverse 0 x = fls-const x*
and *fls-right-inverse 0 y = fls-const y*
using *fls-lr-inverse-const*[*of 0*]
by *auto*

lemma *fls-inverse-zero-conv-fls-const*:
inverse (*0*::*'a*::{*comm-monoid-add*,*mult-zero*,*uminus*,*inverse*} *fls*) = *fls-const* (*inverse 0*)
using *fls-lr-inverse-zero*(*2*)[*of inverse* (*0*::*'a*)] **by** (*simp add*: *fls-inverse-def'*)

lemma *fls-inverse-zero'*:
assumes *inverse* (*0*::*'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}) = *0*
shows *inverse* (*0*::*'a fls*) = *0*
by (*simp add*: *fls-inverse-zero-conv-fls-const assms*)

lemma *fls-inverse-zero* [*simp*]: *inverse* (*0*::*'a*::*division-ring fls*) = *0*
by (*rule fls-inverse-zero'*[*OF inverse-zero*])

lemma *fls-inverse-base2*:
fixes *f* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*,*inverse*} *fls*
shows *inverse f* $$ (−*fls-subdegree f*) = *inverse* (*f* $$ *fls-subdegree f*)
by (*cases f=0*) (*simp-all add*: *fls-inverse-zero-conv-fls-const fls-inverse-def'*)

lemma *fls-lr-inverse-one*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,one\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,mult\text{-}zero,uminus,one\}$
shows *fls-left-inverse 1 x = fls-const x*
and *fls-right-inverse 1 y = fls-const y*
using *fls-lr-inverse-const*[*of 1*]
by *auto*

lemma *fls-lr-inverse-one-one*:
fls-left-inverse 1 1 =
(*1*::*'a*::{*ab-group-add*,*mult-zero*,*one*} *fls*)
fls-right-inverse 1 1 =
(*1*::*'b*::{*comm-monoid-add*,*mult-zero*,*uminus*,*one*} *fls*)
using *fls-lr-inverse-one*[*of 1*] **by** *auto*

lemma *fls-inverse-one*:
assumes *inverse* (*1*::*'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*one*}) = *1*
shows *inverse* (*1*::*'a fls*) = *1*
using *assms fls-lr-inverse-one-one*(*2*)
by (*simp add*: *fls-inverse-def'*)

lemma *fls-left-inverse-delta*:
fixes $b :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero\}$
assumes $b \neq 0$

shows *fls-left-inverse* (*Abs-fls* (λ*n*. *if n=a then b else 0*)) *x* =
Abs-fls (λ*n*. *if n=−a then x else 0*)
proof (*intro fls-eqI*)
fix *n* **from** *assms* **show**
fls-left-inverse (*Abs-fls* (λ*n*. *if n=a then b else 0*)) *x* $$ *n*
= *Abs-fls* (λ*n*. *if n = − a then x else 0*) $$ *n*
using *fls-base-factor-to-fps-delta*[*of a b*]
fls-lr-inverse-const(*1*)[*of b*]
fls-shift-const
by *simp*
qed

lemma *fls-right-inverse-delta*:
fixes *b* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
assumes *b* ≠ *0*
shows *fls-right-inverse* (*Abs-fls* (λ*n*. *if n=a then b else 0*)) *x* =
Abs-fls (λ*n*. *if n=−a then x else 0*)
proof (*intro fls-eqI*)
fix *n* **from** *assms* **show**
fls-right-inverse (*Abs-fls* (λ*n*. *if n=a then b else 0*)) *x* $$ *n*
= *Abs-fls* (λ*n*. *if n = − a then x else 0*) $$ *n*
using *fls-base-factor-to-fps-delta*[*of a b*]
fls-lr-inverse-const(*2*)[*of b*]
fls-shift-const
by *simp*
qed

lemma *fls-inverse-delta-nonzero*:
fixes *b* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*}
assumes *b* ≠ *0*
shows *inverse* (*Abs-fls* (λ*n*. *if n=a then b else 0*)) =
Abs-fls (λ*n*. *if n=−a then inverse b else 0*)
using *assms fls-nonzeroI*[*of Abs-fls* (λ*n*. *if n=a then b else 0*) *a*]
by (*simp add*: *fls-inverse-def' fls-right-inverse-delta*[*symmetric*])

lemma *fls-inverse-delta*:
fixes *b* :: *'a*::*division-ring*
shows *inverse* (*Abs-fls* (λ*n*. *if n=a then b else 0*)) =
Abs-fls (λ*n*. *if n=−a then inverse b else 0*)
by (*cases b=0*) (*simp-all add*: *fls-inverse-delta-nonzero*)

lemma *fls-lr-inverse-X*:
fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*,*zero-neq-one*}
and *y* :: *'b*::{*comm-monoid-add*,*uminus*,*mult-zero*,*zero-neq-one*}
shows *fls-left-inverse fls-X x* = *fls-shift 1* (*fls-const x*)
and *fls-right-inverse fls-X y* = *fls-shift 1* (*fls-const y*)
using *fls-lr-inverse-one*(*1*)[*of x*] *fls-lr-inverse-one*(*2*)[*of y*]
by *auto*

lemma *fls-lr-inverse-X'*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,uminus,mult\text{-}zero,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$
shows *fls-left-inverse fls-X x = fls-const x * fls-X-inv*
and *fls-right-inverse fls-X y = fls-const y * fls-X-inv*
using *fls-lr-inverse-X(1)[of x] fls-lr-inverse-X(2)[of y]*
by (*simp-all add*: *fls-X-inv-times-conv-shift(2)*)

lemma *fls-inverse-X'*:
assumes *inverse 1 = (1::'a::\{comm-monoid-add,inverse,mult-zero,uminus,zero-neq-one\})*
shows *inverse (fls-X::'a fls) = fls-X-inv*
using *assms fls-lr-inverse-X(2)[of 1::'a]*
by (*simp add*: *fls-inverse-def' fls-X-inv-conv-shift-1*)

lemma *fls-inverse-X*: *inverse (fls-X::'a::division-ring fls) = fls-X-inv*
by (*simp add*: *fls-inverse-X'*)

lemma *fls-lr-inverse-X-inv*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,zero\text{-}neq\text{-}one\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,uminus,mult\text{-}zero,zero\text{-}neq\text{-}one\}$
shows *fls-left-inverse fls-X-inv x = fls-shift (−1) (fls-const x)*
and *fls-right-inverse fls-X-inv y = fls-shift (−1) (fls-const y)*
using *fls-lr-inverse-one(1)[of x] fls-lr-inverse-one(2)[of y]*
by *auto*

lemma *fls-lr-inverse-X-inv'*:
fixes $x :: 'a::\{ab\text{-}group\text{-}add,mult\text{-}zero,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$
and $y :: 'b::\{comm\text{-}monoid\text{-}add,uminus,mult\text{-}zero,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$
shows *fls-left-inverse fls-X-inv x = fls-const x * fls-X*
and *fls-right-inverse fls-X-inv y = fls-const y * fls-X*
using *fls-lr-inverse-X-inv(1)[of x] fls-lr-inverse-X-inv(2)[of y]*
by (*simp-all add*: *fls-X-times-conv-shift(2)*)

lemma *fls-inverse-X-inv'*:
assumes *inverse 1 = (1::'a::\{comm-monoid-add,inverse,mult-zero,uminus,zero-neq-one\})*
shows *inverse (fls-X-inv::'a fls) = fls-X*
using *assms fls-lr-inverse-X-inv(2)[of 1::'a]*
by (*simp add*: *fls-inverse-def' fls-X-conv-shift-1*)

lemma *fls-inverse-X-inv*: *inverse (fls-X-inv::'a::division-ring fls) = fls-X*
by (*simp add*: *fls-inverse-X-inv'*)

lemma *fls-lr-inverse-subdegree*:
assumes $x \neq 0$
shows *fls-subdegree (fls-left-inverse f x) = − fls-subdegree f*
and *fls-subdegree (fls-right-inverse f x) = − fls-subdegree f*
by (*auto intro*: *fls-subdegree-eqI simp*: *assms*)

lemma *fls-inverse-subdegree'*:

inverse (*f* $$ *fls-subdegree f*) ≠ *0* ⟹ *fls-subdegree* (*inverse f*) = − *fls-subdegree f*
using *fls-lr-inverse-subdegree*(*2*)[*of inverse* (*f* $$ *fls-subdegree f*)]
by (*simp add*: *fls-inverse-def'*)

lemma *fls-inverse-subdegree* [*simp*]:
fixes *f* :: '*a*::*division-ring fls*
shows *fls-subdegree* (*inverse f*) = − *fls-subdegree f*
by (*cases f*=*0*)
(*auto intro*: *fls-inverse-subdegree'* *simp*: *nonzero-imp-inverse-nonzero*)

lemma *fls-inverse-subdegree-base-nonzero*:
assumes *f* ≠ *0 inverse* (*f* $$ *fls-subdegree f*) ≠ *0*
shows *inverse f* $$ (*fls-subdegree* (*inverse f*)) = *inverse* (*f* $$ *fls-subdegree f*)
using *assms fls-inverse-subdegree'*[*of f*] *fls-inverse-base*[*of f*]
by *simp*

lemma *fls-inverse-subdegree-base*:
fixes *f* :: '*a*::{*ab-group-add*,*inverse*,*mult-zero*} *fls*
shows *inverse f* $$ (*fls-subdegree* (*inverse f*)) = *inverse* (*f* $$ *fls-subdegree f*)
using *fls-inverse-eq-0-iff'*[*of f*] *fls-inverse-subdegree-base-nonzero*[*of f*]
by (*cases f*=*0* ∨ *inverse* (*f* $$ *fls-subdegree f*) = *0*)
(*auto simp*: *fls-inverse-zero-conv-fls-const*)

lemma *fls-lr-inverse-subdegree-0*:
assumes *fls-subdegree f* = *0*
shows *fls-subdegree* (*fls-left-inverse f x*) ≥ *0*
and *fls-subdegree* (*fls-right-inverse f x*) ≥ *0*
using *fls-subdegree-ge0I*[*of fls-left-inverse f x*]
fls-subdegree-ge0I[*of fls-right-inverse f x*]
by (*auto simp*: *assms*)

lemma *fls-inverse-subdegree-0*:
fls-subdegree f = *0* ⟹ *fls-subdegree* (*inverse f*) ≥ *0*
using *fls-lr-inverse-subdegree-0*(*2*)[*of f*] **by** (*simp add*: *fls-inverse-def'*)

lemma *fls-lr-inverse-shift-nonzero*:
fixes *f* :: '*a*::{*comm-monoid-add*,*mult-zero*,*uminus*} *fls*
assumes *f* ≠ *0*
shows *fls-left-inverse* (*fls-shift m f*) *x* = *fls-shift* (−*m*) (*fls-left-inverse f x*)
and *fls-right-inverse* (*fls-shift m f*) *x* = *fls-shift* (−*m*) (*fls-right-inverse f x*)
using *assms fls-base-factor-to-fps-shift*[*of m f*] *fls-shift-subdegree*
by *auto*

lemma *fls-inverse-shift-nonzero*:
fixes *f* :: '*a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
assumes *f* ≠ *0*
shows *inverse* (*fls-shift m f*) = *fls-shift* (−*m*) (*inverse f*)
using *assms fls-lr-inverse-shift-nonzero*(*2*)[*of f m inverse* (*f* $$ *fls-subdegree f*)]

by (*simp add*: *fls-inverse-def'*)

lemma *fls-inverse-shift*:
fixes *f* :: *'a*::*division-ring fls*
shows *inverse* (*fls-shift m f*) = *fls-shift* (−*m*) (*inverse f*)
using *fls-inverse-shift-nonzero*
by (*cases f*=*0*) *simp-all*

lemma *fls-left-inverse-base-factor*:
fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*}
assumes *x* ≠ *0*
shows *fls-left-inverse* (*fls-base-factor f*) *x* = *fls-base-factor* (*fls-left-inverse f x*)
using *assms fls-lr-inverse-zero*(*1*)[*of x*] *fls-lr-inverse-subdegree*(*1*)[*of x*]
by (*cases f*=*0*) *auto*

lemma *fls-right-inverse-base-factor*:
fixes *y* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
assumes *y* ≠ *0*
shows *fls-right-inverse* (*fls-base-factor f*) *y* = *fls-base-factor* (*fls-right-inverse f y*)
using *assms fls-lr-inverse-zero*(*2*)[*of y*] *fls-lr-inverse-subdegree*(*2*)[*of y*]
by (*cases f*=*0*) *auto*

lemma *fls-inverse-base-factor'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
assumes *inverse* (*f* $$ *fls-subdegree f*) ≠ *0*
shows *inverse* (*fls-base-factor f*) = *fls-base-factor* (*inverse f*)
by (*cases f*=*0*)
(*simp-all add*:
assms fls-inverse-shift-nonzero fls-inverse-subdegree'
fls-inverse-zero-conv-fls-const
)

lemma *fls-inverse-base-factor*:
fixes *f* :: *'a*::{*ab-group-add*,*inverse*,*mult-zero*} *fls*
shows *inverse* (*fls-base-factor f*) = *fls-base-factor* (*inverse f*)
using *fls-base-factor-base*[*of f*] *fls-inverse-eq-0-iff'*[*of f*]
fls-inverse-eq-0-iff'[*of fls-base-factor f*] *fls-inverse-base-factor'*[*of f*]
by (*cases inverse* (*f* $$ *fls-subdegree f*) = *0*) *simp-all*

lemma *fls-lr-inverse-regpart*:
assumes *fls-subdegree f* = *0*
shows *fls-regpart* (*fls-left-inverse f x*) = *fps-left-inverse* (*fls-regpart f*) *x*
and *fls-regpart* (*fls-right-inverse f y*) = *fps-right-inverse* (*fls-regpart f*) *y*
using *assms*
by *auto*

lemma *fls-inverse-regpart*:
assumes *fls-subdegree f* = *0*

shows *fls-regpart* (*inverse f*) = *inverse* (*fls-regpart f*)
by (*simp add*: *assms fls-inverse-def*)

lemma *fls-base-factor-to-fps-left-inverse*:
fixes *x* :: *'a*::{*ab-group-add*,*mult-zero*}
shows *fls-base-factor-to-fps* (*fls-left-inverse f x*) =
fps-left-inverse (*fls-base-factor-to-fps f*) *x*
using *fls-left-inverse-base-factor*[*of x f*] *fls-base-factor-subdegree*[*of f*]
by (*cases x=0*) (*simp-all add*: *fls-lr-inverse-starting0*(*1*) *fps-lr-inverse-starting0*(*1*))

lemma *fls-base-factor-to-fps-right-inverse-nonzero*:
fixes *y* :: *'a*::{*comm-monoid-add*,*mult-zero*,*uminus*}
assumes *y* ≠ *0*
shows *fls-base-factor-to-fps* (*fls-right-inverse f y*) =
fps-right-inverse (*fls-base-factor-to-fps f*) *y*
using *assms fls-right-inverse-base-factor*[*of y f*]
fls-base-factor-subdegree[*of f*]
by *simp*

lemma *fls-base-factor-to-fps-right-inverse*:
fixes *y* :: *'a*::{*ab-group-add*,*mult-zero*}
shows *fls-base-factor-to-fps* (*fls-right-inverse f y*) =
fps-right-inverse (*fls-base-factor-to-fps f*) *y*
using *fls-base-factor-to-fps-right-inverse-nonzero*[*of y f*]
by (*cases y=0*) (*simp-all add*: *fls-lr-inverse-starting0*(*2*) *fps-lr-inverse-starting0*(*2*))

lemma *fls-base-factor-to-fps-inverse-nonzero*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
assumes *inverse* (*f* $$ *fls-subdegree f*) ≠ *0*
shows *fls-base-factor-to-fps* (*inverse f*) = *inverse* (*fls-base-factor-to-fps f*)
using *assms fls-base-factor-to-fps-right-inverse-nonzero*
by (*simp add*: *fls-inverse-def' fps-inverse-def*)

lemma *fls-base-factor-to-fps-inverse*:
fixes *f* :: *'a*::{*ab-group-add*,*inverse*,*mult-zero*} *fls*
shows *fls-base-factor-to-fps* (*inverse f*) = *inverse* (*fls-base-factor-to-fps f*)
using *fls-base-factor-to-fps-right-inverse*
by (*simp add*: *fls-inverse-def' fps-inverse-def*)

lemma *fls-lr-inverse-fps-to-fls*:
assumes *subdegree f* = *0*
shows *fls-left-inverse* (*fps-to-fls f*) *x* = *fps-to-fls* (*fps-left-inverse f x*)
and *fls-right-inverse* (*fps-to-fls f*) *x* = *fps-to-fls* (*fps-right-inverse f x*)
using *assms fls-base-factor-to-fps-to-fls*[*of f*]
by (*simp-all add*: *fls-subdegree-fls-to-fps*)

lemma *fls-inverse-fps-to-fls*:
subdegree f = *0* ⟹ *inverse* (*fps-to-fls f*) = *fps-to-fls* (*inverse f*)
using *nth-subdegree-nonzero*[*of f*]

by (*cases f=0*)
(*auto simp add*:
fps-to-fls-nonzeroI fls-inverse-def' fls-subdegree-fls-to-fps fps-inverse-def
fls-lr-inverse-fps-to-fls(*2*)
)

lemma *fls-lr-inverse-X-power*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X* ^ *n*) *x* = *fls-shift n* (*fls-const x*)
and *fls-right-inverse* (*fls-X* ^ *n*) *y* = *fls-shift n* (*fls-const y*)
using *fls-lr-inverse-one*(*1*)[*of x*] *fls-lr-inverse-one*(*2*)[*of y*]
by (*simp-all add*: *fls-X-power-conv-shift-1*)

lemma *fls-lr-inverse-X-power'*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X* ^ *n*) *x* = *fls-const x* ∗ *fls-X-inv* ^ *n*
and *fls-right-inverse* (*fls-X* ^ *n*) *y* = *fls-const y* ∗ *fls-X-inv* ^ *n*
using *fls-lr-inverse-X-power*(*1*)[*of n x*] *fls-lr-inverse-X-power*(*2*)[*of n y*]
by (*simp-all add*: *fls-X-inv-power-times-conv-shift*(*2*))

lemma *fls-inverse-X-power'*:
assumes *inverse 1* = (*1*::*'a*::{*semiring-1*,*uminus*,*inverse*})
shows *inverse* ((*fls-X* ^ *n*)::*'a fls*) = *fls-X-inv* ^ *n*
using *fls-lr-inverse-X-power'*(*2*)[*of n 1*]
by (*simp add*: *fls-inverse-def' assms*)

lemma *fls-inverse-X-power*:
inverse ((*fls-X*::*'a*::*division-ring fls*) ^ *n*) = *fls-X-inv* ^ *n*
by (*simp add*: *fls-inverse-X-power'*)

lemma *fls-lr-inverse-X-inv-power*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-inv* ^ *n*) *x* = *fls-shift* (−*n*) (*fls-const x*)
and *fls-right-inverse* (*fls-X-inv* ^ *n*) *y* = *fls-shift* (−*n*) (*fls-const y*)
using *fls-lr-inverse-one*(*1*)[*of x*] *fls-lr-inverse-one*(*2*)[*of y*]
by (*simp-all add*: *fls-X-inv-power-conv-shift-1*)

lemma *fls-lr-inverse-X-inv-power'*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-inv* ^ *n*) *x* = *fls-const x* ∗ *fls-X* ^ *n*
and *fls-right-inverse* (*fls-X-inv* ^ *n*) *y* = *fls-const y* ∗ *fls-X* ^ *n*
using *fls-lr-inverse-X-inv-power*(*1*)[*of n x*] *fls-lr-inverse-X-inv-power*(*2*)[*of n y*]
by (*simp-all add*: *fls-X-power-times-conv-shift*(*2*))

lemma *fls-inverse-X-inv-power'*:

assumes *inverse 1* = (*1*::*'a*::{*semiring-1*,*uminus*,*inverse*})
shows *inverse* ((*fls-X-inv* ^ *n*)::*'a fls*) = *fls-X* ^ *n*
using *fls-lr-inverse-X-inv-power'*(*2*)[*of n 1*]
by (*simp add*: *fls-inverse-def' assms*)

lemma *fls-inverse-X-inv-power*:
inverse ((*fls-X-inv*::*'a*::*division-ring fls*) ^ *n*) = *fls-X* ^ *n*
by (*simp add*: *fls-inverse-X-inv-power'*)

lemma *fls-lr-inverse-X-intpow*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-intpow i*) *x* = *fls-shift i* (*fls-const x*)
and *fls-right-inverse* (*fls-X-intpow i*) *y* = *fls-shift i* (*fls-const y*)
using *fls-lr-inverse-one*(*1*)[*of x*] *fls-lr-inverse-one*(*2*)[*of y*]
by *auto*

lemma *fls-lr-inverse-X-intpow'*:
fixes *x* :: *'a*::*ring-1*
and *y* :: *'b*::{*semiring-1*,*uminus*}
shows *fls-left-inverse* (*fls-X-intpow i*) *x* = *fls-const x* ∗ *fls-X-intpow* (−*i*)
and *fls-right-inverse* (*fls-X-intpow i*) *y* = *fls-const y* ∗ *fls-X-intpow* (−*i*)
using *fls-lr-inverse-X-intpow*(*1*)[*of i x*] *fls-lr-inverse-X-intpow*(*2*)[*of i y*]
by (*simp-all add*: *fls-shifted-times-simps*(*1*))

lemma *fls-inverse-X-intpow'*:
assumes *inverse 1* = (*1*::*'a*::{*semiring-1*,*uminus*,*inverse*})
shows *inverse* (*fls-X-intpow i* :: *'a fls*) = *fls-X-intpow* (−*i*)
using *fls-lr-inverse-X-intpow'*(*2*)[*of i 1*]
by (*simp add*: *fls-inverse-def' assms*)

lemma *fls-inverse-X-intpow*:
inverse (*fls-X-intpow i* :: *'a*::*division-ring fls*) = *fls-X-intpow* (−*i*)
by (*simp add*: *fls-inverse-X-intpow'*)

lemma *fls-left-inverse*:
fixes *f* :: *'a*::*ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1*
shows *fls-left-inverse f x* ∗ *f* = *1*
proof−
from *assms* **have** *x* ≠ *0 x* ∗ (*fls-base-factor-to-fps f*$*0*) = *1* **by** *auto*
thus *?thesis*
using *fls-base-factor-to-fps-left-inverse*[*of f x*]
fls-lr-inverse-subdegree(*1*)[*of x*] *fps-left-inverse*
by (*fastforce simp*: *fls-times-def*)
qed

lemma *fls-right-inverse*:
fixes *f* :: *'a*::*ring-1 fls*

assumes *f $$ fls-subdegree f * y = 1*
shows *f * fls-right-inverse f y = 1*
proof−
from *assms* **have** *y ≠ 0* (*fls-base-factor-to-fps f$0*) * *y = 1* **by** *auto*
thus *?thesis*
using *fls-base-factor-to-fps-right-inverse*[*of f y*]
fls-lr-inverse-subdegree(*2*)[*of y*] *fps-right-inverse*
by (*fastforce simp*: *fls-times-def*)
qed

— It is possible in a ring for an element to have a left inverse but not a right inverse, or vice versa. But when an element has both, they must be the same.
lemma *fls-left-inverse-eq-fls-right-inverse*:
fixes *f* :: *'a*::*ring-1 fls*
assumes *x * f $$ fls-subdegree f = 1 f $$ fls-subdegree f * y = 1*
— These assumptions imply x equals y, but no need to assume that.
shows *fls-left-inverse f x = fls-right-inverse f y*
using *assms*
by (*simp add*: *fps-left-inverse-eq-fps-right-inverse*)

lemma *fls-left-inverse-eq-inverse*:
fixes *f* :: *'a*::*division-ring fls*
shows *fls-left-inverse f* (*inverse* (*f $$ fls-subdegree f*)) = *inverse f*
proof (*cases f=0*)
case *True*
hence *fls-left-inverse f* (*inverse* (*f $$ fls-subdegree f*)) = *fls-const* (*0*::*'a*)
by (*simp add*: *fls-lr-inverse-zero*(*1*)[*symmetric*])
with *True* **show** *?thesis* **by** *simp*
next
case *False* **thus** *?thesis*
using *fls-left-inverse-eq-fls-right-inverse*[*of inverse* (*f $$ fls-subdegree f*)]
by (*auto simp add*: *fls-inverse-def'*)
qed

lemma *fls-right-inverse-eq-inverse*:
fixes *f* :: *'a*::*division-ring fls*
shows *fls-right-inverse f* (*inverse* (*f $$ fls-subdegree f*)) = *inverse f*
proof (*cases f=0*)
case *True*
hence *fls-right-inverse f* (*inverse* (*f $$ fls-subdegree f*)) = *fls-const* (*0*::*'a*)
by (*simp add*: *fls-lr-inverse-zero*(*2*)[*symmetric*])
with *True* **show** *?thesis* **by** *simp*
qed (*simp add*: *fls-inverse-def'*)

lemma *fls-left-inverse-eq-fls-right-inverse-comm*:
fixes *f* :: *'a*::*comm-ring-1 fls*
assumes *x * f $$ fls-subdegree f = 1*
shows *fls-left-inverse f x = fls-right-inverse f x*
using *assms fls-left-inverse-eq-fls-right-inverse*[*of x f x*]

by (*simp add*: *mult.commute*)

lemma *fls-left-inverse'*:
fixes *f* :: *'a*::*ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *fls-right-inverse f y* ∗ *f* = *1*
using *assms fls-left-inverse-eq-fls-right-inverse*[*of x f y*] *fls-left-inverse*[*of x f*]
by *simp*

lemma *fls-right-inverse'*:
fixes *f* :: *'a*::*ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *f* ∗ *fls-left-inverse f x* = *1*
using *assms fls-left-inverse-eq-fls-right-inverse*[*of x f y*] *fls-right-inverse*[*of f y*]
by *simp*

lemma *fls-mult-left-inverse-base-factor*:
fixes *f* :: *'a*::*ring-1 fls*
assumes *x* ∗ (*f* $$ *fls-subdegree f*) = *1*
shows *fls-left-inverse* (*fls-base-factor f*) *x* ∗ *f* = *fls-X-intpow* (*fls-subdegree f*)
using *assms fls-base-factor-to-fps-base-factor*[*of f*] *fls-base-factor-subdegree*[*of f*]
fls-shifted-times-simps(*2*)[*of* −*fls-subdegree f fls-left-inverse f x f*]
fls-left-inverse[*of x f*]
by *simp*

lemma *fls-mult-right-inverse-base-factor*:
fixes *f* :: *'a*::*ring-1 fls*
assumes (*f* $$ *fls-subdegree f*) ∗ *y* = *1*
shows *f* ∗ *fls-right-inverse* (*fls-base-factor f*) *y* = *fls-X-intpow* (*fls-subdegree f*)
using *assms fls-base-factor-to-fps-base-factor*[*of f*] *fls-base-factor-subdegree*[*of f*]
fls-shifted-times-simps(*1*)[*of f* −*fls-subdegree f fls-right-inverse f y*]
fls-right-inverse[*of f y*]
by *simp*

lemma *fls-mult-inverse-base-factor*:
fixes *f* :: *'a*::*division-ring fls*
assumes *f* ≠ *0*
shows *f* ∗ *inverse* (*fls-base-factor f*) = *fls-X-intpow* (*fls-subdegree f*)
using *fls-mult-right-inverse-base-factor*[*of f inverse* (*f* $$ *fls-subdegree f*)]
fls-base-factor-base[*of f*]
by (*simp add*: *assms fls-right-inverse-eq-inverse*[*symmetric*])

lemma *fls-left-inverse-idempotent-ring1*:
fixes *f* :: *'a*::*ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 y* ∗ *x* = *1*
— These assumptions imply y equals *f* $$ *fls-subdegree f*, but no need to assume that.

shows *fls-left-inverse* (*fls-left-inverse f x*) *y* = *f*
proof−
from *assms*(*1*) **have**
fls-left-inverse (*fls-left-inverse f x*) *y* ∗ *fls-left-inverse f x* ∗ *f* =
fls-left-inverse (*fls-left-inverse f x*) *y*
using *fls-left-inverse*[*of x f*]
by (*simp add*: *mult.assoc*)
moreover have
fls-left-inverse (*fls-left-inverse f x*) *y* ∗ *fls-left-inverse f x* = *1*
using *assms fls-lr-inverse-subdegree*(*1*)[*of x f*] *fls-lr-inverse-base*(*1*)[*of f x*]
by (*fastforce intro*: *fls-left-inverse*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fls-left-inverse-idempotent-comm-ring1*:
fixes *f* :: *'a*::*comm-ring-1 fls*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1*
shows *fls-left-inverse* (*fls-left-inverse f x*) (*f* $$ *fls-subdegree f*) = *f*
using *assms fls-left-inverse-idempotent-ring1*[*of x f f* $$ *fls-subdegree f*]
by (*simp add*: *mult.commute*)

lemma *fls-right-inverse-idempotent-ring1*:
fixes *f* :: *'a*::*ring-1 fls*
assumes *f* $$ *fls-subdegree f* ∗ *x* = *1 x* ∗ *y* = *1*
— These assumptions imply y equals *f* $$ *fls-subdegree f*, but no need to assume that.
shows *fls-right-inverse* (*fls-right-inverse f x*) *y* = *f*
proof−
from *assms*(*1*) **have**
f ∗ (*fls-right-inverse f x* ∗ *fls-right-inverse* (*fls-right-inverse f x*) *y*) =
fls-right-inverse (*fls-right-inverse f x*) *y*
using *fls-right-inverse* [*of f*]
by (*simp add*: *mult.assoc*[*symmetric*])
moreover have
fls-right-inverse f x ∗ *fls-right-inverse* (*fls-right-inverse f x*) *y* = *1*
using *assms fls-lr-inverse-subdegree*(*2*)[*of x f*] *fls-lr-inverse-base*(*2*)[*of f x*]
by (*fastforce intro*: *fls-right-inverse*)
ultimately show *?thesis* **by** *simp*
qed

lemma *fls-right-inverse-idempotent-comm-ring1*:
fixes *f* :: *'a*::*comm-ring-1 fls*
assumes *f* $$ *fls-subdegree f* ∗ *x* = *1*
shows *fls-right-inverse* (*fls-right-inverse f x*) (*f* $$ *fls-subdegree f*) = *f*
using *assms fls-right-inverse-idempotent-ring1*[*of f x f* $$ *fls-subdegree f*]
by (*simp add*: *mult.commute*)

lemma *fls-lr-inverse-unique-ring1*:
fixes *f g* :: *'a* :: *ring-1 fls*

assumes *fg*: *f* ∗ *g* = *1 g* $$ *fls-subdegree g* ∗ *f* $$ *fls-subdegree f* = *1*
shows *fls-left-inverse g* (*f* $$ *fls-subdegree f*) = *f*
and *fls-right-inverse f* (*g* $$ *fls-subdegree g*) = *g*
proof−

have *f* $$ *fls-subdegree f* ∗ *g* $$ *fls-subdegree g* ≠ *0*
proof
assume *f* $$ *fls-subdegree f* ∗ *g* $$ *fls-subdegree g* = *0*
hence *f* $$ *fls-subdegree f* ∗ (*g* $$ *fls-subdegree g* ∗ *f* $$ *fls-subdegree f*) = *0*
by (*simp add*: *mult.assoc*[*symmetric*])
with *fg*(*2*) **show** *False* **by** *simp*
qed
with *fg*(*1*) **have** *subdeg-sum*: *fls-subdegree f* + *fls-subdegree g* = *0*
using *fls-mult-nonzero-base-subdegree-eq*[*of f g*] **by** *simp*
hence *subdeg-sum′*:
fls-subdegree f = −*fls-subdegree g fls-subdegree g* = −*fls-subdegree f*
by *auto*

from *fg*(*1*) **have** *f-ne-0*: *f*≠*0* **by** *auto*
moreover have
fps-left-inverse (*fls-base-factor-to-fps g*) (*fls-regpart* (*fls-shift* (−*fls-subdegree g*) *f*)$*0*)
= *fls-regpart* (*fls-shift* (−*fls-subdegree g*) *f*)
proof (*intro fps-lr-inverse-unique-ring1*(*1*))
from *fg*(*1*) **show**
fls-regpart (*fls-shift* (−*fls-subdegree g*) *f*) ∗ *fls-base-factor-to-fps g* = *1*
using *f-ne-0 fls-times-conv-regpart*[*of fls-shift* (−*fls-subdegree g*) *f fls-base-factor g*]
fls-base-factor-subdegree[*of g*]
by (*simp add*: *fls-times-both-shifted-simp subdeg-sum*)
from *fg*(*2*) **show**
fls-base-factor-to-fps g $ *0* ∗ *fls-regpart* (*fls-shift* (−*fls-subdegree g*) *f*) $ *0* = *1*
by (*simp add*: *subdeg-sum′*(*2*))
qed
ultimately show *fls-left-inverse g* (*f* $$ *fls-subdegree f*) = *f*
by (*simp add*: *subdeg-sum′*(*2*))

from *fg*(*1*) **have** *g-ne-0*: *g*≠*0* **by** *auto*
moreover have
fps-right-inverse (*fls-base-factor-to-fps f*) (*fls-regpart* (*fls-shift* (−*fls-subdegree f*) *g*)$*0*)
= *fls-regpart* (*fls-shift* (−*fls-subdegree f*) *g*)
proof (*intro fps-lr-inverse-unique-ring1*(*2*))
from *fg*(*1*) **show**
fls-base-factor-to-fps f ∗ *fls-regpart* (*fls-shift* (−*fls-subdegree f*) *g*) = *1*
using *g-ne-0 fls-times-conv-regpart*[*of fls-base-factor f fls-shift* (−*fls-subdegree f*) *g*]
fls-base-factor-subdegree[*of f*]
by (*simp add*: *fls-times-both-shifted-simp subdeg-sum add.commute*)

from *fg*(*2*) **show**
fls-regpart (*fls-shift* (−*fls-subdegree f*) *g*) $ *0* ∗ *fls-base-factor-to-fps f* $ *0* = *1*
by (*simp add*: *subdeg-sum′*(*1*))
qed
ultimately show *fls-right-inverse f* (*g* $$ *fls-subdegree g*) = *g*
by (*simp add*: *subdeg-sum′*(*2*))

qed

lemma *fls-lr-inverse-unique-divring*:
fixes *f g* :: *′a* ::*division-ring fls*
assumes *fg*: *f* ∗ *g* = *1*
shows *fls-left-inverse g* (*f* $$ *fls-subdegree f*) = *f*
and *fls-right-inverse f* (*g* $$ *fls-subdegree g*) = *g*
proof−
from *fg* **have** *f* ≠*0 g* ≠ *0* **by** *auto*
with *fg* **have** *fls-subdegree f* + *fls-subdegree g* = *0* **using** *fls-subdegree-mult* **by** *force*
with *fg* **have** *f* $$ *fls-subdegree f* ∗ *g* $$ *fls-subdegree g* = *1*
using *fls-times-base*[*of f g*] **by** *simp*
hence *g* $$ *fls-subdegree g* ∗ *f* $$ *fls-subdegree f* = *1*
using *inverse-unique*[*of f* $$ *fls-subdegree f*] *left-inverse*[*of f* $$ *fls-subdegree f*]
by *force*
thus
fls-left-inverse g (*f* $$ *fls-subdegree f*) = *f*
fls-right-inverse f (*g* $$ *fls-subdegree g*) = *g*
using *fg fls-lr-inverse-unique-ring1*
by *auto*
qed

lemma *fls-lr-inverse-minus*:
fixes *f* :: *′a*::*ring-1 fls*
shows *fls-left-inverse* (−*f*) (−*x*) = − *fls-left-inverse f x*
and *fls-right-inverse* (−*f*) (−*x*) = − *fls-right-inverse f x*
by (*simp-all add*: *fps-lr-inverse-minus*)

lemma *fls-inverse-minus* [*simp*]: *inverse* (−*f*) = −*inverse* (*f* :: *′a* :: *division-ring fls*)
using *fls-lr-inverse-minus*(*2*)[*of f*] **by** (*simp add*: *fls-inverse-def′*)

lemma *fls-lr-inverse-mult-ring1*:
fixes *f g* :: *′a*::*ring-1 fls*
assumes *x*: *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *x* = *1*
and *y*: *y* ∗ *g* $$ *fls-subdegree g* = *1 g* $$ *fls-subdegree g* ∗ *y* = *1*
shows *fls-left-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fls-left-inverse g y* ∗ *fls-left-inverse f x*
and *fls-right-inverse* (*f* ∗ *g*) (*y*∗*x*) = *fls-right-inverse g y* ∗ *fls-right-inverse f x*
proof−
from *x*(*1*) *y*(*2*) **have** *x* ∗ (*f* $$ *fls-subdegree f* ∗ *g* $$ *fls-subdegree g*) ∗ *y* = *1*

```
  by (simp add: mult.assoc)
hence base-prod: f $$ fls-subdegree f * g $$ fls-subdegree g ≠ 0 by auto
hence subdegrees: fls-subdegree (f*g) = fls-subdegree f + fls-subdegree g
  using fls-mult-nonzero-base-subdegree-eq[of f g] by simp

have norm:
  fls-base-factor-to-fps (f * g) = fls-base-factor-to-fps f * fls-base-factor-to-fps g
  using base-prod fls-base-factor-to-fps-mult'[of f g] by simp

have
  fls-left-inverse (f * g) (y*x) =
    fls-shift (fls-subdegree (f * g)) (
      fps-to-fls (
        fps-left-inverse (fls-base-factor-to-fps f * fls-base-factor-to-fps g) (y*x)
      )
    )

  using norm
  by    simp
thus fls-left-inverse (f * g) (y*x) = fls-left-inverse g y * fls-left-inverse f x
  using x y
        fps-lr-inverse-mult-ring1(1)[of
          x fls-base-factor-to-fps f y fls-base-factor-to-fps g
        ]
  by    (simp add:
          fls-times-both-shifted-simp fls-times-fps-to-fls subdegrees algebra-simps
        )

have
  fls-right-inverse (f * g) (y*x) =
    fls-shift (fls-subdegree (f * g)) (
      fps-to-fls (
        fps-right-inverse (fls-base-factor-to-fps f * fls-base-factor-to-fps g) (y*x)
      )
    )

  using norm
  by    simp
thus fls-right-inverse (f * g) (y*x) = fls-right-inverse g y * fls-right-inverse f x
  using x y
        fps-lr-inverse-mult-ring1(2)[of
          x fls-base-factor-to-fps f y fls-base-factor-to-fps g
        ]
  by    (simp add:
          fls-times-both-shifted-simp fls-times-fps-to-fls subdegrees algebra-simps
        )

qed
```

```
lemma fls-lr-inverse-power-ring1:
  fixes   f :: 'a::ring-1 fls
  assumes x: x * f $$ fls-subdegree f = 1 f $$ fls-subdegree f * x = 1
  shows   fls-left-inverse (f ^ n) (x ^ n) = (fls-left-inverse f x) ^ n
          fls-right-inverse (f ^ n) (x ^ n) = (fls-right-inverse f x) ^ n
proof-

  show fls-left-inverse (f ^ n) (x ^ n) = (fls-left-inverse f x) ^ n
  proof (induct n)
    case 0 show ?case using fls-lr-inverse-one(1)[of 1] by simp
  next
    case (Suc n) with assms show ?case
      using fls-lr-inverse-mult-ring1(1)[of x f x^n f^n]
      by    (simp add:
          power-Suc2[symmetric] fls-unit-base-subdegree-power(1) left-right-inverse-power
            )
  qed

  show fls-right-inverse (f ^ n) (x ^ n) = (fls-right-inverse f x) ^ n
  proof (induct n)
    case 0 show ?case using fls-lr-inverse-one(2)[of 1] by simp
  next
    case (Suc n) with assms show ?case
      using fls-lr-inverse-mult-ring1(2)[of x f x^n f^n]
      by    (simp add:
          power-Suc2[symmetric] fls-unit-base-subdegree-power(1) left-right-inverse-power
            )
  qed

qed

lemma fls-divide-convert-times-inverse:
  fixes   f g :: 'a::{comm-monoid-add,inverse,mult-zero,uminus} fls
  shows   f / g = f * inverse g
  using fls-base-factor-to-fps-subdegree[of g] fps-to-fls-base-factor-to-fps[of f]
        fls-times-both-shifted-simp[of -fls-subdegree f fls-base-factor f]
  by    (simp add:
          fls-divide-def fps-divide-unit' fls-times-fps-to-fls
          fls-conv-base-factor-shift-subdegree fls-inverse-def
        )

instance fls :: (division-ring) division-ring
proof
  fix a b :: 'a fls
  show a ≠ 0 ⟹ inverse a * a = 1
    using fls-left-inverse'[of inverse (a $$ fls-subdegree a) a]
    by    (simp add: fls-inverse-def')
  show a ≠ 0 ⟹ a * inverse a = 1
    using fls-right-inverse[of a]
```

```
    by    (simp add: fls-inverse-def')
  show a / b = a * inverse b using fls-divide-convert-times-inverse by fast
  show inverse (0::'a fls) = 0 by simp
qed

lemma fls-lr-inverse-mult-divring:
  fixes   f g   :: 'a::division-ring fls
  and     df dg :: int
  defines df ≡ fls-subdegree f
  and     dg ≡ fls-subdegree g
  shows   fls-left-inverse (f*g) (inverse ((f*g)$$(df+dg))) =
          fls-left-inverse g (inverse (g$$dg)) * fls-left-inverse f (inverse (f$$df))
  and     fls-right-inverse (f*g) (inverse ((f*g)$$(df+dg))) =
          fls-right-inverse g (inverse (g$$dg)) * fls-right-inverse f (inverse (f$$df))
proof -
  show
    fls-left-inverse (f*g) (inverse ((f*g)$$(df+dg))) =
      fls-left-inverse g (inverse (g$$dg)) * fls-left-inverse f (inverse (f$$df))
  proof (cases f=0 ∨ g=0)
    case True thus ?thesis
      using fls-lr-inverse-zero(1)[of inverse (0::'a)] by (auto simp add: assms)
  next
    case False thus ?thesis
      using fls-left-inverse-eq-inverse[of f*g] nonzero-inverse-mult-distrib[of f g]
            fls-left-inverse-eq-inverse[of g] fls-left-inverse-eq-inverse[of f]
      by    (simp add: assms)
  qed
  show
    fls-right-inverse (f*g) (inverse ((f*g)$$(df+dg))) =
      fls-right-inverse g (inverse (g$$dg)) * fls-right-inverse f (inverse (f$$df))
  proof (cases f=0 ∨ g=0)
    case True thus ?thesis
      using fls-lr-inverse-zero(2)[of inverse (0::'a)] by (auto simp add: assms)
  next
    case False thus ?thesis
      using fls-inverse-def'[of f*g] nonzero-inverse-mult-distrib[of f g]
            fls-inverse-def'[of g] fls-inverse-def'[of f]
      by    (simp add: assms)
  qed
qed

lemma fls-lr-inverse-power-divring:
  fls-left-inverse (f ^ n) ((inverse (f $$ fls-subdegree f)) ^ n) =
    (fls-left-inverse f (inverse (f $$ fls-subdegree f))) ^ n (is ?P)
  and fls-right-inverse (f ^ n) ((inverse (f $$ fls-subdegree f)) ^ n) =
    (fls-right-inverse f (inverse (f $$ fls-subdegree f))) ^ n (is ?Q)
  for f :: 'a::division-ring fls
proof -
  note fls-left-inverse-eq-inverse [of f] fls-right-inverse-eq-inverse[of f]
```

```
  moreover have
    fls-right-inverse (f ^ n) ((inverse (f $$ fls-subdegree f)) ^ n) =
      inverse f ^ n
    using fls-right-inverse-eq-inverse [of f ^ n]
    by (simp add: fls-subdegree-pow power-inverse)
  moreover have
    fls-left-inverse (f ^ n) ((inverse (f $$ fls-subdegree f)) ^ n) =
      inverse f ^ n
    using fls-left-inverse-eq-inverse [of f ^ n]
    by (simp add: fls-subdegree-pow power-inverse)
  ultimately show ?P and ?Q
    by simp-all
qed

lemma one-plus-fls-X-powi-eq:
  (1 + fls-X) powi n = fps-to-fls (fps-binomial (of-int n :: 'a :: field-char-0))
proof (cases n ≥ 0)
  case True
  thus ?thesis
    using fps-binomial-of-nat[of nat n, where ?'a = 'a]
    by (simp add: power-int-def fps-to-fls-power)
next
  case False
  thus ?thesis
    using fps-binomial-minus-of-nat[of nat (−n), where ?'a = 'a]
   by (simp add: power-int-def fps-to-fls-power fps-inverse-power flip: fls-inverse-fps-to-fls)
qed

instance fls :: (field) field
  by (standard, simp-all add: field-simps)

instance fls :: ({field-prime-char,comm-semiring-1}) field-prime-char
  by (rule field-prime-charI') auto

instance fls :: (semiring-char-0) semiring-char-0
proof
  show inj (of-nat :: nat ⇒ 'a fls)
    by (metis fls-regpart-of-nat injI of-nat-eq-iff)
qed

instance fls :: (field-char-0) field-char-0 ..


lemma fls-subdegree-power-int [simp]:
  fixes   F :: 'a :: field fls
  shows fls-subdegree (F powi n) = n * fls-subdegree F
  by (auto simp: power-int-def fls-subdegree-pow)
```

7.5.6 Division

lemma *fls-divide-nth-below*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*uminus*,*times*,*inverse*} *fls*
shows *n* < *fls-subdegree f* − *fls-subdegree g* ⟹ (*f div g*) $$ *n* = *0*
by (*simp add*: *fls-divide-def*)

lemma *fls-divide-nth-base*:
fixes *f g* :: *'a*::*division-ring fls*
shows
(*f div g*) $$ (*fls-subdegree f* − *fls-subdegree g*) =
f $$ *fls-subdegree f* / *g* $$ *fls-subdegree g*
using *fps-divide-nth-0'*[*of fls-base-factor-to-fps g fls-base-factor-to-fps f*]
fls-base-factor-to-fps-subdegree[*of g*]
by (*simp add*: *fls-divide-def*)

lemma *fls-div-zero* [*simp*]:
0 div (*g* :: *'a* :: {*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*) = *0*
by (*simp add*: *fls-divide-def*)

lemma *fls-div-by-zero*:
fixes *g* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*} *fls*
assumes *inverse* (*0*::*'a*) = *0*
shows *g div 0* = *0*
by (*simp add*: *fls-divide-def assms fps-div-by-zero'*)

lemma *fls-divide-times*:
fixes *f g* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fls*
shows (*f* ∗ *g*) / *h* = *f* ∗ (*g* / *h*)
by (*simp add*: *fls-divide-convert-times-inverse mult.assoc*)

lemma *fls-divide-times2*:
fixes *f g* :: *'a*::{*comm-semiring-0*,*inverse*,*uminus*} *fls*
shows (*f* ∗ *g*) / *h* = (*f* / *h*) ∗ *g*
using *fls-divide-times*[*of g f h*]
by (*simp add*: *mult.commute*)

lemma *fls-divide-subdegree-ge*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*uminus*,*times*,*inverse*} *fls*
assumes *f* / *g* ≠ *0*
shows *fls-subdegree* (*f* / *g*) ≥ *fls-subdegree f* − *fls-subdegree g*
using *assms fls-divide-nth-below*
by (*intro fls-subdegree-geI*) *simp*

lemma *fls-divide-subdegree*:
fixes *f g* :: *'a*::*division-ring fls*
assumes *f* ≠ *0 g* ≠ *0*
shows *fls-subdegree* (*f* / *g*) = *fls-subdegree f* − *fls-subdegree g*
proof (*intro antisym*)
from *assms* **have** *f* $$ *fls-subdegree f* / *g* $$ *fls-subdegree g* ≠ *0* **by** (*simp add*:

field-simps)
  thus *fls-subdegree* $(f/g) \leq$ *fls-subdegree* $f -$ *fls-subdegree* g
    using *fls-divide-nth-base*[*of f g*] **by** (*intro fls-subdegree-leI*) *simp*
  from *assms* **have** $f\ /\ g \neq 0$ **by** (*simp add*: *field-simps*)
  thus *fls-subdegree* $(f/g) \geq$ *fls-subdegree* $f -$ *fls-subdegree* g
    using *fls-divide-subdegree-ge* **by** *fast*
qed

lemma *fls-divide-shift-numer-nonzero*:
  fixes $f\ g :: 'a :: \{$*comm-monoid-add*,*inverse*,*times*,*uminus*$\}$ *fls*
  assumes $f \neq 0$
  shows *fls-shift* $m\ f\ /\ g =$ *fls-shift* $m\ (f/g)$
  using *assms fls-base-factor-to-fps-shift*[*of m f*]
  by (*simp add*: *fls-divide-def algebra-simps*)

lemma *fls-divide-shift-numer*:
  fixes $f\ g :: 'a :: \{$*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*$\}$ *fls*
  shows *fls-shift* $m\ f\ /\ g =$ *fls-shift* $m\ (f/g)$
  using *fls-divide-shift-numer-nonzero*
  by (*cases f=0*) *auto*

lemma *fls-divide-shift-denom-nonzero*:
  fixes $f\ g :: 'a :: \{$*comm-monoid-add*,*inverse*,*times*,*uminus*$\}$ *fls*
  assumes $g \neq 0$
  shows f / *fls-shift* $m\ g =$ *fls-shift* $(-m)\ (f/g)$
  using *assms fls-base-factor-to-fps-shift*[*of m g*]
  by (*simp add*: *fls-divide-def algebra-simps*)

lemma *fls-divide-shift-denom*:
  fixes $f\ g :: 'a ::$ *division-ring fls*
  shows f / *fls-shift* $m\ g =$ *fls-shift* $(-m)\ (f/g)$
  using *fls-divide-shift-denom-nonzero*
  by (*cases g=0*) *auto*

lemma *fls-divide-shift-both-nonzero*:
  fixes $f\ g :: 'a :: \{$*comm-monoid-add*,*inverse*,*times*,*uminus*$\}$ *fls*
  assumes $f \neq 0\ g \neq 0$
  shows *fls-shift* $n\ f$ / *fls-shift* $m\ g =$ *fls-shift* $(n-m)\ (f/g)$
  by (*simp add*: *assms fls-divide-shift-numer-nonzero fls-divide-shift-denom-nonzero*)

lemma *fls-divide-shift-both* [*simp*]:
  fixes $f\ g :: 'a ::$ *division-ring fls*
  shows *fls-shift* $n\ f$ / *fls-shift* $m\ g =$ *fls-shift* $(n-m)\ (f/g)$
  using *fls-divide-shift-both-nonzero*
  by (*cases f=0* $\vee$ *g=0*) *auto*

lemma *fls-divide-base-factor-numer*:
  fls-base-factor $f\ /\ g =$ *fls-shift* (*fls-subdegree* f) (f/g)
  using *fls-base-factor-to-fps-base-factor*[*of f*]

fls-base-factor-subdegree[*of f*]
by (*simp add*: *fls-divide-def algebra-simps*)

lemma *fls-divide-base-factor-denom*:
f / *fls-base-factor g* = *fls-shift* (−*fls-subdegree g*) (*f*/*g*)
using *fls-base-factor-to-fps-base-factor*[*of g*]
fls-base-factor-subdegree[*of g*]
by (*simp add*: *fls-divide-def*)

lemma *fls-divide-base-factor'*:
fls-base-factor f / *fls-base-factor g* = *fls-shift* (*fls-subdegree f* − *fls-subdegree g*)
(*f*/*g*)
using *fls-divide-base-factor-numer*[*of f fls-base-factor g*]
fls-divide-base-factor-denom[*of f g*]
by *simp*

lemma *fls-divide-base-factor*:
fixes *f g* :: *'a* :: *division-ring fls*
shows *fls-base-factor f* / *fls-base-factor g* = *fls-base-factor* (*f*/*g*)
using *fls-divide-subdegree*[*of f g*] *fls-divide-base-factor'*
by *fastforce*

lemma *fls-divide-regpart*:
fixes *f g* :: *'a*::{*inverse*,*comm-monoid-add*,*uminus*,*mult-zero*} *fls*
assumes *fls-subdegree f* ≥ *0 fls-subdegree g* ≥ *0*
shows *fls-regpart* (*f* / *g*) = *fls-regpart f* / *fls-regpart g*
proof −
have *deg0*:
⋀*g*. *fls-subdegree g* = *0* ⟹
fls-regpart (*f* / *g*) = *fls-regpart f* / *fls-regpart g*
by (*simp add*:
assms(*1*) *fls-divide-convert-times-inverse fls-inverse-subdegree-0*
fls-times-conv-regpart fls-inverse-regpart fls-regpart-subdegree-conv fps-divide-unit'
)
show *?thesis*
proof (*cases fls-subdegree g* = *0*)
case *False*
hence *fls-base-factor g* ≠ *0* **using** *fls-base-factor-nonzero*[*of g*] **by** *force*
with *assms*(*2*) **show** *?thesis*
using *fls-divide-shift-denom-nonzero*[*of fls-base-factor g f* −*fls-subdegree g*]
fps-shift-fls-regpart-conv-fls-shift[*of*
nat (*fls-subdegree g*) *f* / *fls-base-factor g*
]
fls-base-factor-subdegree[*of g*] *deg0*
fls-regpart-subdegree-conv[*of g*] *fps-unit-factor-fls-regpart*[*of g*]
by (*simp add*:
fls-conv-base-factor-shift-subdegree fls-regpart-subdegree-conv fps-divide-def
)
qed (*rule deg0*)

qed

lemma *fls-divide-fls-base-factor-to-fps'*:
fixes *f g* :: *'a*::{*comm-monoid-add*,*uminus*,*inverse*,*mult-zero*} *fls*
shows
fls-base-factor-to-fps f / fls-base-factor-to-fps g =
fls-regpart (*fls-shift* (*fls-subdegree f* − *fls-subdegree g*) (*f / g*))
using *fls-base-factor-subdegree*[*of f*] *fls-base-factor-subdegree*[*of g*]
fls-divide-regpart[*of fls-base-factor f fls-base-factor g*]
fls-divide-base-factor'[*of f g*]
by *simp*

lemma *fls-divide-fls-base-factor-to-fps*:
fixes *f g* :: *'a*::*division-ring fls*
shows *fls-base-factor-to-fps f / fls-base-factor-to-fps g* = *fls-base-factor-to-fps* (*f / g*)
using *fls-divide-fls-base-factor-to-fps' fls-divide-subdegree*[*of f g*]
by *fastforce*

lemma *fls-divide-fps-to-fls*:
fixes *f g* :: *'a*::{*inverse*,*ab-group-add*,*mult-zero*} *fps*
assumes *subdegree f* ≥ *subdegree g*
shows *fps-to-fls f / fps-to-fls g* = *fps-to-fls* (*f*/*g*)
proof−
have *1*:
fps-to-fls f / fps-to-fls g =
fls-shift (*int* (*subdegree g*)) (*fps-to-fls* (*f* ∗ *inverse* (*unit-factor g*)))
using *fls-base-factor-to-fps-to-fls*[*of f*] *fls-base-factor-to-fps-to-fls*[*of g*]
fls-subdegree-fls-to-fps[*of f*] *fls-subdegree-fls-to-fps*[*of g*]
fps-divide-def[*of unit-factor f unit-factor g*]
fls-times-fps-to-fls[*of unit-factor f inverse* (*unit-factor g*)]
fls-shifted-times-simps(*2*)[*of* −*int* (*subdegree f*) *fps-to-fls* (*unit-factor f*)]
fls-times-fps-to-fls[*of f inverse* (*unit-factor g*)]
by (*simp add*: *fls-divide-def*)
with *assms* **show** *?thesis*
using *fps-mult-subdegree-ge*[*of f inverse* (*unit-factor g*)]
fps-shift-to-fls[*of subdegree g f* ∗ *inverse* (*unit-factor g*)]
by (*cases f* ∗ *inverse* (*unit-factor g*) = *0*) (*simp-all add*: *fps-divide-def*)
qed

lemma *fls-divide-1'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f / 1* = *f*
using *assms fls-conv-base-factor-to-fps-shift-subdegree*[*of f*]
by (*simp add*: *fls-divide-def fps-divide-1'*)

lemma *fls-divide-1* [*simp*]: *a / 1* = (*a*::*'a*::*division-ring fls*)

by (*rule fls-divide-1'*[*OF inverse-1*])

lemma *fls-const-divide-const*:
fixes *x y* :: *'a*::*division-ring*
shows *fls-const x / fls-const y* = *fls-const* (*x*/*y*)
by (*simp add*: *fls-divide-def fls-base-factor-to-fps-const fps-const-divide*)

lemma *fls-divide-X'*:
fixes *f* :: *'a*::{*comm-monoid-add*,*inverse*,*mult-zero*,*uminus*,*zero-neq-one*,*monoid-mult*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f / fls-X* = *fls-shift 1 f*
proof−
from *assms* **have**
f / fls-X =
fls-shift 1 (*fls-shift* (−*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*)))
by (*simp add*: *fls-divide-def fps-divide-1'*)
also have ... = *fls-shift 1 f*
using *fls-conv-base-factor-to-fps-shift-subdegree*[*of f*]
by *simp*
finally show *?thesis* **by** *simp*
qed

lemma *fls-divide-X* [*simp*]:
fixes *f* :: *'a*::*division-ring fls*
shows *f / fls-X* = *fls-shift 1 f*
by (*rule fls-divide-X'*[*OF inverse-1*])

lemma *fls-divide-X-power'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f / (fls-X* ^ *n)* = *fls-shift n f*
proof−
have *fls-base-factor-to-fps* ((*fls-X*::*'a fls*) ^ *n*) = *1* **by** (*rule fls-X-power-base-factor-to-fps*)
with *assms* **have**
f / (fls-X ^ *n)* =
fls-shift n (*fls-shift* (−*fls-subdegree f*) (*fps-to-fls* (*fls-base-factor-to-fps f*)))
by (*simp add*: *fls-divide-def fps-divide-1'*)
also have ... = *fls-shift n f*
using *fls-conv-base-factor-to-fps-shift-subdegree*[*of f*] **by** *simp*
finally show *?thesis* **by** *simp*
qed

lemma *fls-divide-X-power* [*simp*]:
fixes *f* :: *'a*::*division-ring fls*
shows *f / (fls-X* ^ *n)* = *fls-shift n f*
by (*rule fls-divide-X-power'*[*OF inverse-1*])

lemma *fls-divide-X-inv'*:

fixes $f :: 'a::\{comm\text{-}monoid\text{-}add,inverse,mult\text{-}zero,uminus,zero\text{-}neq\text{-}one,monoid\text{-}mult\}$ *fls*
assumes *inverse* $(1::'a) = 1$
shows $f \,/\, fls\text{-}X\text{-}inv = fls\text{-}shift\ (-1)\ f$
proof−
from *assms* **have**
$f \,/\, fls\text{-}X\text{-}inv =$
$fls\text{-}shift\ (-1)\ (fls\text{-}shift\ (-fls\text{-}subdegree\ f)\ (fps\text{-}to\text{-}fls\ (fls\text{-}base\text{-}factor\text{-}to\text{-}fps\ f)))$
by (*simp add*: *fls-divide-def fps-divide-1'* *algebra-simps*)
also have $\ldots = fls\text{-}shift\ (-1)\ f$
using *fls-conv-base-factor-to-fps-shift-subdegree*[*of f*]
by *simp*
finally show *?thesis* **by** *simp*
qed

lemma *fls-divide-X-inv* [*simp*]:
fixes $f :: 'a::division\text{-}ring\ fls$
shows $f \,/\, fls\text{-}X\text{-}inv = fls\text{-}shift\ (-1)\ f$
by (*rule fls-divide-X-inv'*[*OF inverse-1*])

lemma *fls-divide-X-inv-power'*:
fixes $f :: 'a::\{semiring\text{-}1,inverse,uminus\}\ fls$
assumes *inverse* $(1::'a) = 1$
shows $f \,/\, (fls\text{-}X\text{-}inv\ \hat{}\ n) = fls\text{-}shift\ (-int\ n)\ f$
proof−
have *fls-base-factor-to-fps* $((fls\text{-}X\text{-}inv::'a\ fls)\ \hat{}\ n) = 1$
by (*rule fls-X-inv-power-base-factor-to-fps*)
with *assms* **have**
$f \,/\, (fls\text{-}X\text{-}inv\ \hat{}\ n) =$
$fls\text{-}shift\ (-int\ n + -fls\text{-}subdegree\ f)\ (fps\text{-}to\text{-}fls\ (fls\text{-}base\text{-}factor\text{-}to\text{-}fps\ f))$
by (*simp add*: *fls-divide-def fps-divide-1'*)
also have
$\ldots = fls\text{-}shift\ (-int\ n)\ (fls\text{-}shift\ (-fls\text{-}subdegree\ f)\ (fps\text{-}to\text{-}fls\ (fls\text{-}base\text{-}factor\text{-}to\text{-}fps\ f)))$
by (*simp add*: *add.commute*)
also have $\ldots = fls\text{-}shift\ (-int\ n)\ f$
using *fls-conv-base-factor-to-fps-shift-subdegree*[*of f*] **by** *simp*
finally show *?thesis* **by** *simp*
qed

lemma *fls-divide-X-inv-power* [*simp*]:
fixes $f :: 'a::division\text{-}ring\ fls$
shows $f \,/\, (fls\text{-}X\text{-}inv\ \hat{}\ n) = fls\text{-}shift\ (-int\ n)\ f$
by (*rule fls-divide-X-inv-power'*[*OF inverse-1*])

lemma *fls-divide-X-intpow'*:
fixes $f :: 'a::\{semiring\text{-}1,inverse,uminus\}\ fls$
assumes *inverse* $(1::'a) = 1$
shows $f \,/\, (fls\text{-}X\text{-}intpow\ i) = fls\text{-}shift\ i\ f$

using *assms*
by (*simp add*: *fls-divide-shift-denom-nonzero fls-divide-1'*)

lemma *fls-divide-X-intpow-conv-times'*:
fixes *f* :: *'a*::{*semiring-1*,*inverse*,*uminus*} *fls*
assumes *inverse* (*1*::*'a*) = *1*
shows *f* / (*fls-X-intpow i*) = *f* ∗ *fls-X-intpow* (−*i*)
using *assms fls-X-intpow-times-conv-shift*(*2*)[*of f* −*i*]
by (*simp add*: *fls-divide-X-intpow'*)

lemma *fls-divide-X-intpow*:
fixes *f* :: *'a*::*division-ring fls*
shows *f* / (*fls-X-intpow i*) = *fls-shift i f*
by (*rule fls-divide-X-intpow'*[*OF inverse-1*])

lemma *fls-divide-X-intpow-conv-times*:
fixes *f* :: *'a*::*division-ring fls*
shows *f* / (*fls-X-intpow i*) = *f* ∗ *fls-X-intpow* (−*i*)
by (*rule fls-divide-X-intpow-conv-times'*[*OF inverse-1*])

lemma *fls-X-intpow-div-fls-X-intpow-semiring1*:
assumes *inverse* (*1*::*'a*::{*semiring-1*,*inverse*,*uminus*}) = *1*
shows (*fls-X-intpow i* :: *'a fls*) / *fls-X-intpow j* = *fls-X-intpow* (*i*−*j*)
by (*simp add*: *assms fls-divide-shift-both-nonzero fls-divide-1'*)

lemma *fls-X-intpow-div-fls-X-intpow*:
(*fls-X-intpow i* :: *'a*::*division-ring fls*) / *fls-X-intpow j* = *fls-X-intpow* (*i*−*j*)
by (*rule fls-X-intpow-div-fls-X-intpow-semiring1*[*OF inverse-1*])

lemma *fls-divide-add*:
fixes *f g h* :: *'a*::{*semiring-0*,*inverse*,*uminus*} *fls*
shows (*f* + *g*) / *h* = *f* / *h* + *g* / *h*
by (*simp add*: *fls-divide-convert-times-inverse algebra-simps*)

lemma *fls-divide-diff*:
fixes *f g h* :: *'a*::{*ring*,*inverse*} *fls*
shows (*f* − *g*) / *h* = *f* / *h* − *g* / *h*
by (*simp add*: *fls-divide-convert-times-inverse algebra-simps*)

lemma *fls-divide-uminus*:
fixes *f g h* :: *'a*::{*ring*,*inverse*} *fls*
shows (− *f*) / *g* = − (*f* / *g*)
by (*simp add*: *fls-divide-convert-times-inverse*)

lemma *fls-divide-uminus'*:
fixes *f g h* :: *'a*::*division-ring fls*
shows *f* / (− *g*) = − (*f* / *g*)
by (*simp add*: *fls-divide-convert-times-inverse*)

7.5.7 Units

lemma *fls-is-left-unit-iff-base-is-left-unit*:
fixes $f :: 'a :: ring\text{-}1\text{-}no\text{-}zero\text{-}divisors\ fls$
shows $(\exists g.\ 1 = f * g) \longleftrightarrow (\exists k.\ 1 = f\ \$\$\ fls\text{-}subdegree\ f * k)$
proof
assume $\exists g.\ 1 = f * g$
then obtain g **where** $1 = f * g$ **by** *fast*
hence $1 = (f\ \$\$\ fls\text{-}subdegree\ f) * (g\ \$\$\ fls\text{-}subdegree\ g)$
using *fls-subdegree-mult*[*of f g*] *fls-times-base*[*of f g*] **by** *fastforce*
thus $\exists k.\ 1 = f\ \$\$\ fls\text{-}subdegree\ f * k$ **by** *fast*
next
assume $\exists k.\ 1 = f\ \$\$\ fls\text{-}subdegree\ f * k$
then obtain k **where** $1 = f\ \$\$\ fls\text{-}subdegree\ f * k$ **by** *fast*
hence $1 = f * fls\text{-}right\text{-}inverse\ f\ k$
using *fls-right-inverse* **by** *simp*
thus $\exists g.\ 1 = f * g$ **by** *fast*
qed

lemma *fls-is-right-unit-iff-base-is-right-unit*:
fixes $f :: 'a :: ring\text{-}1\text{-}no\text{-}zero\text{-}divisors\ fls$
shows $(\exists g.\ 1 = g * f) \longleftrightarrow (\exists k.\ 1 = k * f\ \$\$\ fls\text{-}subdegree\ f)$
proof
assume $\exists g.\ 1 = g * f$
then obtain g **where** $1 = g * f$ **by** *fast*
hence $1 = (g\ \$\$\ fls\text{-}subdegree\ g) * (f\ \$\$\ fls\text{-}subdegree\ f)$
using *fls-subdegree-mult*[*of g f*] *fls-times-base*[*of g f*] **by** *fastforce*
thus $\exists k.\ 1 = k * f\ \$\$\ fls\text{-}subdegree\ f$ **by** *fast*
next
assume $\exists k.\ 1 = k * f\ \$\$\ fls\text{-}subdegree\ f$
then obtain k **where** $1 = k * f\ \$\$\ fls\text{-}subdegree\ f$ **by** *fast*
hence $1 = fls\text{-}left\text{-}inverse\ f\ k * f$
using *fls-left-inverse* **by** *simp*
thus $\exists g.\ 1 = g * f$ **by** *fast*
qed

7.6 Composition

definition $fls\text{-}compose\text{-}fps :: 'a :: field\ fls \Rightarrow 'a\ fps \Rightarrow 'a\ fls$ **where**
$fls\text{-}compose\text{-}fps\ F\ G =$
$fps\text{-}to\text{-}fls\ (fps\text{-}compose\ (fls\text{-}base\text{-}factor\text{-}to\text{-}fps\ F)\ G) * fps\text{-}to\text{-}fls\ G\ powi\ fls\text{-}subdegree\ F$

lemma *fps-compose-of-nat* [*simp*]: $fps\text{-}compose\ (of\text{-}nat\ n :: 'a :: comm\text{-}ring\text{-}1\ fps)\ H = of\text{-}nat\ n$
and *fps-compose-of-int* [*simp*]: $fps\text{-}compose\ (of\text{-}int\ i)\ H = of\text{-}int\ i$
unfolding *fps-of-nat* [*symmetric*] *fps-of-int* [*symmetric*] *numeral-fps-const*
by (*rule fps-const-compose*)+

lemmas [*simp*] = *fps-to-fls-of-nat fps-to-fls-of-int*

lemma *fls-compose-fps-0* [*simp*]: *fls-compose-fps 0 H = 0*
and *fls-compose-fps-1* [*simp*]: *fls-compose-fps 1 H = 1*
and *fls-compose-fps-const* [*simp*]: *fls-compose-fps* (*fls-const c*) *H* = *fls-const c*
and *fls-compose-fps-of-nat* [*simp*]: *fls-compose-fps* (*of-nat n*) *H* = *of-nat n*
and *fls-compose-fps-of-int* [*simp*]: *fls-compose-fps* (*of-int i*) *H* = *of-int i*
and *fls-compose-fps-X* [*simp*]: *fls-compose-fps fls-X F* = *fps-to-fls F*
by (*simp-all add*: *fls-compose-fps-def*)

lemma *fls-compose-fps-0-right*:
fls-compose-fps F 0 = (*if 0* ≤ *fls-subdegree F then fls-const* (*F* $$ *0*) *else 0*)
by (*cases fls-subdegree F* = *0*) (*simp-all add*: *fls-compose-fps-def*)

lemma *fls-compose-fps-shift*:
assumes *H* ≠ *0*
shows *fls-compose-fps* (*fls-shift n F*) *H* = *fls-compose-fps F H* ∗ *fps-to-fls H powi* (−*n*)
proof (*cases F* = *0*)
case *False*
thus *?thesis*
using *assms* **by** (*simp add*: *fls-compose-fps-def power-int-diff power-int-minus field-simps*)
qed *auto*

lemma *fls-compose-fps-to-fls* [*simp*]:
assumes [*simp*]: *G* ≠ *0 fps-nth G 0* = *0*
shows *fls-compose-fps* (*fps-to-fls F*) *G* = *fps-to-fls* (*fps-compose F G*)
proof (*cases F* = *0*)
case *False*
define *n* **where** *n* = *subdegree F*
define *F′* **where** *F′* = *fps-shift n F*
have [*simp*]: *F′* ≠ *0 subdegree F′* = *0*
using *False* **by** (*auto simp*: *F′-def n-def*)
have *F-eq*: *F* = *F′* ∗ *fps-X* ^ *n*
unfolding *F′-def n-def* **using** *subdegree-decompose* **by** *blast*
have *fls-compose-fps* (*fps-to-fls F*) *G* =
fps-to-fls (*fps-shift n* (*fls-regpart* (*fps-to-fls F′* ∗ *fls-X-intpow* (*int n*))) *oo G*) ∗ *fps-to-fls* (*G* ^ *n*)
unfolding *F-eq fls-compose-fps-def*
by (*simp add*: *fls-times-fps-to-fls fls-X-power-conv-shift-1 power-int-add fls-subdegree-fls-to-fps fps-to-fls-power fls-regpart-shift-conv-fps-shift flip*: *fls-times-both-shifted-simp*)
also have *fps-to-fls F′* ∗ *fls-X-intpow* (*int n*) = *fps-to-fls F*
by (*simp add*: *F-eq fls-times-fps-to-fls fps-to-fls-power fls-X-power-conv-shift-1*)
also have *fps-to-fls* (*fps-shift n* (*fls-regpart* (*fps-to-fls F*)) *oo G*) ∗ *fps-to-fls* (*G* ^ *n*) =
fps-to-fls ((*fps-shift n* (*fls-regpart* (*fps-to-fls F*)) ∗ *fps-X* ^ *n*) *oo G*)
by (*simp add*: *fls-times-fps-to-fls flip*: *fps-compose-power add*: *fps-compose-mult-distrib*)
also have *fps-shift n* (*fls-regpart* (*fps-to-fls F*)) ∗ *fps-X* ^ *n* = *F*

by (*simp add*: *F-eq*)
finally show *?thesis* **.**
qed (*auto simp*: *fls-compose-fps-def*)

lemma *fls-compose-fps-mult*:
assumes [*simp*]: $H \neq 0$ *fps-nth* $H\ 0 = 0$
shows *fls-compose-fps* $(F * G)\ H =$ *fls-compose-fps* $F\ H *$ *fls-compose-fps* $G\ H$
using *assms*
proof (*cases* $F * G = 0$)
case *False*
hence [*simp*]: $F \neq 0\ G \neq 0$
by *auto*
define $n\ m$ **where** $n =$ *fls-subdegree* $F\ m =$ *fls-subdegree* G
define F' **where** $F' =$ *fls-regpart* (*fls-shift* $n\ F$)
define G' **where** $G' =$ *fls-regpart* (*fls-shift* $m\ G$)
have *F-eq*: $F =$ *fls-shift* $(-n)$ (*fps-to-fls* F') **and** *G-eq*: $G =$ *fls-shift* $(-m)$ (*fps-to-fls* G')
by (*simp-all add*: F'-*def* G'-*def n-m-def*)
have *fls-compose-fps* $(F * G)\ H =$ *fls-compose-fps* (*fls-shift* $(-(n + m))$ (*fps-to-fls* $(F' * G')$)) H
by (*simp add*: *fls-times-fps-to-fls F-eq G-eq fls-shifted-times-simps*)
also have $\ldots =$ *fps-to-fls* $((F'$ *oo* $H) * (G'$ *oo* $H)) *$ *fps-to-fls* H *powi* $(m + n)$
by (*simp add*: *fls-compose-fps-shift fps-compose-mult-distrib*)
also have $\ldots =$ *fls-compose-fps* $F\ H *$ *fls-compose-fps* $G\ H$
by (*simp add*: *F-eq G-eq fls-compose-fps-shift fls-times-fps-to-fls power-int-add*)
finally show *?thesis* **.**
qed *auto*

lemma *fls-compose-fps-power*:
assumes [*simp*]: $G \neq 0$ *fps-nth* $G\ 0 = 0$
shows *fls-compose-fps* $(F \hat{}\ n)\ G =$ *fls-compose-fps* $F\ G \hat{}\ n$
by (*induction* n) (*auto simp*: *fls-compose-fps-mult*)

lemma *fls-compose-fps-add*:
assumes [*simp*]: $H \neq 0$ *fps-nth* $H\ 0 = 0$
shows *fls-compose-fps* $(F + G)\ H =$ *fls-compose-fps* $F\ H +$ *fls-compose-fps* $G\ H$
proof (*cases* $F = 0 \vee G = 0$)
case *False*
hence [*simp*]: $F \neq 0\ G \neq 0$
by *auto*
define n **where** $n =$ *min* (*fls-subdegree* F) (*fls-subdegree* G)
define F' **where** $F' =$ *fls-regpart* (*fls-shift* $n\ F$)
define G' **where** $G' =$ *fls-regpart* (*fls-shift* $n\ G$)
have *F-eq*: $F =$ *fls-shift* $(-n)$ (*fps-to-fls* F') **and** *G-eq*: $G =$ *fls-shift* $(-n)$ (*fps-to-fls* G')
unfolding *n-def* **by** (*simp-all add*: F'-*def* G'-*def n-def*)
have $F + G =$ *fls-shift* $(-n)$ (*fps-to-fls* $(F' + G')$)

by (*simp add*: *F-eq G-eq*)
also have *fls-compose-fps* ... *H* = *fls-compose-fps* (*fps-to-fls* (*F'* + *G'*)) *H* * *fps-to-fls H powi n*
by (*subst fls-compose-fps-shift*) *auto*
also have ... = *fps-to-fls* (*fps-compose* (*F'* + *G'*) *H*) * *fps-to-fls H powi n*
by (*subst fls-compose-fps-to-fls*) *auto*
also have ... = *fls-compose-fps F H* + *fls-compose-fps G H*
by (*simp add*: *F-eq G-eq fls-compose-fps-shift fps-compose-add-distrib algebra-simps*)
finally show *?thesis* **.**
qed *auto*

lemma *fls-compose-fps-uminus* [*simp*]: *fls-compose-fps* (−*F*) *H* = −*fls-compose-fps F H*
by (*simp add*: *fls-compose-fps-def fps-compose-uminus*)

lemma *fls-compose-fps-diff*:
assumes [*simp*]: *H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*F* − *G*) *H* = *fls-compose-fps F H* − *fls-compose-fps G H*
using *fls-compose-fps-add*[*of H F* −*G*] **by** *simp*

lemma *fls-compose-fps-eq-0-iff*:
assumes *H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps F H* = *0* ⟷ *F* = *0*
using *assms fls-base-factor-to-fps-nonzero*[*of F*]
by (*cases F* = *0*) (*auto simp*: *fls-compose-fps-def fps-compose-eq-0-iff*)

lemma *fls-compose-fps-inverse*:
assumes [*simp*]: *H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*inverse F*) *H* = *inverse* (*fls-compose-fps F H*)
proof (*cases F* = *0*)
case *False*
have *fls-compose-fps* (*inverse F*) *H* * *fls-compose-fps F H* =
fls-compose-fps (*inverse F* * *F*) *H*
by (*subst fls-compose-fps-mult*) *auto*
also have *inverse F* * *F* = *1*
using *False* **by** *simp*
finally show *?thesis*
using *False* **by** (*simp add*: *field-simps fls-compose-fps-eq-0-iff*)
qed *auto*

lemma *fls-compose-fps-divide*:
assumes [*simp*]: *H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*F* / *G*) *H* = *fls-compose-fps F H* / *fls-compose-fps G H*
using *fls-compose-fps-mult*[*of H F inverse G*] *fls-compose-fps-inverse*[*of H G*]
by (*simp add*: *field-simps*)

lemma *fls-compose-fps-powi*:
assumes [*simp*]: *H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*F powi n*) *H* = *fls-compose-fps F H powi n*
by (*simp add*: *power-int-def fls-compose-fps-power fls-compose-fps-inverse*)

lemma *fls-compose-fps-assoc*:
assumes [*simp*]: *G* ≠ *0 fps-nth G 0* = *0 H* ≠ *0 fps-nth H 0* = *0*
shows *fls-compose-fps* (*fls-compose-fps F G*) *H* = *fls-compose-fps F* (*fps-compose G H*)
proof (*cases F* = *0*)
case [*simp*]: *False*
define *n* **where** *n* = *fls-subdegree F*
define *F'* **where** *F'* = *fls-regpart* (*fls-shift n F*)
have *F-eq*: *F* = *fls-shift* (−*n*) (*fps-to-fls F'*)
by (*simp add*: *F'-def n-def*)
show *?thesis*
by (*simp add*: *F-eq fls-compose-fps-shift fls-compose-fps-mult fls-compose-fps-powi fps-compose-eq-0-iff fps-compose-assoc*)
qed *auto*

lemma *subdegree-pos-iff*: *subdegree F* > *0* ⟷ *F* ≠ *0* ∧ *fps-nth F 0* = *0*
using *subdegree-eq-0-iff*[*of F*] **by** *auto*

lemma *fls-X-power-int* [*simp*]: *fls-X powi n* = (*fls-X-intpow n* :: *'a* :: *division-ring fls*)
by (*auto simp*: *power-int-def fls-X-power-conv-shift-1 fls-inverse-X fls-inverse-shift simp flip*: *fls-inverse-X-power*)

lemma *fls-const-power-int*: *fls-const* (*c powi n*) = *fls-const* (*c* :: *'a* :: *division-ring*) *powi n*
by (*auto simp*: *power-int-def fls-const-power fls-inverse-const*)

lemma *fls-nth-fls-compose-fps-linear*:
fixes *c* :: *'a* :: *field*
assumes [*simp*]: *c* ≠ *0*
shows *fls-compose-fps F* (*fps-const c* ∗ *fps-X*) $$ *n* = *F* $$ *n* ∗ *c powi n*
proof −
{
assume ∗: *n* ≥ *fls-subdegree F*
hence *c* ^ *nat* (*n* − *fls-subdegree F*) = *c powi int* (*nat* (*n* − *fls-subdegree F*))
by (*simp add*: *power-int-def*)
also have ... ∗ *c powi fls-subdegree F* = *c powi* (*int* (*nat* (*n* − *fls-subdegree F*)) + *fls-subdegree F*)
using ∗ **by** (*subst power-int-add*) *auto*
also have ... = *c powi n*
using ∗ **by** *simp*
finally have *c* ^ *nat* (*n* − *fls-subdegree F*) ∗ *c powi fls-subdegree F* = *c powi n*
.
}

thus *?thesis*
by (*simp add*: *fls-compose-fps-def fps-compose-linear fls-times-fps-to-fls power-int-mult-distrib fls-shifted-times-simps*
flip: *fls-const-power-int*)
qed

lemma *fls-const-transfer* [*transfer-rule*]:
rel-fun (=) (*pcr-fls* (=))
(λ*c n*. *if n* = *0 then c else 0*) *fls-const*
by (*auto simp*: *fls-const-def rel-fun-def pcr-fls-def OO-def cr-fls-def*)

lemma *fls-shift-transfer* [*transfer-rule*]:
rel-fun (=) (*rel-fun* (*pcr-fls* (=)) (*pcr-fls* (=)))
(λ*n f k*. *f* (*k*+*n*)) *fls-shift*
by (*auto simp*: *fls-const-def rel-fun-def pcr-fls-def OO-def cr-fls-def*)

lift-definition *fls-compose-power* :: *'a* :: *zero fls* ⇒ *nat* ⇒ *'a fls* **is**
λ*f d n*. *if d* > *0* ∧ *int d dvd n then f* (*n div int d*) *else 0*
proof −
fix *f* :: *int* ⇒ *'a* **and** *d* :: *nat*
assume ∗: *eventually* (λ*n*. *f* (−*int n*) = *0*) *cofinite*
show *eventually* (λ*n*. (*if d* > *0* ∧ *int d dvd* −*int n then f* (−*int n div int d*) *else 0*) = *0*) *cofinite*
proof (*cases d* = *0*)
case *False*
from ∗ **have** *eventually* (λ*n*. *f* (−*int n*) = *0*) *at-top*
by (*simp add*: *cofinite-eq-sequentially*)
hence *eventually* (λ*n*. *f* (−*int* (*n div d*)) = *0*) *at-top*
by (*rule eventually-compose-filterlim*[*OF - filterlim-at-top-div-const-nat*]) (*use False* **in** *auto*)
hence *eventually* (λ*n*. (*if d* > *0* ∧ *int d dvd* −*int n then f* (−*int n div int d*) *else 0*) = *0*) *at-top*
by *eventually-elim* (*auto simp*: *zdiv-int dvd-neg-div*)
thus *?thesis*
by (*simp add*: *cofinite-eq-sequentially*)
qed *auto*
qed

lemma *fls-nth-compose-power*:
assumes *d* > *0*
shows *fls-compose-power f d* $$ *n* = (*if int d dvd n then f* $$ (*n div int d*) *else 0*)
by (*simp add*: *assms fls-compose-power.rep-eq*)

lemma *fls-compose-power-0-left* [*simp*]: *fls-compose-power 0 d* = *0*
by *transfer auto*

lemma *fls-compose-power-1-left* [*simp*]: *d* > *0* ⟹ *fls-compose-power 1 d* = *1*

by *transfer* (*auto simp*: *fun-eq-iff*)

lemma *fls-compose-power-const-left* [*simp*]:
d > *0* ⟹ *fls-compose-power* (*fls-const c*) *d* = *fls-const c*
by *transfer* (*auto simp*: *fun-eq-iff*)

lemma *fls-compose-power-shift* [*simp*]:
d > *0* ⟹ *fls-compose-power* (*fls-shift n f*) *d* = *fls-shift* (*d* ∗ *n*) (*fls-compose-power f d*)
by *transfer* (*auto simp*: *fun-eq-iff add-ac mult-ac*)

lemma *fls-compose-power-X-intpow* [*simp*]:
d > *0* ⟹ *fls-compose-power* (*fls-X-intpow n*) *d* = *fls-X-intpow* (*int d* ∗ *n*)
by *simp*

lemma *fls-compose-power-X* [*simp*]:
d > *0* ⟹ *fls-compose-power fls-X d* = *fls-X-intpow* (*int d*)
by *transfer* (*auto simp*: *fun-eq-iff*)

lemma *fls-compose-power-X-inv* [*simp*]:
d > *0* ⟹ *fls-compose-power fls-X-inv d* = *fls-X-intpow* (−*int d*)
by (*simp add*: *fls-X-inv-conv-shift-1*)

lemma *fls-compose-power-0-right* [*simp*]: *fls-compose-power f 0* = *0*
by *transfer auto*

lemma *fls-compose-power-add* [*simp*]:
fls-compose-power (*f* + *g*) *d* = *fls-compose-power f d* + *fls-compose-power g d*
by *transfer auto*

lemma *fls-compose-power-diff* [*simp*]:
fls-compose-power (*f* − *g*) *d* = *fls-compose-power f d* − *fls-compose-power g d*
by *transfer auto*

lemma *fls-compose-power-uminus* [*simp*]:
fls-compose-power (−*f*) *d* = −*fls-compose-power f d*
by *transfer auto*

lemma *fps-nth-compose-X-power*:
fps-nth (*f oo* (*fps-X* ^ *d*)) *n* = (*if d dvd n then fps-nth f* (*n div d*) *else 0*)
proof −
have *fps-nth* (*f oo* (*fps-X* ^ *d*)) *n* = (∑ *i* = *0*..*n*. *f* $ *i* ∗ (*fps-X* ^ (*d* ∗ *i*)) $ *n*)
unfolding *fps-compose-def* **by** (*simp add*: *power-mult*)
also have ... = (∑ *i*∈(*if d dvd n then* {*n div d*} *else* {}). *f* $ *i* ∗ (*fps-X* ^ (*d* ∗ *i*)) $ *n*)
by (*intro sum.mono-neutral-right*) *auto*
also have ... = (*if d dvd n then fps-nth f* (*n div d*) *else 0*)
by *auto*
finally show *?thesis* **.**

qed

lemma *fls-compose-power-fps-to-fls*:
assumes $d > 0$
shows *fls-compose-power* (*fps-to-fls f*) d = *fps-to-fls* (*fps-compose f* (*fps-X* ^ d))
using *assms*
by (*intro fls-eqI*) (*auto simp*: *fls-nth-compose-power fps-nth-compose-X-power pos-imp-zdiv-neg-iff div-neg-pos-less0 nat-div-distrib simp flip*: *int-dvd-int-iff*)

lemma *fls-compose-power-mult* [*simp*]:
fls-compose-power (*f* ∗ *g* :: *'a* :: *idom fls*) d = *fls-compose-power f d* ∗ *fls-compose-power g d*
proof (*cases* $d > 0$)
case *True*
define n **where** n = *nat* (*max 0* (*max* (− *fls-subdegree f*) (− *fls-subdegree g*)))
have *n-ge*: −*fls-subdegree f* ≤ *int n* −*fls-subdegree g* ≤ *int n*
unfolding *n-def* **by** *auto*
obtain *f'* **where** *f'*: *f* = *fls-shift n* (*fps-to-fls f'*)
using *fls-as-fps*[*OF n-ge*(*1*)] **by** (*auto simp*: *n-def*)
obtain *g'* **where** *g'*: *g* = *fls-shift n* (*fps-to-fls g'*)
using *fls-as-fps*[*OF n-ge*(*2*)] **by** (*auto simp*: *n-def*)
show *?thesis* **using** ‹$d > 0$›
by (*simp add*: *f' g' fls-shifted-times-simps mult-ac fls-compose-power-fps-to-fls fps-compose-mult-distrib flip*: *fls-times-fps-to-fls*)
qed *auto*

lemma *fls-compose-power-power* [*simp*]:
assumes $d > 0 \vee n > 0$
shows *fls-compose-power* (*f* ^ n :: *'a* :: *idom fls*) d = *fls-compose-power f d* ^ n
proof (*cases* $d > 0$)
case *True*
thus *?thesis* **by** (*induction n*) *auto*
qed (*use assms* **in** *auto*)

lemma *fls-nth-compose-power'* [*simp*]:
$d = 0 \vee \neg d$ *dvd* $n \Longrightarrow$ *fls-compose-power f d* $$ *int n* = *0*
d *dvd* $n \Longrightarrow d > 0 \Longrightarrow$ *fls-compose-power f d* $$ *int n* = *f* $$ *int* (n *div* d)
by (*transfer*; *force*; *fail*)+

lemma *subdegree-fls-compose-fps* [*simp*]:
fixes *G* :: *'a* :: *field fps*
assumes [*simp*]: *fps-nth G 0* = *0*
shows *fls-subdegree* (*fls-compose-fps F G*) = *fls-subdegree F* ∗ *subdegree G*
proof (*cases F* = *0*; *cases G* = *0*)
assume [*simp*]: *G* ≠ *0 F* ≠ *0*
have *nz1*: *fls-base-factor-to-fps F* ≠ *0*
using ‹*F* ≠ *0*› *fls-base-factor-to-fps-nonzero* **by** *blast*

show *?thesis*
unfolding *fls-compose-fps-def* **using** *nz1*
by (*subst fls-subdegree-mult*) (*simp-all add*: *fps-compose-eq-0-iff fls-subdegree-fls-to-fps*)
qed (*auto simp*: *fls-compose-fps-0-right*)

7.7 Formal differentiation and integration

7.7.1 Derivative

definition *fls-deriv f* = *Abs-fls* (λ*n*. *of-int* (*n*+1) ∗ *f*$$(*n*+1))

lemma *fls-deriv-nth*[*simp*]: *fls-deriv f* $$ *n* = *of-int* (*n*+1) ∗ *f*$$(*n*+1)
proof−
obtain *N* **where** ∀ *n*<*N*. *f*$$*n* = 0 **by** (*elim fls-nth-vanishes-belowE*)
hence ∀ *n*<*N*−1. *of-int* (*n*+1) ∗ *f*$$(*n*+1) = 0 **by** *auto*
thus *?thesis* **using** *nth-Abs-fls-lower-bound* **unfolding** *fls-deriv-def* **by** *simp*
qed

lemma *fls-deriv-residue*: *fls-deriv f* $$ −1 = 0
by *simp*

lemma *fls-deriv-const*[*simp*]: *fls-deriv* (*fls-const x*) = 0
proof (*intro fls-eqI*)
fix *n* **show** *fls-deriv* (*fls-const x*) $$ *n* = 0$$*n*
by (*cases n*+1=0) *auto*
qed

lemma *fls-deriv-of-nat*[*simp*]: *fls-deriv* (*of-nat n*) = 0
by (*simp add*: *fls-of-nat*)

lemma *fls-deriv-of-int*[*simp*]: *fls-deriv* (*of-int i*) = 0
by (*simp add*: *fls-of-int*)

lemma *fls-deriv-zero*[*simp*]: *fls-deriv* 0 = 0
using *fls-deriv-const*[*of* 0] **by** *simp*

lemma *fls-deriv-one*[*simp*]: *fls-deriv* 1 = 0
using *fls-deriv-const*[*of* 1] **by** *simp*

lemma *fls-deriv-numeral* [*simp*]: *fls-deriv* (*numeral n*) = 0
by (*metis fls-deriv-of-int of-int-numeral*)

lemma *fls-deriv-subdegree′*:
assumes *of-int* (*fls-subdegree f*) ∗ *f* $$ *fls-subdegree f* ≠ 0
shows *fls-subdegree* (*fls-deriv f*) = *fls-subdegree f* − 1
by (*auto intro*: *fls-subdegree-eqI simp*: *assms*)

lemma *fls-deriv-subdegree0*:
assumes *fls-subdegree f* = 0
shows *fls-subdegree* (*fls-deriv f*) ≥ 0

proof (*cases fls-deriv f = 0*)
  case *False*
  show *?thesis*
  proof (*intro fls-subdegree-geI*, *rule False*)
    fix *k* :: *int* **assume** *k < 0*
    with *assms* **show** *fls-deriv f $$ k = 0* **by** (*cases k=−1*) *auto*
  qed
qed *simp*

lemma *fls-subdegree-deriv'*:
  fixes *f* :: *'a*::*ring-1-no-zero-divisors fls*
  assumes (*of-int* (*fls-subdegree f*) :: *'a*) ≠ *0*
  shows *fls-subdegree* (*fls-deriv f*) = *fls-subdegree f − 1*
  using *assms nth-fls-subdegree-zero-iff*[*of f*]
  by (*auto intro*: *fls-deriv-subdegree'*)

lemma *fls-subdegree-deriv*:
  fixes *f* :: *'a*::{*ring-1-no-zero-divisors*,*ring-char-0*} *fls*
  assumes *fls-subdegree f ≠ 0*
  shows *fls-subdegree* (*fls-deriv f*) = *fls-subdegree f − 1*
  by (*auto intro*: *fls-subdegree-deriv' simp*: *assms*)

lemma *fps-deriv-fls-regpart*: *fps-deriv* (*fls-regpart F*) = *fls-regpart* (*fls-deriv F*)
  by (*intro fps-ext*) (*auto simp*: *add-ac*)

Shifting is like multiplying by a power of the implied variable, and so satisfies a product-like rule.

lemma *fls-deriv-shift*:
  fls-deriv (*fls-shift n f*) = *of-int* (*−n*) ∗ *fls-shift* (*n+1*) *f* + *fls-shift n* (*fls-deriv f*)
  by (*intro fls-eqI*) (*simp flip*: *fls-shift-fls-shift add*: *algebra-simps*)

lemma *fls-deriv-X* [*simp*]: *fls-deriv fls-X* = *1*
  by (*intro fls-eqI*) *simp*

lemma *fls-deriv-X-inv* [*simp*]: *fls-deriv fls-X-inv* = *−* (*fls-X-inv*^{2})
proof−
  have *fls-deriv fls-X-inv* = *−* (*fls-shift 2 1*)
    by (*simp add*: *fls-X-inv-conv-shift-1 fls-deriv-shift*)
  thus *?thesis* **by** (*simp add*: *fls-X-inv-power-conv-shift-1*)
qed

lemma *fls-deriv-delta*:
  fls-deriv (*Abs-fls* (*λn. if n=m then c else 0*)) =
    Abs-fls (*λn. if n=m−1 then of-int m ∗ c else 0*)
proof−
  have
    fls-deriv (*Abs-fls* (*λn. if n=m then c else 0*)) = *fls-shift* (*1−m*) (*fls-const* (*of-int m ∗ c*))
    using *fls-deriv-shift*[*of −m fls-const c*]

```
    by    (simp
           add: fls-shift-const fls-of-int fls-shifted-times-simps(1)[symmetric]
           fls-const-mult-const[symmetric]
           del: fls-const-mult-const
          )
  thus ?thesis by (simp add: fls-shift-const)
qed

lemma fls-deriv-base-factor:
  fls-deriv (fls-base-factor f) =
    of-int (−fls-subdegree f) ∗ fls-shift (fls-subdegree f + 1) f +
    fls-shift (fls-subdegree f) (fls-deriv f)
  by (simp add: fls-deriv-shift)

lemma fls-regpart-deriv: fls-regpart (fls-deriv f) = fps-deriv (fls-regpart f)
proof (intro fps-ext)
  fix n
  have  1: (of-nat n :: 'a) + 1 = of-nat (n+1)
  and   2: int n + 1 = int (n + 1)
    by  auto
  show fls-regpart (fls-deriv f) $ n = fps-deriv (fls-regpart f) $ n by (simp add: 1
2)
qed

lemma fls-prpart-deriv:
  fixes f :: 'a :: {comm-ring-1,ring-no-zero-divisors} fls
  — Commutivity and no zero divisors are required by the definition of pderiv.
  shows fls-prpart (fls-deriv f) = − pCons 0 (pCons 0 (pderiv (fls-prpart f)))
proof (intro poly-eqI)
  fix n
  show
    coeff (fls-prpart (fls-deriv f)) n =
      coeff (− pCons 0 (pCons 0 (pderiv (fls-prpart f)))) n
  proof (cases n)
    case (Suc m)
    hence n: n = Suc m by fast
    show ?thesis
    proof (cases m)
      case (Suc k)
      with n have
        coeff (− pCons 0 (pCons 0 (pderiv (fls-prpart f)))) n =
          − coeff (pderiv (fls-prpart f)) k
        by (simp flip: coeff-minus)
      with Suc n show ?thesis by (simp add: coeff-pderiv algebra-simps)
    qed (simp add: n)
  qed simp
qed

lemma pderiv-fls-prpart:
```

pderiv (*fls-prpart f*) = − *poly-shift 2* (*fls-prpart* (*fls-deriv f*))
by (*intro poly-eqI*) (*simp add*: *coeff-pderiv coeff-poly-shift algebra-simps*)

lemma *fls-deriv-fps-to-fls*: *fls-deriv* (*fps-to-fls f*) = *fps-to-fls* (*fps-deriv f*)
proof (*intro fls-eqI*)
fix *n*
show *fls-deriv* (*fps-to-fls f*) $$ *n* = *fps-to-fls* (*fps-deriv f*) $$ *n*
proof (*cases n≥0*)
case *True*
from *True* **have** *1*: *nat* (*n* + *1*) = *nat n* + *1* **by** *simp*
from *True* **have** *2*: (*of-int* (*n* + *1*) :: *'a*) = *of-nat* (*nat* (*n+1*)) **by** *simp*
from *True* **show** *?thesis* **using** *arg-cong*[*OF 2*, *of* λ*x*. *x* ∗ *f* $ (*nat n+1*)] **by**
(*simp add*: *1*)
next
case *False* **thus** *?thesis* **by** (*cases n=−1*) *auto*
qed
qed

7.7.2 Algebraic rules of the derivative

lemma *fls-deriv-add* [*simp*]: *fls-deriv* (*f*+*g*) = *fls-deriv f* + *fls-deriv g*
by (*auto intro*: *fls-eqI simp*: *algebra-simps*)

lemma *fls-deriv-sub* [*simp*]: *fls-deriv* (*f*−*g*) = *fls-deriv f* − *fls-deriv g*
by (*auto intro*: *fls-eqI simp*: *algebra-simps*)

lemma *fls-deriv-neg* [*simp*]: *fls-deriv* (−*f*) = − *fls-deriv f*
using *fls-deriv-sub*[*of 0 f*] **by** *simp*

lemma *fls-deriv-mult* [*simp*]:
fls-deriv (*f*∗*g*) = *f* ∗ *fls-deriv g* + *fls-deriv f* ∗ *g*
proof−
define *df dg* :: *int*
where *df* ≡ *fls-subdegree f*
and *dg* ≡ *fls-subdegree g*
define *uf ug* :: *'a fls*
where *uf* ≡ *fls-base-factor f*
and *ug* ≡ *fls-base-factor g*
have
f ∗ *fls-deriv g* =
of-int dg ∗ *fls-shift* (*1* − *dg*) (*f* ∗ *ug*) + *fls-shift* (−*dg*) (*f* ∗ *fls-deriv ug*)
fls-deriv f ∗ *g* =
of-int df ∗ *fls-shift* (*1* − *df*) (*uf* ∗ *g*) + *fls-shift* (−*df*) (*fls-deriv uf* ∗ *g*)
using *fls-deriv-shift*[*of* −*df uf*] *fls-deriv-shift*[*of* −*dg ug*]
mult-of-int-commute[*of dg f*]
mult.assoc[*of of-int dg f*]
fls-shifted-times-simps(*1*)[*of f 1* − *dg ug*]
fls-shifted-times-simps(*1*)[*of f* −*dg fls-deriv ug*]
fls-shifted-times-simps(*2*)[*of 1* − *df uf g*]

```
      fls-shifted-times-simps(2)[of −df fls-deriv uf g]
  by (auto simp add: algebra-simps df-def dg-def uf-def ug-def)
 moreover have
  fls-deriv (f∗g) =
   ( of-int dg ∗ fls-shift (1 − dg) (f ∗ ug) + fls-shift (−dg) (f ∗ fls-deriv ug) ) +
   ( of-int df ∗ fls-shift (1 − df) (uf ∗ g) + fls-shift (−df) (fls-deriv uf ∗ g) )

  using fls-deriv-shift[of
      − (df + dg) fps-to-fls (fls-base-factor-to-fps f ∗ fls-base-factor-to-fps g)
    ]
    fls-deriv-fps-to-fls[of fls-base-factor-to-fps f ∗ fls-base-factor-to-fps g]
    fps-deriv-mult[of fls-base-factor-to-fps f fls-base-factor-to-fps g]
    distrib-right[of
      of-int df of-int dg
      fls-shift (1 − (df + dg)) (
        fps-to-fls (fls-base-factor-to-fps f ∗ fls-base-factor-to-fps g)
      )
    ]
    fls-times-conv-fps-times[of uf ug]
    fls-base-factor-subdegree[of f] fls-base-factor-subdegree[of g]
    fls-regpart-deriv[of ug]
    fls-times-conv-fps-times[of uf fls-deriv ug]
    fls-deriv-subdegree0[of ug]
    fls-regpart-deriv[of uf]
    fls-times-conv-fps-times[of fls-deriv uf ug]
    fls-deriv-subdegree0[of uf]
    fls-shifted-times-simps(1)[of uf −dg ug]
    fls-shifted-times-simps(1)[of fls-deriv uf −dg ug]
    fls-shifted-times-simps(2)[of −df uf ug]
    fls-shifted-times-simps(2)[of −df uf fls-deriv ug]
  by (simp add: fls-times-def algebra-simps df-def dg-def uf-def ug-def)
 ultimately show ?thesis by simp
qed

lemma fls-deriv-mult-const-left:
 fls-deriv (fls-const c ∗ f) = fls-const c ∗ fls-deriv f
 by simp

lemma fls-deriv-linear:
 fls-deriv (fls-const a ∗ f + fls-const b ∗ g) =
   fls-const a ∗ fls-deriv f + fls-const b ∗ fls-deriv g
 by simp

lemma fls-deriv-mult-const-right:
 fls-deriv (f ∗ fls-const c) = fls-deriv f ∗ fls-const c
 by simp

lemma fls-deriv-linear2:
 fls-deriv (f ∗ fls-const a + g ∗ fls-const b) =
```

```
    fls-deriv f * fls-const a + fls-deriv g * fls-const b
  by simp

lemma fls-deriv-sum:
  fls-deriv (sum f S) = sum (λi. fls-deriv (f i)) S
proof (cases finite S)
  case True show ?thesis
    by (induct rule: finite-induct [OF True]) simp-all
qed simp

lemma fls-deriv-power:
  fixes f :: 'a::comm-ring-1 fls
  shows fls-deriv (f^n) = of-nat n * f^(n-1) * fls-deriv f
proof (cases n)
  case (Suc m)
  have fls-deriv (f^Suc m) = of-nat (Suc m) * f^m * fls-deriv f
    by (induct m) (simp-all add: algebra-simps)
  with Suc show ?thesis by simp
qed simp

lemma fls-deriv-X-power:
  fls-deriv (fls-X ^ n) = of-nat n * fls-X ^ (n-1)
proof (cases n)
  case (Suc m)
  have fls-deriv (fls-X^Suc m) = of-nat (Suc m) * fls-X^m
    by (induct m) (simp-all add: mult-of-nat-commute algebra-simps)
  with Suc show ?thesis by simp
qed simp

lemma fls-deriv-X-inv-power:
  fls-deriv (fls-X-inv ^ n) = - of-nat n * fls-X-inv ^ (Suc n)
proof (cases n)
  case (Suc m)
  define iX :: 'a fls where iX ≡ fls-X-inv
  have fls-deriv (iX ^ Suc m) = - of-nat (Suc m) * iX ^ (Suc (Suc m))
  proof (induct m)
    case (Suc m)
    have - of-nat (Suc m + 1) * iX ^ Suc (Suc (Suc m)) =
          iX * (-of-nat (Suc m) * iX ^ Suc (Suc m)) +
              - (iX ^ 2 * iX ^ Suc m)
      using distrib-right[of -of-nat (Suc m) -(1::'a fls) fls-X-inv ^ Suc (Suc (Suc
m))]
    by (simp add: algebra-simps mult-of-nat-commute power2-eq-square Suc iX-def)
    thus ?case using Suc by (simp add: iX-def)
  qed (simp add: numeral-2-eq-2 iX-def)
  with Suc show ?thesis by (simp add: iX-def)
qed simp

lemma fls-deriv-X-intpow:
```

fls-deriv (*fls-X-intpow i*) = *of-int i* ∗ *fls-X-intpow* (*i*−*1*)
by (*simp add*: *fls-deriv-shift*)

lemma *fls-deriv-lr-inverse*:
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1 f* $$ *fls-subdegree f* ∗ *y* = *1*
— These assumptions imply x equals y, but no need to assume that.
shows *fls-deriv* (*fls-left-inverse f x*) =
− *fls-left-inverse f x* ∗ *fls-deriv f* ∗ *fls-left-inverse f x*
and *fls-deriv* (*fls-right-inverse f y*) =
− *fls-right-inverse f y* ∗ *fls-deriv f* ∗ *fls-right-inverse f y*
proof−

define *L* **where** *L* ≡ *fls-left-inverse f x*
hence *fls-deriv* (*L* ∗ *f*) = *0* **using** *fls-left-inverse*[*OF assms*(*1*)] **by** *simp*
with *assms* **show** *fls-deriv L* = − *L* ∗ *fls-deriv f* ∗ *L*
using *fls-right-inverse*′[*OF assms*]
by (*simp add*: *minus-unique mult.assoc L-def*)

define *R* **where** *R* ≡ *fls-right-inverse f y*
hence *fls-deriv* (*f* ∗ *R*) = *0* **using** *fls-right-inverse*[*OF assms*(*2*)] **by** *simp*
hence *1*: *f* ∗ *fls-deriv R* + *fls-deriv f* ∗ *R* = *0* **by** *simp*
have *R* ∗ *f* ∗ *fls-deriv R* = − *R* ∗ *fls-deriv f* ∗ *R*
using *iffD2*[*OF eq-neg-iff-add-eq-0*, *OF 1*] **by** (*simp add*: *mult.assoc*)
thus *fls-deriv R* = − *R* ∗ *fls-deriv f* ∗ *R*
using *fls-left-inverse*′[*OF assms*] **by** (*simp add*: *R-def*)

qed

lemma *fls-deriv-lr-inverse-comm*:
fixes *x y* :: ′*a*::*comm-ring-1*
assumes *x* ∗ *f* $$ *fls-subdegree f* = *1*
shows *fls-deriv* (*fls-left-inverse f x*) = − *fls-deriv f* ∗ (*fls-left-inverse f x*)^2
and *fls-deriv* (*fls-right-inverse f x*) = − *fls-deriv f* ∗ (*fls-right-inverse f x*)^2
using *assms fls-deriv-lr-inverse*[*of x f x*]
by (*simp-all add*: *mult.commute power2-eq-square*)

lemma *fls-inverse-deriv-divring*:
fixes *a* :: ′*a*::*division-ring fls*
shows *fls-deriv* (*inverse a*) = − *inverse a* ∗ *fls-deriv a* ∗ *inverse a*
proof (*cases a*=*0*)
case *False* **thus** *?thesis*
using *fls-deriv-lr-inverse*(*2*)[*of*
inverse (*a* $$ *fls-subdegree a*) *a inverse* (*a* $$ *fls-subdegree a*)
]
by (*auto simp add*: *fls-inverse-def*′)
qed *simp*

lemma *fls-inverse-deriv*:
fixes *a* :: ′*a*::*field fls*

shows *fls-deriv* (*inverse a*) = − *fls-deriv a* ∗ (*inverse a*)^2
by (*simp add*: *fls-inverse-deriv-divring power2-eq-square*)

lemma *fls-inverse-deriv'*:
fixes *a* :: *'a*::*field fls*
shows *fls-deriv* (*inverse a*) = − *fls-deriv a* / *a*^{2}
using *fls-inverse-deriv*[*of a*]
by (*simp add*: *field-simps*)

7.7.3 Equality of derivatives

lemma *fls-deriv-eq-0-iff*:
fls-deriv f = *0* ⟷ *f* = *fls-const* (*f*$$*0* :: *'a*::{*ring-1-no-zero-divisors*,*ring-char-0*})
proof
assume *f*: *fls-deriv f* = *0*
show *f* = *fls-const* (*f*$$*0*)
proof (*intro fls-eqI*)
fix *n*
from *f* **have** *of-int n* ∗ *f*$$ *n* = *0* **using** *fls-deriv-nth*[*of f n*−*1*] **by** *simp*
thus *f*$$*n* = *fls-const* (*f*$$*0*) $$ *n* **by** (*cases n*=*0*) *auto*
qed
next
show *f* = *fls-const* (*f*$$*0*) ⟹ *fls-deriv f* = *0* **using** *fls-deriv-const*[*of f*$$*0*] **by** *simp*
qed

lemma *fls-deriv-eq-iff*:
fixes *f g* :: *'a*::{*ring-1-no-zero-divisors*,*ring-char-0*} *fls*
shows *fls-deriv f* = *fls-deriv g* ⟷ (*f* = *fls-const*(*f*$$*0* − *g*$$*0*) + *g*)
proof −
have *fls-deriv f* = *fls-deriv g* ⟷ *fls-deriv* (*f* − *g*) = *0*
by *simp*
also have ... ⟷ *f* − *g* = *fls-const* ((*f* − *g*) $$ *0*)
unfolding *fls-deriv-eq-0-iff* **..**
finally show *?thesis*
by (*simp add*: *field-simps*)
qed

lemma *fls-deriv-eq-iff-ex*:
fixes *f g* :: *'a*::{*ring-1-no-zero-divisors*,*ring-char-0*} *fls*
shows (*fls-deriv f* = *fls-deriv g*) ⟷ (∃ *c*. *f* = *fls-const c* + *g*)
by (*auto simp*: *fls-deriv-eq-iff*)

7.7.4 Residues

definition *fls-residue-def*[*simp*]: *fls-residue f* ≡ *f* $$ −*1*

lemma *fls-residue-deriv*: *fls-residue* (*fls-deriv f*) = *0*
by *simp*

lemma *fls-residue-add*: *fls-residue* $(f+g)$ = *fls-residue* f + *fls-residue* g
by *simp*

lemma *fls-residue-times-deriv*:
fls-residue (*fls-deriv* $f * g$) = $-$ *fls-residue* (f * *fls-deriv* g)
using *fls-residue-deriv*[*of* $f*g$] *minus-unique*[*of fls-residue* (f * *fls-deriv* g)]
by *simp*

lemma *fls-residue-power-series*: *fls-subdegree* $f \geq 0 \Longrightarrow$ *fls-residue* $f = 0$
by *simp*

lemma *fls-residue-fls-X-intpow*:
fls-residue (*fls-X-intpow* i) = (*if* $i{=}{-}1$ *then* 1 *else* 0)
by *simp*

lemma *fls-residue-shift-nth*:
fixes f :: $'a$::*semiring-1 fls*
shows $f\$\$n$ = *fls-residue* (*fls-X-intpow* $(-n-1)$ * f)
by (*simp add*: *fls-shifted-times-transfer*)

lemma *fls-residue-fls-const-times*:
fixes f :: $'a$::{*comm-monoid-add*, *mult-zero*} *fls*
shows *fls-residue* (*fls-const* c * f) = c * *fls-residue* f
and *fls-residue* (f * *fls-const* c) = *fls-residue* f * c
by *simp-all*

lemma *fls-residue-of-int-times*:
fixes f :: $'a$::*ring-1 fls*
shows *fls-residue* (*of-int* i * f) = *of-int* i * *fls-residue* f
and *fls-residue* (f * *of-int* i) = *fls-residue* f * *of-int* i
by (*simp-all add*: *fls-residue-fls-const-times fls-of-int*)

lemma *fls-residue-deriv-times-lr-inverse-eq-subdegree*:
fixes $f\ g$ :: $'a$::*ring-1 fls*
assumes $y * (f\ \$\$\ \textit{fls-subdegree}\ f) = 1$ $(f\ \$\$\ \textit{fls-subdegree}\ f) * y = 1$
shows *fls-residue* (*fls-deriv* f * *fls-right-inverse* $f\ y$) = *of-int* (*fls-subdegree* f)
and *fls-residue* (*fls-deriv* f * *fls-left-inverse* $f\ y$) = *of-int* (*fls-subdegree* f)
and *fls-residue* (*fls-left-inverse* $f\ y$ * *fls-deriv* f) = *of-int* (*fls-subdegree* f)
and *fls-residue* (*fls-right-inverse* $f\ y$ * *fls-deriv* f) = *of-int* (*fls-subdegree* f)
proof$-$
define *df* :: *int* **where** *df* $\equiv$ *fls-subdegree* f
define $B\ X$ :: $'a$ *fls*
where $B \equiv$ *fls-base-factor* f
and $X \equiv$ (*fls-X-intpow df* :: $'a$ *fls*)
define $D\ L\ R$ :: $'a$ *fls*
where $D \equiv$ *fls-deriv* B
and $L \equiv$ *fls-left-inverse* $B\ y$
and $R \equiv$ *fls-right-inverse* $B\ y$
have *intpow-diff*: *fls-X-intpow* (*df* $-$ 1) = X * *fls-X-inv*

using *fls-X-intpow-diff-conv-times*[*of df 1*] **by** (*simp add*: *X-def fls-X-inv-conv-shift-1*)

show *fls-residue* (*fls-deriv f* ∗ *fls-right-inverse f y*) = *of-int df*
proof−
have *subdegree-DR*: *fls-subdegree* (*D* ∗ *R*) ≥ *0*
using *fls-base-factor-subdegree*[*of f*] *fls-base-factor-subdegree*[*of fls-right-inverse f y*]
assms(*1*) *fls-right-inverse-base-factor*[*of y f*] *fls-mult-subdegree-ge-0*[*of D R*]
by (*force simp*: *fls-deriv-subdegree0 D-def R-def B-def*)
have *decomp*: *f* = *X* ∗ *B*
unfolding *X-def B-def df-def* **by** (*rule fls-base-factor-X-power-decompose*(*2*)[*of f*])
hence *fls-deriv f* = *X* ∗ *D* + *of-int df* ∗ *X* ∗ *fls-X-inv* ∗ *B*
using *intpow-diff fls-deriv-mult*[*of X B*]
by (*simp add*: *fls-deriv-X-intpow X-def B-def D-def mult.assoc*)
moreover from *assms* **have** *fls-right-inverse* (*X* ∗ *B*) *y* = *R* ∗ *fls-right-inverse X 1*
using *fls-base-factor-base*[*of f*] *fls-lr-inverse-mult-ring1*(*2*)[*of 1 X*]
by (*simp add*: *X-def B-def R-def*)
ultimately have
fls-deriv f ∗ *fls-right-inverse f y* =
(*D* + *of-int df* ∗ *fls-X-inv* ∗ *B*) ∗ *R* ∗ (*X* ∗ *fls-right-inverse X 1*)
by (*simp add*: *decomp algebra-simps X-def fls-X-intpow-times-comm*)
also have . . . = *D* ∗ *R* + *of-int df* ∗ *fls-X-inv*
using *fls-right-inverse*[*of X 1*]
assms fls-base-factor-base[*of f*] *fls-right-inverse*[*of B y*]
by (*simp add*: *X-def distrib-right mult.assoc B-def R-def*)
finally show *?thesis* **using** *subdegree-DR* **by** *simp*
qed

with *assms* **show** *fls-residue* (*fls-deriv f* ∗ *fls-left-inverse f y*) = *of-int df*
using *fls-left-inverse-eq-fls-right-inverse*[*of y f*] **by** *simp*

show *fls-residue* (*fls-left-inverse f y* ∗ *fls-deriv f*) = *of-int df*
proof−
have *subdegree-LD*: *fls-subdegree* (*L* ∗ *D*) ≥ *0*
using *fls-base-factor-subdegree*[*of f*] *fls-base-factor-subdegree*[*of fls-left-inverse f y*]
assms(*1*) *fls-left-inverse-base-factor*[*of y f*] *fls-mult-subdegree-ge-0*[*of L D*]
by (*force simp*: *fls-deriv-subdegree0 D-def L-def B-def*)
have *decomp*: *f* = *B* ∗ *X*
unfolding *X-def B-def df-def* **by** (*rule fls-base-factor-X-power-decompose*(*1*)[*of f*])
hence *fls-deriv f* = *D* ∗ *X* + *B* ∗ *of-int df* ∗ *X* ∗ *fls-X-inv*
using *intpow-diff fls-deriv-mult*[*of B X*]
by (*simp add*: *fls-deriv-X-intpow X-def D-def B-def mult.assoc*)

moreover from *assms* **have** *fls-left-inverse* $(B * X)$ y = *fls-left-inverse X 1* $*$ *L*
using *fls-base-factor-base*[*of f*] *fls-lr-inverse-mult-ring1*(*1*)[*of - - 1 X*]
by (*simp add*: *X-def B-def L-def*)
ultimately have
fls-left-inverse f y $*$ *fls-deriv f* =
fls-left-inverse X 1 $*$ *X* $*$ *L* $*$ (*D* + *B* $*$ (*of-int df* $*$ *fls-X-inv*))
by (*simp add*: *decomp algebra-simps X-def fls-X-intpow-times-comm*)
also have ... = *L* $*$ *D* + *of-int df* $*$ *fls-X-inv*
using *assms fls-left-inverse*[*of 1 X*] *fls-base-factor-base*[*of f*] *fls-left-inverse*[*of y B*]
by (*simp add*: *X-def distrib-left mult.assoc*[*symmetric*] *L-def B-def*)
finally show *?thesis* **using** *subdegree-LD* **by** *simp*
qed

with *assms* **show** *fls-residue* (*fls-right-inverse f y* $*$ *fls-deriv f*) = *of-int df*
using *fls-left-inverse-eq-fls-right-inverse*[*of y f*] **by** *simp*

qed

lemma *fls-residue-deriv-times-inverse-eq-subdegree*:
fixes *f g* :: *'a*::*division-ring fls*
shows *fls-residue* (*fls-deriv f* $*$ *inverse f*) = *of-int* (*fls-subdegree f*)
and *fls-residue* (*inverse f* $*$ *fls-deriv f*) = *of-int* (*fls-subdegree f*)
proof−
show *fls-residue* (*fls-deriv f* $*$ *inverse f*) = *of-int* (*fls-subdegree f*)
using *fls-residue-deriv-times-lr-inverse-eq-subdegree*(*1*)[*of - f*]
by (*cases f=0*) (*auto simp*: *fls-inverse-def'*)
show *fls-residue* (*inverse f* $*$ *fls-deriv f*) = *of-int* (*fls-subdegree f*)
using *fls-residue-deriv-times-lr-inverse-eq-subdegree*(*4*)[*of - f*]
by (*cases f=0*) (*auto simp*: *fls-inverse-def'*)
qed

7.7.5 Integral definition and basic properties

definition *fls-integral* :: *'a*::{*ring-1*,*inverse*} *fls* ⇒ *'a fls*
where *fls-integral a* = *Abs-fls* (λ*n*. *if n=0 then 0 else inverse* (*of-int n*) $*$ *a*$$(*n* − *1*))

lemma *fls-integral-nth* [*simp*]:
fls-integral a $$ *n* = (*if n=0 then 0 else inverse* (*of-int n*) $*$ *a*$$(*n*−*1*))
proof−
define *F* **where** *F* ≡ (λ*n*. *if n=0 then 0 else inverse* (*of-int n*) $*$ *a*$$(*n* − *1*))
obtain *N* **where** ∀ *n*<*N*. *a*$$*n* = *0* **by** (*elim fls-nth-vanishes-belowE*)
hence ∀ *n*<*N*. *F n* = *0* **by** (*auto simp add*: *F-def*)
thus *?thesis* **using** *nth-Abs-fls-lower-bound*[*of N F*] **unfolding** *fls-integral-def F-def* **by** *simp*
qed

lemma *fls-integral-conv-fps-zeroth-integral*:
assumes *fls-subdegree a* $\geq$ *0*
shows *fls-integral a* = *fps-to-fls* (*fps-integral0* (*fls-regpart a*))
proof (*rule fls-eqI*)
fix *n*
show *fls-integral a* $$ *n* = *fps-to-fls* (*fps-integral0* (*fls-regpart a*)) $$ *n*
proof (*cases n>0*)
case *False* **with** *assms* **show** *?thesis* **by** *simp*
next
case *True*
hence *int* ((*nat n*) − *1*) = *n* − *1* **by** *simp*
with *True* **show** *?thesis* **by** (*simp add*: *fps-integral-def*)
qed
qed

lemma *fls-integral-zero* [*simp*]: *fls-integral 0* = *0*
by (*intro fls-eqI*) *simp*

lemma *fls-integral-const'*:
fixes *x* :: *'a*::{*ring-1*,*inverse*}
assumes *inverse* (*1*::*'a*) = *1*
shows *fls-integral* (*fls-const x*) = *fls-const x* ∗ *fls-X*
by (*intro fls-eqI*) (*simp add*: *assms*)

lemma *fls-integral-const*:
fixes *x* :: *'a*::*division-ring*
shows *fls-integral* (*fls-const x*) = *fls-const x* ∗ *fls-X*
by (*rule fls-integral-const'*[*OF inverse-1*])

lemma *fls-integral-of-nat'*:
assumes *inverse* (*1*::*'a*::{*ring-1*,*inverse*}) = *1*
shows *fls-integral* (*of-nat n* :: *'a fls*) = *of-nat n* ∗ *fls-X*
by (*simp add*: *assms fls-integral-const' fls-of-nat*)

lemma *fls-integral-of-nat*:
fls-integral (*of-nat n* :: *'a*::*division-ring fls*) = *of-nat n* ∗ *fls-X*
by (*rule fls-integral-of-nat'*[*OF inverse-1*])

lemma *fls-integral-of-int'*:
assumes *inverse* (*1*::*'a*::{*ring-1*,*inverse*}) = *1*
shows *fls-integral* (*of-int i* :: *'a fls*) = *of-int i* ∗ *fls-X*
by (*simp add*: *assms fls-integral-const' fls-of-int*)

lemma *fls-integral-of-int*:
fls-integral (*of-int i* :: *'a*::*division-ring fls*) = *of-int i* ∗ *fls-X*
by (*rule fls-integral-of-int'*[*OF inverse-1*])

lemma *fls-integral-one'*:
assumes *inverse* (*1*::*'a*::{*ring-1*,*inverse*}) = *1*

shows *fls-integral* (*1*::*'a fls*) = *fls-X*
using *fls-integral-const'*[*of 1*]
by (*force simp*: *assms*)

lemma *fls-integral-one*: *fls-integral* (*1*::*'a*::*division-ring fls*) = *fls-X*
by (*rule fls-integral-one'*[*OF inverse-1*])

lemma *fls-subdegree-integral-ge*:
fls-integral f ≠ *0* ⟹ *fls-subdegree* (*fls-integral f*) ≥ *fls-subdegree f* + *1*
by (*intro fls-subdegree-geI*) *simp-all*

lemma *fls-subdegree-integral*:
fixes *f* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
assumes *f* ≠ *0 fls-subdegree f* ≠ −*1*
shows *fls-subdegree* (*fls-integral f*) = *fls-subdegree f* + *1*
using *assms of-int-0-eq-iff*[*of fls-subdegree f* + *1*] *fls-subdegree-integral-ge*
by (*intro fls-subdegree-eqI*) *simp-all*

lemma *fls-integral-X* [*simp*]:
fls-integral (*fls-X*::*'a*::{*ring-1*,*inverse*} *fls*) =
fls-const (*inverse* (*of-int 2*)) ∗ *fls-X*^{2}
proof (*intro fls-eqI*)
fix *n*
show *fls-integral* (*fls-X*::*'a fls*) $$ *n* = (*fls-const* (*inverse* (*of-int 2*)) ∗ *fls-X*^{2}) $$ *n*
using *arg-cong*[*OF fls-X-power-nth*, *of* λ*x*. *inverse* (*of-int 2*) ∗ *x*, *of 2 n*, *symmetric*]
by (*auto simp add*:)
qed

lemma *fls-integral-X-power*:
fls-integral (*fls-X* ^ *n* ::*'a* :: {*ring-1*,*inverse*} *fls*) =
fls-const (*inverse* (*of-nat* (*Suc n*))) ∗ *fls-X* ^ *Suc n*
proof (*intro fls-eqI*)
fix *k*
have (*fls-X* :: *'a fls*) ^ *Suc n* $$ *k* = (*if k*=*Suc n then 1 else 0*)
by (*rule fls-X-power-nth*)
thus
fls-integral ((*fls-X*::*'a fls*) ^ *n*) $$ *k* =
(*fls-const* (*inverse* (*of-nat* (*Suc n*))) ∗ (*fls-X*::*'a fls*) ^ *Suc n*) $$ *k*
by *simp*
qed

lemma *fls-integral-X-power-char0*:
fls-integral (*fls-X* ^ *n* :: *'a* :: {*ring-char-0*,*inverse*} *fls*) =
inverse (*of-nat* (*Suc n*)) ∗ *fls-X* ^ *Suc n*
proof −
have (*of-nat* (*Suc n*) :: *'a*) ≠ *0* **by** (*rule of-nat-neq-0*)
hence *fls-const* (*inverse* (*of-nat* (*Suc n*) :: *'a*)) = *inverse* (*fls-const* (*of-nat* (*Suc*

n)))
  by (*simp add*: *fls-inverse-const*)
 moreover have
  fls-integral ((*fls-X*::*'a fls*) ^ *n*) = *fls-const* (*inverse* (*of-nat* (*Suc n*))) * *fls-X* ^ *Suc n*
  by (*rule fls-integral-X-power*)
 ultimately show *?thesis* **by** (*simp add*: *fls-of-nat*)
qed

lemma *fls-integral-X-inv* [*simp*]: *fls-integral* (*fls-X-inv*::*'a*::{*ring-1*,*inverse*} *fls*) = *0*
 by (*intro fls-eqI*) *simp*

lemma *fls-integral-X-inv-power*:
 assumes *n* ≥ *2*
 shows
  fls-integral (*fls-X-inv* ^ *n* :: *'a* :: {*ring-1*,*inverse*} *fls*) =
   fls-const (*inverse* (*of-int* (*1* − *int n*))) * *fls-X-inv* ^ (*n*−*1*)
proof (*rule fls-eqI*)
 fix *k* **show**
  fls-integral (*fls-X-inv* ^ *n* :: *'a fls*) $$ *k*=
   (*fls-const* (*inverse* (*of-int* (*1* − *int n*))) * *fls-X-inv* ^ (*n*−*1*)) $$ *k*
 proof (*cases k*=*0*)
  case *True* **with** *assms* **show** *?thesis* **by** *simp*
 next
  case *False*
  from *assms* **have** *int* (*n*−*1*) = *int n* − *1* **by** *simp*
  hence
   (*fls-const* (*inverse* (*of-int* (*1* − *int n*))) * (*fls-X-inv*:: *'a fls*) ^ (*n*−*1*)) $$ *k* =
   (*if k* = *1* − *int n then inverse* (*of-int k*) *else 0*)
   by (*simp add*: *fls-X-inv-power-times-conv-shift*(*2*))
  with *False* **show** *?thesis* **by** (*simp add*: *algebra-simps*)
 qed
qed

lemma *fls-integral-X-inv-power-char0*:
 assumes *n* ≥ *2*
 shows
  fls-integral (*fls-X-inv* ^ *n* :: *'a* :: {*ring-char-0*,*inverse*} *fls*) =
   inverse (*of-int* (*1* − *int n*)) * *fls-X-inv* ^ (*n*−*1*)
proof−
 from *assms* **have** (*of-int* (*1* − *int n*) :: *'a*) ≠ *0* **by** *simp*
 hence
  fls-const (*inverse* (*of-int* (*1* − *int n*) :: *'a*)) = *inverse* (*fls-const* (*of-int* (*1* − *int n*)))
  by (*simp add*: *fls-inverse-const*)
 moreover have
  fls-integral (*fls-X-inv* ^ *n* :: *'a fls*) =
   fls-const (*inverse* (*of-int* (*1* − *int n*))) * *fls-X-inv* ^ (*n*−*1*)

```
    using assms by (rule fls-integral-X-inv-power)
  ultimately show ?thesis by (simp add: fls-of-int)
qed

lemma fls-integral-X-inv-power':
  assumes n ≥ 1
  shows
    fls-integral (fls-X-inv ^ n :: 'a :: division-ring fls) =
      − fls-const (inverse (of-nat (n−1))) ∗ fls-X-inv ^ (n−1)
proof (cases n = 1)
  case False
  with assms have n: n ≥ 2 by simp
  hence
    fls-integral (fls-X-inv ^ n :: 'a fls) =
      fls-const (inverse (− of-nat (nat (int n − 1)))) ∗ fls-X-inv ^ (n−1)
    by (simp add: fls-integral-X-inv-power)
  moreover from n have nat (int n − 1) = n − 1 by simp
  ultimately show ?thesis
    using inverse-minus-eq[of of-nat (n−1) :: 'a] by simp
qed simp

lemma fls-integral-X-inv-power-char0':
  assumes n ≥ 1
  shows
    fls-integral (fls-X-inv ^ n :: 'a :: {division-ring,ring-char-0} fls) =
      − inverse (of-nat (n−1)) ∗ fls-X-inv ^ (n−1)
proof (cases n=1)
  case False with assms show ?thesis
    by (simp add: fls-integral-X-inv-power' fls-inverse-const fls-of-nat)
qed simp

lemma fls-integral-delta:
  assumes m ≠ −1
  shows
    fls-integral (Abs-fls (λn. if n=m then c else 0)) =
      Abs-fls (λn. if n=m+1 then inverse (of-int (m+1)) ∗ c else 0)
  using   assms
  by      (intro fls-eqI) auto

lemma fls-regpart-integral:
  fls-regpart (fls-integral f) = fps-integral0 (fls-regpart f)
proof (rule fps-ext)
  fix n
  show fls-regpart (fls-integral f) $ n = fps-integral0 (fls-regpart f) $ n
    by (cases n) (simp-all add: fps-integral-def)
qed

lemma fls-integral-fps-to-fls:
  fls-integral (fps-to-fls f) = fps-to-fls (fps-integral0 f)
```

proof (*intro fls-eqI*)
fix *n* :: *int*
show *fls-integral* (*fps-to-fls f*) $$ *n* = *fps-to-fls* (*fps-integral0 f*) $$ *n*
proof (*cases n<1*)
case *True* **thus** *?thesis* **by** *simp*
next
case *False*
hence *nat* (*n−1*) = *nat n* − *1* **by** *simp*
with *False* **show** *?thesis* **by** (*cases nat n*) *auto*
qed
qed

7.7.6 Algebraic rules of the integral

lemma *fls-integral-add* [*simp*]: *fls-integral* (*f*+*g*) = *fls-integral f* + *fls-integral g*
by (*intro fls-eqI*) (*simp add*: *algebra-simps*)

lemma *fls-integral-sub* [*simp*]: *fls-integral* (*f*−*g*) = *fls-integral f* − *fls-integral g*
by (*intro fls-eqI*) (*simp add*: *algebra-simps*)

lemma *fls-integral-neg* [*simp*]: *fls-integral* (−*f*) = − *fls-integral f*
using *fls-integral-sub*[*of 0 f*] **by** *simp*

lemma *fls-integral-mult-const-left*:
fls-integral (*fls-const c* ∗ *f*) = *fls-const c* ∗ *fls-integral* (*f* :: *'a*::*division-ring fls*)
by (*intro fls-eqI*) (*simp add*: *mult.assoc mult-inverse-of-int-commute*)

lemma *fls-integral-mult-const-left-comm*:
fixes *f* :: *'a*::{*comm-ring-1*,*inverse*} *fls*
shows *fls-integral* (*fls-const c* ∗ *f*) = *fls-const c* ∗ *fls-integral f*
by (*intro fls-eqI*) (*simp add*: *mult.assoc mult.commute*)

lemma *fls-integral-linear*:
fixes *f g* :: *'a*::*division-ring fls*
shows
fls-integral (*fls-const a* ∗ *f* + *fls-const b* ∗ *g*) =
fls-const a ∗ *fls-integral f* + *fls-const b* ∗ *fls-integral g*
by (*simp add*: *fls-integral-mult-const-left*)

lemma *fls-integral-linear-comm*:
fixes *f g* :: *'a*::{*comm-ring-1*,*inverse*} *fls*
shows
fls-integral (*fls-const a* ∗ *f* + *fls-const b* ∗ *g*) =
fls-const a ∗ *fls-integral f* + *fls-const b* ∗ *fls-integral g*
by (*simp add*: *fls-integral-mult-const-left-comm*)

lemma *fls-integral-mult-const-right*:
fls-integral (*f* ∗ *fls-const c*) = *fls-integral f* ∗ *fls-const c*
by (*intro fls-eqI*) (*simp add*: *mult.assoc*)

lemma *fls-integral-linear2*:
  fls-integral (*f* ∗ *fls-const a* + *g* ∗ *fls-const b*) =
    fls-integral f ∗ *fls-const a* + *fls-integral g* ∗ *fls-const b*
 by (*simp add*: *fls-integral-mult-const-right*)

lemma *fls-integral-sum*:
 fls-integral (*sum f S*) = *sum* (λ*i*. *fls-integral* (*f i*)) *S*
proof (*cases finite S*)
 case *True* **show** *?thesis*
  by (*induct rule*: *finite-induct* [*OF True*]) *simp-all*
qed *simp*

7.7.7 Derivatives of integrals and vice versa

lemma *fls-integral-fls-deriv*:
 fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
 shows *fls-integral* (*fls-deriv a*) + *fls-const* (*a*$$*0*) = *a*
 by (*intro fls-eqI*) (*simp add*: *mult.assoc*[*symmetric*])

lemma *fls-deriv-fls-integral*:
 fixes *a* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
 assumes *fls-residue a* = *0*
 shows *fls-deriv* (*fls-integral a*) = *a*
proof (*intro fls-eqI*)
 fix *n* :: *int*
 show *fls-deriv* (*fls-integral a*) $$ *n* = *a* $$ *n*
 proof (*cases n*=−*1*)
  case *True* **with** *assms* **show** *?thesis* **by** *simp*
 next
  case *False*
  hence (*of-int* (*n*+*1*) :: *'a*) ≠ *0* **using** *of-int-eq-0-iff*[*of n*+*1*] **by** *simp*
  hence (*of-int* (*n*+*1*) :: *'a*) ∗ *inverse* (*of-int* (*n*+*1*) :: *'a*) = (*1*::*'a*)
   using *of-int-eq-0-iff*[*of n*+*1*] **by** *simp*
  moreover have
   fls-deriv (*fls-integral a*) $$ *n* =
    (*if n*=−*1 then 0 else of-int* (*n*+*1*) ∗ *inverse* (*of-int* (*n*+*1*)) ∗ *a*$$*n*)
   by (*simp add*: *mult.assoc*)
  ultimately show *?thesis*
   by (*simp add*: *False*)
 qed
qed

Series with zero residue are precisely the derivatives.

lemma *fls-residue-nonzero-ex-antiderivative*:
 fixes *f* :: *'a*::{*division-ring*,*ring-char-0*} *fls*
 assumes *fls-residue f* = *0*
 shows ∃*F*. *fls-deriv F* = *f*
 using *assms fls-deriv-fls-integral*

by *auto*

lemma *fls-ex-antiderivative-residue-nonzero*:
assumes $\exists F.$ *fls-deriv* $F = f$
shows *fls-residue* $f = 0$
using *assms fls-residue-deriv*
by *auto*

lemma *fls-residue-nonzero-ex-anitderivative-iff*:
fixes f :: *'a*::{*division-ring*,*ring-char-0*} *fls*
shows *fls-residue* $f = 0 \longleftrightarrow (\exists F.$ *fls-deriv* $F = f)$
using *fls-residue-nonzero-ex-antiderivative fls-ex-antiderivative-residue-nonzero*
by *fast*

7.8 Topology

instantiation *fls* :: (*group-add*) *metric-space*
begin

definition
dist-fls-def:
dist (a :: *'a fls*) b =
(*if* $a = b$
then 0
else if fls-subdegree $(a-b) \geq 0$
then inverse (2 ^ *nat* (*fls-subdegree* $(a-b)$))
else 2 ^ *nat* ($-$*fls-subdegree* $(a-b)$)
)

lemma *dist-fls-ge0*: *dist* (a :: *'a fls*) $b \geq 0$
by (*simp add*: *dist-fls-def*)

definition *uniformity-fls-def* [*code del*]:
(*uniformity* :: (*'a fls* × *'a fls*) *filter*) = (*INF* $e \in \{0 <..\}.$ *principal* $\{(x, y).$ *dist* $x\ y < e\}$)

definition *open-fls-def'* [*code del*]:
open (U :: *'a fls set*) $\longleftrightarrow$ ($\forall x \in U.$ *eventually* ($\lambda(x', y).$ $x' = x \longrightarrow y \in U$) *uniformity*)

lemma *dist-fls-sym*: *dist* (a :: *'a fls*) b = *dist* b a
by (*cases* $a \neq b$, *cases fls-subdegree* $(a-b) \geq 0$)
(*simp-all add*: *fls-subdegree-minus-sym dist-fls-def*)

context
begin

private lemma *instance-helper*:
fixes a b c :: *'a fls*

```
  assumes neq: a≠b a≠c
  and     dist-ineq: dist a b > dist a c
  shows   fls-subdegree (a − b) < fls-subdegree (a − c)
proof (
  cases fls-subdegree (a−b) ≥ 0 fls-subdegree (a−c) ≥ 0
  rule: case-split[case-product case-split]
)
  case True-True with neq dist-ineq show ?thesis by (simp add: dist-fls-def)
next
  case False-True with dist-ineq show ?thesis by (simp add: dist-fls-def)
next
  case False-False with neq dist-ineq show ?thesis by (simp add: dist-fls-def)
next
  case True-False
  with neq
    have (1::real) > 2 ^(nat (fls-subdegree (a−b)) + nat (−fls-subdegree (a−c)))
    and  nat (fls-subdegree (a−b)) + nat (−fls-subdegree (a−c)) =
           nat (fls-subdegree (a−b) − fls-subdegree (a−c))
    using dist-ineq
    by   (simp-all add: dist-fls-def field-simps power-add)
  hence ¬ (1::real) < 2 ^(nat (fls-subdegree (a−b) − fls-subdegree (a−c))) by
simp
  hence ¬ (0 < nat (fls-subdegree (a − b) − fls-subdegree (a − c))) by auto
  hence fls-subdegree (a − b) ≤ fls-subdegree (a − c) by simp
  with True-False show ?thesis by simp
qed

instance
proof
  show th: dist a b = 0 ⟷ a = b for a b :: 'a fls
    by (simp add: dist-fls-def split: if-split-asm)
  then have th'[simp]: dist a a = 0 for a :: 'a fls by simp

  fix a b c :: 'a fls
  consider a = b | c = a ∨ c = b | a ≠ b a ≠ c b ≠ c by blast
  then show dist a b ≤ dist a c + dist b c
  proof cases
    case 1
    then show ?thesis by (simp add: dist-fls-def)
  next
    case 2
    then show ?thesis
      by (cases c = a) (simp-all add: th dist-fls-sym)
  next
    case neq: 3
    have False if dist a b > dist a c + dist b c
    proof −
      from neq have dist a b > 0 dist b c > 0 dist a c > 0 by (simp-all add:
dist-fls-def)
```

with *that* **have** *dist-ineq*: *dist a b* > *dist a c dist a b* > *dist b c* **by** *simp-all*
have *fls-subdegree* (*a* − *b*) < *fls-subdegree* (*a* − *c*)
and *fls-subdegree* (*a* − *b*) < *fls-subdegree* (*b* − *c*)
using *instance-helper*[*of a b c*] *instance-helper*[*of b a c*] *neq dist-ineq*
by (*simp-all add*: *dist-fls-sym fls-subdegree-minus-sym*)
hence (*a* − *c*) $$ *fls-subdegree* (*a* − *b*) = *0* **and** (*b* − *c*) $$ *fls-subdegree* (*a* − *b*) = *0*
by (*simp-all only*: *fls-eq0-below-subdegree*)
hence (*a* − *b*) $$ *fls-subdegree* (*a* − *b*) = *0* **by** *simp*
moreover from *neq* **have** (*a* − *b*) $$ *fls-subdegree* (*a* − *b*) ≠ *0*
by (*intro nth-fls-subdegree-nonzero*) *simp*
ultimately show *False* **by** *contradiction*
qed
thus *?thesis* **by** (*auto simp*: *not-le*[*symmetric*])
qed
qed (*rule open-fls-def′ uniformity-fls-def*)+

end
end

declare *uniformity-Abort*[**where** *′a*=*′a* :: *group-add fls*, *code*]

lemma *open-fls-def*:
open (*S* :: *′a*::*group-add fls set*) = (∀ *a* ∈ *S*. ∃ *r*. *r* >*0* ∧ {*y*. *dist y a* < *r*} ⊆ *S*)
unfolding *open-dist subset-eq* **by** *simp*

7.9 Notation

bundle *fps-syntax*
begin
notation *fls-nth* (**infixl** ‹$$› *75*)
end

unbundle *no fps-syntax*

end

8 The fraction field of any integral domain

theory *Fraction-Field*
imports *Main*
begin

8.1 General fractions construction

8.1.1 Construction of the type of fractions

context *idom* **begin**

definition *fractrel* :: *'a* × *'a* ⇒ *'a* ∗ *'a* ⇒ *bool* **where**
fractrel = (λ*x y*. *snd x* ≠ *0* ∧ *snd y* ≠ *0* ∧ *fst x* ∗ *snd y* = *fst y* ∗ *snd x*)

lemma *fractrel-iff* [*simp*]:
fractrel x y ⟷ *snd x* ≠ *0* ∧ *snd y* ≠ *0* ∧ *fst x* ∗ *snd y* = *fst y* ∗ *snd x*
by (*simp add*: *fractrel-def*)

lemma *symp-fractrel*: *symp fractrel*
by (*simp add*: *symp-def*)

lemma *transp-fractrel*: *transp fractrel*
proof (*rule transpI*, *unfold split-paired-all*)
fix *a b a' b' a'' b''* :: *'a*
assume *A*: *fractrel* (*a*, *b*) (*a'*, *b'*)
assume *B*: *fractrel* (*a'*, *b'*) (*a''*, *b''*)
have *b'* ∗ (*a* ∗ *b''*) = *b''* ∗ (*a* ∗ *b'*) **by** (*simp add*: *ac-simps*)
also from *A* **have** *a* ∗ *b'* = *a'* ∗ *b* **by** *auto*
also have *b''* ∗ (*a'* ∗ *b*) = *b* ∗ (*a'* ∗ *b''*) **by** (*simp add*: *ac-simps*)
also from *B* **have** *a'* ∗ *b''* = *a''* ∗ *b'* **by** *auto*
also have *b* ∗ (*a''* ∗ *b'*) = *b'* ∗ (*a''* ∗ *b*) **by** (*simp add*: *ac-simps*)
finally have *b'* ∗ (*a* ∗ *b''*) = *b'* ∗ (*a''* ∗ *b*) **.**
moreover from *B* **have** *b'* ≠ *0* **by** *auto*
ultimately have *a* ∗ *b''* = *a''* ∗ *b* **by** *simp*
with *A B* **show** *fractrel* (*a*, *b*) (*a''*, *b''*) **by** *auto*
qed

lemma *part-equivp-fractrel*: *part-equivp fractrel*
using - *symp-fractrel transp-fractrel*
by(*rule part-equivpI*)(*rule exI*[**where** *x*=(*0*, *1*)]; *simp*)

end

quotient-type (**overloaded**) *'a fract* = *'a* :: *idom* × *'a* / *partial*: *fractrel*
by(*rule part-equivp-fractrel*)

8.1.2 Representation and basic operations

lift-definition *Fract* :: *'a* :: *idom* ⇒ *'a* ⇒ *'a fract*
is λ*a b*. *if b* = *0 then* (*0*, *1*) *else* (*a*, *b*)
by *simp*

lemma *Fract-cases* [*cases type*: *fract*]:
obtains (*Fract*) *a b* **where** *q* = *Fract a b b* ≠ *0*
by *transfer simp*

lemma *Fract-induct* [*case-names Fract*, *induct type*: *fract*]:
(⋀*a b*. *b* ≠ *0* ⟹ *P* (*Fract a b*)) ⟹ *P q*
by (*cases q*) *simp*

lemma *eq-fract*:
shows $\bigwedge a\ b\ c\ d.\ b \neq 0 \Longrightarrow d \neq 0 \Longrightarrow Fract\ a\ b = Fract\ c\ d \longleftrightarrow a * d = c * b$
and $\bigwedge a.\ Fract\ a\ 0 = Fract\ 0\ 1$
and $\bigwedge a\ c.\ Fract\ 0\ a = Fract\ 0\ c$
by(*transfer*; *simp*)+

instantiation *fract* :: (*idom*) *comm-ring-1*
begin

lift-definition *zero-fract* :: 'a *fract* **is** (*0*, *1*) **by** *simp*

lemma *Zero-fract-def*: *0* = *Fract 0 1*
by *transfer simp*

lift-definition *one-fract* :: 'a *fract* **is** (*1*, *1*) **by** *simp*

lemma *One-fract-def*: *1* = *Fract 1 1*
by *transfer simp*

lift-definition *plus-fract* :: 'a *fract* ⇒ 'a *fract* ⇒ 'a *fract*
is $\lambda q\ r.$ (*fst q* * *snd r* + *fst r* * *snd q*, *snd q* * *snd r*)
by(*auto simp add*: *algebra-simps*)

lemma *add-fract* [*simp*]:
⟦ *b* ≠ *0*; *d* ≠ *0* ⟧ ⟹ *Fract a b* + *Fract c d* = *Fract* (*a* * *d* + *c* * *b*) (*b* * *d*)
by *transfer simp*

lift-definition *uminus-fract* :: 'a *fract* ⇒ 'a *fract*
is $\lambda x.$ (− *fst x*, *snd x*)
by *simp*

lemma *minus-fract* [*simp*]:
fixes *a b* :: 'a::*idom*
shows − *Fract a b* = *Fract* (− *a*) *b*
by *transfer simp*

lemma *minus-fract-cancel* [*simp*]: *Fract* (− *a*) (− *b*) = *Fract a b*
by (*cases b* = *0*) (*simp-all add*: *eq-fract*)

definition *diff-fract-def*: *q* − *r* = *q* + − (*r*::'a *fract*)

lemma *diff-fract* [*simp*]:
⟦ *b* ≠ *0*; *d* ≠ *0* ⟧ ⟹ *Fract a b* − *Fract c d* = *Fract* (*a* * *d* − *c* * *b*) (*b* * *d*)
by (*simp add*: *diff-fract-def*)

lift-definition *times-fract* :: 'a *fract* ⇒ 'a *fract* ⇒ 'a *fract*
is $\lambda q\ r.$ (*fst q* * *fst r*, *snd q* * *snd r*)
by(*simp add*: *algebra-simps*)

lemma *mult-fract* [*simp*]: *Fract* (*a*::'*a*::*idom*) *b* ∗ *Fract c d* = *Fract* (*a* ∗ *c*) (*b* ∗ *d*)
by *transfer simp*

lemma *mult-fract-cancel*:
c ≠ *0* ⟹ *Fract* (*c* ∗ *a*) (*c* ∗ *b*) = *Fract a b*
by *transfer simp*

instance
proof
fix *q r s* :: '*a fract*
show (*q* ∗ *r*) ∗ *s* = *q* ∗ (*r* ∗ *s*)
by (*cases q*, *cases r*, *cases s*) (*simp add*: *eq-fract algebra-simps*)
show *q* ∗ *r* = *r* ∗ *q*
by (*cases q*, *cases r*) (*simp add*: *eq-fract algebra-simps*)
show *1* ∗ *q* = *q*
by (*cases q*) (*simp add*: *One-fract-def eq-fract*)
show (*q* + *r*) + *s* = *q* + (*r* + *s*)
by (*cases q*, *cases r*, *cases s*) (*simp add*: *eq-fract algebra-simps*)
show *q* + *r* = *r* + *q*
by (*cases q*, *cases r*) (*simp add*: *eq-fract algebra-simps*)
show *0* + *q* = *q*
by (*cases q*) (*simp add*: *Zero-fract-def eq-fract*)
show − *q* + *q* = *0*
by (*cases q*) (*simp add*: *Zero-fract-def eq-fract*)
show *q* − *r* = *q* + − *r*
by (*cases q*, *cases r*) (*simp add*: *eq-fract*)
show (*q* + *r*) ∗ *s* = *q* ∗ *s* + *r* ∗ *s*
by (*cases q*, *cases r*, *cases s*) (*simp add*: *eq-fract algebra-simps*)
show (*0*::'*a fract*) ≠ *1*
by (*simp add*: *Zero-fract-def One-fract-def eq-fract*)
qed

end

lemma *of-nat-fract*: *of-nat k* = *Fract* (*of-nat k*) *1*
by (*induct k*) (*simp-all add*: *Zero-fract-def One-fract-def*)

lemma *Fract-of-nat-eq*: *Fract* (*of-nat k*) *1* = *of-nat k*
by (*rule of-nat-fract* [*symmetric*])

lemma *fract-collapse*:
Fract 0 k = *0*
Fract 1 1 = *1*
Fract k 0 = *0*
by(*transfer*; *simp*)+

lemma *fract-expand*:
0 = *Fract 0 1*
1 = *Fract 1 1*

by (*simp-all add*: *fract-collapse*)

lemma *Fract-cases-nonzero*:
obtains (*Fract*) *a b* **where** $q = Fract\ a\ b$ **and** $b \neq 0$ **and** $a \neq 0$
| (*0*) $q = 0$
proof (*cases* $q = 0$)
case *True*
then show *thesis* **using** *0* **by** *auto*
next
case *False*
then obtain *a b* **where** $q = Fract\ a\ b$ **and** $b \neq 0$ **by** (*cases q*) *auto*
with *False* **have** $0 \neq Fract\ a\ b$ **by** *simp*
with ‹$b \neq 0$› **have** $a \neq 0$ **by** (*simp add*: *Zero-fract-def eq-fract*)
with *Fract* ‹$q = Fract\ a\ b$› ‹$b \neq 0$› **show** *thesis* **by** *auto*
qed

8.1.3 The field of rational numbers

context *idom*
begin

subclass *ring-no-zero-divisors* **..**

end

instantiation *fract* :: (*idom*) *field*
begin

lift-definition *inverse-fract* :: $'a\ fract \Rightarrow 'a\ fract$
is $\lambda x.$ *if fst* $x = 0$ *then* $(0, 1)$ *else* (*snd x*, *fst x*)
by(*auto simp add*: *algebra-simps*)

lemma *inverse-fract* [*simp*]: *inverse* (*Fract a b*) = *Fract* (b::$'a$::*idom*) *a*
by *transfer simp*

definition *divide-fract-def*: *q div r* = *q* ∗ *inverse* (*r*:: $'a$ *fract*)

lemma *divide-fract* [*simp*]: *Fract a b div Fract c d* = *Fract* (*a* ∗ *d*) (*b* ∗ *c*)
by (*simp add*: *divide-fract-def*)

instance
proof
fix *q* :: $'a$ *fract*
assume $q \neq 0$
then show *inverse q* ∗ *q* = *1*
by (*cases q rule*: *Fract-cases-nonzero*)
(*simp-all add*: *fract-expand eq-fract mult.commute*)
next
fix *q r* :: $'a$ *fract*

show *q div r = q * inverse r* **by** (*simp add*: *divide-fract-def*)
next
show *inverse 0 = (0*:: *'a fract*)
by (*simp add*: *fract-expand*) (*simp add*: *fract-collapse*)
qed

end

8.1.4 The ordered field of fractions over an ordered idom

instantiation *fract* :: (*linordered-idom*) *linorder*
begin

lemma *less-eq-fract-respect*:
fixes *a b a' b' c d c' d'* :: *'a*
assumes *neq*: *b ≠ 0 b' ≠ 0 d ≠ 0 d' ≠ 0*
assumes *eq1*: *a * b' = a' * b*
assumes *eq2*: *c * d' = c' * d*
shows *((a * d) * (b * d) ≤ (c * b) * (b * d)) ⟷ ((a' * d') * (b' * d') ≤ (c' * b') * (b' * d'))*
proof −
let *?le = λa b c d. ((a * d) * (b * d) ≤ (c * b) * (b * d))*
{
fix *a b c d x* :: *'a*
assume *x*: *x ≠ 0*
have *?le a b c d = ?le (a * x) (b * x) c d*
proof −
from *x* **have** *0 < x * x*
by (*auto simp add*: *zero-less-mult-iff*)
then have *?le a b c d =*
*((a * d) * (b * d) * (x * x) ≤ (c * b) * (b * d) * (x * x))*
by (*simp add*: *mult-le-cancel-right*)
also have *... = ?le (a * x) (b * x) c d*
by (*simp add*: *ac-simps*)
finally show *?thesis* **.**
qed
} note *le-factor = this*

let *?D = b * d* **and** *?D' = b' * d'*
from *neq* **have** *D*: *?D ≠ 0* **by** *simp*
from *neq* **have** *?D' ≠ 0* **by** *simp*
then have *?le a b c d = ?le (a * ?D') (b * ?D') c d*
by (*rule le-factor*)
also have *... = ((a * b') * ?D * ?D' * d * d' ≤ (c * d') * ?D * ?D' * b * b')*
by (*simp add*: *ac-simps*)
also have *... = ((a' * b) * ?D * ?D' * d * d' ≤ (c' * d) * ?D * ?D' * b * b')*
by (*simp only*: *eq1 eq2*)
also have *... = ?le (a' * ?D) (b' * ?D) c' d'*
by (*simp add*: *ac-simps*)

```
  also from D have ... = ?le a' b' c' d'
    by (rule le-factor [symmetric])
  finally show ?le a b c d = ?le a' b' c' d' .
qed

lift-definition less-eq-fract :: 'a fract ⇒ 'a fract ⇒ bool
  is λq r. (fst q * snd r) * (snd q * snd r) ≤ (fst r * snd q) * (snd q * snd r)
by (clarsimp simp add: less-eq-fract-respect)

definition less-fract-def: z < (w::'a fract) ⟷ z ≤ w ∧ ¬ w ≤ z

lemma le-fract [simp]:
  ⟦ b ≠ 0; d ≠ 0 ⟧ ⟹ Fract a b ≤ Fract c d ⟷ (a * d) * (b * d) ≤ (c * b) * (b
* d)
  by transfer simp

lemma less-fract [simp]:
  ⟦ b ≠ 0; d ≠ 0 ⟧ ⟹ Fract a b < Fract c d ⟷ (a * d) * (b * d) < (c * b) * (b
* d)
  by (simp add: less-fract-def less-le-not-le ac-simps)

instance
proof
  fix q r s :: 'a fract
  assume q ≤ r and r ≤ s
  then show q ≤ s
  proof (induct q, induct r, induct s)
    fix a b c d e f :: 'a
    assume neq: b ≠ 0 d ≠ 0 f ≠ 0
    assume 1: Fract a b ≤ Fract c d
    assume 2: Fract c d ≤ Fract e f
    show Fract a b ≤ Fract e f
    proof −
      from neq obtain bb: 0 < b * b and dd: 0 < d * d and ff: 0 < f * f
        by (auto simp add: zero-less-mult-iff linorder-neq-iff)
      have (a * d) * (b * d) * (f * f) ≤ (c * b) * (b * d) * (f * f)
      proof −
        from neq 1 have (a * d) * (b * d) ≤ (c * b) * (b * d)
          by simp
        with ff show ?thesis by (simp add: mult-le-cancel-right)
      qed
      also have ... = (c * f) * (d * f) * (b * b)
        by (simp only: ac-simps)
      also have ... ≤ (e * d) * (d * f) * (b * b)
      proof −
        from neq 2 have (c * f) * (d * f) ≤ (e * d) * (d * f)
          by simp
        with bb show ?thesis by (simp add: mult-le-cancel-right)
      qed
```

```
    finally have (a * f) * (b * f) * (d * d) ≤ e * b * (b * f) * (d * d)
      by (simp only: ac-simps)
    with dd have (a * f) * (b * f) ≤ (e * b) * (b * f)
      by (simp add: mult-le-cancel-right)
    with neq show ?thesis by simp
  qed
 qed
next
 fix q r :: 'a fract
 assume q ≤ r and r ≤ q
 then show q = r
 proof (induct q, induct r)
  fix a b c d :: 'a
  assume neq: b ≠ 0 d ≠ 0
  assume 1: Fract a b ≤ Fract c d
  assume 2: Fract c d ≤ Fract a b
  show Fract a b = Fract c d
  proof −
   from neq 1 have (a * d) * (b * d) ≤ (c * b) * (b * d)
    by simp
   also have ... ≤ (a * d) * (b * d)
   proof −
    from neq 2 have (c * b) * (d * b) ≤ (a * d) * (d * b)
     by simp
    then show ?thesis by (simp only: ac-simps)
   qed
   finally have (a * d) * (b * d) = (c * b) * (b * d) .
   moreover from neq have b * d ≠ 0 by simp
   ultimately have a * d = c * b by simp
   with neq show ?thesis by (simp add: eq-fract)
  qed
 qed
next
 fix q r :: 'a fract
 show q ≤ q
  by (induct q) simp
 show (q < r) = (q ≤ r ∧ ¬ r ≤ q)
  by (simp only: less-fract-def)
 show q ≤ r ∨ r ≤ q
  by (induct q, induct r)
    (simp add: mult.commute, rule linorder-linear)
qed

end

instantiation fract :: (linordered-idom) linordered-field
begin

definition abs-fract-def2:
```

```
  |q| = (if q < 0 then −q else (q::'a fract))

definition sgn-fract-def:
  sgn (q::'a fract) = (if q = 0 then 0 else if 0 < q then 1 else − 1)

theorem abs-fract [simp]: |Fract a b| = Fract |a| |b|
  unfolding abs-fract-def2 not-le [symmetric]
  by transfer (auto simp add: zero-less-mult-iff le-less)

instance proof
  fix q r s :: 'a fract
  assume q ≤ r
  then show s + q ≤ s + r
  proof (induct q, induct r, induct s)
    fix a b c d e f :: 'a
    assume neq: b ≠ 0 d ≠ 0 f ≠ 0
    assume le: Fract a b ≤ Fract c d
    show Fract e f + Fract a b ≤ Fract e f + Fract c d
    proof −
      let ?F = f * f from neq have F: 0 < ?F
        by (auto simp add: zero-less-mult-iff)
      from neq le have (a * d) * (b * d) ≤ (c * b) * (b * d)
        by simp
      with F have (a * d) * (b * d) * ?F * ?F ≤ (c * b) * (b * d) * ?F * ?F
        by (simp add: mult-le-cancel-right)
      with neq show ?thesis by (simp add: field-simps)
    qed
  qed
next
  fix q r s :: 'a fract
  assume q < r and 0 < s
  then show s * q < s * r
  proof (induct q, induct r, induct s)
    fix a b c d e f :: 'a
    assume neq: b ≠ 0 d ≠ 0 f ≠ 0
    assume le: Fract a b < Fract c d
    assume gt: 0 < Fract e f
    show Fract e f * Fract a b < Fract e f * Fract c d
    proof −
      let ?E = e * f and ?F = f * f
      from neq gt have 0 < ?E
        by (auto simp add: Zero-fract-def order-less-le eq-fract)
      moreover from neq have 0 < ?F
        by (auto simp add: zero-less-mult-iff)
      moreover from neq le have (a * d) * (b * d) < (c * b) * (b * d)
        by simp
      ultimately have (a * d) * (b * d) * ?E * ?F < (c * b) * (b * d) * ?E * ?F
        by (simp add: mult-less-cancel-right)
      with neq show ?thesis
```

```
      by (simp add: ac-simps)
    qed
  qed
qed (fact sgn-fract-def abs-fract-def2)+

end

instantiation fract :: (linordered-idom) distrib-lattice
begin

definition inf-fract-def:
  (inf :: 'a fract ⇒ 'a fract ⇒ 'a fract) = min

definition sup-fract-def:
  (sup :: 'a fract ⇒ 'a fract ⇒ 'a fract) = max

instance
  by standard (simp-all add: inf-fract-def sup-fract-def max-min-distrib2)

end

lemma fract-induct-pos [case-names Fract]:
  fixes P :: 'a::linordered-idom fract ⇒ bool
  assumes step: ⋀a b. 0 < b ⟹ P (Fract a b)
  shows P q
proof (cases q)
  case (Fract a b)
  {
    fix a b :: 'a
    assume b: b < 0
    have P (Fract a b)
    proof -
      from b have 0 < - b by simp
      then have P (Fract (- a) (- b))
        by (rule step)
      then show P (Fract a b)
        by (simp add: order-less-imp-not-eq [OF b])
    qed
  }
  with Fract show P q
    by (auto simp add: linorder-neq-iff step)
qed

lemma zero-less-Fract-iff: 0 < b ⟹ 0 < Fract a b ⟷ 0 < a
  by (auto simp add: Zero-fract-def zero-less-mult-iff)

lemma Fract-less-zero-iff: 0 < b ⟹ Fract a b < 0 ⟷ a < 0
  by (auto simp add: Zero-fract-def mult-less-0-iff)
```

lemma *zero-le-Fract-iff*: $0 < b \Longrightarrow 0 \leq Fract\ a\ b \longleftrightarrow 0 \leq a$
by (*auto simp add*: *Zero-fract-def zero-le-mult-iff*)

lemma *Fract-le-zero-iff*: $0 < b \Longrightarrow Fract\ a\ b \leq 0 \longleftrightarrow a \leq 0$
by (*auto simp add*: *Zero-fract-def mult-le-0-iff*)

lemma *one-less-Fract-iff*: $0 < b \Longrightarrow 1 < Fract\ a\ b \longleftrightarrow b < a$
by (*auto simp add*: *One-fract-def mult-less-cancel-right-disj*)

lemma *Fract-less-one-iff*: $0 < b \Longrightarrow Fract\ a\ b < 1 \longleftrightarrow a < b$
by (*auto simp add*: *One-fract-def mult-less-cancel-right-disj*)

lemma *one-le-Fract-iff*: $0 < b \Longrightarrow 1 \leq Fract\ a\ b \longleftrightarrow b \leq a$
by (*auto simp add*: *One-fract-def mult-le-cancel-right*)

lemma *Fract-le-one-iff*: $0 < b \Longrightarrow Fract\ a\ b \leq 1 \longleftrightarrow a \leq b$
by (*auto simp add*: *One-fract-def mult-le-cancel-right*)

end

9 Fundamental Theorem of Algebra

theory *Fundamental-Theorem-Algebra*
imports *Polynomial Complex-Main*
begin

9.1 More lemmas about module of complex numbers

The triangle inequality for cmod

lemma *complex-mod-triangle-sub*: $cmod\ w \leq cmod\ (w + z) + norm\ z$
by (*metis add-diff-cancel norm-triangle-ineq4*)

9.2 Basic lemmas about polynomials

lemma *poly-bound-exists*:
fixes $p :: 'a::\{comm\text{-}semiring\text{-}0,real\text{-}normed\text{-}div\text{-}algebra\}\ poly$
shows $\exists m.\ m > 0 \wedge (\forall z.\ norm\ z \leq r \longrightarrow norm\ (poly\ p\ z) \leq m)$
proof (*induct p*)
case *0*
then show *?case* **by** (*rule exI*[**where** *x=1*]) *simp*
next
case (*pCons c cs*)
from *pCons.hyps* **obtain** *m* **where** *m*: $\forall z.\ norm\ z \leq r \longrightarrow norm\ (poly\ cs\ z) \leq m$
by *blast*
let *?k* = $1 + norm\ c + |r * m|$
have *kp*: *?k* > *0*
using *abs-ge-zero*[*of r*∗*m*] *norm-ge-zero*[*of c*] **by** *arith*

```
  have norm (poly (pCons c cs) z) ≤ ?k if H: norm z ≤ r for z
  proof −
    from m H have th: norm (poly cs z) ≤ m
      by blast
    from H have rp: r ≥ 0
      using norm-ge-zero[of z] by arith
    have norm (poly (pCons c cs) z) ≤ norm c + norm (z ∗ poly cs z)
      using norm-triangle-ineq[of c z∗ poly cs z] by simp
    also have ... ≤ ?k
      using mult-mono[OF H th rp norm-ge-zero[of poly cs z]]
      by (simp add: norm-mult)
    finally show ?thesis .
  qed
  with kp show ?case by blast
qed
```

Offsetting the variable in a polynomial gives another of same degree

```
definition offset-poly :: 'a::comm-semiring-0 poly ⇒ 'a ⇒ 'a poly
  where offset-poly p h = fold-coeffs (λa q. smult h q + pCons a q) p 0

lemma offset-poly-0: offset-poly 0 h = 0
  by (simp add: offset-poly-def)

lemma offset-poly-pCons:
  offset-poly (pCons a p) h =
    smult h (offset-poly p h) + pCons a (offset-poly p h)
  by (cases p = 0 ∧ a = 0) (auto simp add: offset-poly-def)

lemma offset-poly-single [simp]: offset-poly [:a:] h = [:a:]
  by (simp add: offset-poly-pCons offset-poly-0)

lemma poly-offset-poly: poly (offset-poly p h) x = poly p (h + x)
  by (induct p) (auto simp add: offset-poly-0 offset-poly-pCons algebra-simps)

lemma offset-poly-eq-0-lemma: smult c p + pCons a p = 0 ⟹ p = 0
  by (induct p arbitrary: a) (simp, force)

lemma offset-poly-eq-0-iff [simp]: offset-poly p h = 0 ⟷ p = 0
proof
  show offset-poly p h = 0 ⟹ p = 0
  proof(induction p)
    case 0
    then show ?case by blast
  next
    case (pCons a p)
    then show ?case
      by (metis offset-poly-eq-0-lemma offset-poly-pCons offset-poly-single)
  qed
qed (simp add: offset-poly-0)
```

lemma *degree-offset-poly* [*simp*]: *degree* (*offset-poly p h*) = *degree p*
proof(*induction p*)
case *0*
then show *?case*
by (*simp add*: *offset-poly-0*)
next
case (*pCons a p*)
have $p \neq 0 \Longrightarrow$ *degree* (*offset-poly* (*pCons a p*) *h*) = *Suc* (*degree p*)
by (*metis degree-add-eq-right degree-pCons-eq degree-smult-le le-imp-less-Suc offset-poly-eq-0-iff offset-poly-pCons pCons.IH*)
then show *?case*
by *simp*
qed

definition *psize p* = (*if p* = *0 then 0 else Suc* (*degree p*))

lemma *psize-eq-0-iff* [*simp*]: *psize p* = *0* $\longleftrightarrow$ *p* = *0*
unfolding *psize-def* **by** *simp*

lemma *poly-offset*:
fixes *p* :: *'a*::*comm-ring-1 poly*
shows $\exists\, q.$ *psize q* = *psize p* $\wedge$ ($\forall\, x.$ *poly q x* = *poly p* (*a* + *x*))
by (*metis degree-offset-poly offset-poly-eq-0-iff poly-offset-poly psize-def*)

An alternative useful formulation of completeness of the reals

lemma *real-sup-exists*:
assumes *ex*: $\exists\, x.$ *P x*
and *bz*: $\exists\, z.\ \forall\, x.$ *P x* $\longrightarrow$ *x* < *z*
shows $\exists\, s$::*real*. $\forall\, y.$ ($\exists\, x.$ *P x* $\wedge$ *y* < *x*) $\longleftrightarrow$ *y* < *s*
proof
from *bz* **have** *bdd-above* (*Collect P*)
by (*force intro*: *less-imp-le*)
then show $\forall\, y.$ ($\exists\, x.$ *P x* $\wedge$ *y* < *x*) $\longleftrightarrow$ *y* < *Sup* (*Collect P*)
using *ex bz* **by** (*subst less-cSup-iff*) *auto*
qed

9.3 Fundamental theorem of algebra

lemma *unimodular-reduce-norm*:
assumes *md*: *cmod z* = *1*
shows *cmod* (*z* + *1*) < *1* $\vee$ *cmod* (*z* − *1*) < *1* $\vee$ *cmod* (*z* + i) < *1* $\vee$ *cmod* (*z* − i) < *1*
proof −
obtain *x y* **where** *z*: *z* = *Complex x y*
by (*cases z*) *auto*
from *md z* **have** *xy*: $x^2 + y^2 = 1$
by (*simp add*: *cmod-def*)
have *False* **if** *cmod* (*z* + *1*) ≥ *1 cmod* (*z* − *1*) ≥ *1 cmod* (*z* + i) ≥ *1 cmod* (*z*

$-$ i) $\geq$ *1*
proof $-$
from *that z xy* **have** $*$: *2* $*$ *x* $\leq$ *1 2* $*$ *x* ≥ -1 *2* $*$ *y* $\leq$ *1 2* $*$ *y* ≥ -1
by (*simp-all add*: *cmod-def power2-eq-square algebra-simps*)
then have $|2 * x| \leq 1$ $|2 * y| \leq 1$
by *simp-all*
then have $|2 * x|^2 \leq 1^2$ $|2 * y|^2 \leq 1^2$
by (*metis abs-square-le-1 one-power2 power2-abs*)+
with *xy* $*$ **show** *?thesis*
by (*smt* (*verit*, *best*) *four-x-squared square-le-1*)
qed
then show *?thesis*
by *force*
qed

Hence we can always reduce modulus of *1 + b z^n* if nonzero

lemma *reduce-poly-simple*:
assumes *b*: $b \neq 0$
and *n*: $n \neq 0$
shows $\exists z.$ *cmod* (*1 + b* $*$ *z^n*) $<$ *1*
using *n*
proof (*induct n rule*: *nat-less-induct*)
fix *n*
assume *IH*: $\forall m<n.$ $m \neq 0 \longrightarrow (\exists z.$ *cmod* (*1 + b* $*$ *z* ^ *m*) $<$ *1*)
assume *n*: $n \neq 0$
let *?P* $= \lambda z$ *n. cmod* (*1 + b* $*$ *z* ^ *n*) $<$ *1*
show $\exists z.$ *?P z n*
proof *cases*
assume *even n*
then obtain *m* **where** *m*: $n = 2 * m$ **and** $m \neq 0$ $m < n$
using *n* **by** *auto*
with *IH* **obtain** *z* **where** *z*: *?P z m*
by *blast*
from *z* **have** *?P* (*csqrt z*) *n*
by (*simp add*: *m power-mult*)
then show *?thesis* **..**
next
assume *odd n*
then have $\exists m.$ $n =$ *Suc* (*2* $*$ *m*)
by *presburger*+
then obtain *m* **where** *m*: $n =$ *Suc* (*2* $*$ *m*)
by *blast*
have *0*: *cmod* (*complex-of-real* (*cmod b*) / *b*) = *1*
using *b* **by** (*simp add*: *norm-divide*)
have $\exists v.$ *cmod* (*complex-of-real* (*cmod b*) / *b* + *v^n*) $<$ *1*
proof (*cases cmod* (*complex-of-real* (*cmod b*) / *b* + *1*) $<$ *1*)
case *True*
then show *?thesis*
by (*metis power-one*)

```
next
  case F1: False
  show ?thesis
  proof (cases cmod (complex-of-real (cmod b) / b − 1) < 1)
    case True
    with ‹odd n› show ?thesis
      by (metis add-uminus-conv-diff neg-one-odd-power)
  next
    case F2: False
    show ?thesis
    proof (cases cmod (complex-of-real (cmod b) / b + i) < 1)
      case T1: True
      show ?thesis
      proof (cases even m)
        case True
        with T1 show ?thesis
          by (rule-tac x=i in exI) (simp add: m power-mult)
      next
        case False
        with T1 show ?thesis
          by (rule-tac x=− i in exI) (simp add: m power-mult)
      qed
    next
      case False
      then have lt1: cmod (of-real (cmod b) / b − i) < 1
        using 0 F1 F2 unimodular-reduce-norm by blast
      show ?thesis
      proof (cases even m)
        case True
        with m lt1 show ?thesis
          by (rule-tac x=− i in exI) (simp add: power-mult)
      next
        case False
        with m lt1 show ?thesis
          by (rule-tac x=i in exI) (simp add: power-mult)
      qed
    qed
  qed
qed
then obtain v where v: cmod (complex-of-real (cmod b) / b + v^n) < 1
  by blast
let ?w = v / complex-of-real (root n (cmod b))
from odd-real-root-pow[OF ‹odd n›, of cmod b]
have 1: ?w ^ n = v^n / complex-of-real (cmod b)
  by (simp add: power-divide of-real-power[symmetric])
have 2:cmod (complex-of-real (cmod b) / b) = 1
  using b by (simp add: norm-divide)
then have 3: cmod (complex-of-real (cmod b) / b) ≥ 0
  by simp
```

```
    have 4: cmod (complex-of-real (cmod b) / b) *
        cmod (1 + b * (v ^ n / complex-of-real (cmod b))) <
        cmod (complex-of-real (cmod b) / b) * 1
      apply (simp only: norm-mult[symmetric] distrib-left)
      using b v
      apply (simp add: 2)
      done
    show ?thesis
      by (metis 1 mult-left-less-imp-less[OF 4 3])
  qed
qed
```

Bolzano-Weierstrass type property for closed disc in complex plane.

```
lemma metric-bound-lemma: cmod (x - y) ≤ |Re x - Re y| + |Im x - Im y|
  using real-sqrt-sum-squares-triangle-ineq[of Re x - Re y 0 0 Im x - Im y]
  unfolding cmod-def by simp

lemma Bolzano-Weierstrass-complex-disc:
  assumes r: ∀ n. cmod (s n) ≤ r
  shows ∃ f z. strict-mono (f :: nat ⇒ nat) ∧ (∀ e >0. ∃ N. ∀ n ≥ N. cmod (s (f
n) - z) < e)
proof -
  from seq-monosub[of Re ∘ s]
  obtain f where f: strict-mono f monoseq (λn. Re (s (f n)))
    unfolding o-def by blast
  from seq-monosub[of Im ∘ s ∘ f]
  obtain g where g: strict-mono g monoseq (λn. Im (s (f (g n))))
    unfolding o-def by blast
  let ?h = f ∘ g
  have r ≥ 0
    by (meson norm-ge-zero order-trans r)
  have ∀ n. r + 1 ≥ |Re (s n)|
    by (smt (verit, ccfv-threshold) abs-Re-le-cmod r)
  then have conv1: convergent (λn. Re (s (f n)))
    by (metis Bseq-monoseq-convergent f(2) BseqI' real-norm-def)
  have ∀ n. r + 1 ≥ |Im (s n)|
    by (smt (verit) abs-Im-le-cmod r)
  then have conv2: convergent (λn. Im (s (f (g n))))
    by (metis Bseq-monoseq-convergent g(2) BseqI' real-norm-def)

  obtain x where  x: ∀ r>0. ∃ n0. ∀ n≥n0. |Re (s (f n)) - x| < r
    using conv1[unfolded convergent-def] LIMSEQ-iff real-norm-def by metis
  obtain y where  y: ∀ r>0. ∃ n0. ∀ n≥n0. |Im (s (f (g n))) - y| < r
    using conv2[unfolded convergent-def] LIMSEQ-iff real-norm-def by metis
  let ?w = Complex x y
  from f(1) g(1) have hs: strict-mono ?h
    unfolding strict-mono-def by auto
  have ∃ N. ∀ n≥N. cmod (s (?h n) - ?w) < e if e > 0 for e
  proof -
```

```
    from that have e2: e/2 > 0
      by simp
    from x y e2
    obtain N1 N2 where N1: ∀ n≥N1. |Re (s (f n)) − x| < e / 2
      and N2: ∀ n≥N2. |Im (s (f (g n))) − y| < e / 2
      by blast
    have cmod (s (?h n) − ?w) < e if n ≥ N1 + N2 for n
    proof −
      from that have nN1: g n ≥ N1 and nN2: n ≥ N2
        using seq-suble[OF g(1), of n] by arith+
      show ?thesis
        using metric-bound-lemma[of s (f (g n)) ?w] N1 N2 nN1 nN2 by fastforce
    qed
    then show ?thesis by blast
  qed
  with hs show ?thesis by blast
qed
```

Polynomial is continuous.

```
lemma poly-cont:
  fixes p :: 'a::{comm-semiring-0,real-normed-div-algebra} poly
  assumes ep: e > 0
  shows ∃ d >0. ∀ w. 0 < norm (w − z) ∧ norm (w − z) < d ⟶ norm (poly p
w − poly p z) < e
proof −
  obtain q where degree q = degree p and q: ⋀w. poly p w = poly q (w − z)
    by (metis add.commute degree-offset-poly diff-add-cancel poly-offset-poly)
  show ?thesis unfolding q
  proof (induct q)
    case 0
    then show ?case
      using ep by auto
  next
    case (pCons c cs)
    obtain m where m: m > 0 norm z ≤ 1 ⟹ norm (poly cs z) ≤ m for z
      using poly-bound-exists[of 1 cs] by blast
    with ep have em0: e/m > 0
      by (simp add: field-simps)
    obtain d where d: d > 0 d < 1 d < e / m
      by (meson em0 field-lbound-gt-zero zero-less-one)
    then have ⋀w. norm (w − z) < d ⟹ norm (w − z) * norm (poly cs (w −
z)) < e
      by (smt (verit, del-insts) m mult-left-mono norm-ge-zero pos-less-divide-eq)
    with d show ?case
      by (force simp add: norm-mult)
  qed
qed
```

Hence a polynomial attains minimum on a closed disc in the complex plane.

lemma *poly-minimum-modulus-disc*: $\exists z.\ \forall w.\ cmod\ w \leq r \longrightarrow cmod\ (poly\ p\ z) \leq cmod\ (poly\ p\ w)$
proof −
show *?thesis*
proof (*cases* $r \geq 0$)
case *False*
then show *?thesis*
by (*metis norm-ge-zero order.trans*)
next
case *True*
then have *mth1*: $\exists x\ z.\ cmod\ z \leq r \wedge cmod\ (poly\ p\ z) = -\ x$
by (*metis add.inverse-inverse norm-zero*)
obtain *s* **where** *s*: $\forall y.\ (\exists x.\ (\exists z.\ cmod\ z \leq r \wedge cmod\ (poly\ p\ z) = -\ x) \wedge y < x) \longleftrightarrow y < s$
by (*smt* (*verit*, *del-insts*) *real-sup-exists*[*OF mth1*] *norm-zero zero-less-norm-iff*)

let *?m* = − *s*
have *s1*: $(\exists z.\ cmod\ z \leq r \wedge -\ (-\ cmod\ (poly\ p\ z)) < y) \longleftrightarrow ?m < y$ **for** *y*
by (*metis add.inverse-inverse minus-less-iff s*)
then have *s1m*: $\bigwedge z.\ cmod\ z \leq r \Longrightarrow cmod\ (poly\ p\ z) \geq ?m$
by *force*
have $\exists z.\ cmod\ z \leq r \wedge cmod\ (poly\ p\ z) < -\ s + 1\ /\ real\ (Suc\ n)$ **for** *n*
using *s1*[*of ?m + 1*/*real* (*Suc n*)] **by** *simp*
then obtain *g* **where** *g*: $\forall n.\ cmod\ (g\ n) \leq r\ \forall n.\ cmod\ (poly\ p\ (g\ n)) <?m + 1\ /real(Suc\ n)$
by *metis*
from *Bolzano-Weierstrass-complex-disc*[*OF g*(*1*)]
obtain $f{::}nat \Rightarrow nat$ **and** *z* **where** *fz*: *strict-mono f* $\forall e{>}0.\ \exists N.\ \forall n{\geq}N.\ cmod\ (g\ (f\ n) - z) < e$
by *blast*
{
fix *w*
assume *wr*: $cmod\ w \leq r$
let *?e* = $|cmod\ (poly\ p\ z) - ?m|$
{
assume *e*: *?e* > *0*
then have *e2*: *?e*/*2* > *0*
by *simp*
with *poly-cont* **obtain** *d*
where *d* > *0* **and** *d*: $\bigwedge w.\ 0{<}cmod\ (w - z) \wedge cmod(w - z) < d \longrightarrow cmod(poly\ p\ w - poly\ p\ z) < ?e/2$
by *blast*
have *1*: *cmod*(*poly p w* − *poly p z*) < *?e* / *2* **if** *w*: *cmod* (*w* − *z*) < *d* **for** *w*
using *d*[*of w*] *w e* **by** (*cases w* = *z*) *simp-all*
from *fz*(*2*) ‹*d* > *0*› **obtain** *N1* **where** *N1*: $\forall n{\geq}N1.\ cmod\ (g\ (f\ n) - z) < d$
by *blast*
from *reals-Archimedean2* **obtain** *N2* :: *nat* **where** *N2*: *2*/*?e* < *real N2*
by *blast*

```
      have 2: cmod (poly p (g (f (N1 + N2))) − poly p z) < ?e/2
        using N1 1 by auto
      have 0: a < e2 ⟹ |b − m| < e2 ⟹ 2 ∗ e2 ≤ |b − m| + a ⟹ False
        for a b e2 m :: real
        by arith
      from seq-suble[OF fz(1), of N1 + N2]
      have 00: ?m + 1 / real (Suc (f (N1 + N2))) ≤ ?m + 1 / real (Suc (N1
+ N2))
        by (simp add: frac-le)
      from N2 e2 less-imp-inverse-less[of 2/?e real (Suc (N1 + N2))]
      have ?e/2 > 1/ real (Suc (N1 + N2))
        by (simp add: inverse-eq-divide)
      with  order-less-le-trans[OF - 00]
      have 1: |cmod (poly p (g (f (N1 + N2)))) − ?m| < ?e/2
        using g s1 by (smt (verit))
      with 0[OF 2] have False
        by (smt (verit) field-sum-of-halves norm-triangle-ineq3)
    }
    then have ?e = 0
      by auto
    with s1m[OF wr] have cmod (poly p z) ≤ cmod (poly p w)
      by simp
  }
  then show ?thesis by blast
 qed
qed
```

Nonzero polynomial in z goes to infinity as z does.

```
lemma poly-infinity:
  fixes p:: 'a::{comm-semiring-0,real-normed-div-algebra} poly
  assumes ex: p ≠ 0
  shows ∃ r. ∀ z. r ≤ norm z ⟶ d ≤ norm (poly (pCons a p) z)
  using ex
proof (induct p arbitrary: a d)
  case 0
  then show ?case by simp
next
  case (pCons c cs a d)
  show ?case
  proof (cases cs = 0)
    case False
   with pCons.hyps obtain r where r: ∀ z. r ≤ norm z ⟶ d + norm a ≤ norm
(poly (pCons c cs) z)
     by blast
    let ?r = 1 + |r|
    have d ≤ norm (poly (pCons a (pCons c cs)) z) if 1 + |r| ≤ norm z for z
    proof −
      have d ≤ norm(z ∗ poly (pCons c cs) z) − norm a
       by (smt (verit, best) norm-ge-zero mult-less-cancel-right2 norm-mult r that)
```

```
    with norm-diff-ineq add.commute
    show ?thesis
      by (metis order.trans poly-pCons)
  qed
  then show ?thesis by blast
next
  case True
  have d ≤ norm (poly (pCons a (pCons c cs)) z)
    if (|d| + norm a) / norm c ≤ norm z for z :: 'a
  proof −
    have |d| + norm a ≤ norm (z * c)
    by (metis that True norm-mult pCons.hyps(1) pos-divide-le-eq zero-less-norm-iff)
    also have ... ≤ norm (a + z * c) + norm a
      by (simp add: add.commute norm-add-leD)
    finally show ?thesis
      using True by auto
  qed
  then show ?thesis by blast
qed
qed
```

Hence polynomial's modulus attains its minimum somewhere.

```
lemma poly-minimum-modulus: ∃ z.∀ w. cmod (poly p z) ≤ cmod (poly p w)
proof (induct p)
  case 0
  then show ?case by simp
next
  case (pCons c cs)
  show ?case
  proof (cases cs = 0)
    case False
    from poly-infinity[OF False, of cmod (poly (pCons c cs) 0) c]
    obtain r where r: cmod (poly (pCons c cs) 0) ≤ cmod (poly (pCons c cs) z)
      if r ≤ cmod z for z
      by blast
    from poly-minimum-modulus-disc[of |r| pCons c cs] show ?thesis
      by (smt (verit, del-insts) order.trans linorder-linear r)
  qed (use pCons.hyps in auto)
qed
```

Constant function (non-syntactic characterization).

```
definition constant f ⟷ (∀ x y. f x = f y)

lemma nonconstant-length: ¬ constant (poly p) ⟹ psize p ≥ 2
  by (induct p) (auto simp: constant-def psize-def)

lemma poly-replicate-append: poly (monom 1 n * p) (x::'a::comm-ring-1) = x^n
* poly p x
  by (simp add: poly-monom)
```

Decomposition of polynomial, skipping zero coefficients after the first.

lemma *poly-decompose-lemma*:
assumes *nz*: ¬ (∀ *z*. *z* ≠ *0* ⟶ *poly p z* = (*0*::*'a*::*idom*))
shows ∃ *k a q*. *a* ≠ *0* ∧ *Suc* (*psize q* + *k*) = *psize p* ∧ (∀ *z*. *poly p z* = *z*^*k* ∗ *poly* (*pCons a q*) *z*)
unfolding *psize-def*
using *nz*
proof (*induct p*)
case *0*
then show *?case* **by** *simp*
next
case (*pCons c cs*)
show *?case*
proof (*cases c* = *0*)
case *True*
from *pCons.hyps pCons.prems True* **show** *?thesis*
apply *auto*
apply (*rule-tac x*=*k*+*1* **in** *exI*)
apply (*rule-tac x*=*a* **in** *exI*)
apply *clarsimp*
apply (*rule-tac x*=*q* **in** *exI*)
apply *auto*
done
qed *force*
qed

lemma *poly-decompose*:
fixes *p* :: *'a*::*idom poly*
assumes *nc*: ¬ *constant* (*poly p*)
shows ∃ *k a q*. *a* ≠ *0* ∧ *k* ≠ *0* ∧
psize q + *k* + *1* = *psize p* ∧
(∀ *z*. *poly p z* = *poly p 0* + *z*^*k* ∗ *poly* (*pCons a q*) *z*)
using *nc*
proof (*induct p*)
case *0*
then show *?case*
by (*simp add*: *constant-def*)
next
case (*pCons c cs*)
have ¬ (∀ *z*. *z* ≠ *0* ⟶ *poly cs z* = *0*)
by (*smt* (*verit*) *constant-def mult-eq-0-iff pCons.prems poly-pCons*)
from *poly-decompose-lemma*[*OF this*]
obtain *k a q* **where** ∗: *a* ≠ *0* ∧
Suc (*psize q* + *k*) = *psize cs* ∧ (∀ *z*. *poly cs z* = *z* ^ *k* ∗ *poly* (*pCons a q*) *z*)
by *blast*
then have *psize q* + *k* + *2* = *psize* (*pCons c cs*)
by (*auto simp add*: *psize-def split*: *if-splits*)
then show *?case*
using ∗ **by** *force*

qed

Fundamental theorem of algebra

theorem *fundamental-theorem-of-algebra*:
assumes *nc*: ¬ *constant* (*poly p*)
shows ∃ *z*::*complex*. *poly p z* = *0*
using *nc*
proof (*induct psize p arbitrary*: *p rule*: *less-induct*)
case *less*
let *?p* = *poly p*
let *?ths* = ∃ *z*. *?p z* = *0*

from *nonconstant-length*[*OF less*(*2*)] **have** *n2*: *psize p* ≥ *2* **.**
from *poly-minimum-modulus* **obtain** *c* **where** *c*: ∀ *w*. *cmod* (*?p c*) ≤ *cmod* (*?p w*)
by *blast*

show *?ths*
proof (*cases ?p c* = *0*)
case *True*
then show *?thesis* **by** *blast*
next
case *False*
obtain *q* **where** *q*: *psize q* = *psize p* ∀ *x*. *poly q x* = *?p* (*c* + *x*)
using *poly-offset*[*of p c*] **by** *blast*
then have *qnc*: ¬ *constant* (*poly q*)
by (*metis* (*no-types*, *opaque-lifting*) *add.commute constant-def diff-add-cancel less.prems*)
from *q*(*2*) **have** *pqc0*: *?p c* = *poly q 0*
by *simp*
from *c pqc0* **have** *cq0*: ∀ *w*. *cmod* (*poly q 0*) ≤ *cmod* (*?p w*)
by *simp*
let *?a0* = *poly q 0*
from *False pqc0* **have** *a00*: *?a0* ≠ *0*
by *simp*
from *a00* **have** *qr*: ∀ *z*. *poly q z* = *poly* (*smult* (*inverse ?a0*) *q*) *z* ∗ *?a0*
by *simp*
let *?r* = *smult* (*inverse ?a0*) *q*
have *lgqr*: *psize q* = *psize ?r*
by (*simp add*: *a00 psize-def*)
have *rnc*: ¬ *constant* (*poly ?r*)
using *constant-def qnc qr* **by** *fastforce*
have *r01*: *poly ?r 0* = *1*
by (*simp add*: *a00*)
have *mrmq-eq*: *cmod* (*poly ?r w*) < *1* ⟷ *cmod* (*poly q w*) < *cmod ?a0* **for** *w*
by (*smt* (*verit*, *del-insts*) *a00 mult-less-cancel-right2 norm-mult qr zero-less-norm-iff*)
from *poly-decompose*[*OF rnc*] **obtain** *k a s* **where**
kas: *a* ≠ *0 k* ≠ *0 psize s* + *k* + *1* = *psize ?r*
∀ *z*. *poly ?r z* = *poly ?r 0* + *z*^*k*∗ *poly* (*pCons a s*) *z* **by** *blast*

```
    have ∃ w. cmod (poly ?r w) < 1
    proof (cases psize p = k + 1)
      case True
      with kas q have s0: s = 0
        by (simp add: lgqr)
      with reduce-poly-simple kas show ?thesis
       by (metis mult.commute mult.right-neutral poly-1 poly-smult r01 smult-one)
    next
      case False note kn = this
      from kn kas(3) q(1) lgqr have k1n: k + 1 < psize p
        by simp
      have 01: ¬ constant (poly (pCons 1 (monom a (k − 1))))
        unfolding constant-def poly-pCons poly-monom
          by (metis add-cancel-left-right kas(1) mult.commute mult-cancel-right2
power-one)
      have 02: k + 1 = psize (pCons 1 (monom a (k − 1)))
        using kas by (simp add: psize-def degree-monom-eq)
      from less(1) [OF - 01] k1n 02
      obtain w where w: 1 + w^k * a = 0
        by (metis kas(2) mult.commute mult.left-commute poly-monom poly-pCons
power-eq-if)
      from poly-bound-exists[of cmod w s] obtain m where
        m: m > 0 ∀ z. cmod z ≤ cmod w ⟶ cmod (poly s z) ≤ m by blast
      have w ≠ 0
        using kas(2) w by (auto simp add: power-0-left)
      from w have wm1: w^k * a = − 1
        by (simp add: add-eq-0-iff)
      have inv0: 0 < inverse (cmod w ^ (k + 1) * m)
        by (simp add: ‹w ≠ 0› m(1))
      with field-lbound-gt-zero[OF zero-less-one] obtain t where
        t: t > 0 t < 1 t < inverse (cmod w ^ (k + 1) * m) by blast
      let ?ct = complex-of-real t
      let ?w = ?ct * w
      have 1 + ?w^k * (a + ?w * poly s ?w) = 1 + ?ct^k * (w^k * a) + ?w^k *
?w * poly s ?w
        using kas(1) by (simp add: algebra-simps power-mult-distrib)
      also have ... = complex-of-real (1 − t^k) + ?w^k * ?w * poly s ?w
        unfolding wm1 by simp
      finally have cmod (1 + ?w^k * (a + ?w * poly s ?w)) =
        cmod (complex-of-real (1 − t^k) + ?w^k * ?w * poly s ?w)
        by metis
      with norm-triangle-ineq[of complex-of-real (1 − t^k) ?w^k * ?w * poly s ?w]
       have 11: cmod (1 + ?w^k * (a + ?w * poly s ?w)) ≤ |1 − t^k| + cmod
(?w^k * ?w * poly s ?w)
        unfolding norm-of-real by simp
      have ath: ⋀x t::real. 0 ≤ x ⟹ x < t ⟹ t ≤ 1 ⟹ |1 − t| + x < 1
        by arith
      have tw: cmod ?w ≤ cmod w
       by (smt (verit) mult-le-cancel-right2 norm-ge-zero norm-mult norm-of-real
```

```
t)
    have t * (cmod w ^ (k + 1) * m) < 1
    by (smt (verit, best) inv0 inverse-positive-iff-positive left-inverse mult-strict-right-mono
t(3))
    with zero-less-power[OF t(1), of k] have 30: t^k * (t* (cmod w ^ (k + 1) *
m)) < t^k
      by simp
    have cmod (?w^k * ?w * poly s ?w) = t^k * (t* (cmod w ^ (k + 1) * cmod
(poly s ?w)))
      using ‹w ≠ 0› t(1) by (simp add: algebra-simps norm-power norm-mult)
    with 30 have 120: cmod (?w^k * ?w * poly s ?w) < t^k
        by (smt (verit, ccfv-SIG) m(2) mult-left-mono norm-ge-zero t(1) tw
zero-le-power)
    from power-strict-mono[OF t(2), of k] t(1) kas(2) have 121: t^k ≤ 1
      by auto
    from ath[OF norm-ge-zero[of ?w^k * ?w * poly s ?w] 120 121]
    show ?thesis
      by (smt (verit) 11 kas(4) poly-pCons r01)
  qed
  with cq0 q(2) show ?thesis
    by (smt (verit) mrmq-eq)
 qed
qed
```

Alternative version with a syntactic notion of constant polynomial.

```
lemma fundamental-theorem-of-algebra-alt:
 assumes nc: ¬ (∃ a l. a ≠ 0 ∧ l = 0 ∧ p = pCons a l)
 shows ∃ z. poly p z = (0::complex)
proof (rule ccontr)
 assume N: ∄ z. poly p z = 0
 then have ¬ constant (poly p)
   unfolding constant-def
  by (metis (no-types, opaque-lifting) nc poly-pcompose pcompose-0' pcompose-const
poly-0-coeff-0
      poly-all-0-iff-0 poly-diff right-minus-eq)
 then show False
   using N fundamental-theorem-of-algebra by blast
qed
```

9.4 Nullstellensatz, degrees and divisibility of polynomials

```
lemma nullstellensatz-lemma:
 fixes p :: complex poly
 assumes ∀ x. poly p x = 0 ⟶ poly q x = 0
   and degree p = n
   and n ≠ 0
 shows p dvd (q ^ n)
 using assms
proof (induct n arbitrary: p q rule: nat-less-induct)
```

```
  fix n :: nat
  fix p q :: complex poly
  assume IH: ∀ m<n. ∀ p q.
               (∀ x. poly p x = (0::complex) ⟶ poly q x = 0) ⟶
               degree p = m ⟶ m ≠ 0 ⟶ p dvd (q ^ m)
    and pq0: ∀ x. poly p x = 0 ⟶ poly q x = 0
    and dpn: degree p = n
    and n0: n ≠ 0
  from dpn n0 have pne: p ≠ 0 by auto
  show p dvd (q ^ n)
  proof (cases ∃ a. poly p a = 0)
    case True
    then obtain a where a: poly p a = 0 ..
    have ?thesis if oa: order a p ≠ 0
    proof −
      let ?op = order a p
      from pne have ap: ([:− a, 1:] ^ ?op) dvd p ¬ [:− a, 1:] ^ (Suc ?op) dvd p
        using order by blast+
      note oop = order-degree[OF pne, unfolded dpn]
      show ?thesis
      proof (cases q = 0)
        case True
        with n0 show ?thesis by (simp add: power-0-left)
      next
        case False
        from pq0[rule-format, OF a, unfolded poly-eq-0-iff-dvd]
        obtain r where r: q = [:− a, 1:] * r by (rule dvdE)
        from ap(1) obtain s where s: p = [:− a, 1:] ^ ?op * s
          by (rule dvdE)
        have sne: s ≠ 0
          using s pne by auto
        show ?thesis
        proof (cases degree s = 0)
          case True
          then obtain k where kpn: s = [:k:]
            by (cases s) (auto split: if-splits)
          from sne kpn have k: k ≠ 0 by simp
          let ?w = ([:1/k:] * ([:−a,1:] ^ (n − ?op))) * (r ^ n)
          have q^n = [:− a, 1:] ^ n * r ^ n
            using power-mult-distrib r by blast
          also have ... = [:− a, 1:] ^ order a p * [:k:] * ([:1 / k:] * [:− a, 1:] ^ (n
− order a p) * r ^ n)
            using k oop [of a] by (simp flip: power-add)
          also have ... = p * ?w
            by (metis s kpn)
          finally show ?thesis
            unfolding dvd-def by blast
        next
          case False
```

```
        with sne dpn s oa have dsn: degree s < n
        by (metis add-diff-cancel-right' degree-0 degree-linear-power degree-mult-eq
gr0I zero-less-diff)
        have poly r x = 0 if h: poly s x = 0 for x
        proof −
          have x ≠ a
            by (metis ap(2) dvd-refl mult-dvd-mono poly-eq-0-iff-dvd power-Suc
power-commutes s that)
          moreover have poly p x = 0
            by (metis (no-types) mult-eq-0-iff poly-mult s that)
          ultimately show ?thesis
            using pq0 r by auto
        qed
        with False IH dsn obtain u where u: r ^ (degree s) = s * u
          by blast
        then have u': ⋀x. poly s x * poly u x = poly r x ^ degree s
          by (simp only: poly-mult[symmetric] poly-power[symmetric])
        have q^n = [:− a, 1:] ^ n * r ^ n
          using power-mult-distrib r by blast
        also have ... = [:− a, 1:] ^ order a p * (s * u * ([:− a, 1:] ^ (n − order
a p) * r ^ (n − degree s)))
          by (smt (verit, del-insts) s u mult-ac power-add add-diff-cancel-right'
degree-linear-power degree-mult-eq dpn mult-zero-left)
        also have ... = p * (u * ([:−a,1:] ^ (n − ?op))) * (r ^ (n − degree s))
          using s by force
        finally show ?thesis
          unfolding dvd-def by auto
      qed
    qed
  qed
  then show ?thesis
    using a order-root pne by blast
next
  case False
  then show ?thesis
    using dpn n0 fundamental-theorem-of-algebra-alt[of p]
    by fastforce
qed
qed

lemma nullstellensatz-univariate:
  (∀x. poly p x = (0::complex) ⟶ poly q x = 0) ⟷
    p dvd (q ^ (degree p)) ∨ (p = 0 ∧ q = 0)
proof −
  consider p = 0 | p ≠ 0 degree p = 0 | n where p ≠ 0 degree p = Suc n
    by (cases degree p) auto
  then show ?thesis
  proof cases
    case p: 1
```

```
    then have (∀ x. poly p x = (0::complex) ⟶ poly q x = 0) ⟷ q = 0
      by (auto simp add: poly-all-0-iff-0)
    with p show ?thesis
      by force
  next
    case dp: 2
    then show ?thesis
      by (meson dvd-trans is-unit-iff-degree poly-eq-0-iff-dvd unit-imp-dvd)
  next
    case dp: 3
    have False if p dvd (q ^ (Suc n)) poly p x = 0 poly q x ≠ 0 for x
      by (metis dvd-trans poly-eq-0-iff-dvd poly-power power-eq-0-iff that)
    with dp nullstellensatz-lemma[of p q degree p] show ?thesis
      by auto
  qed
qed
```

Useful lemma

```
lemma constant-degree:
  fixes p :: 'a::{idom,ring-char-0} poly
  shows constant (poly p) ⟷ degree p = 0 (is ?lhs = ?rhs)
proof
  show ?rhs if ?lhs
  proof −
    from that[unfolded constant-def, rule-format, of - 0]
    have poly p = poly [:poly p 0:]
      by auto
    then show ?thesis
      by (metis degree-pCons-0 poly-eq-poly-eq-iff)
  qed
  show ?lhs if ?rhs
    unfolding constant-def
    by (metis degree-eq-zeroE pcompose-const poly-0 poly-pcompose that)
qed

lemma complex-poly-decompose:
  smult (lead-coeff p) (∏ z|poly p z = 0. [:−z, 1:] ^ order z p) = (p :: complex poly)
proof (induction p rule: poly-root-order-induct)
  case (no-roots p)
  show ?case
  proof (cases degree p = 0)
    case False
    hence ¬constant (poly p) by (subst constant-degree)
    with fundamental-theorem-of-algebra and no-roots show ?thesis by blast
  qed (auto elim!: degree-eq-zeroE)
next
  case (root p x n)
  from root have ∗: {z. poly ([:− x, 1:] ^ n ∗ p) z = 0} = insert x {z. poly p z =
0}
```

```
    by auto
  have smult (lead-coeff ([:−x, 1:] ^ n * p))
          (∏ z|poly ([:−x,1:] ^ n * p) z = 0. [:−z, 1:] ^ order z ([:− x, 1:] ^ n *
p)) =
        [:− x, 1:] ^ order x ([:− x, 1:] ^ n * p) *
         smult (lead-coeff p) (∏ z∈{z. poly p z = 0}. [:− z, 1:] ^ order z ([:− x, 1:]
^ n * p))
    by (subst *, subst prod.insert)
     (insert root, auto intro: poly-roots-finite simp: mult-ac lead-coeff-mult lead-coeff-power)
  also have order x ([:− x, 1:] ^ n * p) = n
    using root by (subst order-mult) (auto simp: order-power-n-n order-0I)
  also have (∏ z∈{z. poly p z = 0}. [:− z, 1:] ^ order z ([:− x, 1:] ^ n * p)) =
              (∏ z∈{z. poly p z = 0}. [:− z, 1:] ^ order z p)
  proof (intro prod.cong refl, goal-cases)
    case (1 y)
    with root have order y ([:−x,1:] ^ n) = 0 by (intro order-0I) auto
    thus ?case using root by (subst order-mult) auto
  qed
  also note root.IH
  finally show ?case .
qed simp-all

instance complex :: alg-closed-field
  by standard (use fundamental-theorem-of-algebra constant-degree neq0-conv in
blast)

lemma size-proots-complex: size (proots (p :: complex poly)) = degree p
proof (cases p = 0)
  case [simp]: False
  show size (proots p) = degree p
    by (subst (1 2) complex-poly-decompose [symmetric])
       (simp add: proots-prod proots-power degree-prod-sum-eq degree-power-eq)
qed auto

lemma complex-poly-decompose-multiset:
  smult (lead-coeff p) (∏ x∈#proots p. [:−x, 1:]) = (p :: complex poly)
proof (cases p = 0)
  case False
  hence (∏ x∈#proots p. [:−x, 1:]) = (∏ x | poly p x = 0. [:−x, 1:] ^ order x p)
    by (subst image-prod-mset-multiplicity) simp-all
  also have smult (lead-coeff p) ... = p
    by (rule complex-poly-decompose)
  finally show ?thesis .
qed auto

lemma complex-poly-decompose':
  obtains root where smult (lead-coeff p) (∏ i<degree p. [:−root i, 1:]) = (p ::
complex poly)
proof −
```

obtain *roots* **where** *roots*: *mset roots* = *proots p*
using *ex-mset* **by** *blast*

have *p* = *smult* (*lead-coeff p*) (∏ *x*∈#*proots p*. [:−*x*, *1*:])
by (*rule complex-poly-decompose-multiset* [*symmetric*])
also have (∏ *x*∈#*proots p*. [:−*x*, *1*:]) = (∏ *x*←*roots*. [:−*x*, *1*:])
by (*subst prod-mset-prod-list* [*symmetric*]) (*simp add*: *roots*)
also have ... = (∏ *i*<*length roots*. [:−*roots* ! *i*, *1*:])
by (*subst prod.list-conv-set-nth*) (*auto simp*: *atLeast0LessThan*)
finally have *eq*: *p* = *smult* (*lead-coeff p*) (∏ *i*<*length roots*. [:−*roots* ! *i*, *1*:]) **.**
also have [*simp*]: *degree p* = *length roots*
using *roots* **by** (*subst eq*) (*auto simp*: *degree-prod-sum-eq*)
finally show *?thesis* **by** (*intro that*[*of* λ*i*. *roots* ! *i*]) *auto*
qed

lemma *complex-poly-decompose-rsquarefree*:
assumes *rsquarefree p*
shows *smult* (*lead-coeff p*) (∏ *z*|*poly p z* = *0*. [:−*z*, *1*:]) = (*p* :: *complex poly*)
proof (*cases p* = *0*)
case *False*
have (∏ *z*|*poly p z* = *0*. [:−*z*, *1*:]) = (∏ *z*|*poly p z* = *0*. [:−*z*, *1*:] ^ *order z p*)
using *assms False* **by** (*intro prod.cong*) (*auto simp*: *rsquarefree-root-order*)
also have *smult* (*lead-coeff p*) ... = *p*
by (*rule complex-poly-decompose*)
finally show *?thesis* **.**
qed *auto*

Arithmetic operations on multivariate polynomials.

lemma *mpoly-base-conv*:
fixes *x* :: *'a*::*comm-ring-1*
shows *0* = *poly 0 x c* = *poly* [:*c*:] *x x* = *poly* [:*0*,*1*:] *x*
by *simp-all*

lemma *mpoly-norm-conv*:
fixes *x* :: *'a*::*comm-ring-1*
shows *poly* [:*0*:] *x* = *poly 0 x poly* [:*poly 0 y*:] *x* = *poly 0 x*
by *simp-all*

lemma *mpoly-sub-conv*:
fixes *x* :: *'a*::*comm-ring-1*
shows *poly p x* − *poly q x* = *poly p x* + −*1* ∗ *poly q x*
by *simp*

lemma *poly-pad-rule*: *poly p x* = *0* ⟹ *poly* (*pCons 0 p*) *x* = *0*
by *simp*

lemma *poly-cancel-eq-conv*:
fixes *x* :: *'a*::*field*
shows *x* = *0* ⟹ *a* ≠ *0* ⟹ *y* = *0* ⟷ *a* ∗ *y* − *b* ∗ *x* = *0*

by *auto*

lemma *poly-divides-pad-rule*:
fixes *p*:: (*'a*::*comm-ring-1*) *poly*
assumes *pq*: *p dvd q*
shows *p dvd* (*pCons 0 q*)
by (*metis add-0 dvd-def mult-pCons-right pq smult-0-left*)

lemma *poly-divides-conv0*:
fixes *p*:: *'a*::*field poly*
assumes *lgpq*: *degree q* < *degree p* **and** *lq*: *p* ≠ *0*
shows *p dvd q* ⟷ *q* = *0*
using *lgpq mod-poly-less* **by** *fastforce*

lemma *poly-divides-conv1*:
fixes *p* :: *'a*::*field poly*
assumes *a0*: *a* ≠ *0*
and *pp'*: *p dvd p'*
and *qrp'*: *smult a q* − *p'* = *r*
shows *p dvd q* ⟷ *p dvd r*
by (*metis a0 diff-add-cancel dvd-add-left-iff dvd-smult-iff pp' qrp'*)

lemma *basic-cqe-conv1*:
(∃ *x*. *poly p x* = *0* ∧ *poly 0 x* ≠ *0*) ⟷ *False*
(∃ *x*. *poly 0 x* ≠ *0*) ⟷ *False*
(∃ *x*. *poly* [:*c*:] *x* ≠ *0*) ⟷ *c* ≠ *0*
(∃ *x*. *poly 0 x* = *0*) ⟷ *True*
(∃ *x*. *poly* [:*c*:] *x* = *0*) ⟷ *c* = *0*
by *simp-all*

lemma *basic-cqe-conv2*:
assumes *l*: *p* ≠ *0*
shows ∃ *x*. *poly* (*pCons a* (*pCons b p*)) *x* = (*0*::*complex*)
by (*meson fundamental-theorem-of-algebra-alt l pCons-eq-0-iff pCons-eq-iff*)

lemma *basic-cqe-conv-2b*: (∃ *x*. *poly p x* ≠ (*0*::*complex*)) ⟷ *p* ≠ *0*
by (*metis poly-all-0-iff-0*)

lemma *basic-cqe-conv3*:
fixes *p q* :: *complex poly*
assumes *l*: *p* ≠ *0*
shows (∃ *x*. *poly* (*pCons a p*) *x* = *0* ∧ *poly q x* ≠ *0*) ⟷ ¬ (*pCons a p*) *dvd* (*q* ^ *psize p*)
by (*metis degree-pCons-eq-if l nullstellensatz-univariate pCons-eq-0-iff psize-def*)

lemma *basic-cqe-conv4*:
fixes *p q* :: *complex poly*
assumes *h*: ⋀*x*. *poly* (*q* ^ *n*) *x* = *poly r x*
shows *p dvd* (*q* ^ *n*) ⟷ *p dvd r*

by (*metis* (*no-types*) *basic-cqe-conv-2b h poly-diff right-minus-eq*)

lemma *poly-const-conv*:
fixes $x :: 'a$::*comm-ring-1*
shows *poly* [:*c*:] $x = y \longleftrightarrow c = y$
by *simp*

end

theory *Group-Closure*
imports
Main
begin

context *ab-group-add*
begin

inductive-set *group-closure* :: $'a$ *set* $\Rightarrow$ $'a$ *set* **for** S
where *base*: $s \in$ *insert 0 S* $\Longrightarrow s \in$ *group-closure S*
| *diff*: $s \in$ *group-closure S* $\Longrightarrow t \in$ *group-closure S* $\Longrightarrow s - t \in$ *group-closure S*

lemma *zero-in-group-closure* [*simp*]:
$0 \in$ *group-closure S*
using *group-closure.base* [*of 0 S*] **by** *simp*

lemma *group-closure-minus-iff* [*simp*]:
$- s \in$ *group-closure S* $\longleftrightarrow s \in$ *group-closure S*
using *group-closure.diff* [*of 0 S s*] *group-closure.diff* [*of 0 S* $- s$] **by** *auto*

lemma *group-closure-add*:
$s + t \in$ *group-closure S* **if** $s \in$ *group-closure S* **and** $t \in$ *group-closure S*
using *that group-closure.diff* [*of s S* $- t$] **by** *auto*

lemma *group-closure-empty* [*simp*]:
group-closure $\{\} = \{0\}$
by (*rule ccontr*) (*auto elim*: *group-closure.induct*)

lemma *group-closure-insert-zero* [*simp*]:
group-closure (*insert 0 S*) = *group-closure S*
by (*auto elim*: *group-closure.induct intro*: *group-closure.intros*)

end

context *comm-ring-1*
begin

lemma *group-closure-scalar-mult-left*:
of-nat n $*$ $s \in$ *group-closure S* **if** $s \in$ *group-closure S*

using *that* **by** (*induction n*) (*auto simp add*: *algebra-simps intro*: *group-closure-add*)

lemma *group-closure-scalar-mult-right*:
s ∗ *of-nat n* ∈ *group-closure S* **if** *s* ∈ *group-closure S*
using *that group-closure-scalar-mult-left* [*of s S n*] **by** (*simp add*: *ac-simps*)

end

lemma *group-closure-abs-iff* [*simp*]:
|*s*| ∈ *group-closure S* ⟷ *s* ∈ *group-closure S* **for** *s* :: *int*
by (*simp add*: *abs-if*)

lemma *group-closure-mult-left*:
s ∗ *t* ∈ *group-closure S* **if** *s* ∈ *group-closure S* **for** *s t* :: *int*
proof −
from *that group-closure-scalar-mult-right* [*of s S nat* |*t*|]
have *s* ∗ *int* (*nat* |*t*|) ∈ *group-closure S*
by (*simp only*:)
then show *?thesis*
by (*cases t* ≥ *0*) *simp-all*
qed

lemma *group-closure-mult-right*:
s ∗ *t* ∈ *group-closure S* **if** *t* ∈ *group-closure S* **for** *s t* :: *int*
using *that group-closure-mult-left* [*of t S s*] **by** (*simp add*: *ac-simps*)

context *idom*
begin

lemma *group-closure-mult-all-eq*:
group-closure (*times k* ‘ *S*) = *times k* ‘ *group-closure S*
proof (*rule*; *rule*)
fix *s*
have ∗: *k* ∗ *a* + *k* ∗ *b* = *k* ∗ (*a* + *b*)
k ∗ *a* − *k* ∗ *b* = *k* ∗ (*a* − *b*) **for** *a b*
by (*simp-all add*: *algebra-simps*)
assume *s* ∈ *group-closure* (*times k* ‘ *S*)
then show *s* ∈ *times k* ‘ *group-closure S*
by *induction* (*auto simp add*: ∗ *image-iff intro*: *group-closure.base group-closure.diff bexI* [*of - 0*])
next
fix *s*
assume *s* ∈ *times k* ‘ *group-closure S*
then obtain *r* **where** *r*: *r* ∈ *group-closure S* **and** *s*: *s* = *k* ∗ *r*
by *auto*
from *r* **have** *k* ∗ *r* ∈ *group-closure* (*times k* ‘ *S*)
by (*induction arbitrary*: *s*) (*auto simp add*: *algebra-simps intro*: *group-closure.intros*)
with *s* **show** *s* ∈ *group-closure* (*times k* ‘ *S*)
by *simp*

qed

end

lemma *Gcd-group-closure-eq-Gcd*:
  Gcd (*group-closure S*) = *Gcd S* **for** *S* :: *int set*
proof (*rule associated-eqI*)
  have *Gcd S dvd s* **if** *s* ∈ *group-closure S* **for** *s*
    using *that* **by** *induction auto*
  then show *Gcd S dvd Gcd* (*group-closure S*)
    by *auto*
  have *Gcd* (*group-closure S*) *dvd s* **if** *s* ∈ *S* **for** *s*
  proof −
    from *that* **have** *s* ∈ *group-closure S*
      by (*simp add*: *group-closure.base*)
    then show *?thesis*
      by (*rule Gcd-dvd*)
  qed
  then show *Gcd* (*group-closure S*) *dvd Gcd S*
    by *auto*
qed *simp-all*

lemma *group-closure-sum*:
  fixes *S* :: *int set*
  assumes *X*: *finite X X* ≠ {} *X* ⊆ *S*
  shows (∑ *x*∈*X*. *a x* ∗ *x*) ∈ *group-closure S*
  using *X* **by** (*induction X rule*: *finite-ne-induct*)
    (*auto intro*: *group-closure-mult-right group-closure.base group-closure-add*)

lemma *Gcd-group-closure-in-group-closure*:
  Gcd (*group-closure S*) ∈ *group-closure S* **for** *S* :: *int set*
proof (*cases S* ⊆ {*0*})
  case *True*
  then have *S* = {} ∨ *S* = {*0*}
    by *auto*
  then show *?thesis*
    by *auto*
next
  case *False*
  then obtain *s* **where** *s*: *s* ≠ *0 s* ∈ *S*
    by *auto*
  then have *s'*: |*s*| ≠ *0* |*s*| ∈ *group-closure S*
    by (*auto intro*: *group-closure.base*)
  define *m* **where** *m* = (*LEAST n*. *n* > *0* ∧ *int n* ∈ *group-closure S*)
  have *m* > *0* ∧ *int m* ∈ *group-closure S*
    unfolding *m-def*
    apply (*rule LeastI* [*of* - *nat* |*s*|])
    using *s'*
    by *simp*

```
  then have m: int m ∈ group-closure S and 0 < m
    by auto

  have Gcd (group-closure S) = int m
  proof (rule associated-eqI)
    from m show Gcd (group-closure S) dvd int m
      by (rule Gcd-dvd)
    show int m dvd Gcd (group-closure S)
    proof (rule Gcd-greatest)
      fix s
      assume s: s ∈ group-closure S
      show int m dvd s
      proof (rule ccontr)
        assume ¬ int m dvd s
        then have *: 0 < s mod int m
          using ‹0 < m› le-less by fastforce
        have m ≤ nat (s mod int m)
        proof (subst m-def, rule Least-le, rule)
          from * show 0 < nat (s mod int m)
            by simp
          from minus-div-mult-eq-mod [symmetric, of s int m]
          have s mod int m = s − s div int m * int m
            by auto
          also have s − s div int m * int m ∈ group-closure S
            by (auto intro: group-closure.diff s group-closure-mult-right m)
          finally  show int (nat (s mod int m)) ∈ group-closure S
            by simp
        qed
        with * have int m ≤ s mod int m
          by simp
        moreover have s mod int m < int m
          using ‹0 < m› by simp
        ultimately show False
          by auto
      qed
    qed
  qed simp-all
  with m show ?thesis
    by simp
qed

lemma Gcd-in-group-closure:
  Gcd S ∈ group-closure S for S :: int set
  using Gcd-group-closure-in-group-closure [of S]
  by (simp add: Gcd-group-closure-eq-Gcd)

lemma group-closure-eq:
  group-closure S = range (times (Gcd S)) for S :: int set
proof (auto intro: Gcd-in-group-closure group-closure-mult-left)
```

```
  fix s
  assume s ∈ group-closure S
  then show s ∈ range (times (Gcd S))
  proof induction
    case (base s)
    then have Gcd S dvd s
      by (auto intro: Gcd-dvd)
    then obtain t where s = Gcd S * t ..
    then show ?case
      by auto
  next
    case (diff s t)
    moreover have Gcd S * a - Gcd S * b = Gcd S * (a - b) for a b
      by (simp add: algebra-simps)
    ultimately show ?case
      by auto
  qed
qed

end


theory Normalized-Fraction
imports
  Main
  Euclidean-Algorithm
  Fraction-Field
begin

lemma unit-factor-1-imp-normalized: unit-factor x = 1 ⟹ normalize x = x
  using unit-factor-mult-normalize [of x] by simp

definition quot-to-fract :: 'a × 'a ⇒ 'a :: idom fract where
  quot-to-fract = (λ(a,b). Fraction-Field.Fract a b)

definition normalize-quot :: 'a :: {ring-gcd,idom-divide,semiring-gcd-mult-normalize}
× 'a ⇒ 'a × 'a where
  normalize-quot =
     (λ(a,b). if b = 0 then (0,1) else let d = gcd a b * unit-factor b in (a div d, b
div d))

lemma normalize-quot-zero [simp]:
  normalize-quot (a, 0) = (0, 1)
  by (simp add: normalize-quot-def)

lemma normalize-quot-proj:
  fst (normalize-quot (a, b)) = a div (gcd a b * unit-factor b)
  snd (normalize-quot (a, b)) = normalize b div gcd a b if b ≠ 0
   using that by (simp-all add: normalize-quot-def Let-def mult.commute [of -
```

unit-factor b] *dvd-div-mult2-eq mult-unit-dvd-iff'*)

definition *normalized-fracts* :: ('*a* :: {*ring-gcd*,*idom-divide*} × '*a*) *set* **where**
normalized-fracts = {(*a*,*b*). *coprime a b* ∧ *unit-factor b* = *1*}

lemma *not-normalized-fracts-0-denom* [*simp*]: (*a*, *0*) ∉ *normalized-fracts*
by (*auto simp*: *normalized-fracts-def*)

lemma *unit-factor-snd-normalize-quot* [*simp*]:
unit-factor (*snd* (*normalize-quot x*)) = *1*
by (*simp add*: *normalize-quot-def case-prod-unfold Let-def dvd-unit-factor-div*
mult-unit-dvd-iff unit-factor-mult unit-factor-gcd)

lemma *snd-normalize-quot-nonzero* [*simp*]: *snd* (*normalize-quot x*) ≠ *0*
using *unit-factor-snd-normalize-quot*[*of x*]
by (*auto simp del*: *unit-factor-snd-normalize-quot*)

lemma *normalize-quot-aux*:
fixes *a b*
assumes *b* ≠ *0*
defines *d* ≡ *gcd a b* ∗ *unit-factor b*
shows *a* = *fst* (*normalize-quot* (*a*,*b*)) ∗ *d b* = *snd* (*normalize-quot* (*a*,*b*)) ∗ *d*
d dvd a d dvd b d ≠ *0*
proof −
from *assms* **show** *d dvd a d dvd b*
by (*simp-all add*: *d-def mult-unit-dvd-iff*)
thus *a* = *fst* (*normalize-quot* (*a*,*b*)) ∗ *d b* = *snd* (*normalize-quot* (*a*,*b*)) ∗ *d d* ≠ *0*
by (*auto simp*: *normalize-quot-def Let-def d-def* ‹*b* ≠ *0*›)
qed

lemma *normalize-quotE*:
assumes *b* ≠ *0*
obtains *d* **where** *a* = *fst* (*normalize-quot* (*a*,*b*)) ∗ *d b* = *snd* (*normalize-quot* (*a*,*b*)) ∗ *d*
d dvd a d dvd b d ≠ *0*
using *that*[*OF normalize-quot-aux*[*OF assms*]] **.**

lemma *normalize-quotE'*:
assumes *snd x* ≠ *0*
obtains *d* **where** *fst x* = *fst* (*normalize-quot x*) ∗ *d snd x* = *snd* (*normalize-quot x*) ∗ *d*
d dvd fst x d dvd snd x d ≠ *0*
proof −
from *normalize-quotE*[*OF assms*, *of fst x*] **obtain** *d* **where**
fst x = *fst* (*normalize-quot* (*fst x*, *snd x*)) ∗ *d*
snd x = *snd* (*normalize-quot* (*fst x*, *snd x*)) ∗ *d*
d dvd fst x
d dvd snd x

$d \neq 0$.
then show *?thesis* **unfolding** *prod.collapse* **by** (*intro that*[*of d*])
qed

lemma *coprime-normalize-quot*:
coprime (*fst* (*normalize-quot x*)) (*snd* (*normalize-quot x*))
by (*simp add*: *normalize-quot-def case-prod-unfold div-mult-unit2*)
(*metis coprime-mult-self-right-iff div-gcd-coprime unit-div-mult-self unit-factor-is-unit*)

lemma *normalize-quot-in-normalized-fracts* [*simp*]: *normalize-quot x* ∈ *normalized-fracts*
by (*simp add*: *normalized-fracts-def coprime-normalize-quot case-prod-unfold*)

lemma *normalize-quot-eq-iff*:
assumes $b \neq 0$ $d \neq 0$
shows *normalize-quot* (*a*,*b*) = *normalize-quot* (*c*,*d*) ⟷ $a * d = b * c$
proof −
define *x y* **where** *x* = *normalize-quot* (*a*,*b*) **and** *y* = *normalize-quot* (*c*,*d*)
from *normalize-quotE*[*OF assms*(*1*), *of a*] *normalize-quotE*[*OF assms*(*2*), *of c*]
obtain *d1 d2*
where *a* = *fst x* ∗ *d1 b* = *snd x* ∗ *d1 c* = *fst y* ∗ *d2 d* = *snd y* ∗ *d2 d1* ≠ *0* *d2* ≠ *0*
unfolding *x-def y-def* **by** *metis*
hence $a * d = b * c$ ⟷ *fst x* ∗ *snd y* = *snd x* ∗ *fst y* **by** *simp*
also have ... ⟷ *fst x* = *fst y* ∧ *snd x* = *snd y*
by (*intro coprime-crossproduct′*) (*simp-all add*: *x-def y-def coprime-normalize-quot*)
also have ... ⟷ *x* = *y* **using** *prod-eqI* **by** *blast*
finally show *x* = *y* ⟷ $a * d = b * c$ **..**
qed

lemma *normalize-quot-eq-iff′*:
assumes *snd x* ≠ *0* *snd y* ≠ *0*
shows *normalize-quot x* = *normalize-quot y* ⟷ *fst x* ∗ *snd y* = *snd x* ∗ *fst y*
using *assms* **by** (*cases x*, *cases y*, *hypsubst*) (*subst normalize-quot-eq-iff*, *simp-all*)

lemma *normalize-quot-id*: *x* ∈ *normalized-fracts* ⟹ *normalize-quot x* = *x*
by (*auto simp*: *normalized-fracts-def normalize-quot-def case-prod-unfold*)

lemma *normalize-quot-idem* [*simp*]: *normalize-quot* (*normalize-quot x*) = *normalize-quot x*
by (*rule normalize-quot-id*) *simp-all*

lemma *fractrel-iff-normalize-quot-eq*:
fractrel x y ⟷ *normalize-quot x* = *normalize-quot y* ∧ *snd x* ≠ *0* ∧ *snd y* ≠ *0*
by (*cases x*, *cases y*) (*auto simp*: *fractrel-def normalize-quot-eq-iff*)

lemma *fractrel-normalize-quot-left*:
assumes *snd x* ≠ *0*
shows *fractrel* (*normalize-quot x*) *y* ⟷ *fractrel x y*

using *assms* **by** (*subst* (*1 2*) *fractrel-iff-normalize-quot-eq*) *auto*

lemma *fractrel-normalize-quot-right*:
assumes *snd x* ≠ *0*
shows *fractrel y* (*normalize-quot x*) ⟷ *fractrel y x*
using *assms* **by** (*subst* (*1 2*) *fractrel-iff-normalize-quot-eq*) *auto*

lift-definition *quot-of-fract* ::
'*a* :: {*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *fract* ⇒ '*a* × '*a*
is *normalize-quot*
by (*subst* (*asm*) *fractrel-iff-normalize-quot-eq*) *simp-all*

lemma *quot-to-fract-quot-of-fract* [*simp*]: *quot-to-fract* (*quot-of-fract x*) = *x*
unfolding *quot-to-fract-def*
proof *transfer*
fix *x* :: '*a* × '*a* **assume** *rel*: *fractrel x x*
define *x'* **where** *x'* = *normalize-quot x*
obtain *a b* **where** [*simp*]: *x* = (*a*, *b*) **by** (*cases x*)
from *rel* **have** *b* ≠ *0* **by** *simp*
from *normalize-quotE*[*OF this*, *of a*] **obtain** *d*
where
a = *fst* (*normalize-quot* (*a*, *b*)) ∗ *d*
b = *snd* (*normalize-quot* (*a*, *b*)) ∗ *d*
d dvd a
d dvd b
d ≠ *0* **.**
hence *a* = *fst x'* ∗ *d b* = *snd x'* ∗ *d d* ≠ *0 snd x'* ≠ *0* **by** (*simp-all add*: *x'-def*)
thus *fractrel* (*case x' of* (*a*, *b*) ⇒ *if b* = *0 then* (*0*, *1*) *else* (*a*, *b*)) *x*
by (*auto simp add*: *case-prod-unfold*)
qed

lemma *quot-of-fract-quot-to-fract*: *quot-of-fract* (*quot-to-fract x*) = *normalize-quot x*
proof (*cases snd x* = *0*)
case *True*
thus *?thesis* **unfolding** *quot-to-fract-def*
by *transfer* (*simp add*: *case-prod-unfold normalize-quot-def*)
next
case *False*
thus *?thesis* **unfolding** *quot-to-fract-def* **by** *transfer* (*simp add*: *case-prod-unfold*)
qed

lemma *quot-of-fract-quot-to-fract'*:
x ∈ *normalized-fracts* ⟹ *quot-of-fract* (*quot-to-fract x*) = *x*
unfolding *quot-to-fract-def* **by** *transfer* (*auto simp*: *normalize-quot-id*)

lemma *quot-of-fract-in-normalized-fracts* [*simp*]: *quot-of-fract x* ∈ *normalized-fracts*
by *transfer simp*

lemma *normalize-quotI*:
assumes $a * d = b * c$ $b \neq 0$ $(c, d) \in$ *normalized-fracts*
shows *normalize-quot* $(a, b) = (c, d)$
proof −
from *assms* **have** *normalize-quot* $(a, b) =$ *normalize-quot* (c, d)
by (*subst normalize-quot-eq-iff*) *auto*
also have $\ldots = (c, d)$ **by** (*intro normalize-quot-id*) *fact*
finally show *?thesis* **.**
qed

lemma *td-normalized-fract*:
type-definition quot-of-fract quot-to-fract normalized-fracts
by *standard* (*simp-all add*: *quot-of-fract-quot-to-fract'*)

lemma *quot-of-fract-add-aux*:
assumes *snd* $x \neq 0$ *snd* $y \neq 0$
shows (*fst* $x *$ *snd* $y +$ *fst* $y *$ *snd* $x) *$ (*snd* (*normalize-quot* $x) *$ *snd* (*normalize-quot* $y)) =$
snd $x *$ *snd* $y *$ (*fst* (*normalize-quot* $x) *$ *snd* (*normalize-quot* $y) +$
snd (*normalize-quot* $x) *$ *fst* (*normalize-quot* $y))$
proof −
from *normalize-quotE'*[*OF assms*(*1*)] **obtain** d
where d:
fst $x =$ *fst* (*normalize-quot* $x) * d$
snd $x =$ *snd* (*normalize-quot* $x) * d$
d *dvd fst* x
d *dvd snd* x
$d \neq 0$ **.**
from *normalize-quotE'*[*OF assms*(*2*)] **obtain** e
where e:
fst $y =$ *fst* (*normalize-quot* $y) * e$
snd $y =$ *snd* (*normalize-quot* $y) * e$
e *dvd fst* y
e *dvd snd* y
$e \neq 0$ **.**
show *?thesis* **by** (*simp-all add*: d e *algebra-simps*)
qed

locale *fract-as-normalized-quot*
begin
setup-lifting *td-normalized-fract*
end

lemma *quot-of-fract-add*:
quot-of-fract $(x + y) =$
(*let* $(a,b) =$ *quot-of-fract* x; $(c,d) =$ *quot-of-fract* y

```
    in  normalize-quot (a * d + b * c, b * d))
  by transfer (insert quot-of-fract-add-aux,
            simp-all add: Let-def case-prod-unfold normalize-quot-eq-iff)

lemma quot-of-fract-uminus:
  quot-of-fract (−x) = (let (a,b) = quot-of-fract x in (−a, b))
  by transfer (auto simp: case-prod-unfold Let-def normalize-quot-def dvd-neg-div
mult-unit-dvd-iff)

lemma quot-of-fract-diff:
  quot-of-fract (x − y) =
     (let (a,b) = quot-of-fract x; (c,d) = quot-of-fract y
      in  normalize-quot (a * d − b * c, b * d)) (is - = ?rhs)
proof −
  have x − y = x + −y by simp
  also have quot-of-fract ... = ?rhs
    by (simp only: quot-of-fract-add quot-of-fract-uminus Let-def case-prod-unfold)
simp-all
  finally show ?thesis .
qed

lemma normalize-quot-mult-coprime:
  assumes coprime a b coprime c d unit-factor b = 1 unit-factor d = 1
  defines e ≡ fst (normalize-quot (a, d)) and f ≡ snd (normalize-quot (a, d))
    and  g ≡ fst (normalize-quot (c, b)) and h ≡ snd (normalize-quot (c, b))
  shows   normalize-quot (a * c, b * d) = (e * g, f * h)
proof (rule normalize-quotI)
  from assms have gcd a b = 1 gcd c d = 1
    by simp-all
  from assms have b ≠ 0 d ≠ 0 by auto
  with assms have normalize b = b normalize d = d
    by (auto intro: normalize-unit-factor-eqI)
  from normalize-quotE [OF ‹b ≠ 0›, of c] obtain k
    where
      c = fst (normalize-quot (c, b)) * k
      b = snd (normalize-quot (c, b)) * k
      k dvd c k dvd b k ≠ 0 .
  note k = this [folded ‹gcd a b = 1› ‹gcd c d = 1› assms(3) assms(4)]
  from normalize-quotE [OF ‹d ≠ 0›, of a] obtain l
    where a = fst (normalize-quot (a, d)) * l
      d = snd (normalize-quot (a, d)) * l
      l dvd a l dvd d l ≠ 0 .
  note l = this [folded ‹gcd a b = 1› ‹gcd c d = 1› assms(3) assms(4)]
  from k l show a * c * (f * h) = b * d * (e * g)
    by (metis e-def f-def g-def h-def mult.commute mult.left-commute)
  from assms have [simp]: unit-factor f = 1 unit-factor h = 1
    by simp-all
 from assms have coprime e f coprime g h by (simp-all add: coprime-normalize-quot)
 with k l assms(1,2) ‹b ≠ 0› ‹d ≠ 0› ‹unit-factor b = 1› ‹unit-factor d = 1›
```

‹*normalize b = b*› ‹*normalize d = d*›
show (*e * g, f * h*) ∈ *normalized-fracts*
by (*simp add*: *normalized-fracts-def unit-factor-mult e-def f-def g-def h-def coprime-normalize-quot dvd-unit-factor-div unit-factor-gcd*)
(*metis coprime-mult-left-iff coprime-mult-right-iff*)
qed (*insert assms*(*3*,*4*), *auto*)

lemma *normalize-quot-mult*:
assumes *snd x ≠ 0 snd y ≠ 0*
shows *normalize-quot* (*fst x * fst y, snd x * snd y*) = *normalize-quot*
(*fst* (*normalize-quot x*) * *fst* (*normalize-quot y*),
snd (*normalize-quot x*) * *snd* (*normalize-quot y*))
proof −
from *normalize-quotE'*[*OF assms*(*1*)] **obtain** *d* **where** *d*:
fst x = fst (*normalize-quot x*) * *d*
snd x = snd (*normalize-quot x*) * *d*
d dvd fst x
d dvd snd x
d ≠ 0 **.**
from *normalize-quotE'*[*OF assms*(*2*)] **obtain** *e* **where** *e*:
fst y = fst (*normalize-quot y*) * *e*
snd y = snd (*normalize-quot y*) * *e*
e dvd fst y
e dvd snd y
e ≠ 0 **.**
show *?thesis* **by** (*simp-all add*: *d e algebra-simps normalize-quot-eq-iff*)
qed

lemma *quot-of-fract-mult*:
quot-of-fract (*x * y*) =
(*let* (*a*,*b*) = *quot-of-fract x*; (*c*,*d*) = *quot-of-fract y*;
(*e*,*f*) = *normalize-quot* (*a*,*d*); (*g*,*h*) = *normalize-quot* (*c*,*b*)
in (*e*∗*g*, *f*∗*h*))
by *transfer*
(*simp add*: *split-def Let-def coprime-normalize-quot normalize-quot-mult normalize-quot-mult-coprime*)

lemma *normalize-quot-0* [*simp*]:
normalize-quot (*0*, *x*) = (*0*, *1*) *normalize-quot* (*x*, *0*) = (*0*, *1*)
by (*simp-all add*: *normalize-quot-def*)

lemma *normalize-quot-eq-0-iff* [*simp*]: *fst* (*normalize-quot x*) = *0* ⟷ *fst x = 0* ∨ *snd x = 0*
by (*auto simp*: *normalize-quot-def case-prod-unfold Let-def div-mult-unit2 dvd-div-eq-0-iff*)

lemma *fst-quot-of-fract-0-imp*: *fst* (*quot-of-fract x*) = *0* ⟹ *snd* (*quot-of-fract x*) = *1*
by *transfer auto*

lemma *normalize-quot-swap*:
assumes $a \neq 0$ $b \neq 0$
defines $a' \equiv fst\ (normalize\text{-}quot\ (a,\ b))$ **and** $b' \equiv snd\ (normalize\text{-}quot\ (a,\ b))$
shows $normalize\text{-}quot\ (b,\ a) = (b'\ div\ unit\text{-}factor\ a',\ a'\ div\ unit\text{-}factor\ a')$
proof (*rule normalize-quotI*)
from *normalize-quotE*[*OF assms*(*2*), *of a*] **obtain** d **where**
$a = fst\ (normalize\text{-}quot\ (a,\ b)) * d$
$b = snd\ (normalize\text{-}quot\ (a,\ b)) * d$
$d\ dvd\ a\ d\ dvd\ b\ d \neq 0$.
note $d = this$ [*folded assms*(*3*,*4*)]
show $b * (a'\ div\ unit\text{-}factor\ a') = a * (b'\ div\ unit\text{-}factor\ a')$
using *assms*(*1*,*2*) d
by (*simp add*: *div-unit-factor* [*symmetric*] *unit-div-mult-swap mult-ac del*: *div-unit-factor*)
have *coprime* a' b' **by** (*simp add*: a'*-def* b'*-def coprime-normalize-quot*)
thus $(b'\ div\ unit\text{-}factor\ a',\ a'\ div\ unit\text{-}factor\ a') \in normalized\text{-}fracts$
using *assms*(*1*,*2*) d
by (*auto simp add*: *normalized-fracts-def ac-simps dvd-div-unit-iff elim*: *coprime-imp-coprime*)
qed *fact+*

lemma *quot-of-fract-inverse*:
quot-of-fract (*inverse x*) =
(*let* (*a*,*b*) = *quot-of-fract x*; *d* = *unit-factor a*
in if d = *0 then* (*0*, *1*) *else* (*b div d*, *a div d*))
proof (*transfer*, *goal-cases*)
case (*1 x*)
from *normalize-quot-swap*[*of fst x snd x*] **show** *?case*
by (*auto simp*: *Let-def case-prod-unfold*)
qed

lemma *normalize-quot-div-unit-left*:
fixes *x y u*
assumes *is-unit u*
defines $x' \equiv fst\ (normalize\text{-}quot\ (x,\ y))$ **and** $y' \equiv snd\ (normalize\text{-}quot\ (x,\ y))$
shows $normalize\text{-}quot\ (x\ div\ u,\ y) = (x'\ div\ u,\ y')$
proof (*cases y* = *0*)
case *False*
define *v* **where** *v* = *1 div u*
with ‹*is-unit u*› **have** *is-unit v* **and** *u*: $\bigwedge a.\ a\ div\ u = a * v$
by *simp-all*
from ‹*is-unit v*› **have** *coprime v* = *top*
by (*simp add*: *fun-eq-iff is-unit-left-imp-coprime*)
from *normalize-quotE*[*OF False*, *of x*] **obtain** d **where**
$x = fst\ (normalize\text{-}quot\ (x,\ y)) * d$
$y = snd\ (normalize\text{-}quot\ (x,\ y)) * d$
$d\ dvd\ x\ d\ dvd\ y\ d \neq 0$.
note $d = this$[*folded assms*(*2*,*3*)]
from *assms* **have** *coprime* x' y' *unit-factor* $y' = 1$

```
    by (simp-all add: coprime-normalize-quot)
  with d ‹coprime v = top› have normalize-quot (x * v, y) = (x' * v, y')
    by (auto simp: normalized-fracts-def intro: normalize-quotI)
  then show ?thesis
    by (simp add: u)
qed (simp-all add: assms)

lemma normalize-quot-div-unit-right:
  fixes x y u
  assumes is-unit u
  defines x' ≡ fst (normalize-quot (x, y)) and y' ≡ snd (normalize-quot (x, y))
  shows normalize-quot (x, y div u) = (x' * u, y')
proof (cases y = 0)
  case False
  from normalize-quotE[OF this, of x]
  obtain d where d:
    x = fst (normalize-quot (x, y)) * d
    y = snd (normalize-quot (x, y)) * d
    d dvd x d dvd y d ≠ 0 .
  note d = this[folded assms(2,3)]
 from assms have coprime x' y' unit-factor y' = 1 by (simp-all add: coprime-normalize-quot)
  with d ‹is-unit u› show ?thesis
   by (auto simp add: normalized-fracts-def is-unit-left-imp-coprime unit-div-eq-0-iff
intro: normalize-quotI)
qed (simp-all add: assms)

lemma normalize-quot-normalize-left:
  fixes x y u
  defines x' ≡ fst (normalize-quot (x, y)) and y' ≡ snd (normalize-quot (x, y))
  shows normalize-quot (normalize x, y) = (x' div unit-factor x, y')
  using normalize-quot-div-unit-left[of unit-factor x x y]
  by (cases x = 0) (simp-all add: assms)

lemma normalize-quot-normalize-right:
  fixes x y u
  defines x' ≡ fst (normalize-quot (x, y)) and y' ≡ snd (normalize-quot (x, y))
  shows normalize-quot (x, normalize y) = (x' * unit-factor y, y')
  using normalize-quot-div-unit-right[of unit-factor y x y]
  by (cases y = 0) (simp-all add: assms)

lemma quot-of-fract-0 [simp]: quot-of-fract 0 = (0, 1)
  by transfer auto

lemma quot-of-fract-1 [simp]: quot-of-fract 1 = (1, 1)
  by transfer (rule normalize-quotI, simp-all add: normalized-fracts-def)

lemma quot-of-fract-divide:
  quot-of-fract (x / y) = (if y = 0 then (0, 1) else
    (let (a,b) = quot-of-fract x; (c,d) = quot-of-fract y;
```

```
        (e,f) = normalize-quot (a,c); (g,h) = normalize-quot (d,b)
    in  (e * g, f * h))) (is - = ?rhs)
proof (cases y = 0)
  case False
  hence A: fst (quot-of-fract y) ≠ 0 by transfer auto
  have x / y = x * inverse y by (simp add: divide-inverse)
  also from False A have quot-of-fract ... = ?rhs
    by (simp only: quot-of-fract-mult quot-of-fract-inverse)
       (simp-all add: Let-def case-prod-unfold fst-quot-of-fract-0-imp
          normalize-quot-div-unit-left normalize-quot-div-unit-right
          normalize-quot-normalize-right normalize-quot-normalize-left)
  finally show ?thesis .
qed simp-all

lemma snd-quot-of-fract-nonzero [simp]: snd (quot-of-fract x) ≠ 0
  by transfer simp

lemma Fract-quot-of-fract [simp]: Fract (fst (quot-of-fract x)) (snd (quot-of-fract
x)) = x
  by transfer (simp del: fractrel-iff, subst fractrel-normalize-quot-left, simp)

lemma snd-quot-of-fract-Fract-whole:
  assumes y dvd x
  shows   snd (quot-of-fract (Fract x y)) = 1
  using assms by transfer (auto simp: normalize-quot-def Let-def gcd-proj2-if-dvd)

lemma fst-quot-of-fract-eq-0-iff [simp]: fst (quot-of-fract x) = 0 ⟷ x = 0
  by transfer simp

lemma coprime-quot-of-fract:
  coprime (fst (quot-of-fract x)) (snd (quot-of-fract x))
  by transfer (simp add: coprime-normalize-quot)

lemma unit-factor-snd-quot-of-fract: unit-factor (snd (quot-of-fract x)) = 1
  using quot-of-fract-in-normalized-fracts[of x]
  by (simp add: normalized-fracts-def case-prod-unfold)

lemma normalize-snd-quot-of-fract: normalize (snd (quot-of-fract x)) = snd (quot-of-fract
x)
  by (intro unit-factor-1-imp-normalized unit-factor-snd-quot-of-fract)

end
```

10 n-th powers and roots of naturals

```
theory Nth-Powers
  imports Primes
begin
```

10.1 The set of n-th powers

definition *is-nth-power* :: *nat* ⇒ *'a* :: *monoid-mult* ⇒ *bool* **where**
is-nth-power n x ⟷ (∃ *y*. *x* = *y* ^ *n*)

lemma *is-nth-power-nth-power* [*simp*, *intro*]: *is-nth-power n* (*x* ^ *n*)
by (*auto simp add*: *is-nth-power-def*)

lemma *is-nth-powerI* [*intro?*]: *x* = *y* ^ *n* ⟹ *is-nth-power n x*
by (*auto simp*: *is-nth-power-def*)

lemma *is-nth-powerE*: *is-nth-power n x* ⟹ (⋀*y*. *x* = *y* ^ *n* ⟹ *P*) ⟹ *P*
by (*auto simp*: *is-nth-power-def*)

abbreviation *is-square* **where** *is-square* ≡ *is-nth-power 2*

lemma *is-zeroth-power* [*simp*]: *is-nth-power 0 x* ⟷ *x* = *1*
by (*simp add*: *is-nth-power-def*)

lemma *is-first-power* [*simp*]: *is-nth-power 1 x*
by (*simp add*: *is-nth-power-def*)

lemma *is-first-power'* [*simp*]: *is-nth-power* (*Suc 0*) *x*
by (*simp add*: *is-nth-power-def*)

lemma *is-nth-power-0* [*simp*]: *n* > *0* ⟹ *is-nth-power n* (*0* :: *'a* :: *semiring-1*)
by (*auto simp*: *is-nth-power-def power-0-left intro*!: *exI*[*of - 0*])

lemma *is-nth-power-0-iff* [*simp*]: *is-nth-power n* (*0* :: *'a* :: *semiring-1*) ⟷ *n* > *0*
by (*cases n*) *auto*

lemma *is-nth-power-1* [*simp*]: *is-nth-power n 1*
by (*auto simp*: *is-nth-power-def intro*!: *exI*[*of - 1*])

lemma *is-nth-power-Suc-0* [*simp*]: *is-nth-power n* (*Suc 0*)
by (*metis One-nat-def is-nth-power-1*)

lemma *is-nth-power-conv-multiplicity*:
fixes *x* :: *'a* :: {*factorial-semiring*, *normalization-semidom-multiplicative*}
assumes *n* > *0*
shows *is-nth-power n* (*normalize x*) ⟷ (∀ *p*. *prime p* ⟶ *n dvd multiplicity p x*)
proof (*cases x* = *0*)
case *False*
show *?thesis*
proof (*safe intro*!: *is-nth-powerI elim*!: *is-nth-powerE*)
fix *y p* :: *'a* **assume** ∗: *normalize x* = *y* ^ *n prime p*
with *assms* **and** *False* **have** [*simp*]: *y* ≠ *0* **by** (*auto simp*: *power-0-left*)
have *multiplicity p x* = *multiplicity p* (*y* ^ *n*)

```
    by (metis *(1) multiplicity-normalize-right)
  with False and * and assms show n dvd multiplicity p x
    by (auto simp: prime-elem-multiplicity-power-distrib)
 next
  assume *: ∀p. prime p ⟶ n dvd multiplicity p x
  have multiplicity p ((∏p∈prime-factors x. p ^ (multiplicity p x div n)) ^ n) =
        multiplicity p x if prime p for p
  proof −
    from that and * have n dvd multiplicity p x by blast
    have multiplicity p x = 0 if p ∉ prime-factors x
      using that and ‹prime p› by (simp add: prime-factors-multiplicity)
      with that and * and assms show ?thesis unfolding prod-power-distrib
power-mult [symmetric]
     by (subst multiplicity-prod-prime-powers) (auto simp: in-prime-factors-imp-prime)
  qed
  with assms False
    have normalize x = normalize ((∏p∈prime-factors x. p ^ (multiplicity p x
div n)) ^ n)
     by (intro multiplicity-eq-imp-eq) (auto simp: multiplicity-prod-prime-powers)
  thus normalize x = normalize (∏p∈prime-factors x. p ^ (multiplicity p x div
n)) ^ n
     by (simp add: normalize-power)
 qed
qed (insert assms, auto)

lemma is-nth-power-conv-multiplicity-nat:
 assumes n > 0
 shows   is-nth-power n (x :: nat) ⟷ (∀p. prime p ⟶ n dvd multiplicity p x)
 using is-nth-power-conv-multiplicity[OF assms, of x] by simp

lemma is-nth-power-mult:
 assumes is-nth-power n a is-nth-power n b
 shows   is-nth-power n (a * b :: 'a :: comm-monoid-mult)
 by (metis assms is-nth-power-def power-mult-distrib)

lemma is-nth-power-mult-coprime-natD:
 fixes a b :: nat
 assumes coprime a b is-nth-power n (a * b) a > 0 b > 0
 shows   is-nth-power n a is-nth-power n b
proof −
 have A: is-nth-power n a if coprime a b is-nth-power n (a * b) a ≠ 0 b ≠ 0 n >
0
   for a b :: nat unfolding is-nth-power-conv-multiplicity-nat[OF ‹n > 0›]
 proof safe
   fix p :: nat assume p: prime p
   from ‹coprime a b› have ¬(p dvd a ∧ p dvd b)
     using coprime-common-divisor-nat[of a b p] p by auto
   moreover from that and p
     have n dvd multiplicity p a + multiplicity p b
```

by (*auto simp*: *is-nth-power-conv-multiplicity-nat prime-elem-multiplicity-mult-distrib*)
ultimately show *n dvd multiplicity p a*
by (*auto simp*: *not-dvd-imp-multiplicity-0*)
qed
from *A* [*of a b*] *assms* **show** *is-nth-power n a*
by (*cases n = 0*) *simp-all*
from *A* [*of b a*] *assms* **show** *is-nth-power n b*
by (*cases n = 0*) (*simp-all add*: *ac-simps*)
qed

lemma *is-nth-power-mult-coprime-nat-iff*:
fixes *a b* :: *nat*
assumes *coprime a b*
shows *is-nth-power n* (*a* ∗ *b*) ⟷ *is-nth-power n a* ∧*is-nth-power n b*
using *assms*
by (*cases a = 0*; *cases b = 0*)
(*auto intro*: *is-nth-power-mult dest*: *is-nth-power-mult-coprime-natD*[*of a b n*]
simp del: *One-nat-def*)

lemma *is-nth-power-prime-power-nat-iff*:
fixes *p* :: *nat* **assumes** *prime p*
shows *is-nth-power n* (*p* ^ *k*) ⟷ *n dvd k*
using *assms*
by (*cases n > 0*)
(*auto simp*: *is-nth-power-conv-multiplicity-nat prime-elem-multiplicity-power-distrib*)

lemma *is-nth-power-nth-power*′:
assumes *n dvd n*′
shows *is-nth-power n* (*m* ^ *n*′)
by (*metis assms dvd-div-mult-self is-nth-power-def power-mult*)

definition *is-nth-power-nat* :: *nat* ⇒ *nat* ⇒ *bool*
where [*code-abbrev*]: *is-nth-power-nat* = *is-nth-power*

lemma *is-nth-power-nat-code* [*code*]:
is-nth-power-nat n m =
(*if n* = *0 then m* = *1*
else if m = *0 then n* > *0*
else if n = *1 then True*
else (∃ *k*∈{*1*..*m*}. *k* ^ *n* = *m*))
by (*auto simp*: *is-nth-power-nat-def is-nth-power-def power-eq-iff-eq-base self-le-power*)

lemma *is-nth-power-mult-cancel-left*:
fixes *a b* :: ′*a* :: *semiring-gcd*
assumes *is-nth-power n a a* ≠ *0*
shows *is-nth-power n* (*a* ∗ *b*) ⟷ *is-nth-power n b*
proof (*cases n > 0*)
case *True*
show *?thesis*

```
  proof
    assume is-nth-power n (a * b)
    then obtain x where x: a * b = x ^ n
      by (elim is-nth-powerE)
    obtain y where y: a = y ^ n
      using assms by (elim is-nth-powerE)
    have y ^ n dvd x ^ n
      by (simp flip: x y)
    hence y dvd x
      using ‹n > 0› by simp
    then obtain z where z: x = y * z
      by (elim dvdE)
    with ‹a ≠ 0› show is-nth-power n b
      by (metis  is-nth-powerI mult-left-cancel power-mult-distrib x y)
  qed (use assms in ‹auto intro: is-nth-power-mult›)
qed (use assms in auto)

lemma is-nth-power-mult-cancel-right:
  fixes a b :: 'a :: semiring-gcd
  assumes is-nth-power n b b ≠ 0
  shows   is-nth-power n (a * b) ⟷ is-nth-power n a
  by (metis assms is-nth-power-mult-cancel-left mult.commute)
```

10.2 The n-root of a natural number

```
definition nth-root-nat :: nat ⇒ nat ⇒ nat where
  nth-root-nat k n = (if k = 0 then 0 else Max {m. m ^ k ≤ n})

lemma zeroth-root-nat [simp]: nth-root-nat 0 n = 0
  by (simp add: nth-root-nat-def)

lemma nth-root-nat-aux1:
  assumes k > 0
  shows   {m::nat. m ^ k ≤ n} ⊆ {..n}
proof safe
  fix m assume m ^ k ≤ n
  show m ≤ n
  proof (cases m = 0)
    case False
    with assms have m ^ 1 ≤ m ^ k by (intro power-increasing) simp-all
    also note ‹m ^ k ≤ n›
    finally show ?thesis by simp
  qed simp-all
qed

lemma nth-root-nat-aux2:
  assumes k > 0
  shows   finite {m::nat. m ^ k ≤ n} {m::nat. m ^ k ≤ n} ≠ {}
proof −
```

from *assms* **have** $\{m.\ m \mathbin{\hat{}} k \leq n\} \subseteq \{..n\}$ **by** (*rule nth-root-nat-aux1*)
moreover have *finite* $\{..n\}$ **by** *simp*
ultimately show *finite* $\{m::nat.\ m \mathbin{\hat{}} k \leq n\}$ **by** (*rule finite-subset*)
next
from *assms* **show** $\{m::nat.\ m \mathbin{\hat{}} k \leq n\} \neq \{\}$ **by** (*auto intro*!: *exI*[*of - 0*] *simp*: *power-0-left*)
qed

lemma
assumes $k > 0$
shows *nth-root-nat-power-le*: *nth-root-nat k n* $\mathbin{\hat{}}$ $k \leq n$
and *nth-root-nat-ge*: $x \mathbin{\hat{}} k \leq n \Longrightarrow x \leq$ *nth-root-nat k n*
using *Max-in*[*OF nth-root-nat-aux2*[*OF assms*], *of n*]
Max-ge[*OF nth-root-nat-aux2*(*1*)[*OF assms*], *of x n*] *assms*
by (*auto simp*: *nth-root-nat-def*)

lemma *nth-root-nat-less*:
assumes $k > 0$ $x \mathbin{\hat{}} k > n$
shows *nth-root-nat k n* $< x$
by (*meson assms nth-root-nat-power-le order.strict-trans1 power-less-imp-less-base zero-le*)

lemma *nth-root-nat-unique*:
assumes $m \mathbin{\hat{}} k \leq n$ $(m + 1) \mathbin{\hat{}} k > n$
shows *nth-root-nat k n* $= m$
proof (*cases* $k > 0$)
case *True*
from *nth-root-nat-less*[*OF* ‹$k > 0$› *assms*(*2*)]
have *nth-root-nat k n* $\leq m$ **by** *simp*
moreover from ‹$k > 0$› **and** *assms*(*1*) **have** *nth-root-nat k n* $\geq m$
by (*intro nth-root-nat-ge*)
ultimately show *?thesis* **by** (*rule antisym*)
qed (*insert assms*, *auto*)

lemma *nth-root-nat-0* [*simp*]: *nth-root-nat k 0* $= 0$
by (*simp add*: *nth-root-nat-def*)

lemma *nth-root-nat-1* [*simp*]: $k > 0 \Longrightarrow$ *nth-root-nat k 1* $= 1$
by (*rule nth-root-nat-unique*) (*auto simp del*: *One-nat-def*)

lemma *nth-root-nat-Suc-0* [*simp*]: $k > 0 \Longrightarrow$ *nth-root-nat k* (*Suc 0*) $=$ *Suc 0*
using *One-nat-def is-nth-power-nat-def nth-root-nat-1*
by *presburger*

lemma *first-root-nat* [*simp*]: *nth-root-nat 1 n* $= n$
by (*intro nth-root-nat-unique*) *auto*

lemma *first-root-nat′* [*simp*]: *nth-root-nat* (*Suc 0*) $n = n$
by (*intro nth-root-nat-unique*) *auto*

lemma *nth-root-nat-code-naive'*:
nth-root-nat k n = (if k = 0 then 0 else Max (Set.filter (λm. m ^ k ≤ n) {..n}))
proof (*cases k > 0*)
case *True*
then have {*m. m ^ k ≤ n*} ⊆ {*..n*} **by** (*rule nth-root-nat-aux1*)
then have *Set.filter (λm. m ^ k ≤ n) {..n} = {m. m ^ k ≤ n}*
by (*auto simp:*)
with *True* **show** *?thesis*
by (*simp add*: *nth-root-nat-def*)
qed *simp*

function *nth-root-nat-aux* :: *nat ⇒ nat ⇒ nat ⇒ nat ⇒ nat* **where**
nth-root-nat-aux m k acc n =
(*let acc' = (k + 1) ^ m*
in if k ≥ n ∨ acc' > n then k else nth-root-nat-aux m (k+1) acc' n)
by *auto*
termination by (*relation measure* (*λ*(*-*,*k*,*-*,*n*). *n − k*), *goal-cases*) *auto*

lemma *nth-root-nat-aux-le*:
assumes *k ^ m ≤ n m > 0*
shows *nth-root-nat-aux m k (k ^ m) n ^ m ≤ n*
using *assms*
by (*induction m k k ^ m n rule*: *nth-root-nat-aux.induct*) (*auto simp*: *Let-def*)

lemma *nth-root-nat-aux-gt*:
assumes *m > 0*
shows (*nth-root-nat-aux m k (k ^ m) n + 1*) *^ m > n*
using *assms*
proof (*induction m k k ^ m n rule*: *nth-root-nat-aux.induct*)
case (*1 m k n*)
have *n < Suc k ^ m* **if** *n ≤ k*
proof −
note *that*
also have *k < Suc k ^ 1* **by** *simp*
also from ‹*m > 0*› **have** *... ≤ Suc k ^ m*
by (*intro power-increasing*) *simp-all*
finally show *?thesis* **.**
qed
with *1* **show** *?case* **by** (*auto simp*: *Let-def*)
qed

lemma *nth-root-nat-aux-correct*:
assumes *k ^ m ≤ n m > 0*
shows *nth-root-nat-aux m k (k ^ m) n = nth-root-nat m n*
by (*metis assms nth-root-nat-aux-gt nth-root-nat-aux-le nth-root-nat-unique*)

lemma *nth-root-nat-naive-code* [*code*]:
nth-root-nat m n = (if m = 0 ∨ n = 0 then 0 else if m = 1 ∨ n = 1 then n else

nth-root-nat-aux m 1 1 n)
using *nth-root-nat-aux-correct*[*of 1 m n*] **by** *auto*

lemma *nth-root-nat-nth-power* [*simp*]: $k > 0 \Longrightarrow$ *nth-root-nat k* ($n \mathbin{\hat{}} k$) $= n$
by (*intro nth-root-nat-unique order.refl power-strict-mono*) *simp-all*

lemma *nth-root-nat-nth-power'*:
assumes $k > 0$ *k dvd m*
shows *nth-root-nat k* ($n \mathbin{\hat{}} m$) $= n \mathbin{\hat{}} (m$ *div k*)
by (*metis assms dvd-div-mult-self nth-root-nat-nth-power power-mult*)

lemma *nth-root-nat-mono*:
assumes $m \leq n$
shows *nth-root-nat k m* $\leq$ *nth-root-nat k n*
proof (*cases k* $= 0$)
case *False*
with *assms* **show** *?thesis* **unfolding** *nth-root-nat-def*
using *nth-root-nat-aux2*[*of k m*] *nth-root-nat-aux2*[*of k n*]
by (*auto intro*!: *Max-mono*)
qed *auto*

end

11 Polynomials, fractions and rings

theory *Polynomial-Factorial*
imports
Complex-Main
Polynomial
Normalized-Fraction
begin

11.1 Lifting elements into the field of fractions

definition *to-fract* :: *'a* :: *idom* $\Rightarrow$ *'a fract*
where *to-fract x* = *Fract x 1*
— FIXME: more idiomatic name, abbreviation

lemma *to-fract-0* [*simp*]: *to-fract 0* = *0*
by (*simp add*: *to-fract-def eq-fract Zero-fract-def*)

lemma *to-fract-1* [*simp*]: *to-fract 1* = *1*
by (*simp add*: *to-fract-def eq-fract One-fract-def*)

lemma *to-fract-add* [*simp*]: *to-fract* ($x + y$) = *to-fract x* + *to-fract y*
by (*simp add*: *to-fract-def*)

lemma *to-fract-diff* [*simp*]: *to-fract* ($x - y$) = *to-fract x* − *to-fract y*

by (*simp add*: *to-fract-def*)

lemma *to-fract-uminus* [*simp*]: *to-fract* (−*x*) = −*to-fract x*
by (*simp add*: *to-fract-def*)

lemma *to-fract-mult* [*simp*]: *to-fract* (*x* ∗ *y*) = *to-fract x* ∗ *to-fract y*
by (*simp add*: *to-fract-def*)

lemma *to-fract-eq-iff* [*simp*]: *to-fract x* = *to-fract y* ⟷ *x* = *y*
by (*simp add*: *to-fract-def eq-fract*)

lemma *to-fract-eq-0-iff* [*simp*]: *to-fract x* = *0* ⟷ *x* = *0*
by (*simp add*: *to-fract-def Zero-fract-def eq-fract*)

lemma *to-fract-quot-of-fract*:
assumes *snd* (*quot-of-fract x*) = *1*
shows *to-fract* (*fst* (*quot-of-fract x*)) = *x*
proof −
have *x* = *Fract* (*fst* (*quot-of-fract x*)) (*snd* (*quot-of-fract x*)) **by** *simp*
also note *assms*
finally show *?thesis* **by** (*simp add*: *to-fract-def*)
qed

lemma *Fract-conv-to-fract*: *Fract a b* = *to-fract a* / *to-fract b*
by (*simp add*: *to-fract-def*)

lemma *quot-of-fract-to-fract* [*simp*]: *quot-of-fract* (*to-fract x*) = (*x*, *1*)
unfolding *to-fract-def* **by** *transfer* (*simp add*: *normalize-quot-def*)

lemma *snd-quot-of-fract-to-fract* [*simp*]: *snd* (*quot-of-fract* (*to-fract x*)) = *1*
unfolding *to-fract-def* **by** (*rule snd-quot-of-fract-Fract-whole*) *simp-all*

11.2 Lifting polynomial coefficients to the field of fractions

abbreviation (*input*) *fract-poly* :: ‹*'a*::*idom poly* ⇒ *'a fract poly*›
where *fract-poly* ≡ *map-poly to-fract*

abbreviation (*input*) *unfract-poly* :: ‹*'a*::{*ring-gcd*,*semiring-gcd-mult-normalize*,*idom-divide*}
fract poly ⇒ *'a poly*›
where *unfract-poly* ≡ *map-poly* (*fst* ∘ *quot-of-fract*)

lemma *fract-poly-smult* [*simp*]: *fract-poly* (*smult c p*) = *smult* (*to-fract c*) (*fract-poly*
p)
by (*simp add*: *smult-conv-map-poly map-poly-map-poly o-def*)

lemma *fract-poly-0* [*simp*]: *fract-poly 0* = *0*
by (*simp add*: *poly-eqI coeff-map-poly*)

lemma *fract-poly-1* [*simp*]: *fract-poly 1* = *1*

by (*simp add*: *map-poly-pCons*)

lemma *fract-poly-add* [*simp*]:
fract-poly (*p* + *q*) = *fract-poly p* + *fract-poly q*
by (*intro poly-eqI*) (*simp-all add*: *coeff-map-poly*)

lemma *fract-poly-diff* [*simp*]:
fract-poly (*p* − *q*) = *fract-poly p* − *fract-poly q*
by (*intro poly-eqI*) (*simp-all add*: *coeff-map-poly*)

lemma *to-fract-sum* [*simp*]: *to-fract* (*sum f A*) = *sum* (λ*x*. *to-fract* (*f x*)) *A*
by (*cases finite A*, *induction A rule*: *finite-induct*) *simp-all*

lemma *fract-poly-mult* [*simp*]:
fract-poly (*p* ∗ *q*) = *fract-poly p* ∗ *fract-poly q*
by (*intro poly-eqI*) (*simp-all add*: *coeff-map-poly coeff-mult*)

lemma *fract-poly-eq-iff* [*simp*]: *fract-poly p* = *fract-poly q* ⟷ *p* = *q*
by (*auto simp*: *poly-eq-iff coeff-map-poly*)

lemma *fract-poly-eq-0-iff* [*simp*]: *fract-poly p* = *0* ⟷ *p* = *0*
using *fract-poly-eq-iff*[*of p 0*] **by** (*simp del*: *fract-poly-eq-iff*)

lemma *fract-poly-dvd*: *p dvd q* ⟹ *fract-poly p dvd fract-poly q*
by *auto*

lemma *prod-mset-fract-poly*:
(∏ *x*∈#*A*. *map-poly to-fract* (*f x*)) = *fract-poly* (*prod-mset* (*image-mset f A*))
by (*induct A*) (*simp-all add*: *ac-simps*)

lemma *is-unit-fract-poly-iff*:
p dvd 1 ⟷ *fract-poly p dvd 1* ∧ *content p* = *1*
proof *safe*
assume *A*: *p dvd 1*
with *fract-poly-dvd* [*of p 1*] **show** *is-unit* (*fract-poly p*)
by *simp*
from *A* **show** *content p* = *1*
by (*auto simp*: *is-unit-poly-iff normalize-1-iff*)
next
assume *A*: *fract-poly p dvd 1* **and** *B*: *content p* = *1*
from *A* **obtain** *c* **where** *c*: *fract-poly p* = [:*c*:] **by** (*auto simp*: *is-unit-poly-iff*)
{
fix *n* :: *nat* **assume** *n* > *0*
have *to-fract* (*coeff p n*) = *coeff* (*fract-poly p*) *n* **by** (*simp add*: *coeff-map-poly*)
also note *c*
also from ‹*n* > *0*› **have** *coeff* [:*c*:] *n* = *0* **by** (*simp add*: *coeff-pCons split*: *nat.splits*)
finally have *coeff p n* = *0* **by** *simp*
}

hence *degree p ≤ 0* **by** (*intro degree-le*) *simp-all*
with *B* **show** *p dvd 1* **by** (*auto simp*: *is-unit-poly-iff normalize-1-iff elim*!: *degree-eq-zeroE*)
qed

lemma *fract-poly-is-unit*: *p dvd 1 ⟹ fract-poly p dvd 1*
using *fract-poly-dvd*[*of p 1*] **by** *simp*

lemma *fract-poly-smult-eqE*:
fixes *c* :: *'a* :: {*idom-divide*,*ring-gcd*,*semiring-gcd-mult-normalize*} *fract*
assumes *fract-poly p = smult c* (*fract-poly q*)
obtains *a b*
where *c = to-fract b / to-fract a smult a p = smult b q coprime a b normalize a = a*
proof −
define *a b* **where** *a = fst* (*quot-of-fract c*) **and** *b = snd* (*quot-of-fract c*)
have *smult* (*to-fract a*) (*fract-poly q*) = *smult* (*to-fract b*) (*fract-poly p*)
by (*subst smult-eq-iff*) (*simp-all add*: *a-def b-def Fract-conv-to-fract* [*symmetric*] *assms*)
hence *fract-poly* (*smult a q*) = *fract-poly* (*smult b p*) **by** (*simp del*: *fract-poly-eq-iff*)
hence *smult b p = smult a q* **by** (*simp only*: *fract-poly-eq-iff*)
moreover have *c = to-fract a / to-fract b coprime b a normalize b = b*
by (*simp-all add*: *a-def b-def coprime-quot-of-fract* [*of c*] *ac-simps normalize-snd-quot-of-fract Fract-conv-to-fract* [*symmetric*])
ultimately show *?thesis* **by** (*intro that*[*of a b*])
qed

11.3 Fractional content

abbreviation (*input*) *Lcm-coeff-denoms*
:: *'a* :: {*semiring-Gcd*,*idom-divide*,*ring-gcd*,*semiring-gcd-mult-normalize*} *fract poly ⇒ 'a*
where *Lcm-coeff-denoms p ≡ Lcm* (*snd ‘ quot-of-fract ‘ set* (*coeffs p*))

definition *fract-content* ::
'a :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *fract poly ⇒ 'a fract* **where**
fract-content p =
(*let d = Lcm-coeff-denoms p in Fract* (*content* (*unfract-poly* (*smult* (*to-fract d*) *p*))) *d*)

definition *primitive-part-fract* ::
'a :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *fract poly ⇒ 'a poly* **where**
primitive-part-fract p =
primitive-part (*unfract-poly* (*smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*))

lemma *primitive-part-fract-0* [*simp*]: *primitive-part-fract 0 = 0*
by (*simp add*: *primitive-part-fract-def*)

lemma *fract-content-eq-0-iff* [*simp*]:
fract-content p = 0 ⟷ p = 0
unfolding *fract-content-def Let-def Zero-fract-def*
by (*subst eq-fract*) (*auto simp*: *Lcm-0-iff map-poly-eq-0-iff*)

lemma *content-primitive-part-fract* [*simp*]:
fixes *p* :: *'a* :: {*semiring-gcd-mult-normalize*,
factorial-semiring, *ring-gcd*, *semiring-Gcd*,*idom-divide*} *fract poly*
shows *p ≠ 0 ⟹ content* (*primitive-part-fract p*) = *1*
unfolding *primitive-part-fract-def*
by (*rule content-primitive-part*)
(*auto simp*: *primitive-part-fract-def map-poly-eq-0-iff Lcm-0-iff*)

lemma *content-times-primitive-part-fract*:
smult (*fract-content p*) (*fract-poly* (*primitive-part-fract p*)) = *p*
proof −
define *p'* **where** *p'* = *unfract-poly* (*smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*)
have *fract-poly p'* =
map-poly (*to-fract* ∘ *fst* ∘ *quot-of-fract*) (*smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*)
unfolding *primitive-part-fract-def p'-def*
by (*subst map-poly-map-poly*) (*simp-all add*: *o-assoc*)
also have ... = *smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*
proof (*intro map-poly-idI*, *unfold o-apply*)
fix *c* **assume** *c* ∈ *set* (*coeffs* (*smult* (*to-fract* (*Lcm-coeff-denoms p*)) *p*))
then obtain *c'* **where** *c*: *c'* ∈ *set* (*coeffs p*) *c* = *to-fract* (*Lcm-coeff-denoms p*) ∗ *c'*
by (*auto simp add*: *Lcm-0-iff coeffs-smult split*: *if-splits*)
note *c*(*2*)
also have *c'* = *Fract* (*fst* (*quot-of-fract c'*)) (*snd* (*quot-of-fract c'*))
by *simp*
also have *to-fract* (*Lcm-coeff-denoms p*) ∗ ... =
Fract (*Lcm-coeff-denoms p* ∗ *fst* (*quot-of-fract c'*)) (*snd* (*quot-of-fract c'*))
unfolding *to-fract-def* **by** (*subst mult-fract*) *simp-all*
also have *snd* (*quot-of-fract* ...) = *1*
by (*intro snd-quot-of-fract-Fract-whole dvd-mult2 dvd-Lcm*) (*insert c*(*1*), *auto*)
finally show *to-fract* (*fst* (*quot-of-fract c*)) = *c*
by (*rule to-fract-quot-of-fract*)
qed
also have *p'* = *smult* (*content p'*) (*primitive-part p'*)
by (*rule content-times-primitive-part* [*symmetric*])
also have *primitive-part p'* = *primitive-part-fract p*
by (*simp add*: *primitive-part-fract-def p'-def*)
also have *fract-poly* (*smult* (*content p'*) (*primitive-part-fract p*)) =
smult (*to-fract* (*content p'*)) (*fract-poly* (*primitive-part-fract p*)) **by** *simp*
finally have *smult* (*to-fract* (*content p'*)) (*fract-poly* (*primitive-part-fract p*)) =

smult (*to-fract* (*Lcm-coeff-denoms p*)) *p* .
thus *?thesis*
by (*subst* (*asm*) *smult-eq-iff*)
(*auto simp add*: *Let-def p'-def Fract-conv-to-fract field-simps Lcm-0-iff fract-content-def*)
qed

lemma *fract-content-fract-poly* [*simp*]: *fract-content* (*fract-poly p*) = *to-fract* (*content p*)
proof −
have *Lcm-coeff-denoms* (*fract-poly p*) = *1*
by (*auto simp*: *set-coeffs-map-poly*)
hence *fract-content* (*fract-poly p*) =
to-fract (*content* (*map-poly* (*fst* ∘ *quot-of-fract* ∘ *to-fract*) *p*))
by (*simp add*: *fract-content-def to-fract-def fract-collapse map-poly-map-poly del*: *Lcm-1-iff*)
also have *map-poly* (*fst* ∘ *quot-of-fract* ∘ *to-fract*) *p* = *p*
by (*intro map-poly-idI*) *simp-all*
finally show *?thesis* .
qed

lemma *content-decompose-fract*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,
semiring-gcd-mult-normalize} *fract poly*
obtains *c p'* **where** *p* = *smult c* (*map-poly to-fract p'*) *content p'* = *1*
proof (*cases p* = *0*)
case *True*
hence *p* = *smult 0* (*map-poly to-fract 1*) *content 1* = *1* **by** *simp-all*
thus *?thesis* ..
next
case *False*
thus *?thesis*
by (*rule that*[*OF content-times-primitive-part-fract* [*symmetric*] *content-primitive-part-fract*])
qed

lemma *fract-poly-dvdD*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,
semiring-gcd-mult-normalize} *poly*
assumes *fract-poly p dvd fract-poly q content p* = *1*
shows *p dvd q*
proof −
from *assms*(*1*) **obtain** *r* **where** *r*: *fract-poly q* = *fract-poly p* ∗ *r* **by** (*erule dvdE*)
from *content-decompose-fract*[*of r*]
obtain *c r'* **where** *r'*: *r* = *smult c* (*map-poly to-fract r'*) *content r'* = *1* .
from *r r'* **have** *eq*: *fract-poly q* = *smult c* (*fract-poly* (*p* ∗ *r'*)) **by** *simp*
from *fract-poly-smult-eqE*[*OF this*] **obtain** *a b*
where *ab*:
c = *to-fract b* / *to-fract a*

*smult a q = smult b (p * r')*
coprime a b
normalize a = a **.**
have *content (smult a q) = content (smult b (p * r'))* **by** (*simp only*: *ab(2)*)
hence *eq'*: *normalize b = a * content q* **by** (*simp add*: *assms content-mult r' ab(4)*)
have *1 = gcd a (normalize b)* **by** (*simp add*: *ab*)
also note *eq'*
also have *gcd a (a * content q) = a* **by** (*simp add*: *gcd-proj1-if-dvd ab(4)*)
finally have [*simp*]: *a = 1* **by** *simp*
from *eq ab* **have** *q = p * ([:b:] * r')* **by** *simp*
thus *?thesis* **by** (*rule dvdI*)
qed

11.4 Polynomials over a field are a Euclidean ring

context
begin

interpretation *field-poly*:
normalization-euclidean-semiring-multiplicative **where** *zero = 0 :: 'a :: field poly*
and *one = 1* **and** *plus = plus* **and** *minus = minus*
and *times = times*
and *normalize = λp. smult (inverse (lead-coeff p)) p*
and *unit-factor = λp. [:lead-coeff p:]*
and *euclidean-size = λp. if p = 0 then 0 else 2 ^ degree p*
and *divide = divide* **and** *modulo = modulo*
rewrites *dvd.dvd (times :: 'a poly ⇒ -) = Rings.dvd*
and *comm-monoid-mult.prod-mset times 1 = prod-mset*
and *comm-semiring-1.irreducible times 1 0 = irreducible*
and *comm-semiring-1.prime-elem times 1 0 = prime-elem*
proof −
show *dvd.dvd (times :: 'a poly ⇒ -) = Rings.dvd*
by (*simp add*: *dvd-dict*)
show *comm-monoid-mult.prod-mset times 1 = prod-mset*
by (*simp add*: *prod-mset-dict*)
show *comm-semiring-1.irreducible times 1 0 = irreducible*
by (*simp add*: *irreducible-dict*)
show *comm-semiring-1.prime-elem times 1 0 = prime-elem*
by (*simp add*: *prime-elem-dict*)
show *class.normalization-euclidean-semiring-multiplicative divide plus minus (0 :: 'a poly) times 1*
modulo (λp. if p = 0 then 0 else 2 ^ degree p)
(λp. [:lead-coeff p:]) (λp. smult (inverse (lead-coeff p)) p)
proof (*standard*, *fold dvd-dict*)
fix *p :: 'a poly*
show *[:lead-coeff p:] * smult (inverse (lead-coeff p)) p = p*
by (*cases p = 0*) *simp-all*
next

fix $p :: 'a\ poly$ **assume** *is-unit p*
then show [:*lead-coeff p*:] = *p*
by (*elim is-unit-polyE*) (*auto simp*: *monom-0 one-poly-def field-simps*)
next
fix $p :: 'a\ poly$ **assume** $p \neq 0$
then show *is-unit* [:*lead-coeff p*:]
by (*simp add*: *is-unit-pCons-iff*)
next
fix *a b* :: $'a\ poly$ **assume** *is-unit a*
thus [:*lead-coeff* (*a* ∗ *b*):] = *a* ∗ [:*lead-coeff b*:]
by (*auto elim*!: *is-unit-polyE*)
qed (*auto simp*: *lead-coeff-mult Rings.div-mult-mod-eq intro*!: *degree-mod-less'* *degree-mult-right-le*)
qed

lemma *field-poly-irreducible-imp-prime*:
prime-elem p **if** *irreducible p* **for** *p* :: $'a$ :: *field poly*
using *that* **by** (*fact field-poly.irreducible-imp-prime-elem*)

lemma *field-poly-prod-mset-prime-factorization*:
prod-mset (*field-poly.prime-factorization p*) = *smult* (*inverse* (*lead-coeff p*)) *p*
if $p \neq 0$ **for** *p* :: $'a$ :: *field poly*
using *that* **by** (*fact field-poly.prod-mset-prime-factorization*)

lemma *field-poly-in-prime-factorization-imp-prime*:
prime-elem p **if** *p* ∈# *field-poly.prime-factorization x*
for *p* :: $'a$ :: *field poly*
by (*rule field-poly.prime-imp-prime-elem*, *rule field-poly.in-prime-factors-imp-prime*) (*fact that*)

11.5 Primality and irreducibility in polynomial rings

lemma *nonconst-poly-irreducible-iff*:
fixes *p* :: $'a$ :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *poly*
assumes *degree p* ≠ *0*
shows *irreducible p* ⟷ *irreducible* (*fract-poly p*) ∧ *content p* = *1*
proof *safe*
assume *p*: *irreducible p*

from *content-decompose*[*of p*] **obtain** *p'* **where** *p'*: *p* = *smult* (*content p*) *p'* *content p'* = *1* .
hence *p* = [:*content p*:] ∗ *p'* **by** *simp*
from *p this* **have** [:*content p*:] *dvd 1* ∨ *p' dvd 1* **by** (*rule irreducibleD*)
moreover have ¬*p' dvd 1*
proof
assume *p' dvd 1*
hence *degree p* = *0* **by** (*subst p'*) (*auto simp*: *is-unit-poly-iff*)
with *assms* **show** *False* **by** *contradiction*

```
 qed
 ultimately show [simp]: content p = 1 by (simp add: is-unit-const-poly-iff)

 show irreducible (map-poly to-fract p)
 proof (rule irreducibleI)
  have fract-poly p = 0 ⟷ p = 0 by (intro map-poly-eq-0-iff) auto
  with assms show map-poly to-fract p ≠ 0 by auto
 next
  show ¬is-unit (fract-poly p)
  proof
   assume is-unit (map-poly to-fract p)
   hence degree (map-poly to-fract p) = 0
    by (auto simp: is-unit-poly-iff)
   hence degree p = 0 by (simp add: degree-map-poly)
   with assms show False by contradiction
 qed
next
 fix q r assume qr: fract-poly p = q * r
 from content-decompose-fract[of q]
 obtain cg q' where q: q = smult cg (map-poly to-fract q') content q' = 1 .
 from content-decompose-fract[of r]
 obtain cr r' where r: r = smult cr (map-poly to-fract r') content r' = 1 .
 from qr q r p have nz: cg ≠ 0 cr ≠ 0 by auto
 from qr have eq: fract-poly p = smult (cr * cg) (fract-poly (q' * r'))
  by (simp add: q r)
 from fract-poly-smult-eqE[OF this] obtain a b
  where ab: cr * cg = to-fract b / to-fract a
    smult a p = smult b (q' * r') coprime a b normalize a = a .
 hence content (smult a p) = content (smult b (q' * r')) by (simp only:)
 with ab(4) have a: a = normalize b by (simp add: content-mult q r)
 then have normalize b = gcd a b
  by simp
 with ‹coprime a b› have normalize b = 1
  by simp
 then have a = 1 is-unit b
  by (simp-all add: a normalize-1-iff)

 note eq
 also from ab(1) ‹a = 1› have cr * cg = to-fract b by simp
  also have smult ... (fract-poly (q' * r')) = fract-poly (smult b (q' * r')) by
simp
 finally have p = ([:b:] * q') * r' by (simp del: fract-poly-smult)
 from p and this have ([:b:] * q') dvd 1 ∨ r' dvd 1 by (rule irreducibleD)
 hence q' dvd 1 ∨ r' dvd 1 by (auto dest: dvd-mult-right simp del: mult-pCons-left)
 hence fract-poly q' dvd 1 ∨ fract-poly r' dvd 1 by (auto simp: fract-poly-is-unit)
 with q r show is-unit q ∨ is-unit r
  by (auto simp add: is-unit-smult-iff dvd-field-iff nz)
qed
```

next

assume *irred*: *irreducible* (*fract-poly p*) **and** *primitive*: *content p* = *1*
show *irreducible p*
proof (*rule irreducibleI*)
from *irred* **show** *p* ≠ *0* **by** *auto*
next
from *irred* **show** ¬*p dvd 1*
by (*auto simp*: *irreducible-def dest*: *fract-poly-is-unit*)
next
fix *q r* **assume** *qr*: *p* = *q* ∗ *r*
hence *fract-poly p* = *fract-poly q* ∗ *fract-poly r* **by** *simp*
from *irred* **and** *this* **have** *fract-poly q dvd 1* ∨ *fract-poly r dvd 1*
by (*rule irreducibleD*)
with *primitive qr* **show** *q dvd 1* ∨ *r dvd 1*
by (*auto simp*: *content-prod-eq-1-iff is-unit-fract-poly-iff*)
qed
qed

lemma *irreducible-imp-prime-poly*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*}
poly
assumes *irreducible p*
shows *prime-elem p*
proof (*cases degree p* = *0*)
case *True*
with *assms* **show** *?thesis*
by (*auto simp*: *prime-elem-const-poly-iff irreducible-const-poly-iff*
intro!: *irreducible-imp-prime-elem elim*!: *degree-eq-zeroE*)
next
case *False*
from *assms False* **have** *irred*: *irreducible* (*fract-poly p*) **and** *primitive*: *content p*
= *1*
by (*simp-all add*: *nonconst-poly-irreducible-iff*)
from *irred* **have** *prime*: *prime-elem* (*fract-poly p*) **by** (*rule field-poly-irreducible-imp-prime*)
show *?thesis*
proof (*rule prime-elemI*)
fix *q r* **assume** *p dvd q* ∗ *r*
hence *fract-poly p dvd fract-poly* (*q* ∗ *r*) **by** (*rule fract-poly-dvd*)
hence *fract-poly p dvd fract-poly q* ∗ *fract-poly r* **by** *simp*
from *prime* **and** *this* **have** *fract-poly p dvd fract-poly q* ∨ *fract-poly p dvd*
fract-poly r
by (*rule prime-elem-dvd-multD*)
with *primitive* **show** *p dvd q* ∨ *p dvd r* **by** (*auto dest*: *fract-poly-dvdD*)
qed (*insert assms*, *auto simp*: *irreducible-def*)
qed

lemma *degree-primitive-part-fract* [*simp*]:
degree (*primitive-part-fract p*) = *degree p*

proof −
have p = *smult* (*fract-content p*) (*fract-poly* (*primitive-part-fract p*))
by (*simp add*: *content-times-primitive-part-fract*)
also have *degree* ... = *degree* (*primitive-part-fract p*)
by (*auto simp*: *degree-map-poly*)
finally show *?thesis* **..**
qed

lemma *irreducible-primitive-part-fract*:
fixes p :: $'a$ :: {*idom-divide*, *ring-gcd*, *factorial-semiring*, *semiring-Gcd*,*semiring-gcd-mult-normalize*} *fract poly*
assumes *irreducible p*
shows *irreducible* (*primitive-part-fract p*)
proof −
from *assms* **have** *deg*: *degree* (*primitive-part-fract p*) ≠ *0*
by (*intro notI*)
(*auto elim*!: *degree-eq-zeroE simp*: *irreducible-def is-unit-poly-iff dvd-field-iff*)
hence [*simp*]: $p \neq 0$ **by** *auto*

note ‹*irreducible p*›
also have p = [:*fract-content p*:] ∗ *fract-poly* (*primitive-part-fract p*)
by (*simp add*: *content-times-primitive-part-fract*)
also have *irreducible* ... ⟷ *irreducible* (*fract-poly* (*primitive-part-fract p*))
by (*intro irreducible-mult-unit-left*) (*simp-all add*: *is-unit-poly-iff dvd-field-iff*)
finally show *?thesis* **using** *deg*
by (*simp add*: *nonconst-poly-irreducible-iff*)
qed

lemma *prime-elem-primitive-part-fract*:
fixes p :: $'a$ :: {*idom-divide*, *ring-gcd*, *factorial-semiring*, *semiring-Gcd*,*semiring-gcd-mult-normalize*} *fract poly*
shows *irreducible p* ⟹ *prime-elem* (*primitive-part-fract p*)
by (*intro irreducible-imp-prime-poly irreducible-primitive-part-fract*)

lemma *irreducible-linear-field-poly*:
fixes *a b* :: $'a$::*field*
assumes $b \neq 0$
shows *irreducible* [:*a*,*b*:]
proof (*rule irreducibleI*)
fix *p q* **assume** *pq*: [:*a*,*b*:] = *p* ∗ *q*
also from *pq assms* **have** *degree* ... = *degree p* + *degree q*
by (*intro degree-mult-eq*) *auto*
finally have *degree p* = *0* ∨ *degree q* = *0* **using** *assms* **by** *auto*
with *assms pq* **show** *is-unit p* ∨ *is-unit q*
by (*auto simp*: *is-unit-const-poly-iff dvd-field-iff elim*!: *degree-eq-zeroE*)
qed (*insert assms*, *auto simp*: *is-unit-poly-iff*)

lemma *prime-elem-linear-field-poly*:
(*b* :: $'a$ :: *field*) ≠ *0* ⟹ *prime-elem* [:*a*,*b*:]

by (*rule field-poly-irreducible-imp-prime*, *rule irreducible-linear-field-poly*)

lemma *irreducible-linear-poly*:
fixes *a b* :: *'a*::{*idom-divide*,*ring-gcd*,*factorial-semiring*,*semiring-Gcd*,*semiring-gcd-mult-normalize*}
shows $b \neq 0 \Longrightarrow$ *coprime a b* $\Longrightarrow$ *irreducible* [:*a*,*b*:]
by (*auto intro*!: *irreducible-linear-field-poly*
simp: *nonconst-poly-irreducible-iff content-def map-poly-pCons*)

lemma *prime-elem-linear-poly*:
fixes *a b* :: *'a*::{*idom-divide*,*ring-gcd*,*factorial-semiring*,*semiring-Gcd*,*semiring-gcd-mult-normalize*}
shows $b \neq 0 \Longrightarrow$ *coprime a b* $\Longrightarrow$ *prime-elem* [:*a*,*b*:]
by (*rule irreducible-imp-prime-poly*, *rule irreducible-linear-poly*)

11.6 Prime factorisation of polynomials

lemma *poly-prime-factorization-exists-content-1*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*}
poly
assumes $p \neq 0$ *content p* = *1*
shows $\exists A.\ (\forall p.\ p \in\# A \longrightarrow$ *prime-elem p*$) \wedge$ *prod-mset A* = *normalize p*
proof −
let *?P* = *field-poly.prime-factorization* (*fract-poly p*)
define *c* **where** *c* = *prod-mset* (*image-mset fract-content ?P*)
define *c'* **where** *c'* = *c* ∗ *to-fract* (*lead-coeff p*)
define *e* **where** *e* = *prod-mset* (*image-mset primitive-part-fract ?P*)
define *A* **where** *A* = *image-mset* (*normalize* ∘ *primitive-part-fract*) *?P*
have *content e* = (∏*x*∈#*field-poly.prime-factorization* (*map-poly to-fract p*).
content (*primitive-part-fract x*))
by (*simp add*: *e-def content-prod-mset multiset.map-comp o-def*)
also have *image-mset* (λ*x*. *content* (*primitive-part-fract x*)) *?P* = *image-mset* (λ-. *1*) *?P*
by (*intro image-mset-cong content-primitive-part-fract*) *auto*
finally have *content-e*: *content e* = *1*
by *simp*

from ‹*p* ≠ *0*› **have** *fract-poly p* = [:*lead-coeff* (*fract-poly p*):] ∗
smult (*inverse* (*lead-coeff* (*fract-poly p*))) (*fract-poly p*)
by *simp*
also have [:*lead-coeff* (*fract-poly p*):] = [:*to-fract* (*lead-coeff p*):]
by (*simp add*: *monom-0 degree-map-poly coeff-map-poly*)
also from *assms* **have** *smult* (*inverse* (*lead-coeff* (*fract-poly p*))) (*fract-poly p*) = *prod-mset ?P*
by (*subst field-poly-prod-mset-prime-factorization*) *simp-all*
also have ... = *prod-mset* (*image-mset id ?P*) **by** *simp*
also have *image-mset id ?P* =
image-mset (λ*x*. [:*fract-content x*:] ∗ *fract-poly* (*primitive-part-fract x*)) *?P*
by (*intro image-mset-cong*) (*auto simp*: *content-times-primitive-part-fract*)
also have *prod-mset* ... = *smult c* (*fract-poly e*)

by (*subst prod-mset.distrib*) (*simp-all add*: *prod-mset-fract-poly prod-mset-const-poly c-def e-def*)
also have [:*to-fract* (*lead-coeff p*):] ∗ ... = *smult c'* (*fract-poly e*)
by (*simp add*: *c'-def*)
finally have *eq*: *fract-poly p* = *smult c'* (*fract-poly e*) **.**
also obtain *b* **where** *b*: *c'* = *to-fract b is-unit b*
proof −
from *fract-poly-smult-eqE*[*OF eq*]
obtain *a b* **where** *ab*:
c' = *to-fract b* / *to-fract a*
smult a p = *smult b e*
coprime a b
normalize a = *a* **.**
from *ab*(*2*) **have** *content* (*smult a p*) = *content* (*smult b e*) **by** (*simp only*:)
with *assms content-e* **have** *a* = *normalize b* **by** (*simp add*: *ab*(*4*))
with *ab* **have** *ab'*: *a* = *1 is-unit b*
by (*simp-all add*: *normalize-1-iff*)
with *ab ab'* **have** *c'* = *to-fract b* **by** *auto*
from *this* **and** ‹*is-unit b*› **show** *?thesis* **by** (*rule that*)
qed
hence *smult c'* (*fract-poly e*) = *fract-poly* (*smult b e*) **by** *simp*
finally have *p* = *smult b e* **by** (*simp only*: *fract-poly-eq-iff*)
hence *p* = [:*b*:] ∗ *e* **by** *simp*
with *b* **have** *normalize p* = *normalize e*
by (*simp only*: *normalize-mult*) (*simp add*: *is-unit-normalize is-unit-poly-iff*)
also have *normalize e* = *prod-mset A*
by (*simp add*: *multiset.map-comp e-def A-def normalize-prod-mset*)
finally have *prod-mset A* = *normalize p* **..**

have *prime-elem p* **if** *p* ∈# *A* **for** *p*
using *that* **by** (*auto simp*: *A-def prime-elem-primitive-part-fract prime-elem-imp-irreducible*

dest!: *field-poly-in-prime-factorization-imp-prime*)
from *this* **and** ‹*prod-mset A* = *normalize p*› **show** *?thesis*
by (*intro exI*[*of* - *A*]) *blast*
qed

lemma *poly-prime-factorization-exists*:
fixes *p* :: *'a* :: {*factorial-semiring*,*semiring-Gcd*,*ring-gcd*,*idom-divide*,*semiring-gcd-mult-normalize*} *poly*
assumes *p* ≠ *0*
shows ∃ *A*. (∀ *p*. *p* ∈# *A* ⟶ *prime-elem p*) ∧ *normalize* (*prod-mset A*) = *normalize p*
proof −
define *B* **where** *B* = *image-mset* (λ*x*. [:*x*:]) (*prime-factorization* (*content p*))
have ∃ *A*. (∀ *p*. *p* ∈# *A* ⟶ *prime-elem p*) ∧ *prod-mset A* = *normalize* (*primitive-part p*)
by (*rule poly-prime-factorization-exists-content-1*) (*insert assms*, *simp-all*)
then obtain *A* **where** *A*: ∀ *p*. *p* ∈# *A* ⟶ *prime-elem p* $\prod_{\#}$ *A* = *normalize*

(*primitive-part p*)
by *blast*
have *normalize* (*prod-mset* (*A* + *B*)) = *normalize* (*prod-mset A* ∗ *normalize* (*prod-mset B*))
by *simp*
also from *assms* **have** *normalize* (*prod-mset B*) = *normalize* [:*content p*:]
by (*simp add*: *prod-mset-const-poly normalize-const-poly prod-mset-prime-factorization-weak B-def*)
also have *prod-mset A* = *normalize* (*primitive-part p*)
using *A* **by** *simp*
finally have *normalize* (*prod-mset* (*A* + *B*)) = *normalize* (*primitive-part p* ∗ [:*content p*:])
by *simp*
moreover have ∀ *p*. *p* ∈# *B* ⟶ *prime-elem p*
by (*auto simp*: *B-def intro*!: *lift-prime-elem-poly dest*: *in-prime-factors-imp-prime*)
ultimately show *?thesis* **using** *A* **by** (*intro exI*[*of* - *A* + *B*]) (*auto*)
qed

end

11.7 Typeclass instances

instance *poly* :: ({*factorial-ring-gcd*,*semiring-gcd-mult-normalize*}) *factorial-semiring*
by *standard* (*rule poly-prime-factorization-exists*)

instantiation *poly* :: ({*factorial-ring-gcd*, *semiring-gcd-mult-normalize*}) *factorial-ring-gcd*
begin

definition *gcd-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly* **where**
[*code del*]: *gcd-poly* = *gcd-factorial*

definition *lcm-poly* :: *'a poly* ⇒ *'a poly* ⇒ *'a poly* **where**
[*code del*]: *lcm-poly* = *lcm-factorial*

definition *Gcd-poly* :: *'a poly set* ⇒ *'a poly* **where**
[*code del*]: *Gcd-poly* = *Gcd-factorial*

definition *Lcm-poly* :: *'a poly set* ⇒ *'a poly* **where**
[*code del*]: *Lcm-poly* = *Lcm-factorial*

instance by *standard* (*simp-all add*: *gcd-poly-def lcm-poly-def Gcd-poly-def Lcm-poly-def*)

end

instance *poly* :: ({*factorial-ring-gcd*, *semiring-gcd-mult-normalize*}) *semiring-gcd-mult-normalize*
..

instance *poly* :: ({*field*,*factorial-ring-gcd*,*semiring-gcd-mult-normalize*})
normalization-euclidean-semiring **..**

instance *poly* :: ({*field*, *normalization-euclidean-semiring*, *factorial-ring-gcd*, *semiring-gcd-mult-normalize*}) *euclidean-ring-gcd*
by (*rule euclidean-ring-gcd-class.intro*, *rule factorial-euclidean-semiring-gcdI*) *standard*

instance *poly* :: ({*field*, *normalization-euclidean-semiring*, *factorial-ring-gcd*, *semiring-gcd-mult-normalize*}) *factorial-semiring-multiplicative* **..**

11.8 Polynomial GCD

lemma *gcd-poly-decompose*:
fixes *p q* :: *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*
shows *gcd p q* =
smult (*gcd* (*content p*) (*content q*)) (*gcd* (*primitive-part p*) (*primitive-part q*))
proof (*rule sym*, *rule gcdI*)
have [:*gcd* (*content p*) (*content q*):] ∗ *gcd* (*primitive-part p*) (*primitive-part q*) *dvd*
[:*content p*:] ∗ *primitive-part p* **by** (*intro mult-dvd-mono*) *simp-all*
thus *smult* (*gcd* (*content p*) (*content q*)) (*gcd* (*primitive-part p*) (*primitive-part q*)) *dvd p*
by *simp*
next
have [:*gcd* (*content p*) (*content q*):] ∗ *gcd* (*primitive-part p*) (*primitive-part q*) *dvd*
[:*content q*:] ∗ *primitive-part q* **by** (*intro mult-dvd-mono*) *simp-all*
thus *smult* (*gcd* (*content p*) (*content q*)) (*gcd* (*primitive-part p*) (*primitive-part q*)) *dvd q*
by *simp*
next
fix *d* **assume** *d dvd p d dvd q*
hence [:*content d*:] ∗ *primitive-part d dvd*
[:*gcd* (*content p*) (*content q*):] ∗ *gcd* (*primitive-part p*) (*primitive-part q*)
by (*intro mult-dvd-mono*) *auto*
thus *d dvd smult* (*gcd* (*content p*) (*content q*)) (*gcd* (*primitive-part p*) (*primitive-part q*))
by *simp*
qed (*auto simp*: *normalize-smult*)

lemma *gcd-poly-pseudo-mod*:
fixes *p q* :: *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*
assumes *nz*: *q* ≠ *0* **and** *prim*: *content p* = *1 content q* = *1*
shows *gcd p q* = *gcd q* (*primitive-part* (*pseudo-mod p q*))
proof −
define *r s* **where** *r* = *fst* (*pseudo-divmod p q*) **and** *s* = *snd* (*pseudo-divmod p q*)
define *a* **where** *a* = [:*coeff q* (*degree q*) ^ (*Suc* (*degree p*) − *degree q*):]

```
  have [simp]: primitive-part a = unit-factor a
    by (simp add: a-def unit-factor-poly-def unit-factor-power monom-0)
  from nz have [simp]: a ≠ 0 by (auto simp: a-def)

  have rs: pseudo-divmod p q = (r, s) by (simp add: r-def s-def)
  have gcd (q * r + s) q = gcd q s
    using gcd-add-mult[of q r s] by (simp add: gcd.commute add-ac mult-ac)
  with pseudo-divmod(1)[OF nz rs]
    have gcd (p * a) q = gcd q s by (simp add: a-def)
  also from prim have gcd (p * a) q = gcd p q
    by (subst gcd-poly-decompose)
       (auto simp: primitive-part-mult gcd-mult-unit1 primitive-part-prim
            simp del: mult-pCons-right )
  also from prim have gcd q s = gcd q (primitive-part s)
    by (subst gcd-poly-decompose) (simp-all add: primitive-part-prim)
  also have s = pseudo-mod p q by (simp add: s-def pseudo-mod-def)
  finally show ?thesis .
qed

lemma degree-pseudo-mod-less:
  assumes q ≠ 0 pseudo-mod p q ≠ 0
  shows   degree (pseudo-mod p q) < degree q
  using pseudo-mod(2)[of q p] assms by auto

function gcd-poly-code-aux :: 'a :: factorial-ring-gcd poly ⇒ 'a poly ⇒ 'a poly
where
  gcd-poly-code-aux p q =
    (if q = 0 then normalize p else gcd-poly-code-aux q (primitive-part (pseudo-mod
p q)))
by auto
termination
  by (relation measure ((λp. if p = 0 then 0 else Suc (degree p)) ∘ snd))
     (auto simp: degree-pseudo-mod-less)

declare gcd-poly-code-aux.simps [simp del]

lemma gcd-poly-code-aux-correct:
  assumes content p = 1 q = 0 ∨ content q = 1
  shows   gcd-poly-code-aux p q = gcd p q
  using assms
proof (induction p q rule: gcd-poly-code-aux.induct)
  case (1 p q)
  show ?case
  proof (cases q = 0)
    case True
    thus ?thesis by (subst gcd-poly-code-aux.simps) auto
  next
    case False
   hence gcd-poly-code-aux p q = gcd-poly-code-aux q (primitive-part (pseudo-mod
```

p q))
by (*subst gcd-poly-code-aux.simps*) *simp-all*
also from *1.prems False*
have *primitive-part* (*pseudo-mod p q*) = *0* ∨
content (*primitive-part* (*pseudo-mod p q*)) = *1*
by (*cases pseudo-mod p q* = *0*) *auto*
with *1.prems False*
have *gcd-poly-code-aux q* (*primitive-part* (*pseudo-mod p q*)) =
gcd q (*primitive-part* (*pseudo-mod p q*))
by (*intro 1*) *simp-all*
also from *1.prems False*
have ... = *gcd p q* **by** (*intro gcd-poly-pseudo-mod* [*symmetric*]) *auto*
finally show *?thesis* **.**
qed
qed

definition *gcd-poly-code*
:: *'a* :: *factorial-ring-gcd poly* ⇒ *'a poly* ⇒ *'a poly*
where *gcd-poly-code p q* =
(*if p* = *0 then normalize q else if q* = *0 then normalize p else*
smult (*gcd* (*content p*) (*content q*))
(*gcd-poly-code-aux* (*primitive-part p*) (*primitive-part q*)))

lemma *gcd-poly-code* [*code*]: *gcd p q* = *gcd-poly-code p q*
by (*simp add*: *gcd-poly-code-def gcd-poly-code-aux-correct gcd-poly-decompose* [*symmetric*])

lemma *lcm-poly-code* [*code*]:
fixes *p q* :: *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*} *poly*
shows *lcm p q* = *normalize* (*p* ∗ *q div gcd p q*)
by (*fact lcm-gcd*)

lemmas *Gcd-poly-set-eq-fold* [*code*] =
Gcd-set-eq-fold [**where** *?'a* = *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*}
poly]
lemmas *Lcm-poly-set-eq-fold* [*code*] =
Lcm-set-eq-fold [**where** *?'a* = *'a* :: {*factorial-ring-gcd*,*semiring-gcd-mult-normalize*}
poly]

end

12 Squarefreeness

theory *Squarefree*
imports *Primes*
begin

definition *squarefree* :: *'a* :: *comm-monoid-mult* ⇒ *bool* **where**

squarefree n $\longleftrightarrow$ $(\forall x.\ x \mathbin{\hat{}} 2\ dvd\ n \longrightarrow x\ dvd\ 1)$

lemma *squarefreeI*: $(\bigwedge x.\ x \mathbin{\hat{}} 2\ dvd\ n \Longrightarrow x\ dvd\ 1) \Longrightarrow$ *squarefree n*
by (*auto simp*: *squarefree-def*)

lemma *squarefreeD*: *squarefree n* $\Longrightarrow x \mathbin{\hat{}} 2\ dvd\ n \Longrightarrow x\ dvd\ 1$
by (*auto simp*: *squarefree-def*)

lemma *not-squarefreeI*: $x \mathbin{\hat{}} 2\ dvd\ n \Longrightarrow \neg x\ dvd\ 1 \Longrightarrow \neg$*squarefree n*
by (*auto simp*: *squarefree-def*)

lemma *not-squarefreeE* [*case-names square-dvd*]:
$\neg$*squarefree n* $\Longrightarrow (\bigwedge x.\ x \mathbin{\hat{}} 2\ dvd\ n \Longrightarrow \neg x\ dvd\ 1 \Longrightarrow P) \Longrightarrow P$
by (*auto simp*: *squarefree-def*)

lemma *not-squarefree-0* [*simp*]: $\neg$*squarefree* $(0 :: 'a :: comm\text{-}semiring\text{-}1)$
by (*rule not-squarefreeI*[*of 0*]) *auto*

lemma *squarefree-factorial-semiring*:
assumes $n \neq 0$
shows *squarefree* $(n :: 'a :: factorial\text{-}semiring) \longleftrightarrow (\forall p.\ prime\ p \longrightarrow \neg p \mathbin{\hat{}} 2\ dvd\ n)$
unfolding *squarefree-def*
proof *safe*
assume $*$: $\forall p.\ prime\ p \longrightarrow \neg p \mathbin{\hat{}} 2\ dvd\ n$
fix $x :: 'a$ **assume** x: $x \mathbin{\hat{}} 2\ dvd\ n$
{
assume $\neg$*is-unit x*
moreover from *assms* **and** x **have** $x \neq 0$ **by** *auto*
ultimately obtain p **where** $p\ dvd\ x\ prime\ p$
using *prime-divisor-exists* **by** *blast*
with $*$ **have** $\neg p \mathbin{\hat{}} 2\ dvd\ n$ **by** *blast*
moreover from $\langle p\ dvd\ x \rangle$ **have** $p \mathbin{\hat{}} 2\ dvd\ x \mathbin{\hat{}} 2$ **by** (*rule dvd-power-same*)
ultimately have $\neg x \mathbin{\hat{}} 2\ dvd\ n$ **by** (*blast dest*: *dvd-trans*)
with x **have** *False* **by** *contradiction*
}
thus *is-unit x* **by** *blast*
qed *auto*

lemma *squarefree-factorial-semiring'*:
assumes $n \neq 0$
shows *squarefree* $(n :: 'a :: factorial\text{-}semiring) \longleftrightarrow$
$(\forall p \in prime\text{-}factors\ n.\ multiplicity\ p\ n = 1)$
proof (*subst squarefree-factorial-semiring* [*OF assms*], *safe*)
fix p **assume** $\forall p \in \#prime\text{-}factorization\ n.\ multiplicity\ p\ n = 1\ prime\ p\ p \hat{} 2\ dvd\ n$
with *assms* **show** *False*
by (*cases p dvd n*)
(*auto simp*: *prime-factors-dvd power-dvd-iff-le-multiplicity not-dvd-imp-multiplicity-0*)

qed (*auto intro*!: *multiplicity-eqI simp*: *power2-eq-square* [*symmetric*])

lemma *squarefree-factorial-semiring''*:
assumes $n \neq 0$
shows *squarefree* (n :: $'a$ :: *factorial-semiring*) $\longleftrightarrow$
($\forall p$. *prime* $p \longrightarrow$ *multiplicity* p $n \leq 1$)
by (*subst squarefree-factorial-semiring'*[*OF assms*]) (*auto simp*: *prime-factors-multiplicity*)

lemma *squarefree-unit* [*simp*]: *is-unit* $n \Longrightarrow$ *squarefree* n
proof (*rule squarefreeI*)
fix x **assume** $x\hat{}2$ *dvd* n n *dvd* 1
hence *is-unit* ($x\hat{}2$) **by** (*rule dvd-unit-imp-unit*)
thus *is-unit* x **by** (*simp add*: *is-unit-power-iff*)
qed

lemma *squarefree-1* [*simp*]: *squarefree* (1 :: $'a$ :: *algebraic-semidom*)
by *simp*

lemma *squarefree-minus* [*simp*]: *squarefree* ($-n$ :: $'a$ :: *comm-ring-1*) $\longleftrightarrow$ *squarefree* n
by (*simp add*: *squarefree-def*)

lemma *squarefree-mono*: a *dvd* $b \Longrightarrow$ *squarefree* $b \Longrightarrow$ *squarefree* a
by (*auto simp*: *squarefree-def intro*: *dvd-trans*)

lemma *squarefree-multD*:
assumes *squarefree* ($a * b$)
shows *squarefree* a *squarefree* b
by (*rule squarefree-mono*[*OF - assms*], *simp*)+

lemma *squarefree-prime-elem*:
assumes *prime-elem* (p :: $'a$ :: *factorial-semiring*)
shows *squarefree* p
proof −
from *assms* **have** $p \neq 0$ **by** *auto*
show *?thesis*
proof (*subst squarefree-factorial-semiring* [*OF* ‹$p \neq 0$›]; *safe*)
fix q **assume** $*$: *prime* q $q\hat{}2$ *dvd* p
with *assms* **have** *multiplicity* q $p \geq 2$ **by** (*intro multiplicity-geI*) *auto*
thus *False* **using** *assms* ‹*prime q*› *prime-multiplicity-other*[*of q normalize p*]
by (*cases* q = *normalize* p) *simp-all*
qed
qed

lemma *squarefree-prime*:
assumes *prime* (p :: $'a$ :: *factorial-semiring*)
shows *squarefree* p
using *assms* **by** (*intro squarefree-prime-elem*) *auto*

lemma *squarefree-mult-coprime*:
fixes *a b* :: *'a* :: *factorial-semiring-gcd*
assumes *coprime a b squarefree a squarefree b*
shows *squarefree* (*a* ∗ *b*)
proof −
from *assms* **have** *nz*: *a* ∗ *b* ≠ *0* **by** *auto*
show *?thesis* **unfolding** *squarefree-factorial-semiring'*[*OF nz*]
proof
fix *p* **assume** *p*: *p* ∈ *prime-factors* (*a* ∗ *b*)
with *nz* **have** *prime p*
by (*simp add*: *prime-factors-dvd*)
have ¬ (*p dvd a* ∧ *p dvd b*)
proof
assume *p dvd a* ∧ *p dvd b*
with ‹*coprime a b*› **have** *is-unit p*
by (*auto intro*: *coprime-common-divisor*)
with ‹*prime p*› **show** *False*
by *simp*
qed
moreover from *p* **have** *p dvd a* ∨ *p dvd b* **using** *nz*
by (*auto simp*: *prime-factors-dvd prime-dvd-mult-iff*)
ultimately show *multiplicity p* (*a* ∗ *b*) = *1* **using** *nz p assms*(*2*,*3*)
by (*auto simp*: *prime-elem-multiplicity-mult-distrib prime-factors-multiplicity*
not-dvd-imp-multiplicity-0 squarefree-factorial-semiring')
qed
qed

lemma *squarefree-prod-coprime*:
fixes *f* :: *'a* ⇒ *'b* :: *factorial-semiring-gcd*
assumes ⋀*a b*. *a* ∈ *A* ⟹ *b* ∈ *A* ⟹ *a* ≠ *b* ⟹ *coprime* (*f a*) (*f b*)
assumes ⋀*a*. *a* ∈ *A* ⟹ *squarefree* (*f a*)
shows *squarefree* (*prod f A*)
using *assms*
by (*induction A rule*: *infinite-finite-induct*)
(*auto intro*!: *squarefree-mult-coprime prod-coprime-right*)

lemma *squarefree-powerD*: *m* > *0* ⟹ *squarefree* (*n* ^ *m*) ⟹ *squarefree n*
by (*cases m*) (*auto dest*: *squarefree-multD*)

lemma *squarefree-power-iff*:
squarefree (*n* ^ *m*) ⟷ *m* = *0* ∨ *is-unit n* ∨ (*squarefree n* ∧ *m* = *1*)
proof *safe*
assume *squarefree* (*n* ^ *m*) *m* > *0* ¬*is-unit n*
show *m* = *1*
proof (*rule ccontr*)
assume *m* ≠ *1*
with ‹*m* > *0*› **have** *n* ^ *2 dvd n* ^ *m* **by** (*intro le-imp-power-dvd*) *auto*
from *this* **and** ‹¬*is-unit n*› **have** ¬*squarefree* (*n* ^ *m*) **by** (*rule not-squarefreeI*)
with ‹*squarefree* (*n* ^ *m*)› **show** *False* **by** *contradiction*

```
  qed
qed (auto simp: is-unit-power-iff dest: squarefree-powerD)

definition squarefree-nat :: nat ⇒ bool where
  [code-abbrev]: squarefree-nat = squarefree

lemma squarefree-nat-code-naive [code]:
  squarefree-nat n ⟷ n ≠ 0 ∧ (∀ k∈{2..n}. ¬k ^ 2 dvd n)
proof safe
  assume ∗: ∀ k∈{2..n}. ¬ k^2 dvd n and n: n > 0
  show squarefree-nat n unfolding squarefree-nat-def
  proof (rule squarefreeI)
    fix k assume k: k ^ 2 dvd n
    have k dvd n by (rule dvd-trans[OF - k]) auto
    with n have k ≤ n by (intro dvd-imp-le)
    with bspec[OF ∗, of k] k have ¬k > 1 by (intro notI) auto
    moreover from k and n have k ≠ 0 by (intro notI) auto
    ultimately have k = 1 by presburger
    thus is-unit k by simp
  qed
qed (auto simp: squarefree-nat-def squarefree-def intro!: Nat.gr0I)



definition square-part :: 'a :: factorial-semiring ⇒ 'a where
  square-part n = (if n = 0 then 0 else
    normalize (∏ p∈prime-factors n. p ^ (multiplicity p n div 2)))

lemma square-part-nonzero:
  n ≠ 0 ⟹ square-part n = normalize (∏ p∈prime-factors n. p ^ (multiplicity p
n div 2))
  by (simp add: square-part-def)

lemma square-part-0 [simp]: square-part 0 = 0
  by (simp add: square-part-def)

lemma square-part-unit [simp]: is-unit x ⟹ square-part x = 1
  by (auto simp: square-part-def prime-factorization-unit)

lemma square-part-1 [simp]: square-part 1 = 1
  by simp

lemma square-part-0-iff [simp]: square-part n = 0 ⟷ n = 0
  by (simp add: square-part-def)

lemma normalize-uminus [simp]:
  normalize (−x :: 'a :: {normalization-semidom, comm-ring-1}) = normalize x
  by (rule associatedI) auto
```

lemma *multiplicity-uminus-right* [*simp*]:
multiplicity ($x :: 'a :: \{$*factorial-semiring*, *comm-ring-1*$\}$) $(-y)$ = *multiplicity x y*
proof −
have *multiplicity x* $(-y)$ = *multiplicity x* (*normalize* $(-y)$)
by (*rule multiplicity-normalize-right* [*symmetric*])
also have ... = *multiplicity x y* **by** *simp*
finally show *?thesis* **.**
qed

lemma *multiplicity-uminus-left* [*simp*]:
multiplicity ($-x :: 'a :: \{$*factorial-semiring*, *comm-ring-1*$\}$) *y* = *multiplicity x y*
proof −
have *multiplicity* $(-x)$ *y* = *multiplicity* (*normalize* $(-x)$) *y*
by (*rule multiplicity-normalize-left* [*symmetric*])
also have ... = *multiplicity x y* **by** *simp*
finally show *?thesis* **.**
qed

lemma *prime-factorization-uminus* [*simp*]:
prime-factorization ($-x :: 'a :: \{$*factorial-semiring*, *comm-ring-1*$\}$) = *prime-factorization x*
by (*rule prime-factorization-cong*) *simp-all*

lemma *square-part-uminus* [*simp*]:
square-part ($-x :: 'a :: \{$*factorial-semiring*, *comm-ring-1*$\}$) = *square-part x*
by (*simp add*: *square-part-def*)

lemma *prime-multiplicity-square-part*:
assumes *prime p*
shows *multiplicity p* (*square-part n*) = *multiplicity p n div 2*
proof (*cases n* = *0*)
case *False*
thus *?thesis* **unfolding** *square-part-nonzero*[*OF False*] *multiplicity-normalize-right*
using *finite-prime-divisors*[*of n*] *assms*
by (*subst multiplicity-prod-prime-powers*)
(*auto simp*: *not-dvd-imp-multiplicity-0 prime-factors-dvd multiplicity-prod-prime-powers*)
qed *auto*

lemma *square-part-square-dvd* [*simp*, *intro*]: *square-part n* ^ *2 dvd n*
proof (*cases n* = *0*)
case *False*
thus *?thesis*
by (*intro multiplicity-le-imp-dvd*)
(*auto simp*: *prime-multiplicity-square-part prime-elem-multiplicity-power-distrib*)
qed *auto*

lemma *prime-multiplicity-le-imp-dvd*:
assumes *x* ≠ *0 y* ≠ *0*
shows *x dvd y* ⟷ (∀ *p*. *prime p* ⟶ *multiplicity p x* ≤ *multiplicity p y*)

using *assms* **by** (*auto intro*: *multiplicity-le-imp-dvd dvd-imp-multiplicity-le*)

lemma *dvd-square-part-iff*: *x dvd square-part n* $\longleftrightarrow$ *x* $\hat{}$ *2 dvd n*
proof (*cases x = 0*; *cases n = 0*)
assume *nz*: *x* $\neq$ *0 n* $\neq$ *0*
thus *?thesis*
by (*subst* (*1 2*) *prime-multiplicity-le-imp-dvd*)
(*auto simp*: *prime-multiplicity-square-part prime-elem-multiplicity-power-distrib*)
qed *auto*

definition *squarefree-part* :: *'a* :: *factorial-semiring* $\Rightarrow$ *'a* **where**
squarefree-part n = (*if n = 0 then 1 else n div square-part n* $\hat{}$ *2*)

lemma *squarefree-part-0* [*simp*]: *squarefree-part 0 = 1*
by (*simp add*: *squarefree-part-def*)

lemma *squarefree-part-unit* [*simp*]: *is-unit n* $\Longrightarrow$ *squarefree-part n = n*
by (*auto simp add*: *squarefree-part-def*)

lemma *squarefree-part-1* [*simp*]: *squarefree-part 1 = 1*
by *simp*

lemma *squarefree-decompose*: *n = squarefree-part n* $*$ *square-part n* $\hat{}$ *2*
by (*simp add*: *squarefree-part-def*)

lemma *squarefree-part-uminus* [*simp*]:
assumes *x* $\neq$ *0*
shows *squarefree-part* (*−x* :: *'a* :: {*factorial-semiring*, *comm-ring-1*}) = *−squarefree-part x*
proof −
have *−*(*squarefree-part x* $*$ *square-part x* $\hat{}$ *2*) = *−x*
by (*subst squarefree-decompose* [*symmetric*]) *auto*
also have ... = *squarefree-part* (*−x*) $*$ *square-part* (*−x*) $\hat{}$ *2* **by** (*rule squarefree-decompose*)
finally have (*−* *squarefree-part x*) $*$ *square-part x* $\hat{}$ *2* =
squarefree-part (*−x*) $*$ *square-part x* $\hat{}$ *2* **by** *simp*
thus *?thesis* **using** *assms* **by** (*subst* (*asm*) *mult-right-cancel*) *auto*
qed

lemma *squarefree-part-nonzero* [*simp*]: *squarefree-part n* $\neq$ *0*
using *squarefree-decompose*[*of n*] **by** (*cases n* $\neq$ *0*) *auto*

lemma *prime-multiplicity-squarefree-part*:
assumes *prime p*
shows *multiplicity p* (*squarefree-part n*) = *multiplicity p n mod 2*
proof (*cases n = 0*)
case *False*
hence *n*: *n* $\neq$ *0* **by** *auto*

have *multiplicity p n mod 2 + 2 * (multiplicity p n div 2) = multiplicity p n* **by** *simp*
also have ... = *multiplicity p (squarefree-part n * square-part n ^ 2)*
by (*subst squarefree-decompose*[*of n*]) *simp*
also from *assms n* **have** ... = *multiplicity p (squarefree-part n) + 2 * (multiplicity p n div 2)*
by (*subst prime-elem-multiplicity-mult-distrib*)
(*auto simp*: *prime-elem-multiplicity-power-distrib prime-multiplicity-square-part*)
finally show *?thesis* **by** (*subst* (*asm*) *add-right-cancel*) *simp*
qed *auto*

lemma *prime-multiplicity-squarefree-part-le-Suc-0* [*intro*]:
assumes *prime p*
shows *multiplicity p (squarefree-part n) ≤ Suc 0*
by (*simp add*: *assms prime-multiplicity-squarefree-part*)

lemma *squarefree-squarefree-part* [*simp*, *intro*]: *squarefree (squarefree-part n)*
by (*subst squarefree-factorial-semiring''*)
(*auto simp*: *prime-multiplicity-squarefree-part-le-Suc-0*)

lemma *squarefree-decomposition-unique*:
assumes *square-part m = square-part n*
assumes *squarefree-part m = squarefree-part n*
shows *m = n*
by (*subst* (*1 2*) *squarefree-decompose*) (*simp-all add*: *assms*)

lemma *normalize-square-part* [*simp*]: *normalize (square-part x) = square-part x*
by (*simp add*: *square-part-def*)

lemma *square-part-even-power'*: *square-part (x ^ (2 * n)) = normalize (x ^ n)*
proof (*cases x = 0*)
case *False*
have *normalize (square-part (x ^ (2 * n))) = normalize (x ^ n)* **using** *False*
by (*intro multiplicity-eq-imp-eq*)
(*auto simp*: *prime-multiplicity-square-part prime-elem-multiplicity-power-distrib*)
thus *?thesis* **by** *simp*
qed (*auto simp*: *power-0-left*)

lemma *square-part-even-power*: *even n ⟹ square-part (x ^ n) = normalize (x ^ (n div 2))*
by (*subst square-part-even-power'* [*symmetric*]) *auto*

lemma *square-part-odd-power'*: *square-part (x ^ (Suc (2 * n))) = normalize (x ^ n * square-part x)*
proof (*cases x = 0*)
case *False*
have *normalize (square-part (x ^ (Suc (2 * n)))) = normalize (square-part x * x ^ n)*
proof (*rule multiplicity-eq-imp-eq*, *goal-cases*)

case (*3 p*)
hence *multiplicity p* (*square-part* (*x* ^ *Suc* (*2* ∗ *n*))) =
(*2* ∗ (*n* ∗ *multiplicity p x*) + *multiplicity p x*) *div 2*
by (*subst prime-multiplicity-square-part*)
(*auto simp*: *False prime-elem-multiplicity-power-distrib algebra-simps simp del*: *power-Suc*)
also from *3 False* **have** ... = *multiplicity p* (*square-part x* ∗ *x* ^ *n*)
by (*subst div-mult-self4*) (*auto simp*: *prime-multiplicity-square-part prime-elem-multiplicity-mult-distrib prime-elem-multiplicity-power-distrib*)
finally show *?case* **.**
qed (*insert False*, *auto*)
thus *?thesis* **by** (*simp add*: *mult-ac*)
qed *auto*

lemma *square-part-odd-power*:
odd n ⟹ *square-part* (*x* ^ *n*) = *normalize* (*x* ^ (*n div 2*) ∗ *square-part x*)
by (*subst square-part-odd-power'* [*symmetric*]) *auto*

end

13 Pieces of computational Algebra

theory *Computational-Algebra*
imports
Euclidean-Algorithm
Factorial-Ring
Formal-Laurent-Series
Fraction-Field
Fundamental-Theorem-Algebra
Group-Closure
Normalized-Fraction
Nth-Powers
Polynomial-FPS
Polynomial
Polynomial-Factorial
Primes
Squarefree
begin

end

theory *Field-as-Ring*
imports
Complex-Main
Euclidean-Algorithm
begin

context *field*

begin

subclass *idom-divide* **..**

definition *normalize-field* :: *'a* ⇒ *'a*
  where [*simp*]: *normalize-field x* = (*if x* = *0 then 0 else 1*)
definition *unit-factor-field* :: *'a* ⇒ *'a*
  where [*simp*]: *unit-factor-field x* = *x*
definition *euclidean-size-field* :: *'a* ⇒ *nat*
  where [*simp*]: *euclidean-size-field x* = (*if x* = *0 then 0 else 1*)
definition *mod-field* :: *'a* ⇒ *'a* ⇒ *'a*
  where [*simp*]: *mod-field x y* = (*if y* = *0 then x else 0*)

end

instantiation *real* ::
 {*unique-euclidean-ring*, *normalization-euclidean-semiring*, *normalization-semidom-multiplicative*}
begin

definition [*simp*]: *normalize-real* = (*normalize-field* :: *real* ⇒ -)
definition [*simp*]: *unit-factor-real* = (*unit-factor-field* :: *real* ⇒ -)
definition [*simp*]: *modulo-real* = (*mod-field* :: *real* ⇒ -)
definition [*simp*]: *euclidean-size-real* = (*euclidean-size-field* :: *real* ⇒ -)
definition [*simp*]: *division-segment* (*x* :: *real*) = *1*

instance
  by *standard*
    (*simp-all add*: *dvd-field-iff field-split-simps split*: *if-splits*)

end

instantiation *real* :: *euclidean-ring-gcd*
begin

definition *gcd-real* :: *real* ⇒ *real* ⇒ *real* **where**
  gcd-real = *Euclidean-Algorithm.gcd*
definition *lcm-real* :: *real* ⇒ *real* ⇒ *real* **where**
  lcm-real = *Euclidean-Algorithm.lcm*
definition *Gcd-real* :: *real set* ⇒ *real* **where**
 Gcd-real = *Euclidean-Algorithm.Gcd*
definition *Lcm-real* :: *real set* ⇒ *real* **where**
 Lcm-real = *Euclidean-Algorithm.Lcm*

instance by *standard* (*simp-all add*: *gcd-real-def lcm-real-def Gcd-real-def Lcm-real-def*)

end

instance *real* :: *field-gcd* **..**

instantiation *rat* ::
{*unique-euclidean-ring*, *normalization-euclidean-semiring*, *normalization-semidom-multiplicative*}
begin

definition [*simp*]: *normalize-rat* = (*normalize-field* :: *rat* ⇒ -)
definition [*simp*]: *unit-factor-rat* = (*unit-factor-field* :: *rat* ⇒ -)
definition [*simp*]: *modulo-rat* = (*mod-field* :: *rat* ⇒ -)
definition [*simp*]: *euclidean-size-rat* = (*euclidean-size-field* :: *rat* ⇒ -)
definition [*simp*]: *division-segment* (*x* :: *rat*) = *1*

instance
by *standard*
(*simp-all add*: *dvd-field-iff field-split-simps split*: *if-splits*)

end

instantiation *rat* :: *euclidean-ring-gcd*
begin

definition *gcd-rat* :: *rat* ⇒ *rat* ⇒ *rat* **where**
gcd-rat = *Euclidean-Algorithm.gcd*
definition *lcm-rat* :: *rat* ⇒ *rat* ⇒ *rat* **where**
lcm-rat = *Euclidean-Algorithm.lcm*
definition *Gcd-rat* :: *rat set* ⇒ *rat* **where**
Gcd-rat = *Euclidean-Algorithm.Gcd*
definition *Lcm-rat* :: *rat set* ⇒ *rat* **where**
Lcm-rat = *Euclidean-Algorithm.Lcm*

instance by *standard* (*simp-all add*: *gcd-rat-def lcm-rat-def Gcd-rat-def Lcm-rat-def*)

end

instance *rat* :: *field-gcd* **..**

instantiation *complex* ::
{*unique-euclidean-ring*, *normalization-euclidean-semiring*, *normalization-semidom-multiplicative*}
begin

definition [*simp*]: *normalize-complex* = (*normalize-field* :: *complex* ⇒ -)
definition [*simp*]: *unit-factor-complex* = (*unit-factor-field* :: *complex* ⇒ -)
definition [*simp*]: *modulo-complex* = (*mod-field* :: *complex* ⇒ -)
definition [*simp*]: *euclidean-size-complex* = (*euclidean-size-field* :: *complex* ⇒ -)
definition [*simp*]: *division-segment* (*x* :: *complex*) = *1*

instance
by *standard*
(*simp-all add*: *dvd-field-iff field-split-simps split*: *if-splits*)

end

instantiation *complex* :: *euclidean-ring-gcd*
begin

definition *gcd-complex* :: *complex* ⇒ *complex* ⇒ *complex* **where**
gcd-complex = *Euclidean-Algorithm.gcd*
definition *lcm-complex* :: *complex* ⇒ *complex* ⇒ *complex* **where**
lcm-complex = *Euclidean-Algorithm.lcm*
definition *Gcd-complex* :: *complex set* ⇒ *complex* **where**
Gcd-complex = *Euclidean-Algorithm.Gcd*
definition *Lcm-complex* :: *complex set* ⇒ *complex* **where**
Lcm-complex = *Euclidean-Algorithm.Lcm*

instance by *standard* (*simp-all add*: *gcd-complex-def lcm-complex-def Gcd-complex-def Lcm-complex-def*)

end

instance *complex* :: *field-gcd* **..**

end

14 Computation checks

theory *Computation-Checks*
imports *Primes Polynomial-Factorial HOL−Library.Discrete-Functions HOL−Library.Code-Target-Numeral*
begin

floor-sqrt 16476148165462159 = *128359449*

prime 97

prime 97

prime 9973

prime 9973

Gcd {[:*1*, *2*, *3*:], [:*2*, *3*, *4*:]} = *1*

Lcm {[:*1*, *2*, *3*:], [:*2*, *3*, *4*:]} = [:[:*2*:], [:*7*:], [:*16*:], [:*17*:], [:*12*:]:]
end

References

[1] K. J. Nowak. Some elementary proofs of Puiseuxs theorems. *Univ. Iagel. Acta Math*, 38:279–282, 2000.