Complex Analysis

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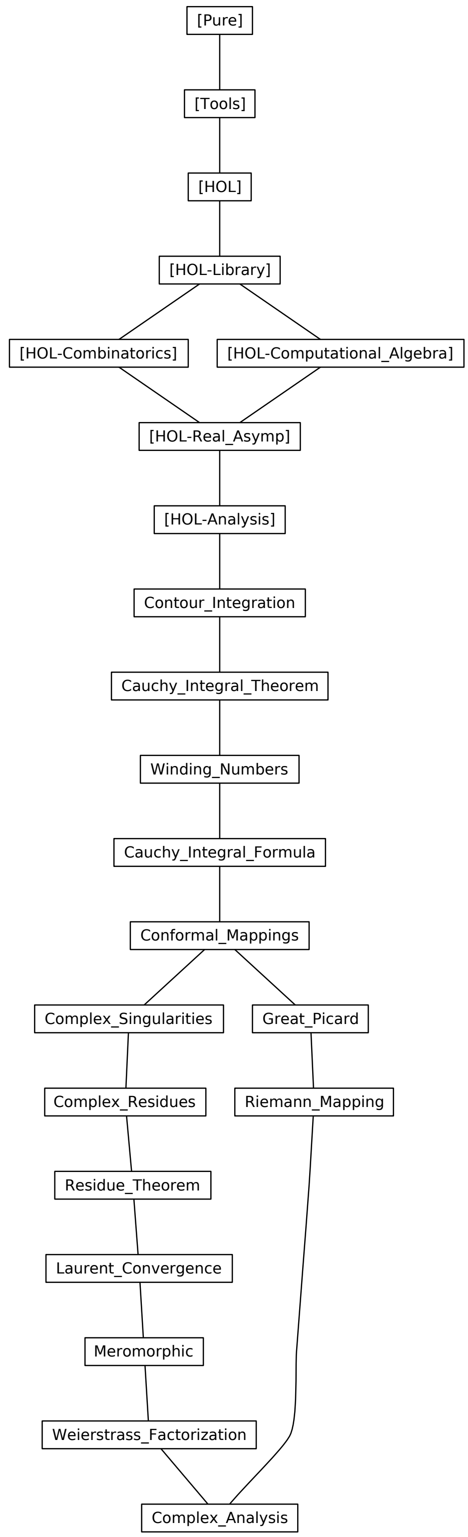
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1 Contour integration

theory *Contour_Integration*
imports *HOL−Analysis.Analysis*
begin

1.1 Definition

definition *has_contour_integral* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ (*real* ⇒ *complex*) ⇒ *bool*
(**infixr** ‹*has′_contour′_integral*› *50*)
where (*f has_contour_integral i*) *g* ≡
((λ*x*. *f*(*g x*) ∗ *vector_derivative g* (*at x within* {*0..1*}))
has_integral i) {*0..1*}

definition *contour_integrable_on*
(**infixr** ‹*contour′_integrable′_on*› *50*)
where *f contour_integrable_on g* ≡ ∃ *i*. (*f has_contour_integral i*) *g*

definition *contour_integral*
where *contour_integral g f* ≡ *SOME i*. (*f has_contour_integral i*) *g* ∨ ¬ *f contour_integrable_on g* ∧ *i=0*

1.2 Relation to subpath construction

1.3 Cauchy's theorem where there's a primitive

corollary *Cauchy_theorem_primitive*:
assumes ⋀*x*. *x* ∈ *S* ⟹ (*f has_field_derivative f′ x*) (*at x within S*)
and *valid_path g* *path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f′ has_contour_integral 0*) *g*

1.4 Reversing the order in a double path integral

proposition *contour_integral_swap*:
assumes *fcon*: *continuous_on* (*path_image g* × *path_image h*) (λ(*y1*,*y2*). *f y1 y2*)
and *vp*: *valid_path g valid_path h*
and *gvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative g* (*at t*))
and *hvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative h* (*at t*))

shows *contour_integral g* $(\lambda w.$ *contour_integral h* $(f\ w)) =$
contour_integral h $(\lambda z.$ *contour_integral g* $(\lambda w.\ f\ w\ z))$

1.5 Partial circle path

definition *part_circlepath* :: [*complex*, *real*, *real*, *real*, *real*] ⇒ *complex*
where *part_circlepath z r s t* ≡ λ*x*. *z* + *of_real r* ∗ *exp* (i ∗ *of_real* (*linepath s t x*))

proposition *path_image_part_circlepath*:
assumes $s \le t$
shows *path_image* (*part_circlepath z r s t*) = {*z* + *r* ∗ *exp*(i ∗ *of_real x*) | *x*. $s \le x \wedge x \le t$}

corollary *contour_integral_bound_part_circlepath_strong*:
assumes *f contour_integrable_on part_circlepath z r s t*
and *finite k* **and** $0 \le B$ $0 < r$ $s \le t$
and ⋀*x*. *x* ∈ *path_image*(*part_circlepath z r s t*) − *k* ⟹ *norm*(*f x*) ≤ *B*
shows *cmod* (*contour_integral* (*part_circlepath z r s t*) *f*) ≤ *B* ∗ *r* ∗ (*t* − *s*)

1.6 Special case of one complete circle

definition *circlepath* :: [*complex*, *real*, *real*] ⇒ *complex*
where *circlepath z r* ≡ *part_circlepath z r 0* (*2*∗*pi*)

1.7 Uniform convergence of path integral

proposition *contour_integral_uniform_limit*:
assumes *ev_fint*: *eventually* (λ*n*::′*a*. (*f n*) *contour_integrable_on* γ) *F*
and *ul_f*: *uniform_limit* (*path_image* γ) *f l F*
and *noleB*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *norm* (*vector_derivative* γ (*at t*)) ≤ *B*
and γ: *valid_path* γ
and [*simp*]: ¬ *trivial_limit F*
shows *l contour_integrable_on* γ ((λ*n*. *contour_integral* γ (*f n*)) ⟶ *contour_integral* γ *l*) *F*

end

2 Complex Path Integrals and Cauchy's Integral Theorem

theory *Cauchy_Integral_Theorem*
imports
HOL−Analysis.Analysis
Contour_Integration
begin

proposition *Cauchy_theorem_triangle_interior*:
assumes *contf*: *continuous_on* (*convex hull* {*a*,*b*,*c*}) *f*
and *holf*: *f holomorphic_on interior* (*convex hull* {*a*,*b*,*c*})
shows (*f has_contour_integral 0*) (*linepath a b* +++ *linepath b c* +++ *linepath c a*)

2.1 Cauchy's theorem for a convex set

corollary *Cauchy_theorem_convex_simple*:
assumes *holf*: *f holomorphic_on S*
and *convex S valid_path g path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f has_contour_integral 0*) *g*

2.2 Homotopy forms of Cauchy's theorem

proposition *Cauchy_theorem_homotopic_paths*:
assumes *hom*: *homotopic_paths S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

proposition *Cauchy_theorem_homotopic_loops*:
assumes *hom*: *homotopic_loops S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

end

3 Winding numbers

theory *Winding_Numbers*
imports *Cauchy_Integral_Theorem*
begin

3.1 Definition

definition *winding_number_prop* :: [*real* ⇒ *complex*, *complex*, *real*, *real* ⇒ *complex*, *complex*] ⇒ *bool* **where**
winding_number_prop γ *z e p n* ≡

$valid\_path\ p \wedge z \notin path\_image\ p \wedge$
$pathstart\ p = pathstart\ \gamma \wedge$
$pathfinish\ p = pathfinish\ \gamma \wedge$
$(\forall t \in \{0..1\}.\ norm(\gamma\ t - p\ t) < e) \wedge$
$contour\_integral\ p\ (\lambda w.\ 1/(w - z)) = 2 * pi * i * n$

definition *winding_number*:: $[real \Rightarrow complex, complex] \Rightarrow complex$ **where**
$winding\_number\ \gamma\ z \equiv SOME\ n.\ \forall e > 0.\ \exists p.\ winding\_number\_prop\ \gamma\ z\ e\ p\ n$

proposition *winding_number_valid_path*:
assumes $valid\_path\ \gamma\ z \notin path\_image\ \gamma$
shows $winding\_number\ \gamma\ z = 1/(2*pi*i) * contour\_integral\ \gamma\ (\lambda w.\ 1/(w - z))$

proposition *has_contour_integral_winding_number*:
assumes γ: $valid\_path\ \gamma\ z \notin path\_image\ \gamma$
shows $((\lambda w.\ 1/(w - z))\ has\_contour\_integral\ (2*pi*i*winding\_number\ \gamma\ z))\ \gamma$

3.2 The winding number is an integer

theorem *integer_winding_number*:
$[\![path\ \gamma;\ pathfinish\ \gamma = pathstart\ \gamma;\ z \notin path\_image\ \gamma]\!] \Longrightarrow winding\_number\ \gamma\ z \in \mathbb{Z}$

3.3 Continuity of winding number and invariance on connected sets

theorem *continuous_at_winding_number*:
fixes z::*complex*
assumes γ: $path\ \gamma$ **and** z: $z \notin path\_image\ \gamma$
shows $continuous\ (at\ z)\ (winding\_number\ \gamma)$

corollary *continuous_on_winding_number*:
$path\ \gamma \Longrightarrow continuous\_on\ (-\ path\_image\ \gamma)\ (\lambda w.\ winding\_number\ \gamma\ w)$

3.4 Winding number is zero "outside" a curve

proposition *winding_number_zero_in_outside*:
assumes γ: $path\ \gamma$ **and** *loop*: $pathfinish\ \gamma = pathstart\ \gamma$ **and** z: $z \in outside\ (path\_image\ \gamma)$
shows $winding\_number\ \gamma\ z = 0$

proposition *winding_number_part_circlepath_pos_less*:
assumes $s < t$ **and** *no*: $norm(w - z) < r$

shows *0 < Re (winding_number(part_circlepath z r s t) w)*

proposition *winding_number_circlepath*:
assumes *norm(w − z) < r* **shows** *winding_number(circlepath z r) w = 1*

3.5 Winding number for a triangle

proposition *winding_number_triangle*:
assumes *z*: *z ∈ interior(convex hull {a,b,c})*
shows *winding_number(linepath a b +++ linepath b c +++ linepath c a) z =
(if 0 < Im((b − a) ∗ cnj (b − z)) then 1 else −1)*

3.6 Winding numbers for simple closed paths

proposition *simple_closed_path_winding_number_inside*:
assumes *simple_path γ*
obtains *⋀z. z ∈ inside(path_image γ) ⟹ winding_number γ z = 1*
| ⋀z. z ∈ inside(path_image γ) ⟹ winding_number γ z = −1

3.7 Winding number for rectangular paths

proposition *winding_number_rectpath*:
assumes *z ∈ box a1 a3*
shows *winding_number (rectpath a1 a3) z = 1*

proposition *winding_number_rectpath_outside*:
assumes *Re a1 ≤ Re a3 Im a1 ≤ Im a3*
assumes *z ∉ cbox a1 a3*
shows *winding_number (rectpath a1 a3) z = 0*

end

4 Cauchy's Integral Formula

theory *Cauchy_Integral_Formula*
imports *Winding_Numbers*
begin

4.1 Proof

theorem *Cauchy_integral_formula_convex_simple*:
assumes *convex S* **and** *holf*: *f holomorphic_on S* **and** *z ∈ interior S valid_path γ path_image γ ⊆ S − {z}*

pathfinish γ = *pathstart* γ
shows (($\lambda w.$ *f w* / (*w*−*z*)) *has_contour_integral* (*2*∗*pi* ∗ i ∗ *winding_number* γ *z* ∗ *f z*)) γ
theorem *Cauchy_integral_circlepath*:
assumes *contf*: *continuous_on* (*cball z r*) *f* **and** *holf*: *f holomorphic_on* (*ball z r*) **and** *wz*: *norm*(*w*−*z*) < *r*
shows (($\lambda u.$ *f u*/(*u*−*w*)) *has_contour_integral* (*2* ∗ *of_real pi* ∗ i ∗ *f w*)) (*circlepath z r*)

4.2 Existence of all higher derivatives

proposition *derivative_is_holomorphic*:
assumes *open S*
and *fder*: ⋀*z*. *z* ∈ *S* ⟹ (*f has_field_derivative f′ z*) (*at z*)
shows *f′ holomorphic_on S*

4.3 Morera's theorem

proposition *Morera_triangle*:
⟦*continuous_on S f*; *open S*;
⋀*a b c*. *convex hull* {*a*,*b*,*c*} ⊆ *S*
⟶ *contour_integral* (*linepath a b*) *f* +
contour_integral (*linepath b c*) *f* +
contour_integral (*linepath c a*) *f* = *0*⟧
⟹ *f analytic_on S*

4.4 Combining theorems for higher derivatives including Leibniz rule

proposition *no_isolated_singularity*:
fixes *z*::*complex*
assumes *f*: *continuous_on S f* **and** *holf*: *f holomorphic_on* (*S*−*K*) **and** *S*: *open S* **and** *K*: *finite K*
shows *f holomorphic_on S*

proposition *Cauchy_integral_formula_convex*:
assumes *S*: *convex S* **and** *K*: *finite K* **and** *contf*: *continuous_on S f*
and *fcd*: (⋀*x*. *x* ∈ *interior S* − *K* ⟹ *f field_differentiable at x*)
and *z*: *z* ∈ *interior S* **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* γ ⊆ *S* − {*z*} **and** *loop*: *pathfinish* γ = *pathstart* γ
shows (($\lambda w.$ *f w* / (*w*−*z*)) *has_contour_integral* (*2*∗*pi* ∗ i ∗ *winding_number* γ *z* ∗ *f z*)) γ

corollary *Cauchy_contour_integral_circlepath*:
assumes *continuous_on* (*cball z r*) *f f holomorphic_on ball z r w* ∈ *ball z r*

shows $contour\_integral(circlepath\ z\ r)\ (\lambda u.\ f\ u/(u-w)\hat{}(Suc\ k)) = (2 * pi * \mathrm{i}) * (deriv\ \hat{}\hat{}\ k)\ f\ w\ /\ (fact\ k)$

4.5 A holomorphic function is analytic, i.e. has local power series

theorem *holomorphic_power_series*:
assumes *holf*: f *holomorphic_on ball* $z\ r$
and w: $w \in ball\ z\ r$
shows $((\lambda n.\ (deriv\ \hat{}\hat{}\ n)\ f\ z\ /\ (fact\ n) * (w-z)\hat{}n)\ sums\ f\ w)$

4.6 The Liouville theorem and the Fundamental Theorem of Algebra

proposition *Liouville_weak*:
assumes f *holomorphic_on UNIV* **and** $(f \longrightarrow l)$ *at_infinity*
shows $f\ z = l$

proposition *Liouville_weak_inverse*:
assumes f *holomorphic_on UNIV* **and** *unbounded*: $\bigwedge B.\ eventually\ (\lambda x.\ norm\ (f\ x) \geq B)\ at\_infinity$
obtains z **where** $f\ z = 0$
theorem *fundamental_theorem_of_algebra*:
fixes $a :: nat \Rightarrow complex$
assumes $a\ 0 = 0 \vee (\exists\, i \in \{1..n\}.\ a\ i \neq 0)$
obtains z **where** $(\sum i{\leq}n.\ a\ i * z\hat{}i) = 0$

4.7 Weierstrass convergence theorem

proposition *has_complex_derivative_uniform_limit*:
fixes $z{::}complex$
assumes *cont*: $eventually\ (\lambda n.\ continuous\_on\ (cball\ z\ r)\ (f\ n) \wedge (\forall\, w \in ball\ z\ r.\ ((f\ n)\ has\_field\_derivative\ (f'\ n\ w))\ (at\ w)))\ F$
and *ulim*: $uniform\_limit\ (cball\ z\ r)\ f\ g\ F$
and F: $\neg\ trivial\_limit\ F$ **and** $0 < r$
obtains g' **where**
$continuous\_on\ (cball\ z\ r)\ g$
$\bigwedge w.\ w \in ball\ z\ r \Longrightarrow (g\ has\_field\_derivative\ (g'\ w))\ (at\ w) \wedge ((\lambda n.\ f'\ n\ w) \longrightarrow g'\ w)\ F$

4.8 On analytic functions defined by a series

corollary *holomorphic_iff_power_series*:

f holomorphic_on ball z r ⟷
(∀ *w* ∈ *ball z r*. (λ*n*. (*deriv* ^^ *n*) *f z* / (*fact n*) ∗ (*w*−*z*)^*n*) *sums f w*)

4.9 General, homology form of Cauchy's theorem

theorem *Cauchy_integral_formula_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *z*: *z* ∈ *S* **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* γ ⊆ *S* − {*z*} **and** *loop*: *pathfinish* γ = *pathstart* γ
and *zero*: ⋀*w*. *w* ∉ *S* ⟹ *winding_number* γ *w* = *0*
shows ((λ*w*. *f w* / (*w*−*z*)) *has_contour_integral* (*2*∗*pi* ∗ i ∗ *winding_number* γ *z* ∗ *f z*)) γ

theorem *Cauchy_theorem_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *vpg*: *valid_path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *pas*: *path_image* γ ⊆ *S*
and *zero*: ⋀*w*. *w* ∉ *S* ⟹ *winding_number* γ *w* = *0*
shows (*f has_contour_integral 0*) γ

corollary *Cauchy_theorem_global_outside*:
assumes *open S f holomorphic_on S valid_path* γ *pathfinish* γ = *pathstart* γ *path_image* γ ⊆ *S*
⋀*w*. *w* ∉ *S* ⟹ *w* ∈ *outside*(*path_image* γ)
shows (*f has_contour_integral 0*) γ

4.10 Cauchy's inequality and more versions of Liouville

theorem *Liouville_theorem*:
assumes *holf*: *f holomorphic_on UNIV*
and *bf*: *bounded* (*range f*)
shows *f constant_on UNIV*

4.11 Complex functions and power series

definition *fps_expansion* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex fps*
where
fps_expansion f z0 = *Abs_fps* (λ*n*. (*deriv* ^^ *n*) *f z0* / *fact n*)

end

5 Conformal Mappings and Consequences of Cauchy's Integral Theorem

theory *Conformal_Mappings*

imports *Cauchy_Integral_Formula*

begin

5.1 Analytic continuation

proposition *isolated_zeros*:
 assumes *holf*: f *holomorphic_on* S
  and *open* S *connected* S $\xi \in S$ $f\ \xi = 0$ $\beta \in S$ $f\ \beta \neq 0$
 obtains r **where** $0 < r$ **and** *ball* $\xi\ r \subseteq S$ **and**
  $\bigwedge z.\ z \in ball\ \xi\ r - \{\xi\} \Longrightarrow f\ z \neq 0$

proposition *analytic_continuation*:
 assumes *holf*: f *holomorphic_on* S
  and *open* S **and** *connected* S
  and $U \subseteq S$ **and** $\xi \in S$
  and ξ *islimpt* U
  and *fU0* [*simp*]: $\bigwedge z.\ z \in U \Longrightarrow f\ z = 0$
  and $w \in S$
 shows $f\ w = 0$

corollary *analytic_continuation_open*:
 assumes *open* s **and** *open* s' **and** $s \neq \{\}$ **and** *connected* s'
  and $s \subseteq s'$
 assumes f *holomorphic_on* s' **and** g *holomorphic_on* s'
  and $\bigwedge z.\ z \in s \Longrightarrow f\ z = g\ z$
 assumes $z \in s'$
 shows $f\ z = g\ z$

corollary *analytic_continuation'*:
 assumes f *holomorphic_on* S *open* S *connected* S
  and $U \subseteq S$ $\xi \in S$ ξ *islimpt* U
  and f *constant_on* U
 shows f *constant_on* S

5.2 Open mapping theorem

theorem *open_mapping_thm*:
 assumes *holf*: f *holomorphic_on* S
  and S: *open* S **and** *connected* S
  and *open* U **and** $U \subseteq S$
  and *fne*: $\neg$ f *constant_on* S
 shows *open* $(f\ `\ U)$

5.3 Maximum modulus principle

proposition *maximum_modulus_principle*:

assumes *holf*: f *holomorphic_on* S
and S: *open* S **and** *connected* S
and *open* U **and** $U \subseteq S$ **and** $\xi \in U$
and *no*: $\bigwedge z.\ z \in U \Longrightarrow norm(f\ z) \le norm(f\ \xi)$
shows f *constant_on* S

proposition *maximum_modulus_frontier*:
assumes *holf*: f *holomorphic_on* (*interior* S)
and *contf*: *continuous_on* (*closure* S) f
and *bos*: *bounded* S
and *leB*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow norm(f\ z) \le B$
and $\xi \in S$
shows $norm(f\ \xi) \le B$

5.4 Relating invertibility and nonvanishing of derivative

proposition *holomorphic_has_inverse*:
assumes *holf*: f *holomorphic_on* S
and *open* S **and** *injf*: *inj_on* f S
obtains g **where** g *holomorphic_on* (f ‘ S)
$\bigwedge z.\ z \in S \Longrightarrow deriv\ f\ z * deriv\ g\ (f\ z) = 1$
$\bigwedge z.\ z \in S \Longrightarrow g(f\ z) = z$

5.5 The Schwarz Lemma

proposition *Schwarz_Lemma*:
assumes *holf*: f *holomorphic_on* (*ball* 0 1) **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ norm\ z < 1 \Longrightarrow norm\ (f\ z) < 1$
and ξ: $norm\ \xi < 1$
shows $norm\ (f\ \xi) \le norm\ \xi$ **and** $norm(deriv\ f\ 0) \le 1$
and $((\exists z.\ norm\ z < 1 \wedge z \ne 0 \wedge norm(f\ z) = norm\ z)$
$\vee\ norm(deriv\ f\ 0) = 1)$
$\Longrightarrow \exists \alpha.\ (\forall z.\ norm\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge norm\ \alpha = 1$
(**is** $?P \Longrightarrow ?Q$)

corollary *Schwarz_Lemma'*:
assumes *holf*: f *holomorphic_on* (*ball* 0 1) **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ norm\ z < 1 \Longrightarrow norm\ (f\ z) < 1$
shows $((\forall \xi.\ norm\ \xi < 1 \longrightarrow norm\ (f\ \xi) \le norm\ \xi)$
$\wedge\ norm(deriv\ f\ 0) \le 1)$
$\wedge\ (((\exists z.\ norm\ z < 1 \wedge z \ne 0 \wedge norm(f\ z) = norm\ z)$
$\vee\ norm(deriv\ f\ 0) = 1)$
$\longrightarrow (\exists \alpha.\ (\forall z.\ norm\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge norm\ \alpha = 1))$

5.6 The Schwarz reflection principle

proposition *Schwarz_reflection*:
  assumes *open S* **and** *cnjs*: *cnj ‘ S* $\subseteq$ *S*
    and *holf*: *f holomorphic_on* ($S \cap \{z.\ 0 < Im\ z\}$)
    and *contf*: *continuous_on* ($S \cap \{z.\ 0 \le Im\ z\}$) *f*
    and *f*: $\bigwedge z.\ [\![z \in S;\ z \in \mathbb{R}]\!] \Longrightarrow (f\ z) \in \mathbb{R}$
   shows ($\lambda z.$ *if* $0 \le Im\ z$ *then f z else cnj(f(cnj z))*) *holomorphic_on S*

5.7 Bloch's theorem

proposition *Bloch_unit*:
  assumes *holf*: *f holomorphic_on ball a 1* **and** [*simp*]: *deriv f a* = *1*
  obtains *b r* **where** $1/12 < r$ **and** *ball b r* $\subseteq$ *f ‘* (*ball a 1*)

theorem *Bloch*:
  assumes *holf*: *f holomorphic_on ball a r* **and** $0 < r$
    and r': $r' \le r * norm\ (deriv\ f\ a)\ /\ 12$
  obtains *b* **where** *ball b* r' $\subseteq$ *f ‘* (*ball a r*)

corollary *Bloch_general*:
  assumes *holf*: *f holomorphic_on S* **and** $a \in S$
    and *tle*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow t \le dist\ a\ z$
    and *rle*: $r \le t * norm(deriv\ f\ a)\ /\ 12$
  obtains *b* **where** *ball b r* $\subseteq$ *f ‘ S*

end

6 The Great Picard Theorem and its Applications

theory *Great_Picard*
  imports *Conformal_Mappings*
begin

6.1 Schottky's theorem

theorem *Schottky*:
  assumes *holf*: *f holomorphic_on cball 0 1*
    and *nof0*: $norm(f\ 0) \le r$

and *not01*: $\bigwedge z.\ z \in cball\ 0\ 1 \Longrightarrow \neg(f\ z = 0 \vee f\ z = 1)$
and $0 < t$ $t < 1$ $norm\ z \leq t$
shows $norm(f\ z) \leq exp(pi * exp(pi * (2 + 2 * r + 12 * t / (1 - t))))$

6.2 The Little Picard Theorem

theorem *Landau_Picard*:
obtains *R*
where $\bigwedge z.\ 0 < R\ z$
$\bigwedge f.\ [\![f\ holomorphic\_on\ cball\ 0\ (R(f\ 0));$
$\bigwedge z.\ norm\ z \leq R(f\ 0) \Longrightarrow f\ z \neq 0 \wedge f\ z \neq 1]\!] \Longrightarrow norm(deriv\ f\ 0)$
< 1

theorem *little_Picard*:
assumes *holf*: $f\ holomorphic\_on\ UNIV$
and $a \neq b$ $range\ f \cap \{a,b\} = \{\}$
obtains *c* **where** $f = (\lambda x.\ c)$

6.3 The Arzelà–Ascoli theorem

theorem *Arzela_Ascoli*:
fixes $\mathcal{F} :: [nat,'a::euclidean\_space] \Rightarrow 'b::\{real\_normed\_vector,heine\_borel\}$
assumes *compact S*
and *M*: $\bigwedge n\ x.\ x \in S \Longrightarrow norm(\mathcal{F}\ n\ x) \leq M$
and *equicont*:
$\bigwedge x\ e.\ [\![x \in S;\ 0 < e]\!]$
$\Longrightarrow \exists\, d.\ 0 < d \wedge (\forall\, n\ y.\ y \in S \wedge norm(x - y) < d \longrightarrow norm(\mathcal{F}\ n$
$x - \mathcal{F}\ n\ y) < e)$
obtains *g k* **where** $continuous\_on\ S\ g$ $strict\_mono\ (k :: nat \Rightarrow nat)$
$\bigwedge e.\ 0 < e \Longrightarrow \exists\, N.\ \forall\, n\ x.\ n \geq N \wedge x \in S \longrightarrow norm(\mathcal{F}(k\ n)\ x -$
$g\ x) < e$

6.3.1 Montel's theorem

theorem *Montel*:
fixes $\mathcal{F} :: [nat,complex] \Rightarrow complex$
assumes *open S*
and $\mathcal{H}$: $\bigwedge h.\ h \in \mathcal{H} \Longrightarrow h\ holomorphic\_on\ S$
and *bounded*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow \exists\, B.\ \forall\, h \in \mathcal{H}.\ \forall\ z \in K.\ norm(h$
$z) \leq B$
and *rng_f*: $range\ \mathcal{F} \subseteq \mathcal{H}$
obtains *g r*
where $g\ holomorphic\_on\ S$ $strict\_mono\ (r :: nat \Rightarrow nat)$

$\bigwedge x.\ x \in S \Longrightarrow ((\lambda n.\ \mathcal{F}\ (r\ n)\ x) \longrightarrow g\ x)$ *sequentially*
$\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow uniform\_limit\ K\ (\mathcal{F} \circ r)\ g$ *sequentially*

6.4 Some simple but useful cases of Hurwitz's theorem

proposition *Hurwitz_no_zeros*:
assumes *S*: *open S connected S*
and *holf*: $\bigwedge n::nat.\ \mathcal{F}\ n$ *holomorphic_on S*
and *holg*: *g holomorphic_on S*
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow uniform\_limit\ K\ \mathcal{F}\ g$ *sequentially*
and *nonconst*: $\neg$ *g constant_on S*
and *nz*: $\bigwedge n\ z.\ z \in S \Longrightarrow \mathcal{F}\ n\ z \neq 0$
and $z0 \in S$
shows $g\ z0 \neq 0$

corollary *Hurwitz_injective*:
assumes *S*: *open S connected S*
and *holf*: $\bigwedge n::nat.\ \mathcal{F}\ n$ *holomorphic_on S*
and *holg*: *g holomorphic_on S*
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow uniform\_limit\ K\ \mathcal{F}\ g$ *sequentially*
and *nonconst*: $\neg$ *g constant_on S*
and *inj*: $\bigwedge n.\ inj\_on\ (\mathcal{F}\ n)\ S$
shows *inj_on g S*

6.5 The Great Picard theorem

theorem *great_Picard*:
assumes *open M* $z \in M$ $a \neq b$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
and *fab*: $\bigwedge w.\ w \in M - \{z\} \Longrightarrow f\ w \neq a \wedge f\ w \neq b$
obtains *l* **where** $(f \longrightarrow l)\ (at\ z) \vee ((inverse \circ f) \longrightarrow l)\ (at\ z)$

corollary *great_Picard_alt*:
assumes *M*: *open M* $z \in M$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
obtains *a* **where** $- \{a\} \subseteq f\ `\ (M - \{z\})$

corollary *great_Picard_infinite*:
assumes *M*: *open M* $z \in M$ **and** *holf*: *f holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$

obtains *a* **where** $\bigwedge w.\ w \neq a \Longrightarrow \mathit{infinite}\ \{x.\ x \in M - \{z\} \land f\ x = w\}$

theorem *Casorati_Weierstrass*:
assumes $\mathit{open}\ M\ z \in M\ f\ \mathit{holomorphic\_on}\ (M - \{z\})$
and $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (\mathit{at}\ z)\ \bigwedge l.\ \neg\ ((\mathit{inverse} \circ f) \longrightarrow l)\ (\mathit{at}\ z)$
shows $\mathit{closure}(f\ `\ (M - \{z\})) = \mathit{UNIV}$

end

7 Moebius functions, Equivalents of Simply Connected Sets, Riemann Mapping Theorem

theory *Riemann_Mapping*
imports *Great_Picard*
begin

7.1 Moebius functions are biholomorphisms of the unit disc

definition *Moebius_function* :: [*real*,*complex*,*complex*] ⇒ *complex* **where**
$\mathit{Moebius\_function} \equiv \lambda t\ w\ z.\ \mathit{exp}(\mathrm{i} * \mathit{of\_real}\ t) * (z - w)\ /\ (1 - \mathit{cnj}\ w * z)$

7.2 A big chain of equivalents of simple connectedness for an open set

proposition
assumes *open S*
shows *simply_connected_eq_winding_number_zero*:
simply_connected S ⟷
connected S ∧
(∀ *g z*. *path g* ∧ *path_image g* ⊆ *S* ∧
pathfinish g = *pathstart g* ∧ (*z* ∉ *S*)
⟶ *winding_number g z* = *0*) (**is** *?wn0*)
and *simply_connected_eq_contour_integral_zero*:
simply_connected S ⟷
connected S ∧
(∀ *g f*. *valid_path g* ∧ *path_image g* ⊆ *S* ∧
pathfinish g = *pathstart g* ∧ *f holomorphic_on S*
⟶ (*f has_contour_integral 0*) *g*) (**is** *?ci0*)
and *simply_connected_eq_global_primitive*:
simply_connected S ⟷
connected S ∧
(∀ *f*. *f holomorphic_on S* ⟶
(∃ *h*. ∀ *z*. *z* ∈ *S* ⟶ (*h has_field_derivative f z*) (*at z*))) (**is** *?gp*)
and *simply_connected_eq_holomorphic_log*:
simply_connected S ⟷

$\quad$ *connected* $S \wedge$
$\quad (\forall f.\ f$ *holomorphic_on* $S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\qquad \longrightarrow (\exists g.\ g$ *holomorphic_on* $S \wedge (\forall z \in S.\ f\ z = exp(g\ z))))$ (**is** *?log*)
and *simply_connected_eq_holomorphic_sqrt*:
simply_connected $S \longleftrightarrow$
$\quad$ *connected* $S \wedge$
$\quad (\forall f.\ f$ *holomorphic_on* $S \wedge (\forall z \in S.\ f\ z \neq 0)$
$\qquad \longrightarrow (\exists g.\ g$ *holomorphic_on* $S \wedge (\forall z \in S.\ \ f\ z = (g\ z)^2)))$ (**is** *?sqrt*)
and *simply_connected_eq_biholomorphic_to_disc*:
simply_connected $S \longleftrightarrow$
$\quad S = \{\} \vee S = UNIV \vee$
$\quad (\exists f\ g.\ f$ *holomorphic_on* $S \wedge g$ *holomorphic_on ball 0 1* $\wedge$
$\qquad (\forall z \in S.\ f\ z \in ball\ 0\ 1 \wedge g(f\ z) = z) \wedge$
$\qquad (\forall z \in ball\ 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))$ (**is** *?bih*)
and *simply_connected_eq_homeomorphic_to_disc*:
simply_connected $S \longleftrightarrow S = \{\} \vee S$ *homeomorphic ball* $(0::complex)\ 1$ (**is** *?disc*)

corollary *contractible_eq_simply_connected_2d*:
fixes S :: *complex set*
assumes *open* S
shows *contractible* $S \longleftrightarrow$ *simply_connected* S

7.3 A further chain of equivalences about components of the complement of a simply connected set

proposition
fixes S :: *complex set*
assumes *open* S
shows *simply_connected_eq_frontier_properties*:
simply_connected $S \longleftrightarrow$
$\quad$ *connected* $S \wedge$
$\qquad$ (*if bounded* S *then connected*(*frontier* S)
$\qquad$ *else* $(\forall C \in$ *components*(*frontier* S). $\neg$*bounded* C)) (**is** *?fp*)
and *simply_connected_eq_unbounded_complement_components*:
simply_connected $S \longleftrightarrow$
$\quad$ *connected* $S \wedge (\forall C \in$ *components*$(-\ S)$. $\neg$*bounded* C) (**is** *?ucc*)
and *simply_connected_eq_empty_inside*:
simply_connected $S \longleftrightarrow$
$\quad$ *connected* $S \wedge$ *inside* $S = \{\}$ (**is** *?ei*)

7.4 Further equivalences based on continuous logs and sqrts

proposition

fixes S :: *complex set*
assumes *open* S
shows *simply_connected_eq_continuous_log*:
simply_connected $S \longleftrightarrow$
connected $S \wedge$
$(\forall f::complex \Rightarrow complex.\ continuous\_on\ S\ f \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.\ continuous\_on\ S\ g \wedge (\forall z \in S.\ f\ z = exp\ (g\ z))))$ (**is** *?log*)
and *simply_connected_eq_continuous_sqrt*:
simply_connected $S \longleftrightarrow$
connected $S \wedge$
$(\forall f::complex \Rightarrow complex.\ continuous\_on\ S\ f \wedge (\forall z \in S.\ f\ z \neq 0)$
$\longrightarrow (\exists g.\ continuous\_on\ S\ g \wedge (\forall z \in S.\ f\ z = (g\ z)^2)))$ (**is** *?sqrt*)

7.5 Finally, the Riemann Mapping Theorem

theorem *Riemann_mapping_theorem*:
open $S \wedge$ *simply_connected* $S \longleftrightarrow$
$S = \{\} \vee S = UNIV \vee$
$(\exists f\ g.\ f\ holomorphic\_on\ S \wedge g\ holomorphic\_on\ ball\ 0\ 1 \wedge$
$(\forall z \in S.\ f\ z \in ball\ 0\ 1 \wedge g(f\ z) = z) \wedge$
$(\forall z \in ball\ 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))$
(**is** _ = *?rhs*)

7.6 Applications to Winding Numbers

7.7 Winding number equality is the same as path/loop homotopy in C - 0

proposition *winding_number_homotopic_paths_eq*:
assumes *path* p **and** ζp: $\zeta \notin path\_image\ p$
and *path* q **and** ζq: $\zeta \notin path\_image\ q$
and *qp*: *pathstart* q = *pathstart* p *pathfinish* q = *pathfinish* p
shows *winding_number* $p\ \zeta$ = *winding_number* $q\ \zeta \longleftrightarrow$ *homotopic_paths* $(-\{\zeta\})\ p\ q$
(**is** *?lhs* = *?rhs*)

end
theory *Complex_Singularities*
imports *Conformal_Mappings*
begin

7.8 Non-essential singular points

definition
is_pole :: $('a::topological\_space \Rightarrow 'b::real\_normed\_vector) \Rightarrow 'a \Rightarrow bool$

where *is_pole f a* = (*LIM x* (*at a*). *f x* :> *at_infinity*)

7.9 Isolated singularities

7.10 The order of non-essential singularities (i.e. removable singularities or poles)

definition *zorder* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *int* **where**
zorder f z = (*THE n*. (∃ *h r*. *r*>*0* ∧ *h holomorphic_on cball z r* ∧ *h z*≠*0*
∧ (∀ *w*∈*cball z r* − {*z*}. *f w* = *h w* ∗ (*w*−*z*) *powi n*
∧ *h w* ≠*0*)))

definition *zor_poly*
:: [*complex* ⇒ *complex*, *complex*] ⇒ *complex* ⇒ *complex* **where**
zor_poly f z = (*SOME h*. ∃ *r*. *r* > *0* ∧ *h holomorphic_on cball z r* ∧ *h z* ≠ *0*
∧ (∀ *w*∈*cball z r* − {*z*}. *f w* = *h w* ∗ (*w*−*z*) *powi* (*zorder f z*)
∧ *h w* ≠*0*))

7.11 Isolated points

7.12 Isolated zeros

end
theory *Complex_Residues*
imports *Complex_Singularities*
begin

7.13 Definition of residues

definition *residue* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex* **where**
residue f z = (*SOME int*. ∃ *e*>*0*. ∀ *ε*>*0*. *ε*<*e*
⟶ (*f has_contour_integral 2*∗*pi*∗ i ∗*int*) (*circlepath z ε*))

theorem *residue_fps_expansion_over_power_at_0*:
assumes *f has_fps_expansion F*
shows *residue* (λ*z*. *f z* / *z* ^ *Suc n*) *0* = *fps_nth F n*

7.14 Poles and residues of some well-known functions

end

8 The Residue Theorem, the Argument Principle and Rouché's Theorem

theory *Residue_Theorem*
  imports *Complex_Residues HOL−Library.Landau_Symbols*
begin

8.1 Cauchy's residue theorem

theorem *Residue_theorem*:
  fixes *S pts*::*complex set* **and** *f*::*complex* ⇒ *complex*
    and *g*::*real* ⇒ *complex*
  assumes *open S connected S finite pts* **and**
        holo:*f holomorphic_on S−pts* **and**
        valid_path g **and**
        loop:*pathfinish g* = *pathstart g* **and**
        path_image g ⊆ *S−pts* **and**
        homo:∀ *z*. (*z* ∉ *S*) ⟶ *winding_number g z* = *0*
  shows *contour_integral g f* = *2* ∗ *pi* ∗ i ∗(∑ *p*∈*pts*. *winding_number g p* ∗ *residue f p*)

8.2 The argument principle

theorem *argument_principle*:
  fixes *f*::*complex* ⇒ *complex* **and** *poles S*:: *complex set*
  defines *pz* ≡ {*w*∈*S*. *f w* = *0* ∨ *w* ∈ *poles*} — *pz* is the set of poles and zeros
  assumes *open S connected S* **and**
        f_holo:*f holomorphic_on S−poles* **and**
        h_holo:*h holomorphic_on S* **and**
        valid_path g **and**
        loop:*pathfinish g* = *pathstart g* **and**
        path_img:*path_image g* ⊆ *S* − *pz* **and**
        homo:∀ *z*. (*z* ∉ *S*) ⟶ *winding_number g z* = *0* **and**
        finite:*finite pz* **and**
        poles:∀ *p*∈*S*∩*poles*. *is_pole f p*
  shows *contour_integral g* (λ*x*. *deriv f x* ∗ *h x* / *f x*) = *2* ∗ *pi* ∗ i ∗
        (∑ *p*∈*pz*. *winding_number g p* ∗ *h p* ∗ *zorder f p*)
    (**is** *?L*=*?R*)

8.3 Coefficient asymptotics for generating functions

theorem
  fixes *f* :: *complex* ⇒ *complex* **and** *n* :: *nat* **and** *r* :: *real*
  defines *g* ≡ (λ*w*. *f w* / *w* ^ *Suc n*) **and** γ ≡ *circlepath 0 r*
  assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
  assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*

shows *fps_coeff_conv_residues*:
(*deriv* ^^ *n*) *f 0* / *fact n* =
contour_integral γ *g* / (*2* ∗ *pi* ∗ i) − (∑ *z*∈*S*. *residue g z*) (**is** *?thesis1*)
and *fps_coeff_residues_bound*:
(⋀*z*. *norm z* = *r* ⟹ *z* ∉ *k* ⟹ *norm* (*f z*) ≤ *C*) ⟹ *C* ≥ *0* ⟹ *finite k* ⟹
norm ((*deriv* ^^ *n*) *f 0* / *fact n* + (∑ *z*∈*S*. *residue g z*)) ≤ *C* / *r* ^ *n*

corollary *fps_coeff_residues_bigo*:
fixes *f* :: *complex* ⇒ *complex* **and** *r* :: *real*
assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*
assumes *g*: *eventually* (λ*n*. *g n* = −(∑ *z*∈*S*. *residue* (λ*z*. *f z* / *z* ^ *Suc n*) *z*)) *sequentially*
(**is** *eventually* (λ*n*. _ = −*?g′ n*) _)
shows (λ*n*. (*deriv* ^^ *n*) *f 0* / *fact n* − *g n*) ∈ *O*(λ*n*. *1* / *r* ^ *n*) (**is** (λ*n*. *?c n* − _) ∈ *O*(_))

corollary *fps_coeff_residues_bigo′*:
fixes *f* :: *complex* ⇒ *complex* **and** *r* :: *real*
assumes *exp*: *f has_fps_expansion F*
assumes *open A connected A cball 0 r* ⊆ *A r* > *0*
assumes *f holomorphic_on A* − *S S* ⊆ *ball 0 r finite S 0* ∉ *S*
assumes *eventually* (λ*n*. *g n* = −(∑ *z*∈*S*. *residue* (λ*z*. *f z* / *z* ^ *Suc n*) *z*)) *sequentially*
(**is** *eventually* (λ*n*. _ = −*?g′ n*) _)
shows (λ*n*. *fps_nth F n* − *g n*) ∈ *O*(λ*n*. *1* / *r* ^ *n*) (**is** (λ*n*. *?c n* − _) ∈ *O*(_))

8.4 Rouche's theorem

theorem *Rouche_theorem*:
fixes *f g*::*complex* ⇒ *complex* **and** *s*:: *complex set*
defines *fg*≡(λ*p*. *f p* + *g p*)
defines *zeros_fg*≡{*p*∈*s*. *fg p* = *0*} **and** *zeros_f*≡{*p*∈*s*. *f p* = *0*}
assumes
open s **and** *connected s* **and**
finite zeros_fg **and**
finite zeros_f **and**
f_holo:*f holomorphic_on s* **and**
g_holo:*g holomorphic_on s* **and**
valid_path γ **and**
loop:*pathfinish* γ = *pathstart* γ **and**
path_img:*path_image* γ ⊆ *s* **and**
path_less:∀ *z*∈*path_image* γ. *cmod*(*f z*) > *cmod*(*g z*) **and**
homo:∀ *z*. (*z* ∉ *s*) ⟶ *winding_number* γ *z* = *0*
shows (∑ *p*∈*zeros_fg*. *winding_number* γ *p* ∗ *zorder fg p*)
= (∑ *p*∈*zeros_f*. *winding_number* γ *p* ∗ *zorder f p*)

end
theory *Laurent_Convergence*
imports *HOL−Computational_Algebra.Formal_Laurent_Series HOL−Library.Landau_Symbols Residue_Theorem*

begin

definition *fls_conv_radius* :: *complex fls* ⇒ *ereal* **where**
fls_conv_radius f = *fps_conv_radius* (*fls_regpart f*)

definition *eval_fls* :: *complex fls* ⇒ *complex* ⇒ *complex* **where**
eval_fls F z = *eval_fps* (*fls_base_factor_to_fps F*) *z* ∗ *z powi fls_subdegree F*

definition
has_laurent_expansion :: (*complex* ⇒ *complex*) ⇒ *complex fls* ⇒ *bool*
(**infixl** ‹*has′_laurent′_expansion*› *60*)
where (*f has_laurent_expansion F*) ⟷
fls_conv_radius F > *0* ∧ *eventually* (λ*z*. *eval_fls F z* = *f z*) (*at 0*)

theorem *sums_eval_fls*:
fixes *f*
defines *n* ≡ *fls_subdegree f*
assumes *norm z* < *fls_conv_radius f* **and** *z* ≠ *0* ∨ *n* ≥ *0*
shows (λ*k*. *fls_nth f* (*int k* + *n*) ∗ *z powi* (*int k* + *n*)) *sums eval_fls f z*

theorem *not_essential_has_laurent_expansion_0*:
assumes *isolated_singularity_at f 0 not_essential f 0*
shows *f has_laurent_expansion laurent_expansion f 0*

8.5 More Laurent expansions

8.6 Formal convergence versus analytic convergence

proposition *uniform_limit_imp_fps_expansion_eq*:
fixes *f* :: ′*a* ⇒ *complex fps*
assumes *lim1*: (*f* ⟶ *h*) *F*
assumes *lim2*: *uniform_limit A* (λ*x z*. *f′ x z*) *g′ F*
assumes *expansions*: *eventually* (λ*x*. *f′ x has_fps_expansion f x*) *F g′ has_fps_expansion g*
assumes *holo*: *eventually* (λ*x*. *f′ x holomorphic_on A*) *F*
assumes *A*: *open A 0* ∈ *A*

assumes *nontriv* [*simp*]: $F \neq bot$
shows $g = h$

end

theory *Meromorphic* **imports**
Laurent_Convergence
Cauchy_Integral_Formula
HOL−Analysis.Sparse_In
begin

8.7 Remove singular points

definition *remove_sings* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex* **where**
remove_sings f z = (*if* ∃ *c*. *f* −*z*→ *c then Lim* (*at z*) *f else 0*)

8.8 Meromorphicity

definition *meromorphic_on* :: (*complex* ⇒ *complex*) ⇒ *complex set* ⇒ *bool*
(**infixl** ‹(*meromorphic′_on*)› *50*) **where**
f meromorphic_on A ⟷ (∀ *z*∈*A*. ∃ *F*. (λ*w*. *f* (*z* + *w*)) *has_laurent_expansion* *F*)

8.9 Nice meromorphicity

8.10 Closure properties and proofs for individual functions

8.11 Meromorphic functions and zorder

8.12 More on poles and zeros

end

9 The Weierstraß Factorisation Theorem

theory *Weierstrass_Factorization*
imports *Meromorphic*
begin

9.1 The elementary factors

9.2 Infinite products of elementary factors

9.3 Writing a quotient as an exponential

9.4 Constructing the sequence of zeros

9.5 The factorisation theorem for holomorphic functions

theorem *weierstrass_factorization*:
  assumes *g holomorphic_on A open A connected A*
  assumes $\bigwedge z.\ z \in frontier\ A \Longrightarrow \neg z\ islimpt\ \{w{\in}A.\ g\ w = 0\}$
  obtains *h f* **where**
    h holomorphic_on A f holomorphic_on UNIV
    $\forall z.\ f\ z = 0 \longleftrightarrow (\forall z{\in}A.\ g\ z = 0) \lor (z \in A \land g\ z = 0)$
    $\forall z{\in}A.\ zorder\ f\ z = zorder\ g\ z$
    $\forall z{\in}A.\ h\ z \neq 0$
    $\forall z{\in}A.\ g\ z = h\ z * f\ z$
theorem *weierstrass_factorization_UNIV*:
  assumes *g holomorphic_on UNIV*
  obtains *h f* **where**
    h holomorphic_on UNIV f holomorphic_on UNIV
    $\forall z.\ f\ z = 0 \longleftrightarrow g\ z = 0$
    $\forall z.\ zorder\ f\ z = zorder\ g\ z$
    $\forall z.\ h\ z \neq 0$
    $\forall z.\ g\ z = h\ z * f\ z$

9.6 The factorisation theorem for meromorphic functions

theorem *weierstrass_factorization_meromorphic*:
  assumes *mero*: *g nicely_meromorphic_on A* **and** *A*: *open A connected A*
  assumes *no_limpt*: $\bigwedge z.\ z \in frontier\ A \Longrightarrow \neg z\ islimpt\ \{w{\in}A.\ g\ w = 0 \lor is\_pole\ g\ w\}$
  obtains *h f1 f2* **where**
    h holomorphic_on A f1 holomorphic_on UNIV f2 holomorphic_on UNIV
    $\forall z{\in}A.\ f1\ z = 0 \longleftrightarrow \neg is\_pole\ g\ z \land g\ z = 0$
    $\forall z{\in}A.\ f2\ z = 0 \longleftrightarrow is\_pole\ g\ z$
    $\forall z{\in}A.\ \neg is\_pole\ g\ z \longrightarrow zorder\ f1\ z = zorder\ g\ z$
    $\forall z{\in}A.\ is\_pole\ g\ z \longrightarrow zorder\ f2\ z = -zorder\ g\ z$
    $\forall z{\in}A.\ h\ z \neq 0$
    $\forall z{\in}A.\ g\ z = h\ z * f1\ z\ /\ f2\ z$
theorem *weierstrass_factorization_meromorphic_UNIV*:
  assumes *g nicely_meromorphic_on UNIV*
  obtains *h f1 f2* **where**
    h holomorphic_on UNIV f1 holomorphic_on UNIV f2 holomorphic_on UNIV

$\forall z.\ f1\ z = 0 \longleftrightarrow \neg is\_pole\ g\ z \wedge g\ z = 0$
$\forall z.\ f2\ z = 0 \longleftrightarrow is\_pole\ g\ z$
$\forall z.\ \neg is\_pole\ g\ z \longrightarrow zorder\ f1\ z = zorder\ g\ z$
$\forall z.\ is\_pole\ g\ z \longrightarrow zorder\ f2\ z = -zorder\ g\ z$
$\forall z.\ h\ z \neq 0$
$\forall z.\ g\ z = h\ z * f1\ z\ /\ f2\ z$

end
theory *Complex_Analysis*
imports
Riemann_Mapping
Residue_Theorem
Weierstrass_Factorization
begin

end

References

[1]