Analysis

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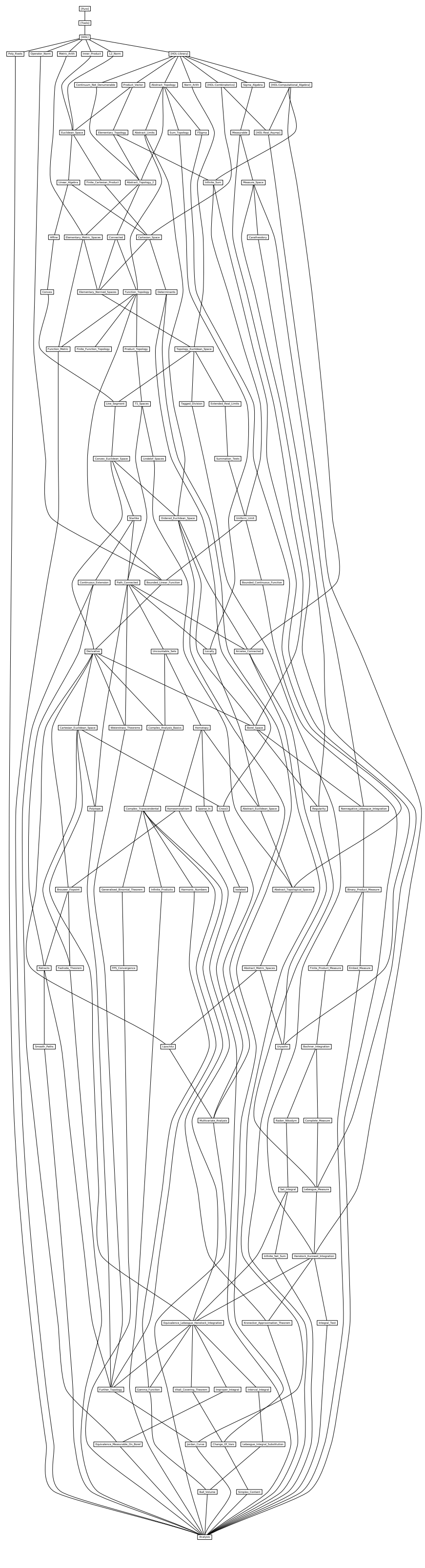
Contents



Chapter 1

Linear Algebra

theory *L2_Norm*
imports *Complex_Main*
begin

1.1 L2 Norm

definition *L2_set* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow real$ **where**
$L2\_set\ f\ A = sqrt\ (\sum i{\in}A.\ (f\ i)^2)$

proposition *L2_set_triangle_ineq*:
$L2\_set\ (\lambda i.\ f\ i + g\ i)\ A \leq L2\_set\ f\ A + L2\_set\ g\ A$

end

1.2 Inner Product Spaces and Gradient Derivative

theory *Inner_Product*
imports *Complex_Main*
begin

1.2.1 Real inner product spaces

class *real_inner* = *real_vector* + *sgn_div_norm* + *dist_norm* + *uniformity_dist* + *open_uniformity* +
fixes *inner* :: $'a \Rightarrow 'a \Rightarrow real$
assumes *inner_commute*: *inner x y* = *inner y x*
and *inner_add_left*: *inner* (*x* + *y*) *z* = *inner x z* + *inner y z*
and *inner_scaleR_left* [*simp*]: *inner* (*scaleR r x*) *y* = *r* ∗ (*inner x y*)
and *inner_ge_zero* [*simp*]: *0* ≤ *inner x x*
and *inner_eq_zero_iff* [*simp*]: *inner x x* = *0* ⟷ *x* = *0*
and *norm_eq_sqrt_inner*: *norm x* = *sqrt* (*inner x x*)
begin

1.2.2 Class instances

instantiation *real* :: *real_inner*
begin

instantiation *complex* :: *real_inner*
begin

1.2.3 Gradient derivative

definition
gderiv :: [′*a*::*real_inner* ⇒ *real*, ′*a*, ′*a*] ⇒ *bool*
(‹(‹*notation*=‹*mixfix GDERIV*››*GDERIV* (_)/ (_)/ :> (_))› [*1000*, *1000*, *60*] *60*)
where
GDERIV f x :> *D* ⟷ *FDERIV f x* :> (λ*h*. *inner h D*)

end

1.3 Cartesian Products as Vector Spaces

theory *Product_Vector*
imports
Complex_Main
HOL−Library.Product_Plus
begin

1.3.1 Product is a Module

lemma *scale_prod*: *scale x* (*a*, *b*) = (*s1 x a*, *s2 x b*)

sublocale *p*: *module scale*

1.3.2 Product is a Real Vector Space

instantiation *prod* :: (*real_vector*, *real_vector*) *real_vector*
begin

proposition *scaleR_Pair* [*simp*]: *scaleR r* (*a*, *b*) = (*scaleR r a*, *scaleR r b*)

1.3.3 Product is a Metric Space

class *uniform_topological_monoid_add* = *topological_monoid_add* + *uniform_space* +
assumes *uniformly_continuous_add'*:
filterlim $(\lambda((a,b), (c,d)).\ (a + c,\ b + d))$ *uniformity* (*uniformity* $\times_F$ *uniformity*)

class *uniform_topological_group_add* = *topological_group_add* + *uniform_topological_monoid_add* +
assumes *uniformly_continuous_uminus'*: *filterlim* $(\lambda(a, b).\ (-a, -b))$ *uniformity* *uniformity*
begin

instantiation *prod* :: (*metric_space*, *metric_space*) *metric_space*
begin

proposition *dist_Pair_Pair*: *dist* (*a*, *b*) (*c*, *d*) = *sqrt* ((*dist a c*)^2 + (*dist b d*)^2)

1.3.4 Product is a Complete Metric Space

instance *prod* :: (*complete_space*, *complete_space*) *complete_space*

1.3.5 Product is a Normed Vector Space

instantiation *prod* :: (*real_normed_vector*, *real_normed_vector*) *real_normed_vector*
begin

proposition *norm_Pair*: *norm* (*a*, *b*) = *sqrt* ((*norm a*)^2 + (*norm b*)^2)

instance *prod* :: (*banach*, *banach*) *banach*

proposition *has_derivative_Pair* [*derivative_intros*]:
assumes *f*: (*f has_derivative f'*) (*at x within s*)
and *g*: (*g has_derivative g'*) (*at x within s*)
shows $((\lambda x.\ (f\ x,\ g\ x))$ *has_derivative* $(\lambda h.\ (f'\ h,\ g'\ h)))$ (*at x within s*)

1.3.6 Product is Finite Dimensional

proposition *dim_Times*:
assumes *vs1.subspace S vs2.subspace T*
shows *p.dim*($S \times T$) = *vs1.dim S* + *vs2.dim T*

end

1.4 Finite-Dimensional Inner Product Spaces

theory *Euclidean_Space*
imports
  L2_Norm
  Inner_Product
  Product_Vector
begin

1.4.1 Type class of Euclidean spaces

class *euclidean_space* = *real_inner* +
  fixes *Basis* :: *'a set*
  assumes *nonempty_Basis* [*simp*]: *Basis* ≠ {}
  assumes *finite_Basis* [*simp*]: *finite Basis*
  assumes *inner_Basis*:
    ⟦*u* ∈ *Basis*; *v* ∈ *Basis*⟧ ⟹ *inner u v* = (*if u* = *v then 1 else 0*)
  assumes *euclidean_all_zero_iff*:
    (∀ *u*∈*Basis*. *inner x u* = *0*) ⟷ (*x* = *0*)

1.4.2 Class instances

instantiation *real* :: *euclidean_space*
begin
instantiation *complex* :: *euclidean_space*
begin
instantiation *prod* :: (*real_inner*, *real_inner*) *real_inner*
begin

instantiation *prod* :: (*euclidean_space*, *euclidean_space*) *euclidean_space*
begin

1.4.3 Locale instances

end

1.5 Elementary Linear Algebra on Euclidean Spaces

theory *Linear_Algebra*
imports
  Euclidean_Space
  HOL−Library.Infinite_Set
begin

1.5.1 Substandard Basis

1.5.2 Orthogonality

definition (**in** *real_inner*) *orthogonal x y* $\longleftrightarrow$ *x* $\cdot$ *y* = *0*

1.5.3 Orthogonality of a transformation

definition *orthogonal_transformation f* $\longleftrightarrow$ *linear f* $\wedge$ ($\forall$ *v w*. *f v* $\cdot$ *f w* = *v* $\cdot$ *w*)

1.5.4 Bilinear functions

definition
bilinear :: (*'a*::*real_vector* $\Rightarrow$ *'b*::*real_vector* $\Rightarrow$ *'c*::*real_vector*) $\Rightarrow$ *bool* **where**
bilinear f $\longleftrightarrow$ ($\forall$ *x*. *linear* (λ*y*. *f x y*)) $\wedge$ ($\forall$ *y*. *linear* (λ*x*. *f x y*))

1.5.5 Adjoints

definition *adjoint* :: ((*'a*::*real_inner*) $\Rightarrow$ (*'b*::*real_inner*)) $\Rightarrow$ *'b* $\Rightarrow$ *'a* **where**
adjoint f = (*SOME f'*. $\forall$ *x y*. *f x* $\cdot$ *y* = *x* $\cdot$ *f' y*)

1.5.6 Infinity norm

definition *infnorm* (*x*::*'a*::*euclidean_space*) = *Sup* {|*x* $\cdot$ *b*| |*b*. *b* $\in$ *Basis*}

1.5.7 Collinearity

definition *collinear* :: *'a*::*real_vector set* $\Rightarrow$ *bool*
where *collinear S* $\longleftrightarrow$ ($\exists$ *u*. $\forall$ *x* $\in$ *S*. $\forall$ *y* $\in$ *S*. $\exists$ *c*. *x* − *y* = *c* $*_R$ *u*)

1.5.8 Properties of special hyperplanes

proposition *dim_hyperplane*:
fixes *a* :: *'a*::*euclidean_space*
assumes *a* $\neq$ *0*
shows *dim* {*x*. *a* $\cdot$ *x* = *0*} = *DIM*(*'a*) − *1*

1.5.9 Orthogonal bases and Gram-Schmidt process

proposition *Gram_Schmidt_step*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *S*: *pairwise orthogonal S* **and** *x*: *x* $\in$ *span S*

shows *orthogonal x* $(a - (\sum b{\in}S.\ (b \cdot a\ /\ (b \cdot b))\ *_R\ b))$

proposition *orthogonal_extension*:
fixes S :: *'a::euclidean_space set*
assumes S: *pairwise orthogonal S*
obtains U **where** *pairwise orthogonal* $(S \cup U)$ *span* $(S \cup U) =$ *span* $(S \cup T)$

1.5.10 Decomposing a vector into parts in orthogonal subspaces

proposition *orthonormal_basis_subspace*:
fixes S :: *'a :: euclidean_space set*
assumes *subspace S*
obtains B **where** $B \subseteq S$ *pairwise orthogonal B*
and $\bigwedge x.\ x \in B \Longrightarrow$ *norm* $x = 1$
and *independent B card* $B =$ *dim S span* $B = S$

proposition *dim_orthogonal_sum*:
fixes A :: *'a::euclidean_space set*
assumes $\bigwedge x\ y.\ [\![x \in A;\ y \in B]\!] \Longrightarrow x \cdot y = 0$
shows *dim*$(A \cup B) =$ *dim* A + *dim* B

1.5.11 Linear functions are (uniformly) continuous on any set

end

1.6 Affine Sets

theory *Affine*
imports *Linear_Algebra*
begin

1.6.1 Affine set and affine hull

definition *affine* :: *'a::real_vector set* $\Rightarrow$ *bool*
where *affine* $S \longleftrightarrow (\forall x{\in}S.\ \forall y{\in}S.\ \forall u\ v.\ u + v = 1 \longrightarrow u *_R x + v *_R y \in S)$

1.6.2 Affine Dependence

definition *affine_dependent* :: $'a$::*real_vector set* ⇒ *bool*
where *affine_dependent S* ⟷ (∃ *x*∈*S*. *x* ∈ *affine hull* (*S* − {*x*}))

proposition *affine_dependent_explicit*:
affine_dependent p ⟷
(∃ *S U*. *finite S* ∧ *S* ⊆ *p* ∧ *sum U S* = *0* ∧ (∃ *v*∈*S*. *U v* ≠ *0*) ∧ *sum* (λ*v*. *U v* $*_R$ *v*) *S* = *0*)

proposition *extend_to_affine_basis*:
fixes *S V* :: $'n$::*real_vector set*
assumes ¬ *affine_dependent S S* ⊆ *V*
obtains *T* **where** ¬ *affine_dependent T S* ⊆ *T T* ⊆ *V affine hull T* = *affine hull V*

1.6.3 Affine Dimension of a Set

definition *aff_dim* :: ($'a$::*euclidean_space*) *set* ⇒ *int*
where *aff_dim V* =
(*SOME d* :: *int*.
∃ *B*. *affine hull B* = *affine hull V* ∧ ¬ *affine_dependent B* ∧ *of_nat* (*card B*) = *d* + *1*)

end

1.7 Convex Sets and Functions

theory *Convex*
imports
Affine HOL−Library.Set_Algebras HOL−Library.FuncSet
begin

1.7.1 Convex Sets

definition *convex* :: $'a$::*real_vector set* ⇒ *bool*
where *convex s* ⟷ (∀ *x*∈*s*. ∀ *y*∈*s*. ∀ *u*≥*0*. ∀ *v*≥*0*. *u* + *v* = *1* ⟶ *u* $*_R$ *x* + *v* $*_R$ *y* ∈ *s*)

1.7.2 Convex Functions on a Set

definition *convex_on* :: $'a$::*real_vector set* ⇒ ($'a$ ⇒ *real*) ⇒ *bool*
where *convex_on S f* ⟷ *convex S* ∧
(∀ *x*∈*S*. ∀ *y*∈*S*. ∀ *u*≥*0*. ∀ *v*≥*0*. *u* + *v* = *1* ⟶ *f* (*u* $*_R$ *x* + *v* $*_R$ *y*) ≤ *u* ∗ *f x* + *v* ∗ *f y*)

definition *concave_on* :: $'a$::*real_vector set* ⇒ ($'a$ ⇒ *real*) ⇒ *bool*
  where *concave_on S f* ≡ *convex_on S* (λ*x*. − *f x*)

1.7.3 Convexity of the generalised binomial

1.7.4 Some inequalities: Applications of convexity

1.7.5 Misc related lemmas

1.7.6 Cones

definition *cone* :: $'a$::*real_vector set* ⇒ *bool*
  where *cone s* ⟷ (∀ *x*∈*s*. ∀ *c*≥*0*. $c *_R x$ ∈ *s*)

proposition *cone_hull_expl*: *cone hull S* = {$c *_R x$ | *c x*. *c* ≥ *0* ∧ *x* ∈ *S*}
  (**is** *?lhs* = *?rhs*)

1.7.7 Convex hull

proposition *convex_hull_indexed*:
  fixes *S* :: $'a$::*real_vector set*
  shows *convex hull S* =
    {*y*. ∃ *k u x*. (∀ *i*∈{*1*::*nat* .. *k*}. *0* ≤ *u i* ∧ *x i* ∈ *S*) ∧
          (*sum u* {*1*..*k*} = *1*) ∧ (∑ *i* = *1*..*k*. $u\ i *_R x\ i$) = *y*}
    (**is** *?xyz* = *?hull*)

1.7.8 Caratheodory's theorem

theorem *caratheodory*:
  convex hull p =
    {*x*::$'a$::*euclidean_space*. ∃ *S*. *finite S* ∧ *S* ⊆ *p* ∧ *card S* ≤ *DIM*($'a$) + *1* ∧ *x* ∈ *convex hull S*}

1.8 Conic sets and conic hull

1.9 Convex cones and corresponding hulls

1.9.1 Radon's theorem

theorem *Radon*:
  assumes *affine_dependent c*

obtains $M\ P$ **where** $M \subseteq c\ P \subseteq c\ M \cap P = \{\}$ $(convex\ hull\ M) \cap (convex\ hull\ P) \neq \{\}$

1.9.2 Helly's theorem

theorem *Helly*:
fixes $\mathcal{F}$ :: $'a::euclidean\_space\ set\ set$
assumes $card\ \mathcal{F} \geq DIM('a) + 1\ \forall s \in \mathcal{F}.\ convex\ s$
and $\bigwedge t.\ [\![t \subseteq \mathcal{F};\ card\ t = DIM('a) + 1]\!] \Longrightarrow \bigcap t \neq \{\}$
shows $\bigcap \mathcal{F} \neq \{\}$

1.9.3 Epigraphs of convex functions

definition $epigraph\ S\ (f :: \_ \Rightarrow real) = \{xy.\ fst\ xy \in S \wedge f\ (fst\ xy) \leq snd\ xy\}$

end

1.10 Definition of Finite Cartesian Product Type

theory *Finite_Cartesian_Product*
imports
Euclidean_Space
L2_Norm
HOL−Library.Numeral_Type
HOL−Library.Countable_Set
HOL−Library.FuncSet
begin

1.10.1 Cardinality of vectors

proposition *CARD_vec* [*simp*]:
$CARD('a^{\frown}'b) = CARD('a)\ \hat{}\ CARD('b)$
instantiation *vec* :: (*zero*, *finite*) *zero*
begin

instantiation *vec* :: (*plus*, *finite*) *plus*
begin

instantiation *vec* :: (*minus*, *finite*) *minus*
begin

instantiation *vec* :: (*uminus*, *finite*) *uminus*
begin
instantiation *vec* :: (*times*, *finite*) *times*
begin

instantiation *vec* :: (*one*, *finite*) *one*
begin

instantiation *vec* :: (*ord*, *finite*) *ord*
begin

1.10.2 Real vector space

definition *scaleR* ≡ (λ *r x*. (χ *i*. *scaleR r* (*x*$*i*)))

1.10.3 Topological space

definition [*code del*]:
open (*S* :: (′*a* ^ ′*b*) *set*) ⟷
(∀ *x*∈*S*. ∃ *A*. (∀ *i*. *open* (*A i*) ∧ *x*$*i* ∈ *A i*) ∧
(∀ *y*. (∀ *i*. *y*$*i* ∈ *A i*) ⟶ *y* ∈ *S*))

1.10.4 Metric space

definition
dist x y = *L2_set* (λ*i*. *dist* (*x*$*i*) (*y*$*i*)) *UNIV*

definition [*code del*]:
(*uniformity* :: ((′*a*^′*b*::_) × (′*a*^′*b*::_)) *filter*) =
(*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

proposition *dist_vec_nth_le*: *dist* (*x* $ *i*) (*y* $ *i*) ≤ *dist x y*

1.10.5 Normed vector space

definition *norm x* = *L2_set* (λ*i*. *norm* (*x*$*i*)) *UNIV*

definition *sgn* (*x*::′*a*^′*b*) = *scaleR* (*inverse* (*norm x*)) *x*

1.10.6 Inner product space

definition *inner x y* = *sum* (λ*i*. *inner* (*x*$*i*) (*y*$*i*)) *UNIV*

1.10.7 Euclidean space

definition $axis\ k\ x = (\chi\ i.\ if\ i = k\ then\ x\ else\ 0)$

definition $Basis = (\bigcup i.\ \bigcup u{\in}Basis.\ \{axis\ i\ u\})$

proposition DIM_cart [$simp$]: $DIM('a^{\frown}'b) = CARD('b) * DIM('a)$

1.10.8 Matrix operations

definition $map\_matrix{::}('a \Rightarrow 'b) \Rightarrow (('a,\ 'i{::}finite)vec,\ 'j{::}finite)\ vec \Rightarrow (('b,\ 'i)vec,\ 'j)\ vec$ **where**
$map\_matrix\ f\ x = (\chi\ i\ j.\ f\ (x\ \$\ i\ \$\ j))$

definition $matrix\_matrix\_mult :: ('a{::}semiring\_1)\ ^{\frown}'n^{\frown}'m \Rightarrow 'a\ ^{\frown}'p^{\frown}'n \Rightarrow 'a\ ^{\frown}'p\ ^{\frown}'m$
(**infixl** ‹$**$› 70)
where $m ** m' == (\chi\ i\ j.\ sum\ (\lambda k.\ ((m\$i)\$k) * ((m'\$k)\$j))\ (UNIV :: 'n\ set)) ::'a\ ^{\frown}'p\ ^{\frown}'m$

definition $matrix\_vector\_mult :: ('a{::}semiring\_1)\ ^{\frown}'n^{\frown}'m \Rightarrow 'a\ ^{\frown}'n \Rightarrow 'a\ ^{\frown}'m$
(**infixl** ‹$*v$› 70)
where $m *v\ x \equiv (\chi\ i.\ sum\ (\lambda j.\ ((m\$i)\$j) * (x\$j))\ (UNIV ::'n\ set)) :: 'a^{\frown}'m$

definition $vector\_matrix\_mult :: 'a\ ^{\frown}'m \Rightarrow ('a{::}semiring\_1)\ ^{\frown}'n^{\frown}'m \Rightarrow 'a\ ^{\frown}'n$
(**infixl** ‹$v*$› 70)
where $v\ v*\ m == (\chi\ j.\ sum\ (\lambda i.\ ((v\$i) * (m\$i)\$j))\ (UNIV :: 'm\ set)) :: 'a^{\frown}'n$
definition $matrix :: ('a{::}\{plus,times,\ one,\ zero\}^{\frown}'m \Rightarrow 'a\ ^{\frown}'n) \Rightarrow 'a^{\frown}'m^{\frown}'n$
where $matrix\ f = (\chi\ i\ j.\ (f(axis\ j\ 1))\$i)$

1.10.9 Inverse matrices (not necessarily square)

definition
$invertible(A{::}'a{::}semiring\_1^{\frown}'n^{\frown}'m) \longleftrightarrow (\exists A'{::}'a^{\frown}'m^{\frown}'n.\ A ** A' = mat\ 1 \wedge A' ** A = mat\ 1)$

definition
$matrix\_inv(A{::}\ 'a{::}semiring\_1^{\frown}'n^{\frown}'m) =$
$(SOME\ A'{::}'a^{\frown}'m^{\frown}'n.\ A ** A' = mat\ 1 \wedge A' ** A = mat\ 1)$

end

1.11 Linear Algebra on Finite Cartesian Products

theory *Cartesian_Space*
imports
HOL−Combinatorics.Transposition
Finite_Cartesian_Product

Linear_Algebra
begin

1.11.1 Some interesting theorems and interpretations

1.11.2 Rank of a matrix

definition *rank* :: *'a::field^'n^'m=>nat*
where *row_rank_def_gen*: *rank A* ≡ *vec.dim*(*rows A*)

1.11.3 Orthogonality of a matrix

definition *orthogonal_matrix* (*Q::'a::semiring_1^'n^'n*) ⟷
transpose Q ** *Q* = *mat 1* ∧ *Q* ** *transpose Q* = *mat 1*

proposition *orthogonal_matrix_mul*:
fixes *A* :: *real ^'n^'n*
assumes *orthogonal_matrix A orthogonal_matrix B*
shows *orthogonal_matrix*(*A* ** *B*)

proposition *orthogonal_transformation_matrix*:
fixes *f*:: *real^'n* ⇒ *real^'n*
shows *orthogonal_transformation f* ⟷ *linear f* ∧ *orthogonal_matrix*(*matrix f*)
(**is** *?lhs* ⟷ *?rhs*)

1.11.4 Finding an Orthogonal Matrix

proposition *orthogonal_matrix_exists_basis*:
fixes *a* :: *real^'n*
assumes *norm a* = *1*
obtains *A* **where** *orthogonal_matrix A A* *v (*axis k 1*) = *a*

proposition *orthogonal_transformation_exists*:
fixes *a b* :: *real^'n*
assumes *norm a* = *norm b*
obtains *f* **where** *orthogonal_transformation f f a* = *b*

1.11.5 Scaling and isometry

proposition *scaling_linear*:
  fixes f :: $'a::real\_inner \Rightarrow 'a::real\_inner$
  assumes *f0*: $f\ 0 = 0$
    and *fd*: $\forall x\ y.\ dist\ (f\ x)\ (f\ y) = c * dist\ x\ y$
  shows *linear f*
proposition *orthogonal_transformation_isometry*:
  $orthogonal\_transformation\ f \longleftrightarrow f(0::'a::real\_inner) = (0::'a) \land (\forall x\ y.\ dist(f\ x)\ (f\ y) = dist\ x\ y)$

1.11.6 Induction on matrix row operations

end

1.12 Traces and Determinants of Square Matrices

theory *Determinants*
imports
  HOL−Combinatorics.Permutations
  Cartesian_Space
begin

1.12.1 Trace

definition $trace :: 'a::semiring\_1^{\frown}'n^{\frown}'n \Rightarrow 'a$
  where $trace\ A = sum\ (\lambda i.\ ((A\$i)\$i))\ (UNIV::'n\ set)$

Definition of determinant

definition $det:: 'a::comm\_ring\_1^{\frown}'n^{\frown}'n \Rightarrow 'a$ **where**
  $det\ A =$
    $sum\ (\lambda p.\ of\_int\ (sign\ p) * prod\ (\lambda i.\ A\$i\$p\ i)\ (UNIV :: 'n\ set))$
      $\{p.\ p\ permutes\ (UNIV :: 'n\ set)\}$
proposition *det_diagonal*:
  fixes $A :: 'a::comm\_ring\_1^{\frown}'n^{\frown}'n$
  assumes *ld*: $\bigwedge i\ j.\ i \neq j \Longrightarrow A\$i\$j = 0$
  shows $det\ A = prod\ (\lambda i.\ A\$i\$i)\ (UNIV::'n\ set)$

proposition *det_matrix_scaleR* [*simp*]: $det\ (matrix\ (((*_R)\ r)) :: real^{\frown}'n^{\frown}'n) = r\ ^\frown CARD('n::finite)$

proposition *det_mul*:
  fixes $A\ B :: 'a::comm\_ring\_1^{\frown}'n^{\frown}'n$
  shows $det\ (A ** B) = det\ A * det\ B$

1.12.2 Relation to invertibility

proposition *invertible_det_nz*:
fixes $A{::}'a{::}\{field\}^\frown 'n^\frown 'n$
shows *invertible* $A \longleftrightarrow det\ A \neq 0$

Invertibility of matrices and corresponding linear functions

1.12.3 Cramer's rule

proposition *cramer_lemma*:
fixes $A :: 'a{::}\{field\}^\frown 'n^\frown 'n$
shows $det((\chi\ i\ j.\ if\ j = k\ then\ (A *v\ x)\$i\ else\ A\$i\$j){::}\ 'a{::}\{field\}^\frown 'n^\frown 'n) = x\$k * det\ A$

proposition *cramer*:
fixes $A\ {::}'a{::}\{field\}^\frown 'n^\frown 'n$
assumes *d0*: $det\ A \neq 0$
shows $A *v\ x = b \longleftrightarrow x = (\chi\ k.\ det(\chi\ i\ j.\ if\ j{=}k\ then\ b\$i\ else\ A\$i\$j)\ /\ det\ A)$

proposition *det_orthogonal_matrix*:
fixes $Q{::}\ 'a{::}linordered\_idom^\frown 'n^\frown 'n$
assumes *oQ*: *orthogonal_matrix Q*
shows $det\ Q = 1 \lor det\ Q = -\ 1$

proposition *orthogonal_transformation_det* [*simp*]:
fixes $f :: real^\frown 'n \Rightarrow real^\frown 'n$
shows *orthogonal_transformation f* $\Longrightarrow |det\ (matrix\ f)| = 1$

1.12.4 Rotation, reflection, rotoinversion

definition *rotation_matrix Q* $\longleftrightarrow$ *orthogonal_matrix Q* $\land\ det\ Q = 1$
definition *rotoinversion_matrix Q* $\longleftrightarrow$ *orthogonal_matrix Q* $\land\ det\ Q = -\ 1$

end

1.13 Operators involving abstract topology

theory *Abstract_Topology*
imports
Complex_Main
HOL−Library.Set_Idioms
HOL−Library.FuncSet
begin

1.13.1 General notion of a topology as a value

definition *istopology* :: ('*a set* ⇒ *bool*) ⇒ *bool* **where**
istopology $L \equiv (\forall S\ T.\ L\ S \longrightarrow L\ T \longrightarrow L\ (S \cap T)) \wedge (\forall \mathcal{K}.\ (\forall K{\in}\mathcal{K}.\ L\ K) \longrightarrow L\ (\bigcup \mathcal{K}))$

typedef '*a topology* = {*L*::('*a set*) ⇒ *bool*. *istopology L*}
morphisms *openin topology*
proposition *openin_clauses*:
fixes *U* :: '*a topology*
shows
openin U {}
$\bigwedge S\ T.$ *openin U S* ⟹ *openin U T* ⟹ *openin U* (*S*∩*T*)
$\bigwedge K.$ (∀ *S* ∈ *K*. *openin U S*) ⟹ *openin U* (⋃ *K*)
definition *closedin* :: '*a topology* ⇒ '*a set* ⇒ *bool* **where**
closedin U S ⟷ *S* ⊆ *topspace U* ∧ *openin U* (*topspace U* − *S*)

1.13.2 The discrete topology

1.13.3 Subspace topology

definition *subtopology* :: '*a topology* ⇒ '*a set* ⇒ '*a topology*
where *subtopology U V* = *topology* (λ*T*. ∃*S*. *T* = *S* ∩ *V* ∧ *openin U S*)

1.13.4 The canonical topology from the underlying type class

abbreviation *euclidean* :: '*a*::*topological_space topology*
where *euclidean* ≡ *topology open*

1.13.5 Basic "localization" results are handy for connectedness.

1.13.6 Derived set (set of limit points)

1.13.7 Closure with respect to a topological space

1.13.8 Frontier with respect to topological space

1.13.9 Locally finite collections

1.13.10 Continuous maps

lemma *continuous_map_alt*:
continuous_map T1 T2 f
$= ((\forall\, U.\ openin\ T2\ U \longrightarrow openin\ T1\ (f -` U \cap topspace\ T1)) \wedge f \in topspace\ T1 \to topspace\ T2)$

1.13.11 Open and closed maps (not a priori assumed continuous)

1.13.12 Quotient maps

1.13.13 Separated Sets

1.13.14 Homeomorphisms

1.13.15 Relation of homeomorphism between topological spaces

1.13.16 Connected topological spaces

1.13.17 Compact sets

proposition *compact_space_fip*:
compact_space X $\longleftrightarrow$
$(\forall \mathcal{U}.\ (\forall\, C{\in}\mathcal{U}.\ closedin\ X\ C) \wedge (\forall \mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow \bigcap \mathcal{F} \neq \{\}) \longrightarrow \bigcap \mathcal{U} \neq \{\})$
(**is** _ = *?rhs*)

corollary *compactin_fip*:
compactin X S $\longleftrightarrow$
$S \subseteq topspace\ X\ \wedge$
$(\forall \mathcal{U}.\ (\forall\, C{\in}\mathcal{U}.\ closedin\ X\ C) \wedge (\forall \mathcal{F}.\ finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow S \cap \bigcap \mathcal{F} \neq \{\}) \longrightarrow S \cap \bigcap \mathcal{U} \neq \{\})$

corollary *compact_space_imp_nest*:
fixes $C :: nat \Rightarrow 'a\ set$
assumes *compact_space X* **and** *clo*: $\bigwedge n.\ closedin\ X\ (C\ n)$
and *ne*: $\bigwedge n.\ C\ n \neq \{\}$ **and** *dec*: *decseq C*
shows $(\bigcap n.\ C\ n) \neq \{\}$

1.13.18 Embedding maps

1.13.19 Retraction and section maps

1.13.20 Continuity

1.13.21 The topology generated by some (open) subsets

1.13.22 Topology bases and sub-bases

1.13.23 Continuity via bases/subbases, hence upper and lower semicontinuity

1.13.24 Pullback topology

definition *pullback_topology*::(*'a set*) ⇒ (*'a* ⇒ *'b*) ⇒ (*'b topology*) ⇒ (*'a topology*)
  where *pullback_topology A f T* = *topology* (λ*S*. ∃ *U*. *openin T U* ∧ *S* = *f*−'*U* ∩ *A*)

proposition *continuous_map_pullback* [*intro*]:
  assumes *continuous_map T1 T2 g*
  shows *continuous_map* (*pullback_topology A f T1*) *T2* (*g o f*)

proposition *continuous_map_pullback'* [*intro*]:
  assumes *continuous_map T1 T2* (*f o g*) *topspace T1* ⊆ *g*−'*A*
  shows *continuous_map T1* (*pullback_topology A f T2*) *g*

1.13.25 Proper maps (not a priori assumed continuous)

1.13.26 Perfect maps (proper, continuous and surjective)

end

1.14 *F*-Sigma and *G*-Delta sets in a Topological Space

theory *FSigma*
  imports *Abstract_Topology*
begin

end

1.15 Disjoint sum of arbitarily many spaces

theory *Sum_Topology*
  imports *Abstract_Topology*
begin

end

Chapter 2

Topology

theory *Elementary_Topology*
imports
  HOL−Library.Set_Idioms
  HOL−Library.Disjoint_Sets
  Product_Vector
begin

2.1 Elementary Topology

2.1.1 Topological Basis

definition *topological_basis B* $\longleftrightarrow$
  $(\forall\, b{\in}B.\ open\ b) \wedge (\forall\, x.\ open\ x \longrightarrow (\exists\, B'.\ B' \subseteq B \wedge \bigcup B' = x))$

2.1.2 Countable Basis

locale *countable_basis* = *topological_space p* **for** *p*::*'a set* ⇒ *bool* +
  fixes *B* :: *'a set set*
  assumes *is_basis*: *topological_basis B*
    and *countable_basis*: *countable B*
begin

class *second_countable_topology* = *topological_space* +
  assumes *ex_countable_subbasis*:
    $\exists\, B{::}'a\ set\ set.\ countable\ B \wedge open = generate\_topology\ B$
begin

proposition *Lindelof*:
  fixes $\mathcal{F}$ :: *'a::second_countable_topology set set*
  assumes $\mathcal{F}$: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow open\ S$
  obtains $\mathcal{F}'$ **where** $\mathcal{F}' \subseteq \mathcal{F}$ *countable* $\mathcal{F}'$ $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$

2.1.3 Polish spaces

class *polish_space* = *complete_space* + *second_countable_topology*

2.1.4 Limit Points

definition (**in** *topological_space*) *islimpt*:: $'a \Rightarrow 'a$ *set* $\Rightarrow$ *bool* (**infixr** ‹*islimpt*› *60*)
where x *islimpt* $S \longleftrightarrow (\forall\, T.\ x{\in}T \longrightarrow$ *open* $T \longrightarrow (\exists\, y{\in}S.\ y{\in}T \wedge y{\neq}x))$

2.1.5 Interior of a Set

definition *interior* :: $('a$::*topological_space*) *set* $\Rightarrow 'a$ *set* **where**
interior $S = \bigcup\{T.$ *open* $T \wedge T \subseteq S\}$

2.1.6 Closure of a Set

definition *closure* :: $('a$::*topological_space*) *set* $\Rightarrow 'a$ *set* **where**
closure $S = S \cup \{x\ .\ x$ *islimpt* $S\}$

2.1.7 Frontier (also known as boundary)

definition *frontier* :: $('a$::*topological_space*) *set* $\Rightarrow 'a$ *set* **where**
frontier S = *closure* S − *interior* S

2.1.8 Limits

2.1.9 Compactness

proposition *Heine_Borel_imp_Bolzano_Weierstrass*:
assumes *compact* S
and *infinite* T
and $T \subseteq S$
shows $\exists\, x \in S.\ x$ *islimpt* T

definition *countably_compact* :: $('a$::*topological_space*) *set* $\Rightarrow$ *bool* **where**
countably_compact $U \longleftrightarrow$
$(\forall\, A.$ *countable* $A \longrightarrow (\forall\, a{\in}A.$ *open* $a) \longrightarrow U \subseteq \bigcup A$
$\longrightarrow (\exists\, T{\subseteq}A.$ *finite* $T \wedge U \subseteq \bigcup T))$

proposition *countably_compact_imp_compact_second_countable*:
countably_compact $U \Longrightarrow$ *compact* $(U :: 'a ::$ *second_countable_topology set*)

definition *seq_compact* :: *'a::topological_space set* ⇒ *bool* **where**
seq_compact S ⟷
(∀ *f*. (∀ *n*. *f n* ∈ *S*) ⟶ (∃ *l*∈*S*. ∃ *r*::*nat*⇒*nat*. *strict_mono r* ∧ (*f* ∘ *r*) ⟶ *l*))

proposition *Bolzano_Weierstrass_imp_seq_compact*:
fixes *S* :: *'a*::{*t1_space*, *first_countable_topology*} *set*
shows (⋀*T*. ⟦*infinite T*; *T* ⊆ *S*⟧ ⟹∃ *x* ∈ *S*. *x islimpt T*) ⟹ *seq_compact S*

2.1.10 Continuity

2.1.11 Homeomorphisms

definition *homeomorphism S T f g* ⟷
(∀ *x*∈*S*. (*g*(*f x*) = *x*)) ∧ (*f* ' *S* = *T*) ∧ *continuous_on S f* ∧
(∀ *y*∈*T*. (*f*(*g y*) = *y*)) ∧ (*g* ' *T* = *S*) ∧ *continuous_on T g*

definition *homeomorphic* :: *'a::topological_space set* ⇒ *'b::topological_space set* ⇒ *bool*
(**infixr** ‹*homeomorphic*› *60*)
where *s homeomorphic t* ≡ (∃ *f g*. *homeomorphism s t f g*)

end
theory *Abstract_Limits*
imports
Abstract_Topology
begin

2.1.12 nhdsin and atin

2.1.13 Limits in a topological space

2.1.14 Pointwise continuity in topological spaces

2.1.15 Combining theorems for continuous functions into the reals

end

2.2 Non-Denumerability of the Continuum

theory *Continuum_Not_Denumerable*
imports
Complex_Main

HOL−Library.Countable_Set
begin

theorem *real_non_denum*: $\nexists f :: nat \Rightarrow real.\ surj\ f$

corollary *complex_non_denum*: $\nexists f :: nat \Rightarrow complex.\ surj\ f$

end

2.3 Abstract Topology 2

theory *Abstract_Topology_2*
imports
Elementary_Topology Abstract_Topology Continuum_Not_Denumerable
HOL−Library.Indicator_Function
HOL−Library.Equipollence
begin

2.3.1 Closure

corollary *infinite_openin*:
fixes $S :: 'a :: t1\_space\ set$
shows $[\![openin\ (top\_of\_set\ U)\ S;\ x \in S;\ x\ islimpt\ U]\!] \Longrightarrow infinite\ S$

2.3.2 Frontier

2.3.3 Compactness

2.3.4 Continuity

2.3.5 Retractions

definition *retraction* :: $('a::topological\_space)\ set \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow 'a) \Rightarrow bool$
where *retraction S T r* $\longleftrightarrow$
$T \subseteq S \wedge continuous\_on\ S\ r \wedge r \in S \rightarrow T \wedge (\forall x{\in}T.\ r\ x = x)$

definition *retract_of* (**infixl** ‹*retract′_of*› *50*) **where**
$T\ retract\_of\ S \longleftrightarrow (\exists r.\ retraction\ S\ T\ r)$

2.3.6 Retractions on a topological space

2.3.7 Paths and path-connectedness

2.3.8 Connected components

2.3.9 Combining theorems for continuous functions into the reals

2.3.10 A few cardinality results

end

Chapter 3

Connected Components

theory *Connected*
  imports
    Abstract_Topology_2
begin

3.0.1 Connected components, considered as a connectedness relation or a set

definition *connected_component* $S\ x\ y \equiv \exists\, T.\ connected\ T \wedge T \subseteq S \wedge x \in T \wedge y \in T$

3.0.2 The set of connected components of a set

definition *components*:: $'a$::*topological_space set* $\Rightarrow$ $'a$ *set set*
  where *components* $S \equiv$ *connected_component_set* S ‘ S

3.0.3 Lemmas about components

proposition *component_diff_connected*:
  fixes S :: $'a$::*metric_space set*
  assumes *connected* S *connected* U $S \subseteq U$ **and** C: $C \in$ *components* $(U - S)$
  shows *connected*$(U - C)$

end

theory *Function_Topology*
  imports

Elementary_Topology
Abstract_Limits
Connected
begin

3.1 Function Topology

3.1.1 The product topology

definition *product_topology*::$('i \Rightarrow ('a\ topology)) \Rightarrow ('i\ set) \Rightarrow (('i \Rightarrow 'a)\ topology)$
where *product_topology T I* =
topology_generated_by $\{(\Pi_E\ i{\in}I.\ X\ i)\ |X.\ (\forall i.\ openin\ (T\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (T\ i)\}\}$

proposition *product_topology*:
product_topology X I =
topology
(*arbitrary union_of*
((*finite intersection_of*
$(\lambda F.\ \exists i\ U.\ F = \{f.\ f\ i \in U\} \wedge i \in I \wedge openin\ (X\ i)\ U))$
relative_to $(\Pi_E\ i{\in}I.\ topspace\ (X\ i))))$
(**is** _ = *topology* (_ *union_of* ((_ *intersection_of* ?Ψ) *relative_to* ?TOP)))

proposition *product_topology_open_contains_basis*:
assumes *openin* (*product_topology T I*) $U\ x \in U$
shows $\exists X.\ x \in (\Pi_E\ i{\in}I.\ X\ i) \wedge (\forall i.\ openin\ (T\ i)\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq topspace\ (T\ i)\} \wedge (\Pi_E\ i{\in}I.\ X\ i) \subseteq U$

corollary *openin_product_topology_alt*:
openin (*product_topology X I*) $S \longleftrightarrow$
$(\forall x \in S.\ \exists U.\ finite\ \{i \in I.\ U\ i \neq topspace(X\ i)\} \wedge$
$(\forall i \in I.\ openin\ (X\ i)\ (U\ i)) \wedge x \in Pi_E\ I\ U \wedge Pi_E\ I\ U \subseteq S)$

corollary *closedin_product_topology*:
closedin (*product_topology X I*) (*PiE I S*) $\longleftrightarrow$ *PiE I S* = {} ∨ $(\forall i \in I.\ closedin\ (X\ i)\ (S\ i))$

corollary *closedin_product_topology_singleton*:
$f \in extensional\ I \Longrightarrow$ *closedin* (*product_topology X I*) $\{f\} \longleftrightarrow (\forall i \in I.\ closedin\ (X\ i)\ \{f\ i\})$

Powers of a single topological space as a topological space, using type classes

instantiation *fun* :: (*type*, *topological_space*) *topological_space*
begin

definition *open_fun_def*:
open U = *openin* (*product_topology* ($\lambda i.$ *euclidean*) *UNIV*) *U*

proposition *product_topology_basis'*:
fixes $x::'i \Rightarrow 'a$ **and** $U::'i \Rightarrow ('b::topological\_space)\ set$
assumes $finite\ I\ \bigwedge i.\ i \in I \Longrightarrow open\ (U\ i)$
shows $open\ \{f.\ \forall i{\in}I.\ f\ (x\ i) \in U\ i\}$

Topological countability for product spaces

proposition *product_topology_countable_basis*:
shows $\exists K::(('a::countable \Rightarrow 'b::second\_countable\_topology)\ set\ set).$
$topological\_basis\ K \wedge countable\ K \wedge$
$(\forall k{\in}K.\ \exists X.\ (k = Pi_E\ UNIV\ X) \wedge (\forall i.\ open\ (X\ i)) \wedge finite\ \{i.\ X\ i \neq UNIV\})$

3.1.2 The Alexander subbase theorem

theorem *Alexander_subbase*:
assumes X: $topology\ (arbitrary\ union\_of\ (finite\ intersection\_of\ (\lambda x.\ x \in \mathcal{B})\ relative\_to\ \bigcup \mathcal{B})) = X$
and fin: $\bigwedge C.\ [\![C \subseteq \mathcal{B};\ \bigcup C = topspace\ X]\!] \Longrightarrow \exists C'.\ finite\ C' \wedge C' \subseteq C \wedge \bigcup C' = topspace\ X$
shows $compact\_space\ X$

corollary *Alexander_subbase_alt*:
assumes $U \subseteq \bigcup \mathcal{B}$
and fin: $\bigwedge C.\ [\![C \subseteq \mathcal{B};\ U \subseteq \bigcup C]\!] \Longrightarrow \exists C'.\ finite\ C' \wedge C' \subseteq C \wedge U \subseteq \bigcup C'$
and X: $topology$
$(arbitrary\ union\_of$
$(finite\ intersection\_of\ (\lambda x.\ x \in \mathcal{B})\ relative\_to\ U)) = X$
shows $compact\_space\ X$

proposition *continuous_map_componentwise*:
$continuous\_map\ X\ (product\_topology\ Y\ I)\ f \longleftrightarrow$
$f\ `\ (topspace\ X) \subseteq extensional\ I \wedge (\forall k \in I.\ continuous\_map\ X\ (Y\ k)\ (\lambda x.\ f\ x\ k))$
(**is** $?lhs \longleftrightarrow \_ \wedge ?rhs$)

proposition *open_map_product_projection*:
assumes $i \in I$
shows $open\_map\ (product\_topology\ Y\ I)\ (Y\ i)\ (\lambda f.\ f\ i)$

3.1.3 Open Pi-sets in the product topology

proposition *openin_PiE_gen*:

openin (*product_topology X I*) (*PiE I S*) ⟷
PiE I S = {} ∨
finite {*i* ∈ *I*. *S i* ≠ *topspace* (*X i*)} ∧ (∀ *i* ∈ *I*. *openin* (*X i*) (*S i*))
(**is** *?lhs* ⟷ _ ∨ *?rhs*)

corollary *openin_PiE*:
finite I ⟹ *openin* (*product_topology X I*) (*PiE I S*) ⟷ *PiE I S* = {} ∨ (∀ *i* ∈ *I*. *openin* (*X i*) (*S i*))

proposition *compact_space_product_topology*:
compact_space(*product_topology X I*) ⟷
(*product_topology X I*) = *trivial_topology* ∨ (∀ *i* ∈ *I*. *compact_space*(*X i*))
(**is** *?lhs* = *?rhs*)

corollary *compactin_PiE*:
compactin (*product_topology X I*) (*PiE I S*) ⟷
PiE I S = {} ∨ (∀ *i* ∈ *I*. *compactin* (*X i*) (*S i*))

3.1.4 Relationship with connected spaces, paths, etc.

proposition *connected_space_product_topology*:
connected_space(*product_topology X I*) ⟷
(∃ *i* ∈ *I*. *X i* = *trivial_topology*) ∨ (∀ *i* ∈ *I*. *connected_space*(*X i*))
(**is** *?lhs* ⟷ *?eq* ∨ *?rhs*)

3.1.5 Projections from a function topology to a component

3.1.6 Limits

end

3.2 The binary product topology

theory *Product_Topology*
imports *Function_Topology*
begin

3.3 Product Topology

3.3.1 Definition

3.3.2 Continuity

proposition *compact_space_prod_topology*:
compact_space(*prod_topology* X Y) $\longleftrightarrow$ (*prod_topology* X Y) = *trivial_topology* $\vee$ *compact_space* X $\wedge$ *compact_space* Y

3.3.3 Homeomorphic maps

proposition *connected_space_prod_topology*:
connected_space(*prod_topology* X Y) $\longleftrightarrow$
(*prod_topology* X Y) = *trivial_topology* $\vee$ *connected_space* X $\wedge$ *connected_space* Y (**is** *?lhs*=*?rhs*)

end

3.4 T1 and Hausdorff spaces

theory *T1_Spaces*
imports *Product_Topology*
begin

3.5 T1 spaces with equivalences to many naturally "nice" properties.

proposition *t1_space_product_topology*:
t1_space (*product_topology* X I)
$\longleftrightarrow$ (*product_topology* X I) = *trivial_topology* $\vee$ ($\forall i \in I$. *t1_space* (X i))

3.5.1 Hausdorff Spaces

end

3.6 Lindelöf spaces

theory *Lindelof_Spaces*
imports *T1_Spaces*
begin

end

Chapter 4

Functional Analysis

theory *Metric_Arith*
  imports *HOL.Real_Vector_Spaces*
begin
theorem *metric_eq_thm* [*THEN HOL.eq_reflection*]:
  $x \in s \Longrightarrow y \in s \Longrightarrow x = y \longleftrightarrow (\forall\, a{\in}s.\ dist\ x\ a = dist\ y\ a)$

end

Chapter 5

Elementary Metric Spaces

theory *Elementary_Metric_Spaces*
  imports
    Abstract_Topology_2
    Metric_Arith
begin

5.1 Open and closed balls

definition *ball* :: $'a$::*metric_space* ⇒ *real* ⇒ $'a$ *set*
  where *ball* x ε = $\{y.\ dist\ x\ y < \varepsilon\}$

definition *cball* :: $'a$::*metric_space* ⇒ *real* ⇒ $'a$ *set*
  where *cball* x ε = $\{y.\ dist\ x\ y \leq \varepsilon\}$

definition *sphere* :: $'a$::*metric_space* ⇒ *real* ⇒ $'a$ *set*
  where *sphere* x ε = $\{y.\ dist\ x\ y = \varepsilon\}$

5.2 Limit Points

5.3 Perfect Metric Spaces

5.4 Finite and discrete

5.5 Interior

5.6 Frontier

5.7 Limits

proposition *Lim*: $(f \longrightarrow l)$ *net* $\longleftrightarrow$ *trivial_limit net* $\vee$ $(\forall \varepsilon{>}0.$ *eventually* $(\lambda x.$ *dist* $(f\,x)\ l < \varepsilon)$ *net*$)$
proposition *Lim_within_le*: $(f \longrightarrow l)(at\ a\ within\ S) \longleftrightarrow$
$(\forall \varepsilon{>}0.\ \exists \delta{>}0.\ \forall x{\in}S.\ 0 < dist\ x\ a \wedge dist\ x\ a \leq \delta \longrightarrow dist\ (f\,x)\ l < \varepsilon)$

proposition *Lim_within*: $(f \longrightarrow l)\ (at\ a\ within\ S) \longleftrightarrow$
$(\forall \varepsilon > 0.\ \exists \delta{>}0.\ \forall x \in S.\ 0 < dist\ x\ a \wedge dist\ x\ a < \delta \longrightarrow dist\ (f\,x)\ l < \varepsilon)$

corollary *Lim_withinI* [*intro?*]:
assumes $\bigwedge\varepsilon.\ \varepsilon > 0 \Longrightarrow \exists \delta{>}0.\ \forall x \in S.\ 0 < dist\ x\ a \wedge dist\ x\ a < \delta \longrightarrow dist\ (f\ x)\ l \leq \varepsilon$
shows $(f \longrightarrow l)\ (at\ a\ within\ S)$

proposition *Lim_at*: $(f \longrightarrow l)\ (at\ a) \longleftrightarrow$
$(\forall \varepsilon > 0.\ \exists \delta{>}0.\ \forall x.\ 0 < dist\ x\ a \wedge dist\ x\ a < \delta \longrightarrow dist\ (f\,x)\ l < \varepsilon)$

5.8 Continuity

proposition *continuous_within_eps_delta*:
continuous $(at\ x\ within\ s)\ f \longleftrightarrow (\forall \varepsilon{>}0.\ \exists \delta{>}0.\ \forall x'{\in}\ s.\ \ dist\ x'\ x < \delta\ {-}{-}{>}\ dist\ (f\,x')\ (f\,x) < \varepsilon)$

corollary *continuous_at_eps_delta*:
continuous $(at\ x)\ f \longleftrightarrow (\forall \varepsilon > 0.\ \exists \delta > 0.\ \forall x'.\ dist\ x'\ x < \delta \longrightarrow dist\ (f\,x')\ (f\ x) < \varepsilon)$

corollary *continuous_at_ball*:
continuous $(at\ x)\ f \longleftrightarrow (\forall \varepsilon{>}0.\ \exists \delta{>}0.\ f\ `\ (ball\ x\ \delta) \subseteq ball\ (f\,x)\ \varepsilon)$

5.9 Closure and Limit Characterization

5.10 Boundedness

definition (**in** *metric_space*) *bounded* :: *'a set* $\Rightarrow$ *bool*
where *bounded* $S \longleftrightarrow (\exists x\ \varepsilon.\ \forall y{\in}S.\ dist\ x\ y \leq \varepsilon)$

5.11 Compactness

proposition *seq_compact_imp_totally_bounded*:
assumes *seq_compact S*
shows $\forall \varepsilon>0.\ \exists k.\ finite\ k \wedge k \subseteq S \wedge S \subseteq (\bigcup x\in k.\ ball\ x\ \varepsilon)$
proposition *seq_compact_imp_Heine_Borel*:
fixes S :: $'a$ :: *metric_space set*
assumes *seq_compact S*
shows *compact S*

proposition *compact_eq_seq_compact_metric*:
compact (S :: $'a$::*metric_space set*) $\longleftrightarrow$ *seq_compact S*

proposition *compact_def*: — this is the definition of compactness in HOL Light
compact (S :: $'a$::*metric_space set*) $\longleftrightarrow$
$(\forall f.\ (\forall n.\ f\ n \in S) \longrightarrow (\exists l\in S.\ \exists r::nat\Rightarrow nat.\ strict\_mono\ r \wedge (f \circ r) \longrightarrow l))$
proposition *compact_eq_Bolzano_Weierstrass*:
fixes S :: $'a$::*metric_space set*
shows *compact S* $\longleftrightarrow (\forall T.\ infinite\ T \wedge T \subseteq S \longrightarrow (\exists x \in S.\ x\ islimpt\ T))$

proposition *Bolzano_Weierstrass_imp_bounded*:
$(\bigwedge T.\ [\![infinite\ T;\ T \subseteq S]\!] \Longrightarrow (\exists x \in S.\ x\ islimpt\ T)) \Longrightarrow bounded\ S$

5.12 Banach fixed point theorem

theorem *Banach_fix*:
assumes S: *complete S* $S \neq \{\}$
and c: $0 \leq c\ c < 1$
and f: f ‘ $S \subseteq S$
and *lipschitz*: $\bigwedge x\ y.\ [\![x\in S;\ y\in S]\!] \Longrightarrow dist\ (f\ x)\ (f\ y) \leq c * dist\ x\ y$
shows $\exists!x\in S.\ f\ x = x$

5.13 Edelstein fixed point theorem

theorem *Edelstein_fix*:
fixes S :: $'a$::*metric_space set*
assumes S: *compact S* $S \neq \{\}$
and *gs*: (g ‘ S) $\subseteq S$
and *dist*: $\bigwedge x\ y.\ [\![x\in S;\ y\in S]\!] \Longrightarrow x \neq y \longrightarrow dist\ (g\ x)\ (g\ y) < dist\ x\ y$
shows $\exists!x\in S.\ g\ x = x$

5.14 The diameter of a set

definition *diameter* :: $'a$::*metric_space set* $\Rightarrow$ *real* **where**

diameter $S = (\textit{if } S = \{\} \textit{ then } 0 \textit{ else } SUP\ (x,y){\in}S{\times}S.\ dist\ x\ y)$

proposition *Lebesgue_number_lemma*:
assumes *compact* S $\mathcal{C} \neq \{\}$ $S \subseteq \bigcup\mathcal{C}$ **and** *ope*: $\bigwedge B.\ B \in \mathcal{C} \Longrightarrow$ *open* B
obtains δ **where** $0 < \delta$ $\bigwedge T.\ [\![T \subseteq S;\ diameter\ T < \delta]\!] \Longrightarrow \exists B \in \mathcal{C}.\ T \subseteq B$

5.15 Metric spaces with the Heine-Borel property

class *heine_borel* = *metric_space* +
assumes *bounded_imp_convergent_subsequence*:
bounded (*range f*) $\Longrightarrow \exists\, l\ r.$ *strict_mono* (r::*nat*⇒*nat*) $\wedge$ (($f \circ r$) $\longrightarrow l$) *sequentially*

proposition *bounded_closed_imp_seq_compact*:
fixes *S*::*'a*::*heine_borel set*
assumes *bounded S*
and *closed S*
shows *seq_compact S*

instance *real* :: *heine_borel*

instance *prod* :: (*heine_borel*, *heine_borel*) *heine_borel*

5.16 Completeness

proposition (**in** *metric_space*) *completeI*:
assumes $\bigwedge f.\ \forall n.\ f\ n \in s \Longrightarrow Cauchy\ f \Longrightarrow \exists l{\in}s.\ f \longrightarrow l$
shows *complete s*

proposition (**in** *metric_space*) *completeE*:
assumes *complete s* **and** $\forall n.\ f\ n \in s$ **and** *Cauchy f*
obtains l **where** $l \in s$ **and** $f \longrightarrow l$

proposition *compact_eq_totally_bounded*:
compact $S \longleftrightarrow$ *complete* $S \wedge (\forall \varepsilon{>}0.\ \exists k.\ finite\ k \wedge S \subseteq (\bigcup x{\in}k.\ ball\ x\ \varepsilon))$
(**is** _ $\longleftrightarrow$ *?rhs*)

5.17 Cauchy continuity

5.18 Properties of Balls and Spheres

5.19 Distance from a Set

5.20 Infimum Distance

definition *infdist x A* = (*if A* = {} *then 0 else INF a*∈*A. dist x a*)

5.21 Separation between Points and Sets

proposition *separate_point_closed*:
  fixes *S* :: ′*a*::*heine_borel set*
  assumes *closed S* **and** *a* ∉ *S*
  shows ∃*δ*>*0*. ∀*x*∈*S*. *δ* ≤ *dist a x*

proposition *separate_compact_closed*:
  fixes *S T* :: ′*a*::*heine_borel set*
  assumes *compact S*
    and *T*: *closed T S* ∩ *T* = {}
  shows ∃*δ*>*0*. ∀*x*∈*S*. ∀*y*∈*T*. *δ* ≤ *dist x y*

proposition *separate_closed_compact*:
  fixes *S T* :: ′*a*::*heine_borel set*
  assumes *S*: *closed S*
    and *T*: *compact T*
    and *dis*: *S* ∩ *T* = {}
  shows ∃*δ*>*0*. ∀*x*∈*S*. ∀*y*∈*T*. *δ* ≤ *dist x y*

proposition *compact_in_open_separated*:
  fixes *A*::′*a*::*heine_borel set*
  assumes *A*: *A* ≠ {} *compact A*
  assumes *open B*
  assumes *A* ⊆ *B*
  obtains *ε* **where** *ε* > *0* {*x*. *infdist x A* ≤ *ε*} ⊆ *B*

5.22 Uniform Continuity

5.23 Continuity on a Compact Domain Implies Uniform Continuity

corollary *compact_uniformly_continuous*:
  fixes *f* :: ′*a* :: *metric_space* ⇒ ′*b* :: *metric_space*

assumes *f*: *continuous_on S f* **and** *S*: *compact S*
shows *uniformly_continuous_on S f*

5.24 With Abstract Topology (TODO: move and remove dependency?)

5.25 Closed Nest

5.26 Consequences for Real Numbers

5.27 The infimum of the distance between two sets

definition *setdist* :: *'a::metric_space set* ⇒ *'a set* ⇒ *real* **where**
setdist S T ≡
(*if S* = {} ∨ *T* = {} *then 0*
else Inf {*dist x y*| *x y*. *x* ∈ *S* ∧ *y* ∈ *T*})

proposition *setdist_attains_inf*:
assumes *compact B B* ≠ {}
obtains *y* **where** *y* ∈ *B setdist A B* = *infdist y A*

5.28 Diameter Lemma

end

5.29 Elementary Normed Vector Spaces

theory *Elementary_Normed_Spaces*
imports
HOL−Library.FuncSet
Elementary_Metric_Spaces Cartesian_Space
Connected
begin

5.29.1 Orthogonal Transformation of Balls

5.29.2 Support

5.29.3 Intervals

5.29.4 Limit Points

5.29.5 Balls and Spheres in Normed Spaces

corollary *compact_sphere* [*simp*]:
  fixes *a* :: *'a::{real_normed_vector,perfect_space,heine_borel}*
  shows *compact* (*sphere a r*)

corollary *bounded_sphere* [*simp*]:
  fixes *a* :: *'a::{real_normed_vector,perfect_space,heine_borel}*
  shows *bounded* (*sphere a r*)

corollary *closed_sphere* [*simp*]:
  fixes *a* :: *'a::{real_normed_vector,perfect_space,heine_borel}*
  shows *closed* (*sphere a r*)

5.29.6 Filters

5.29.7 Trivial Limits

5.29.8 Limits

proposition *Lim_at_infinity*: $(f \longrightarrow l)$ *at_infinity* $\longleftrightarrow (\forall e>0. \exists b. \forall x.$ *norm* $x \geq b \longrightarrow$ *dist* $(f\ x)\ l < e)$

corollary *Lim_at_infinityI* [*intro?*]:
  assumes $\bigwedge e.\ e > 0 \Longrightarrow \exists B. \forall x.$ *norm* $x \geq B \longrightarrow$ *dist* $(f\ x)\ l \leq e$
  shows $(f \longrightarrow l)$ *at_infinity*

5.29.9 Boundedness

corollary *cobounded_imp_unbounded*:
    fixes *S* :: *'a::{real_normed_vector, perfect_space} set*
    shows *bounded* $(- S) \Longrightarrow \neg$ *bounded S*

5.29.10 Normed spaces with the Heine-Borel property

5.29.11 Intersecting chains of compact sets and the Baire property

proposition *bounded_closed_chain*:
  fixes $\mathcal{F}$ :: *'a::heine_borel set set*
  assumes $B \in \mathcal{F}$ *bounded B* **and** $\mathcal{F}$: $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *closed S* **and** $\{\} \notin \mathcal{F}$
    and *chain*: $\bigwedge S\ T.\ S \in \mathcal{F} \wedge T \in \mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$

shows $\bigcap \mathcal{F} \neq \{\}$

corollary *compact_chain*:
fixes $\mathcal{F}$ :: *'a::heine_borel set set*
assumes $\bigwedge S.\ S \in \mathcal{F} \Longrightarrow$ *compact* S $\{\} \notin \mathcal{F}$
$\bigwedge S\ T.\ S \in \mathcal{F} \wedge T \in \mathcal{F} \Longrightarrow S \subseteq T \vee T \subseteq S$
shows $\bigcap\ \mathcal{F} \neq \{\}$
theorem *Baire*:
fixes *S::'a::{real_normed_vector,heine_borel} set*
assumes *closed S countable* $\mathcal{G}$
and *ope*: $\bigwedge T.\ T \in \mathcal{G} \Longrightarrow$ *openin* (*top_of_set S*) $T \wedge S \subseteq$ *closure* T
shows $S \subseteq$ *closure*($\bigcap \mathcal{G}$)

5.29.12 Continuity

proposition *homeomorphic_ball_UNIV*:
fixes *a ::'a::real_normed_vector*
assumes *0* $<$ *r* **shows** *ball a r homeomorphic* (*UNIV:: 'a set*)

5.29.13 Connected Normed Spaces

end

5.30 Linear Decision Procedure for Normed Spaces

theory *Norm_Arith*
imports *HOL−Library.Sum_of_Squares*
begin

method_setup *norm* = ‹
Scan.succeed (*SIMPLE_METHOD' o NormArith.norm_arith_tac*)
› *prove simple linear statements about vector norms*

proposition *dist_triangle_add*:
fixes *x y x' y' :: 'a::real_normed_vector*
shows *dist* $(x + y)$ $(x' + y') \leq$ *dist* $x\ x'$ + *dist* $y\ y'$

end

Chapter 6

Vector Analysis

theory *Topology_Euclidean_Space*
  imports
    Elementary_Normed_Spaces
    Linear_Algebra
    Norm_Arith
begin

6.1 Elementary Topology in Euclidean Space

6.1.1 Boxes

abbreviation *One* :: *'a::euclidean_space* **where**
One ≡ ∑ *Basis*

definition (**in** *euclidean_space*) *eucl_less* (**infix** ‹<e› *50*) **where**
eucl_less a b ⟷ (∀ *i*∈*Basis*. *a* • *i* < *b* • *i*)

definition *box_eucl_less*: *box a b* = {*x*. *a* <*e x* ∧ *x* <*e b*}
definition *cbox a b* = {*x*. ∀ *i*∈*Basis*. *a* • *i* ≤ *x* • *i* ∧ *x* • *i* ≤ *b* • *i*}

corollary *open_countable_Union_open_box*:
  fixes *S* :: *'a* :: *euclidean_space set*
  assumes *open S*
  obtains 𝒟 **where** *countable* 𝒟 𝒟 ⊆ *Pow S* ⋀*X*. *X* ∈ 𝒟 ⟹ ∃ *a b*. *X* = *box a b*
⋃𝒟 = *S*

corollary *open_countable_Union_open_cbox*:
  fixes *S* :: *'a* :: *euclidean_space set*
  assumes *open S*
  obtains 𝒟 **where** *countable* 𝒟 𝒟 ⊆ *Pow S* ⋀*X*. *X* ∈ 𝒟 ⟹ ∃ *a b*. *X* = *cbox a*
b ⋃𝒟 = *S*

6.1.2 General Intervals

definition *is_interval* $(s::('a::euclidean\_space)\ set) \longleftrightarrow$
$(\forall\, a{\in}s.\ \forall\, b{\in}s.\ \forall\, x.\ (\forall\, i{\in}Basis.\ ((a{\bullet}i \le x{\bullet}i \wedge x{\bullet}i \le b{\bullet}i) \vee (b{\bullet}i \le x{\bullet}i \wedge x{\bullet}i \le a{\bullet}i)))$
$\longrightarrow x \in s)$

6.1.3 Limit Component Bounds

6.1.4 Class Instances

instance *euclidean_space* ⊆ *heine_borel*

instance *euclidean_space* ⊆ *banach*

6.1.5 Compact Boxes

proposition *is_interval_compact*:
$is\_interval\ S \wedge compact\ S \longleftrightarrow (\exists\, a\ b.\ S = cbox\ a\ b)$ (**is** *?lhs = ?rhs*)

proposition *tendsto_componentwise_iff*:
fixes $f :: \_ \Rightarrow 'b::euclidean\_space$
shows $(f \longrightarrow l)\ F \longleftrightarrow (\forall\, i \in Basis.\ ((\lambda x.\ (f\ x \cdot i)) \longrightarrow (l \cdot i))\ F)$
(**is** *?lhs = ?rhs*)

corollary *continuous_componentwise*:
$continuous\ F\ f \longleftrightarrow (\forall\, i \in Basis.\ continuous\ F\ (\lambda x.\ (f\ x \cdot i)))$

corollary *continuous_on_componentwise*:
fixes $S :: 'a :: t2\_space\ set$
shows $continuous\_on\ S\ f \longleftrightarrow (\forall\, i \in Basis.\ continuous\_on\ S\ (\lambda x.\ (f\ x \cdot i)))$

6.1.6 Separability

proposition *separable*:
fixes $S :: 'a::\{metric\_space,\ second\_countable\_topology\}\ set$
obtains T **where** $countable\ T\ T \subseteq S\ S \subseteq closure\ T$
proposition *open_surjective_linear_image*:
fixes $f :: 'a::real\_normed\_vector \Rightarrow 'b::euclidean\_space$
assumes $open\ A\ linear\ f\ surj\ f$
shows $open(f\ `\ A)$

corollary *open_bijective_linear_image_eq*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *linear f bij f*
shows *open*(f ‘ A) ⟷ *open A*

corollary *interior_bijective_linear_image*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *linear f bij f*
shows *interior* (f ‘ S) = f ‘ *interior S*
proposition *injective_imp_isometric*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *s*: *closed s subspace s*
and *f*: *bounded_linear f* ∀ *x*∈*s*. *f x* = *0* ⟶ *x* = *0*
shows ∃ *e*>*0*. ∀ *x*∈*s*. *norm* (*f x*) ≥ *e* ∗ *norm x*

proposition *closed_injective_image_subspace*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *subspace s bounded_linear f* ∀ *x*∈*s*. *f x* = *0* ⟶ *x* = *0 closed s*
shows *closed*(f ‘ s)

6.1.7 Set Distance

corollary *setdist_gt_0_compact_closed*:
assumes *S*: *compact S* **and** *T*: *closed T*
shows *setdist S T* > *0* ⟷ (S ≠ {} ∧ T ≠ {} ∧ S ∩ T = {})

end

6.2 Line Segment

theory *Line_Segment*
imports
Convex
Topology_Euclidean_Space
begin

corollary *component_complement_connected*:
fixes S :: *'a::real_normed_vector set*
assumes *connected S C* ∈ *components* (−S)
shows *connected*(−C)

proposition *clopen*:
fixes S :: *'a* :: *real_normed_vector set*
shows *closed S* ∧ *open S* ⟷ S = {} ∨ S = *UNIV*

corollary *compact_open*:

fixes S :: $'a$ :: *euclidean_space set*
shows *compact* $S \wedge$ *open* $S \longleftrightarrow S = \{\}$

corollary *finite_imp_not_open*:
fixes S :: $'a$::{*real_normed_vector*, *perfect_space*} *set*
shows $[\![$*finite* S; *open* $S]\!] \Longrightarrow S=\{\}$

corollary *empty_interior_finite*:
fixes S :: $'a$::{*real_normed_vector*, *perfect_space*} *set*
shows *finite* $S \Longrightarrow$ *interior* $S = \{\}$

6.2.1 Midpoint

definition *midpoint* :: $'a$::*real_vector* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$
where *midpoint* $a\ b = ($*inverse* $(2$::*real*$)) *_R (a + b)$

6.2.2 Open and closed segments

definition *closed_segment* :: $'a$::*real_vector* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *set*
where *closed_segment* $a\ b = \{(1 - u) *_R a + u *_R b \mid u$::*real*. $0 \le u \wedge u \le 1\}$

definition *open_segment* :: $'a$::*real_vector* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *set* **where**
open_segment $a\ b \equiv$ *closed_segment* $a\ b - \{a,b\}$

proposition *dist_decreases_open_segment*:
fixes a :: $'a$ :: *euclidean_space*
assumes $x \in$ *open_segment* $a\ b$
shows *dist* $c\ x <$ *dist* $c\ a \vee$ *dist* $c\ x <$ *dist* $c\ b$

corollary *open_segment_furthest_le*:
fixes $a\ b\ x\ y$ :: $'a$::*euclidean_space*
assumes $x \in$ *open_segment* $a\ b$
shows *norm* $(y - x) <$ *norm* $(y - a) \vee$ *norm* $(y - x) <$ *norm* $(y - b)$

corollary *dist_decreases_closed_segment*:
fixes a :: $'a$ :: *euclidean_space*
assumes $x \in$ *closed_segment* $a\ b$
shows *dist* $c\ x \le$ *dist* $c\ a \vee$ *dist* $c\ x \le$ *dist* $c\ b$

corollary *segment_furthest_le*:
fixes $a\ b\ x\ y$ :: $'a$::*euclidean_space*
assumes $x \in$ *closed_segment* $a\ b$
shows *norm* $(y - x) \le$ *norm* $(y - a) \vee$ *norm* $(y - x) \le$ *norm* $(y - b)$

6.2.3 Betweenness

definition *between* = (λ(*a*,*b*) *x*. *x* ∈ *closed_segment a b*)

end

6.3 Convex Sets and Functions on (Normed) Euclidean Spaces

theory *Convex_Euclidean_Space*
imports
  Convex Topology_Euclidean_Space Line_Segment
begin
corollary *empty_interior_lowdim*:
  fixes *S* :: *'n*::*euclidean_space set*
  shows *dim S* < *DIM* (*'n*) ⟹ *interior S* = {}

corollary *aff_dim_nonempty_interior*:
  fixes *S* :: *'a*::*euclidean_space set*
  shows *interior S* ≠ {} ⟹ *aff_dim S* = *DIM*(*'a*)

6.3.1 Relative interior of a set

definition *rel_interior S* =
  {*x*. ∃ *T*. *openin* (*top_of_set* (*affine hull S*)) *T* ∧ *x* ∈ *T* ∧ *T* ⊆ *S*}
definition *rel_open S* ⟷ *rel_interior S* = *S*

6.3.2 Closest point of a convex set is unique, with a continuous projection

definition *closest_point* :: *'a*::{*real_inner*,*heine_borel*} *set* ⇒ *'a* ⇒ *'a*
  where *closest_point S a* = (*SOME x*. *x* ∈ *S* ∧ (∀ *y*∈*S*. *dist a x* ≤ *dist a y*))

proposition *closest_point_in_rel_interior*:
  assumes *closed S S* ≠ {} **and** *x*: *x* ∈ *affine hull S*
    shows *closest_point S x* ∈ *rel_interior S* ⟷ *x* ∈ *rel_interior S*

end

Chapter 7

Unsorted

theory *Starlike*
  imports
    Convex_Euclidean_Space
    Line_Segment
begin

7.0.1 The relative frontier of a set

definition $rel\_frontier\ S = closure\ S - rel\_interior\ S$

proposition *ray_to_rel_frontier*:
  fixes $a :: 'a::real\_inner$
  assumes $bounded\ S$
    and a: $a \in rel\_interior\ S$
    and aff: $(a + l) \in affine\ hull\ S$
    and $l \neq 0$
  obtains d **where** $0 < d\ (a + d *_R l) \in rel\_frontier\ S$
    $\bigwedge e.\ [\![0 \leq e;\ e < d]\!] \Longrightarrow (a + e *_R l) \in rel\_interior\ S$

corollary *ray_to_frontier*:
  fixes $a :: 'a::euclidean\_space$
  assumes $bounded\ S$
    and a: $a \in interior\ S$
    and $l \neq 0$
  obtains d **where** $0 < d\ (a + d *_R l) \in frontier\ S$
    $\bigwedge e.\ [\![0 \leq e;\ e < d]\!] \Longrightarrow (a + e *_R l) \in interior\ S$

proposition *rel_frontier_not_sing*:
  fixes $a :: 'a::euclidean\_space$
  assumes $bounded\ S$
    shows $rel\_frontier\ S \neq \{a\}$

7.0.2 Coplanarity, and collinearity in terms of affine hull

definition *coplanar* **where**
coplanar S $\equiv$ $\exists$ *u v w. S* $\subseteq$ *affine hull* {*u*,*v*,*w*}

7.0.3 Connectedness of the intersection of a chain

proposition *connected_chain*:
fixes $\mathcal{F}$ :: *'a* :: *euclidean_space set set*
assumes *cc*: $\bigwedge$*S. S* $\in \mathcal{F} \Longrightarrow$ *compact S* $\wedge$ *connected S*
and *linear*: $\bigwedge$*S T. S* $\in \mathcal{F}$ $\wedge$ *T* $\in \mathcal{F} \Longrightarrow$ *S* $\subseteq$ *T* $\vee$ *T* $\subseteq$ *S*
shows *connected*($\bigcap\mathcal{F}$)

7.0.4 Proper maps, including projections out of compact sets

proposition *proper_map*:
fixes *f* :: *'a*::*heine_borel* $\Rightarrow$ *'b*::*heine_borel*
assumes *closedin* (*top_of_set S*) *K*
and *com*: $\bigwedge$*U*. ⟦*U* $\subseteq$ *T*; *compact U*⟧ $\Longrightarrow$ *compact* (*S* $\cap$ *f* −' *U*)
and *f* ' *S* $\subseteq$ *T*
shows *closedin* (*top_of_set T*) (*f* ' *K*)

7.0.5 Closure of conic hulls

proposition *closedin_conic_hull*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *compact T 0* $\notin$ *T T* $\subseteq$ *S*
shows *closedin* (*top_of_set* (*conic hull S*)) (*conic hull T*)

corollary *affine_hull_convex_Int_open*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *convex S open T S* $\cap$ *T* $\neq$ {}
shows *affine hull* (*S* $\cap$ *T*) = *affine hull S*

corollary *affine_hull_affine_Int_nonempty_interior*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *affine S S* $\cap$ *interior T* $\neq$ {}
shows *affine hull* (*S* $\cap$ *T*) = *affine hull S*

corollary *affine_hull_affine_Int_open*:
fixes *S* :: *'a*::*real_normed_vector set*

assumes *affine S open T S* $\cap$ *T* $\neq$ {}
shows *affine hull* (*S* $\cap$ *T*) = *affine hull S*

corollary *affine_hull_convex_Int_openin*:
fixes *S* :: *'a::real_normed_vector set*
assumes *convex S openin* (*top_of_set* (*affine hull S*)) *T S* $\cap$ *T* $\neq$ {}
shows *affine hull* (*S* $\cap$ *T*) = *affine hull S*

corollary *affine_hull_openin*:
fixes *S* :: *'a::real_normed_vector set*
assumes *openin* (*top_of_set* (*affine hull T*)) *S S* $\neq$ {}
shows *affine hull S* = *affine hull T*

corollary *affine_hull_open*:
fixes *S* :: *'a::real_normed_vector set*
assumes *open S S* $\neq$ {}
shows *affine hull S* = *UNIV*

proposition *aff_dim_eq_hyperplane*:
fixes *S* :: *'a::euclidean_space set*
shows *aff_dim S* = *DIM('a)* − *1* $\longleftrightarrow$ ($\exists$ *a b. a* $\neq$ *0* $\wedge$ *affine hull S* = {*x. a* $\cdot$ *x* = *b*})
(**is** *?lhs* = *?rhs*)

corollary *aff_dim_hyperplane* [*simp*]:
fixes *a* :: *'a::euclidean_space*
shows *a* $\neq$ *0* $\Longrightarrow$ *aff_dim* {*x. a* $\cdot$ *x* = *r*} = *DIM('a)* − *1*

proposition *aff_dim_sums_Int*:
assumes *affine S*
and *affine T*
and *S* $\cap$ *T* $\neq$ {}
shows *aff_dim* {*x* + *y*| *x y. x* $\in$ *S* $\wedge$ *y* $\in$ *T*} = (*aff_dim S* + *aff_dim T*) − *aff_dim*(*S* $\cap$ *T*)

7.0.6 Lower-dimensional affine subsets are nowhere dense

proposition *dense_complement_subspace*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *dim_less*: *dim T* < *dim S* **and** *subspace S* **shows** *closure*(*S* − *T*) = *S*

7.0.7 Paracompactness

proposition *paracompact*:
fixes *S* :: *'a* :: {*metric_space*,*second_countable_topology*} *set*
assumes $S \subseteq \bigcup\mathcal{C}$ **and** *opC*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow$ *open* T
obtains $\mathcal{C}'$ **where** $S \subseteq \bigcup\ \mathcal{C}'$
and $\bigwedge U.\ U \in \mathcal{C}' \Longrightarrow$ *open* $U \wedge (\exists\, T.\ T \in \mathcal{C} \wedge U \subseteq T)$
and $\bigwedge x.\ x \in S$
$\Longrightarrow \exists\, V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U.\ U \in \mathcal{C}' \wedge (U \cap V \neq \{\})\}$

corollary *paracompact_closedin*:
fixes *S* :: *'a* :: {*metric_space*,*second_countable_topology*} *set*
assumes *cin*: *closedin* (*top_of_set* *U*) *S*
and *oin*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow$ *openin* (*top_of_set* *U*) *T*
and $S \subseteq \bigcup\mathcal{C}$
obtains $\mathcal{C}'$ **where** $S \subseteq \bigcup\ \mathcal{C}'$
and $\bigwedge V.\ V \in \mathcal{C}' \Longrightarrow$ *openin* (*top_of_set* *U*) $V \wedge (\exists\, T.\ T \in \mathcal{C} \wedge V$
$\subseteq T)$
and $\bigwedge x.\ x \in U$
$\Longrightarrow \exists\, V.$ *openin* (*top_of_set* *U*) $V \wedge x \in V \wedge$
finite $\{X.\ X \in \mathcal{C}' \wedge (X \cap V \neq \{\})\}$

7.0.8 Covering an open set by a countable chain of compact sets

proposition *open_Union_compact_subsets*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *open S*
obtains *C* **where** $\bigwedge n.$ *compact*$(C\ n)$ $\bigwedge n.\ C\ n \subseteq S$
$\bigwedge n.\ C\ n \subseteq$ *interior*$(C(Suc\ n))$
$\bigcup$(*range C*) $= S$
$\bigwedge K.\ [\![$*compact* $K;\ K \subseteq S]\!] \Longrightarrow \exists\, N.\ \forall\, n \geq N.\ K \subseteq (C\ n)$

7.0.9 Orthogonal complement

definition *orthogonal_comp* (‹(‹open_block notation=‹postfix ⊥››_$^\perp$)› [80] 80)
where *orthogonal_comp* $W \equiv \{x.\ \forall\, y \in W.$ *orthogonal* $y\ x\}$

proposition *subspace_orthogonal_comp*: *subspace* $(W^\perp)$

proposition *subspace_sum_orthogonal_comp*:
fixes *U* :: *'a* :: *euclidean_space set*
assumes *subspace U*
shows $U + U^\perp = UNIV$

end

7.1 Path-Connectedness

theory *Path_Connected*
imports
Starlike
T1_Spaces
begin

7.1.1 Paths and Arcs

definition *path* :: (*real* ⇒ ′*a*::*topological_space*) ⇒ *bool*
where *path g* ≡ *continuous_on* {*0*..*1*} *g*

definition *pathstart* :: (*real* ⇒ ′*a*::*topological_space*) ⇒ ′*a*
where *pathstart g* ≡ *g 0*

definition *pathfinish* :: (*real* ⇒ ′*a*::*topological_space*) ⇒ ′*a*
where *pathfinish g* ≡ *g 1*

definition *path_image* :: (*real* ⇒ ′*a*::*topological_space*) ⇒ ′*a set*
where *path_image g* ≡ *g* ‘ {*0* .. *1*}

definition *reversepath* :: (*real* ⇒ ′*a*::*topological_space*) ⇒ *real* ⇒ ′*a*
where *reversepath g* ≡ (λ*x*. *g*(*1* − *x*))

definition *joinpaths* :: (*real* ⇒ ′*a*::*topological_space*) ⇒ (*real* ⇒ ′*a*) ⇒ *real* ⇒ ′*a*
(**infixr** ‹+++› *75*)
where *g1* +++ *g2* ≡ (λ*x*. *if x* ≤ *1*/*2 then g1* (*2* ∗ *x*) *else g2* (*2* ∗ *x* − *1*))

definition *loop_free* :: (*real* ⇒ ′*a*::*topological_space*) ⇒ *bool*
where *loop_free g* ≡ ∀ *x*∈{*0*..*1*}. ∀ *y*∈{*0*..*1*}. *g x* = *g y* ⟶ *x* = *y* ∨ *x* = *0* ∧ *y* = *1* ∨ *x* = *1* ∧ *y* = *0*

definition *simple_path* :: (*real* ⇒ ′*a*::*topological_space*) ⇒ *bool*
where *simple_path g* ≡ *path g* ∧ *loop_free g*

definition *arc* :: (*real* ⇒ ′*a* :: *topological_space*) ⇒ *bool*
where *arc g* ≡ *path g* ∧ *inj_on g* {*0*..*1*}

7.1.2 Subpath

definition *subpath* :: *real* ⇒ *real* ⇒ (*real* ⇒ ′*a*) ⇒ *real* ⇒ ′*a*::*real_normed_vector*
where *subpath a b g* ≡ λ*x*. *g*((*b* − *a*) ∗ *x* + *a*)

7.1.3 Shift Path to Start at Some Given Point

definition *shiftpath* :: *real* ⇒ (*real* ⇒ ′*a*::*topological_space*) ⇒ *real* ⇒ ′*a*
where *shiftpath a f* = (λ*x*. *if* (*a* + *x*) ≤ *1 then f* (*a* + *x*) *else f* (*a* + *x* − *1*))

7.1.4 Straight-Line Paths

definition *linepath* :: *'a::real_normed_vector* ⇒ *'a* ⇒ *real* ⇒ *'a*
where *linepath a b* = $(\lambda x.\ (1 - x) *_R a + x *_R b)$
proposition *injective_eq_1d_open_map_UNIV*:
fixes *f* :: *real* ⇒ *real*
assumes *contf*: *continuous_on S f* **and** *S*: *is_interval S*
shows *inj_on f S* ⟷ (∀ *T*. *open T* ∧ *T* ⊆ *S* ⟶ *open*(*f* ' *T*))
(**is** *?lhs* = *?rhs*)

7.1.5 Path component

definition *path_component S x y* ≡
(∃ *g*. *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*)

abbreviation
path_component_set S x ≡ *Collect* (*path_component S x*)

7.1.6 Path connectedness of a space

definition *path_connected S* ⟷
(∀ *x*∈*S*. ∀ *y*∈*S*. ∃ *g*. *path g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*)

7.1.7 Path components

7.1.8 Paths and path-connectedness

7.1.9 Path components

7.1.10 Sphere is path-connected

corollary *connected_punctured_universe*:
2 ≤ *DIM*(*'N::euclidean_space*) ⟹ *connected*(− {*a::'N*})

proposition *path_connected_sphere*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *2* ≤ *DIM*(*'a*)
shows *path_connected*(*sphere a r*)

corollary *path_connected_complement_bounded_convex*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *bounded S convex S* **and** *2*: *2* ≤ *DIM*(*'a*)

shows *path_connected* $(-\ S)$

proposition *connected_open_delete*:
assumes *open S connected S* **and** *2*: $2 \le DIM('N::euclidean\_space)$
shows $connected(S - \{a::'N\})$

corollary *path_connected_open_delete*:
assumes *open S connected S* **and** *2*: $2 \le DIM('N::euclidean\_space)$
shows $path\_connected(S - \{a::'N\})$

corollary *path_connected_punctured_ball*:
$2 \le DIM('N::euclidean\_space) \Longrightarrow path\_connected(ball\ a\ r - \{a::'N\})$

corollary *connected_punctured_ball*:
$2 \le DIM('N::euclidean\_space) \Longrightarrow connected(ball\ a\ r - \{a::'N\})$

corollary *connected_open_delete_finite*:
fixes *S T::'a::euclidean_space set*
assumes *S*: *open S connected S* **and** *2*: $2 \le DIM('a)$ **and** *finite T*
shows $connected(S - T)$

7.1.11 Every annulus is a connected set

proposition *path_connected_annulus*:
fixes *a :: 'N::euclidean_space*
assumes $2 \le DIM('N)$
shows $path\_connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) < r2\}$
$path\_connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) \le r2\}$
$path\_connected\ \{x.\ r1 \le norm(x - a) \wedge norm(x - a) < r2\}$
$path\_connected\ \{x.\ r1 \le norm(x - a) \wedge norm(x - a) \le r2\}$

proposition *connected_annulus*:
fixes *a :: 'N::euclidean_space*
assumes $2 \le DIM('N::euclidean\_space)$
shows $connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) < r2\}$
$connected\ \{x.\ r1 < norm(x - a) \wedge norm(x - a) \le r2\}$
$connected\ \{x.\ r1 \le norm(x - a) \wedge norm(x - a) < r2\}$
$connected\ \{x.\ r1 \le norm(x - a) \wedge norm(x - a) \le r2\}$

corollary *open_components*:
fixes *S :: 'a::real_normed_vector set*
shows $[\![open\ u;\ S \in components\ u]\!] \Longrightarrow open\ S$

proposition *components_open_unique*:
fixes *S :: 'a::real_normed_vector set*
assumes *pairwise disjnt A* $\bigcup A = S$
$\bigwedge X.\ X \in A \Longrightarrow open\ X \wedge connected\ X \wedge X \ne \{\}$

shows *components* $S = A$

7.1.12 The *inside* and *outside* of a Set

The inside comprises the points in a bounded connected component of the set's complement. The outside comprises the points in unbounded connected component of the complement.

definition *inside* **where**
inside $S \equiv \{x.\ (x \notin S) \wedge bounded(connected\_component\_set\ (-\ S)\ x)\}$

definition *outside* **where**
outside $S \equiv -S \cap \{x.\ \neg\ bounded(connected\_component\_set\ (-\ S)\ x)\}$

7.1.13 Condition for an open map's image to contain a ball

proposition *ball_subset_open_map_image*:
fixes f :: $'a$::*heine_borel* $\Rightarrow$ $'b$ :: {*real_normed_vector*,*heine_borel*}
assumes *contf*: *continuous_on* (*closure* S) f
and *oint*: *open* (f ‘ *interior* S)
and *le_no*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow r \leq norm(f\ z - f\ a)$
and *bounded* S $a \in S$ $0 < r$
shows *ball* $(f\ a)$ $r \subseteq f$ ‘ S

proposition *embedding_map_into_euclideanreal*:
assumes *path_connected_space* X
shows *embedding_map* X *euclideanreal* $f \longleftrightarrow$
continuous_map X *euclideanreal* $f \wedge$ *inj_on* f (*topspace* X)

end

7.2 Neighbourhood bases and Locally path-connected spaces

theory *Locally*
imports
Path_Connected Function_Topology Sum_Topology
begin

7.2.1 Neighbourhood Bases

7.2.2 Locally path-connected spaces

7.2.3 Locally connected spaces

7.2.4 Dimension of a topological space

end

7.3 Some Uncountable Sets

theory *Uncountable_Sets*
  imports *Path_Connected Continuum_Not_Denumerable*
begin

end

7.4 Homotopy of Maps

theory *Homotopy*
  imports *Path_Connected Product_Topology Uncountable_Sets*
begin

definition *homotopic_with*
where
 $homotopic\_with\ P\ X\ Y\ f\ g \equiv$
  $(\exists h.\ continuous\_map\ (prod\_topology\ (top\_of\_set\ \{0..1::real\})\ X)\ Y\ h \wedge$
    $(\forall x.\ h(0,\ x) = f\ x) \wedge$
    $(\forall x.\ h(1,\ x) = g\ x) \wedge$
    $(\forall t \in \{0..1\}.\ P(\lambda x.\ h(t,x))))$

proposition *homotopic_with*:
  assumes $\bigwedge h\ k.\ (\bigwedge x.\ x \in topspace\ X \Longrightarrow h\ x = k\ x) \Longrightarrow (P\ h \longleftrightarrow P\ k)$
  shows $homotopic\_with\ P\ X\ Y\ p\ q \longleftrightarrow$
      $(\exists h.\ continuous\_map\ (prod\_topology\ (subtopology\ euclideanreal\ \{0..1\})$
$X)\ Y\ h \wedge$
       $(\forall x \in topspace\ X.\ h(0,x) = p\ x) \wedge$
       $(\forall x \in topspace\ X.\ h(1,x) = q\ x) \wedge$
       $(\forall t \in \{0..1\}.\ P(\lambda x.\ h(t,\ x))))$

7.4.1 Homotopy with P is an equivalence relation

proposition *homotopic_with_trans*:
   assumes $homotopic\_with\ P\ X\ Y\ f\ g\ \ homotopic\_with\ P\ X\ Y\ g\ h$

shows *homotopic_with P X Y f h*

7.4.2 Continuity lemmas

corollary *homotopic_compose*:
assumes *homotopic_with* (λx. *True*) *X Y f f'* *homotopic_with* (λx. *True*) *Y Z g g'*
shows *homotopic_with* (λx. *True*) *X Z* ($g \circ f$) ($g' \circ f'$)

proposition *homotopic_with_compose_continuous_right*:
⟦*homotopic_with_canon* (λf. *p* ($f \circ h$)) *X Y f g*; *continuous_on W h*; $h \in W \to X$⟧
$\Longrightarrow$ *homotopic_with_canon p W Y* ($f \circ h$) ($g \circ h$)

proposition *homotopic_with_compose_continuous_left*:
⟦*homotopic_with_canon* (λf. *p* ($h \circ f$)) *X Y f g*; *continuous_on Y h*; $h \in Y \to Z$⟧
$\Longrightarrow$ *homotopic_with_canon p X Z* ($h \circ f$) ($h \circ g$)

proposition *homotopic_with_eq*:
assumes *h*: *homotopic_with P X Y f g*
and *f'*: $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow f'\ x = f\ x$
and *g'*: $\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow g'\ x = g\ x$
and *P*: ($\bigwedge h\ k.$ ($\bigwedge x.\ x \in$ *topspace X* $\Longrightarrow h\ x = k\ x$) $\Longrightarrow P\ h \longleftrightarrow P\ k$)
shows *homotopic_with P X Y f' g'*

7.4.3 Homotopy of paths, maintaining the same endpoints

definition *homotopic_paths* :: ['*a set*, *real* ⇒ '*a*, *real* ⇒ '*a*::*topological_space*] ⇒ *bool*
where
homotopic_paths S p q ≡
homotopic_with_canon (λr. *pathstart r* = *pathstart p* ∧ *pathfinish r* = *pathfinish p*) {*0..1*} *S p q*

proposition *homotopic_paths_imp_pathstart*:
homotopic_paths S p q $\Longrightarrow$ *pathstart p* = *pathstart q*

proposition *homotopic_paths_imp_pathfinish*:
homotopic_paths S p q $\Longrightarrow$ *pathfinish p* = *pathfinish q*

proposition *homotopic_paths_refl* [*simp*]: *homotopic_paths S p p* $\longleftrightarrow$ *path p* ∧ *path_image p* ⊆ *S*

proposition *homotopic_paths_sym*: *homotopic_paths S p q* $\Longrightarrow$ *homotopic_paths S q p*

proposition *homotopic_paths_sym_eq*: $homotopic\_paths\ S\ p\ q \longleftrightarrow homotopic\_paths\ S\ q\ p$

proposition *homotopic_paths_trans* [*trans*]:
assumes $homotopic\_paths\ S\ p\ q$ $homotopic\_paths\ S\ q\ r$
shows $homotopic\_paths\ S\ p\ r$

proposition *homotopic_paths_eq*:
$[\![path\ p;\ path\_image\ p \subseteq S;\ \bigwedge t.\ t \in \{0..1\} \Longrightarrow p\ t = q\ t]\!] \Longrightarrow homotopic\_paths\ S\ p\ q$

proposition *homotopic_paths_reparametrize*:
assumes $path\ p$
and *pips*: $path\_image\ p \subseteq S$
and *contf*: $continuous\_on\ \{0..1\}\ f$
and *f01* :$f \in \{0..1\} \rightarrow \{0..1\}$
and [*simp*]: $f(0) = 0$ $f(1) = 1$
and *q*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow q(t) = p(f\ t)$
shows $homotopic\_paths\ S\ p\ q$

proposition *homotopic_paths_reversepath*:
$homotopic\_paths\ S\ (reversepath\ p)\ (reversepath\ q) \longleftrightarrow homotopic\_paths\ S\ p\ q$

proposition *homotopic_paths_join*:
$[\![homotopic\_paths\ S\ p\ p';\ homotopic\_paths\ S\ q\ q';\ pathfinish\ p = pathstart\ q]\!] \Longrightarrow homotopic\_paths\ S\ (p +\!+\!+ q)\ (p' +\!+\!+ q')$

proposition *homotopic_paths_continuous_image*:
$[\![homotopic\_paths\ S\ f\ g;\ continuous\_on\ S\ h;\ h \in S \rightarrow t]\!] \Longrightarrow homotopic\_paths\ t\ (h \circ f)\ (h \circ g)$

7.4.4 Group properties for homotopy of paths

So taking equivalence classes under homotopy would give the fundamental group

proposition *homotopic_paths_rid*:
assumes $path\ p$ $path\_image\ p \subseteq S$
shows $homotopic\_paths\ S\ (p +\!+\!+ linepath\ (pathfinish\ p)\ (pathfinish\ p))\ p$

proposition *homotopic_paths_lid*:
$[\![path\ p;\ path\_image\ p \subseteq S]\!] \Longrightarrow homotopic\_paths\ S\ (linepath\ (pathstart\ p)\ (pathstart\ p) +\!+\!+ p)\ p$

proposition *homotopic_paths_assoc*:
$[\![path\ p;\ path\_image\ p \subseteq S;\ path\ q;\ path\_image\ q \subseteq S;\ path\ r;\ path\_image\ r \subseteq$

S; *pathfinish p* = *pathstart q*;
pathfinish q = *pathstart r*⟧
⟹ *homotopic_paths S* (*p* +++ (*q* +++ *r*)) ((*p* +++ *q*) +++ *r*)

proposition *homotopic_paths_rinv*:
assumes *path p path_image p* ⊆ *S*
shows *homotopic_paths S* (*p* +++ *reversepath p*) (*linepath* (*pathstart p*) (*pathstart p*))

proposition *homotopic_paths_linv*:
assumes *path p path_image p* ⊆ *S*
shows *homotopic_paths S* (*reversepath p* +++ *p*) (*linepath* (*pathfinish p*) (*pathfinish p*))

7.4.5 Homotopy of loops without requiring preservation of endpoints

definition *homotopic_loops* :: *'a::topological_space set* ⇒ (*real* ⇒ *'a*) ⇒ (*real* ⇒ *'a*) ⇒ *bool* **where**
homotopic_loops S p q ≡
homotopic_with_canon (λ*r*. *pathfinish r* = *pathstart r*) {*0..1*} *S p q*

proposition *homotopic_loops_imp_loop*:
homotopic_loops S p q ⟹ *pathfinish p* = *pathstart p* ∧ *pathfinish q* = *pathstart q*

proposition *homotopic_loops_imp_path*:
homotopic_loops S p q ⟹ *path p* ∧ *path q*

proposition *homotopic_loops_imp_subset*:
homotopic_loops S p q ⟹ *path_image p* ⊆ *S* ∧ *path_image q* ⊆ *S*

proposition *homotopic_loops_refl*:
homotopic_loops S p p ⟷
path p ∧ *path_image p* ⊆ *S* ∧ *pathfinish p* = *pathstart p*

proposition *homotopic_loops_sym*: *homotopic_loops S p q* ⟹ *homotopic_loops S q p*

proposition *homotopic_loops_sym_eq*: *homotopic_loops S p q* ⟷ *homotopic_loops S q p*

proposition *homotopic_loops_trans*:
⟦*homotopic_loops S p q*; *homotopic_loops S q r*⟧ ⟹ *homotopic_loops S p r*

proposition *homotopic_loops_subset*:
⟦*homotopic_loops S p q*; *S* ⊆ *t*⟧ ⟹ *homotopic_loops t p q*

proposition *homotopic_loops_eq*:
⟦*path p*; *path_image p* ⊆ *S*; *pathfinish p* = *pathstart p*; ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*t*) = *q*(*t*)⟧
⟹ *homotopic_loops S p q*

proposition *homotopic_loops_continuous_image*:
⟦*homotopic_loops S f g*; *continuous_on S h*; *h* ∈ *S* → *t*⟧ ⟹ *homotopic_loops t* (*h* ∘ *f*) (*h* ∘ *g*)

7.4.6 Relations between the two variants of homotopy

proposition *homotopic_paths_imp_homotopic_loops*:
⟦*homotopic_paths S p q*; *pathfinish p* = *pathstart p*; *pathfinish q* = *pathstart p*⟧ ⟹ *homotopic_loops S p q*

proposition *homotopic_loops_imp_homotopic_paths_null*:
assumes *homotopic_loops S p* (*linepath a a*)
shows *homotopic_paths S p* (*linepath* (*pathstart p*) (*pathstart p*))

proposition *homotopic_loops_conjugate*:
fixes *S* :: *'a::real_normed_vector set*
assumes *path p path q* **and** *pip*: *path_image p* ⊆ *S* **and** *piq*: *path_image q* ⊆ *S*
and *pq*: *pathfinish p* = *pathstart q* **and** *qloop*: *pathfinish q* = *pathstart q*
shows *homotopic_loops S* (*p* +++ *q* +++ *reversepath p*) *q*

7.4.7 Homotopy and subpaths

proposition *homotopic_join_subpaths*:
⟦*path g*; *path_image g* ⊆ *S*; *u* ∈ {*0..1*}; *v* ∈ {*0..1*}; *w* ∈ {*0..1*}⟧
⟹ *homotopic_paths S* (*subpath u v g* +++ *subpath v w g*) (*subpath u w g*)

7.4.8 Simply connected sets

defined as "all loops are homotopic (as loops)

definition *simply_connected* **where**
simply_connected S ≡
∀ *p q*. *path p* ∧ *pathfinish p* = *pathstart p* ∧ *path_image p* ⊆ *S* ∧
path q ∧ *pathfinish q* = *pathstart q* ∧ *path_image q* ⊆ *S*
⟶ *homotopic_loops S p q*

proposition *simply_connected_Times*:
fixes *S* :: *'a::real_normed_vector set* **and** *T* :: *'b::real_normed_vector set*
assumes *S*: *simply_connected S* **and** *T*: *simply_connected T*
shows *simply_connected*(*S* × *T*)

7.4.9 Contractible sets

definition *contractible* **where**
contractible S ≡ ∃ *a. homotopic_with_canon* (λ*x. True*) *S S id* (λ*x. a*)

proposition *contractible_imp_simply_connected*:
fixes *S* :: _::*real_normed_vector set*
assumes *contractible S* **shows** *simply_connected S*

corollary *contractible_imp_connected*:
fixes *S* :: _::*real_normed_vector set*
shows *contractible S* ⟹ *connected S*

7.4.10 Starlike sets

definition *starlike S* ⟷ (∃ *a*∈*S*. ∀ *x*∈*S*. *closed_segment a x* ⊆ *S*)

7.4.11 The slotted complex plane

7.4.12 Contractible sets

7.4.13 Local versions of topological properties in general

definition *locally* :: ('*a*::*topological_space set* ⇒ *bool*) ⇒ '*a set* ⇒ *bool*
where
locally P S ≡
∀ *w x. openin* (*top_of_set S*) *w* ∧ *x* ∈ *w*
⟶ (∃ *U V*. *openin* (*top_of_set S*) *U* ∧ *P V* ∧ *x* ∈ *U* ∧ *U* ⊆ *V* ∧ *V*
⊆ *w*)

proposition *homeomorphism_locally_imp*:
fixes *S* :: '*a*::*metric_space set* **and** *T* :: '*b*::*t2_space set*
assumes *S*: *locally P S* **and** *hom*: *homeomorphism S T f g*
and *Q*: ⋀*S S*'. ⟦*P S*; *homeomorphism S S*' *f g*⟧ ⟹ *Q S*'
shows *locally Q T*

7.4.14 An induction principle for connected sets

proposition *connected_induction*:
assumes *connected S*
and *opD*: ⋀*T a*. ⟦*openin* (*top_of_set S*) *T*; *a* ∈ *T*⟧ ⟹ ∃ *z*. *z* ∈ *T* ∧ *P z*
and *opI*: ⋀*a*. *a* ∈ *S*

$\Longrightarrow \exists\, T.\ openin\ (top\_of\_set\ S)\ T \wedge a \in T \wedge$
$(\forall x \in T.\ \forall y \in T.\ P\ x \wedge P\ y \wedge Q\ x \longrightarrow Q\ y)$
and *etc*: $a \in S$ $b \in S$ $P\ a$ $P\ b$ $Q\ a$
shows $Q\ b$

7.4.15 Basic properties of local compactness

proposition *locally_compact*:
fixes $S :: 'a :: metric\_space\ set$
shows
$locally\ compact\ S \longleftrightarrow$
$(\forall x \in S.\ \exists u\ v.\ x \in u \wedge u \subseteq v \wedge v \subseteq S \wedge$
$openin\ (top\_of\_set\ S)\ u \wedge compact\ v)$
(**is** $?lhs = ?rhs$)

7.4.16 Sura-Bura's results about compact components of sets

proposition *Sura_Bura_compact*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $compact\ S$ **and** C: $C \in components\ S$
shows $C = \bigcap \{T.\ C \subseteq T \wedge openin\ (top\_of\_set\ S)\ T \wedge$
$closedin\ (top\_of\_set\ S)\ T\}$
(**is** $C = \bigcap\ ?\mathcal{T}$)

corollary *Sura_Bura_clopen_subset*:
fixes $S :: 'a::euclidean\_space\ set$
assumes S: $locally\ compact\ S$ **and** C: $C \in components\ S$ **and** $compact\ C$
and U: $open\ U$ $C \subseteq U$
obtains K **where** $openin\ (top\_of\_set\ S)\ K$ $compact\ K$ $C \subseteq K$ $K \subseteq U$

corollary *Sura_Bura_clopen_subset_alt*:
fixes $S :: 'a::euclidean\_space\ set$
assumes S: $locally\ compact\ S$ **and** C: $C \in components\ S$ **and** $compact\ C$
and $opeSU$: $openin\ (top\_of\_set\ S)\ U$ **and** $C \subseteq U$
obtains K **where** $openin\ (top\_of\_set\ S)\ K$ $compact\ K$ $C \subseteq K$ $K \subseteq U$

corollary *Sura_Bura*:
fixes $S :: 'a::euclidean\_space\ set$
assumes $locally\ compact\ S$ $C \in components\ S$ $compact\ C$
shows $C = \bigcap\ \{K.\ C \subseteq K \wedge compact\ K \wedge openin\ (top\_of\_set\ S)\ K\}$
(**is** $C = ?rhs$)

7.4.17 Special cases of local connectedness and path connectedness

proposition *locally_path_connected*:
locally path_connected S $\longleftrightarrow$
$(\forall\, V\, x.\ openin\ (top\_of\_set\ S)\ V \wedge x \in V$
$\longrightarrow (\exists\, U.\ openin\ (top\_of\_set\ S)\ U \wedge path\_connected\ U \wedge x \in U \wedge U \subseteq V))$

proposition *locally_path_connected_open_path_component*:
locally path_connected S $\longleftrightarrow$
$(\forall\, t\, x.\ openin\ (top\_of\_set\ S)\ t \wedge x \in t$
$\longrightarrow openin\ (top\_of\_set\ S)\ (path\_component\_set\ t\ x))$

proposition *locally_connected_im_kleinen*:
locally connected S $\longleftrightarrow$
$(\forall\, v\, x.\ openin\ (top\_of\_set\ S)\ v \wedge x \in v$
$\longrightarrow (\exists\, u.\ openin\ (top\_of\_set\ S)\ u \wedge$
$x \in u \wedge u \subseteq v \wedge$
$(\forall\, y.\ y \in u \longrightarrow (\exists\, c.\ connected\ c \wedge c \subseteq v \wedge x \in c \wedge y \in c))))$
(**is** *?lhs* = *?rhs*)

proposition *locally_path_connected_im_kleinen*:
locally path_connected S $\longleftrightarrow$
$(\forall\, v\, x.\ openin\ (top\_of\_set\ S)\ v \wedge x \in v$
$\longrightarrow (\exists\, u.\ openin\ (top\_of\_set\ S)\ u \wedge$
$x \in u \wedge u \subseteq v \wedge$
$(\forall\, y.\ y \in u \longrightarrow (\exists\, p.\ path\ p \wedge path\_image\ p \subseteq v \wedge$
$pathstart\ p = x \wedge pathfinish\ p = y))))$
(**is** *?lhs* = *?rhs*)

7.4.18 Relations between components and path components

proposition *locally_connected_quotient_image*:
assumes *lcS*: *locally connected S*
and *oo*: $\bigwedge T.\ T \subseteq f\ `\ S$
$\Longrightarrow openin\ (top\_of\_set\ S)\ (S \cap f\ -`\ T) \longleftrightarrow$
$openin\ (top\_of\_set\ (f\ `\ S))\ T$
shows *locally connected* $(f\ `\ S)$
proposition *locally_path_connected_quotient_image*:
assumes *lcS*: *locally path_connected S*
and *oo*: $\bigwedge T.\ T \subseteq f\ `\ S$
$\Longrightarrow openin\ (top\_of\_set\ S)\ (S \cap f\ -`\ T) \longleftrightarrow openin\ (top\_of\_set\ (f\ `\ S))\ T$
shows *locally path_connected* $(f\ `\ S)$

7.4.19 Existence of isometry between subspaces of same dimension

proposition *isometries_subspaces*:
  fixes S :: $'a$::*euclidean_space set*
    and T :: $'b$::*euclidean_space set*
  assumes S: *subspace* S
      and T: *subspace* T
      and d: *dim* S = *dim* T
  obtains f g **where** *linear* f *linear* g f ` $S$ = $T$ $g$ ` T = S
    $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) = norm\ x$
    $\bigwedge x.\ x \in T \Longrightarrow norm(g\ x) = norm\ x$
    $\bigwedge x.\ x \in S \Longrightarrow g(f\ x) = x$
    $\bigwedge x.\ x \in T \Longrightarrow f(g\ x) = x$

corollary *isometry_subspaces*:
  fixes S :: $'a$::*euclidean_space set*
    and T :: $'b$::*euclidean_space set*
  assumes S: *subspace* S
      and T: *subspace* T
      and d: *dim* S = *dim* T
  obtains f **where** *linear* f f ` S = T $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) = norm\ x$

corollary *isomorphisms_UNIV_UNIV*:
  assumes $DIM('M) = DIM('N)$
  obtains f::$'M$::*euclidean_space* ⇒$'N$::*euclidean_space* **and** g
  where *linear* f *linear* g
    $\bigwedge x.\ norm(f\ x) = norm\ x$ $\bigwedge y.\ norm(g\ y) = norm\ y$
    $\bigwedge x.\ g\ (f\ x) = x$ $\bigwedge y.\ f(g\ y) = y$

7.4.20 Retracts, in a general sense, preserve (co)homotopic triviality)

locale *Retracts* =
  fixes S h t k
  assumes *conth*: *continuous_on* S h
      and *imh*: h ` S = t
      and *contk*: *continuous_on* t k
      and *imk*: $k \in t \rightarrow S$
      and *idhk*: $\bigwedge y.\ y \in t \Longrightarrow h(k\ y) = y$

begin

7.4.21 Homotopy equivalence

7.4.22 Homotopy equivalence of topological spaces.

definition *homotopy_equivalent_space*
(**infix** ‹*homotopy'_equivalent'_space*› *50*)
where *X homotopy_equivalent_space Y* ≡
(∃*f g*. *continuous_map X Y f* ∧
continuous_map Y X g ∧
homotopic_with (λ*x*. *True*) *X X* (*g* ∘ *f*) *id* ∧
homotopic_with (λ*x*. *True*) *Y Y* (*f* ∘ *g*) *id*)

7.4.23 Contractible spaces

corollary *contractible_space_euclideanreal*: *contractible_space euclideanreal*

abbreviation *homotopy_eqv* :: *'a::topological_space set* ⇒ *'b::topological_space set* ⇒ *bool*
(**infix** ‹*homotopy'_eqv*› *50*)
where *S homotopy_eqv T* ≡ *top_of_set S homotopy_equivalent_space top_of_set T*

corollary *bounded_path_connected_Compl_real*:
fixes *S* :: *real set*
assumes *bounded S path_connected*(− *S*) **shows** *S* = {}
proposition *path_connected_convex_diff_countable*:
fixes *U* :: *'a::euclidean_space set*
assumes *convex U* ¬ *collinear U countable S*
shows *path_connected*(*U* − *S*)

corollary *connected_convex_diff_countable*:
fixes *U* :: *'a::euclidean_space set*
assumes *convex U* ¬ *collinear U countable S*
shows *connected*(*U* − *S*)

proposition *path_connected_openin_diff_countable*:
fixes *S* :: *'a::euclidean_space set*
assumes *connected S* **and** *ope*: *openin* (*top_of_set* (*affine hull S*)) *S*
and ¬ *collinear S countable T*
shows *path_connected*(*S* − *T*)

corollary *connected_openin_diff_countable*:
fixes S :: $'a$::*euclidean_space set*
assumes *connected* S **and** *ope*: *openin* (*top_of_set* (*affine hull* S)) S
and ¬ *collinear* S *countable* T
shows *connected*($S - T$)

corollary *path_connected_open_diff_countable*:
fixes S :: $'a$::*euclidean_space set*
assumes $2 \leq DIM('a)$ *open* S *connected* S *countable* T
shows *path_connected*($S - T$)

corollary *connected_open_diff_countable*:
fixes S :: $'a$::*euclidean_space set*
assumes $2 \leq DIM('a)$ *open* S *connected* S *countable* T
shows *connected*($S - T$)

7.4.24 Nullhomotopic mappings

proposition *nullhomotopic_from_sphere_extension*:
fixes f :: $'M$::*euclidean_space* ⇒ $'a$::*real_normed_vector*
shows (∃ *c*. *homotopic_with_canon* (λ*x*. *True*) (*sphere a r*) *S f* (λ*x*. *c*)) ⟷
(∃ *g*. *continuous_on* (*cball a r*) *g* ∧ *g* ‘ (*cball a r*) ⊆ *S* ∧
(∀ *x* ∈ *sphere a r*. *g x* = *f x*))
(**is** *?lhs* = *?rhs*)

end

7.5 Euclidean space and n-spheres, as subtopologies of n-dimensional space

theory *Abstract_Euclidean_Space*
imports *Homotopy Locally*
begin

7.5.1 Euclidean spaces as abstract topologies

7.5.2 n-dimensional spheres

proposition *contractible_space_upper_hemisphere*:
assumes $k \leq n$

shows *contractible_space*(*subtopology* (*nsphere* *n*) {*x*. *x* *k* ≥ *0*})

corollary *contractible_space_lower_hemisphere*:
assumes $k \leq n$
shows *contractible_space*(*subtopology* (*nsphere* *n*) {*x*. *x* *k* ≤ *0*})

proposition *nullhomotopic_nonsurjective_sphere_map*:
assumes *f*: *continuous_map* (*nsphere* *p*) (*nsphere* *p*) *f*
and *fim*: *f* ‘ (*topspace*(*nsphere* *p*)) ≠ *topspace*(*nsphere* *p*)
obtains *a* **where** *homotopic_with* (λ*x*. *True*) (*nsphere* *p*) (*nsphere* *p*) *f* (λ*x*. *a*)

end

7.6 Various Forms of Topological Spaces

theory *Abstract_Topological_Spaces*
imports *Lindelof_Spaces Locally Abstract_Euclidean_Space Sum_Topology FSigma*
begin

7.6.1 Connected topological spaces

7.6.2 The notion of "separated between" (complement of "connected between)"

7.6.3 Connected components

7.6.4 Monotone maps (in the general topological sense)

proposition *connected_space_monotone_quotient_map_preimage*:
assumes *f*: *monotone_map X Y f quotient_map X Y f* **and** *connected_space Y*
shows *connected_space X*

7.6.5 Other countability properties

7.6.6 Neigbourhood bases EXTRAS

7.6.7 T_0 spaces and the Kolmogorov quotient

proposition *t0_space_product_topology*:
  t0_space (*product_topology X I*) ⟷ *product_topology X I* = *trivial_topology* ∨ (∀ *i* ∈ *I*. *t0_space* (*X i*))
  (**is** *?lhs=?rhs*)

7.6.8 Kolmogorov quotients

7.6.9 Closed diagonals and graphs

7.6.10 KC spaces, those where all compact sets are closed.

proposition *kc_space_prod_topology_left*:
 assumes *X*: *kc_space X* **and** *Y*: *Hausdorff_space Y*
 shows *kc_space* (*prod_topology X Y*)

7.6.11 Technical results about proper maps, perfect maps, etc

7.6.12 Regular spaces

proposition *regular_space_continuous_proper_map_image*:
  assumes *regular_space X* **and** *contf*: *continuous_map X Y f* **and** *pmapf*: *proper_map X Y f*
  and *fim*: *f* ‘ (*topspace X*) = *topspace Y*
 shows *regular_space Y*

proposition *regular_space_perfect_map_image_eq*:
 assumes *Hausdorff_space X* **and** *perf*: *perfect_map X Y f*

shows *regular_space X* ⟷ *regular_space Y* (**is** *?lhs=?rhs*)

7.6.13 Locally compact spaces

proposition *quotient_map_prod_right*:
assumes *loc*: *locally_compact_space Z*
and *reg*: *Hausdorff_space Z* ∨ *regular_space Z*
and *f*: *quotient_map X Y f*
shows *quotient_map* (*prod_topology Z X*) (*prod_topology Z Y*) (λ(*x*,*y*). (*x*,*f y*))

7.6.14 Special characterizations of classes of functions into and out of R

7.6.15 Normal spaces

7.6.16 Hereditary topological properties

7.6.17 Limits in a topological space

7.6.18 Quasi-components

7.6.19 Additional quasicomponent and continuum properties like Boundary Bumping

7.6.20 Compactly generated spaces (k-spaces)

end

7.7 Abstract Metric Spaces

theory *Abstract_Metric_Spaces*
imports *Elementary_Metric_Spaces Abstract_Limits Abstract_Topological_Spaces*
begin

7.7.1 Metric topology

7.7.2 Bounded sets

7.7.3 Subspace of a metric space

7.7.4 Abstract type of metric spaces

7.7.5 The discrete metric

7.7.6 Metrizable spaces

7.7.7 Limits at a point in a topological space

7.7.8 Normal spaces and metric spaces

7.7.9 Topological limitin in metric spaces

7.7.10 Cauchy sequences and complete metric spaces

7.7.11 Totally bounded subsets of metric spaces

7.7.12 Compactness in metric spaces

7.7.13 Continuous functions on metric spaces

7.7.14 Completely metrizable spaces

7.7.15 Product metric

7.7.16 More sequential characterizations in a metric space

7.7.17 Three strong notions of continuity for metric spaces

7.7.18 Isometries

7.7.19 "Capped" equivalent bounded metrics and general product metrics

proposition *metrizable_space_product_topology*:
  metrizable_space (*product_topology X I*) ⟷
    (*product_topology X I*) = *trivial_topology* ∨
    countable {*i* ∈ *I*. ¬ (∃ *a*. *topspace*(*X i*) ⊆ {*a*})} ∧
    (∀ *i* ∈ *I*. *metrizable_space* (*X i*))

proposition *completely_metrizable_space_product_topology*:
  completely_metrizable_space (*product_topology X I*) ⟷
    (*product_topology X I*) = *trivial_topology* ∨
    countable {*i* ∈ *I*. ¬ (∃ *a*. *topspace*(*X i*) ⊆ {*a*})} ∧
    (∀ *i* ∈ *I*. *completely_metrizable_space* (*X i*))

end

7.8 Infinite sums

theory *Infinite_Sum*
  imports
    Elementary_Topology
    HOL−Library.Extended_Nonnegative_Real
    HOL−Library.Complex_Order
    HOL−Computational_Algebra.Formal_Power_Series
begin

7.8.1 Definition and syntax

7.8.2 General properties

7.8.3 Absolute convergence

7.8.4 Extended reals and nats

7.8.5 Real numbers

7.8.6 Complex numbers

class *complete_uniform_space* = *uniform_space* +
assumes *cauchy_filter_convergent′*: *cauchy_filter* (*F* :: ′*a filter*) $\Longrightarrow$ *F* $\neq$ *bot* $\Longrightarrow$ *convergent_filter F*

theorem (**in** *uniform_space*) *controlled_sequences_convergent_imp_complete*:
fixes *U* :: *nat* $\Rightarrow$ (′*a* $\times$ ′*a*) *set*
assumes *gen*: *countably_generated_filter* (*uniformity* :: (′*a* $\times$ ′*a*) *filter*)
assumes *U*: $\bigwedge$*n*. *eventually* (λ*z*. *z* $\in$ *U n*) *uniformity*
assumes *conv*: $\bigwedge$(*u* :: *nat* $\Rightarrow$ ′*a*). ($\bigwedge$*N m n*. *N* $\leq$ *m* $\Longrightarrow$ *N* $\leq$ *n* $\Longrightarrow$ (*u m*, *u n*) $\in$ *U N*) $\Longrightarrow$ *convergent u*
shows *class.complete_uniform_space open uniformity*

theorem (**in** *uniform_space*) *controlled_seq_imp_Cauchy_seq*:
fixes *U* :: *nat* $\Rightarrow$ (′*a* $\times$ ′*a*) *set*
assumes *U*: $\bigwedge$*P*. *eventually P uniformity* $\Longrightarrow$ ($\exists$ *n*. $\forall$ *x*$\in$*U n*. *P x*)
assumes *controlled*: $\bigwedge$*N m n*. *N* $\leq$ *m* $\Longrightarrow$ *N* $\leq$ *n* $\Longrightarrow$ (*f m*, *f n*) $\in$ *U N*
shows *Cauchy f*

theorem (**in** *uniform_space*) *Cauchy_seq_convergent_imp_complete*:
fixes *U* :: *nat* $\Rightarrow$ (′*a* $\times$ ′*a*) *set*
assumes *gen*: *countably_generated_filter* (*uniformity* :: (′*a* $\times$ ′*a*) *filter*)
assumes *conv*: $\bigwedge$(*u* :: *nat* $\Rightarrow$ ′*a*). *Cauchy u* $\Longrightarrow$ *convergent u*
shows *class.complete_uniform_space open uniformity*

7.8.7 Infinite sums of formal power series

end

7.9 Ordered Euclidean Space

theory *Ordered_Euclidean_Space*
imports
  Convex_Euclidean_Space Abstract_Limits
  HOL−Library.Product_Order
beginclass *ordered_euclidean_space* = *ord* + *inf* + *sup* + *abs* + *Inf* + *Sup* + *euclidean_space* +
  assumes *eucl_le*: $x \leq y \longleftrightarrow (\forall i{\in}Basis.\ x \cdot i \leq y \cdot i)$
  assumes *eucl_less_le_not_le*: $x < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
  assumes *eucl_inf*: $inf\ x\ y = (\sum i{\in}Basis.\ inf\ (x \cdot i)\ (y \cdot i) *_R i)$
  assumes *eucl_sup*: $sup\ x\ y = (\sum i{\in}Basis.\ sup\ (x \cdot i)\ (y \cdot i) *_R i)$
  assumes *eucl_Inf*: $Inf\ X = (\sum i{\in}Basis.\ (INF\ x{\in}X.\ x \cdot i) *_R i)$
  assumes *eucl_Sup*: $Sup\ X = (\sum i{\in}Basis.\ (SUP\ x{\in}X.\ x \cdot i) *_R i)$
  assumes *eucl_abs*: $|x| = (\sum i{\in}Basis.\ |x \cdot i| *_R i)$
begin

proposition *compact_attains_Inf_componentwise*:
  fixes $b::'a::ordered\_euclidean\_space$
  assumes $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
  obtains x **where** $x \in X\ x \cdot b = Inf\ X \cdot b\ \bigwedge y.\ y \in X \Longrightarrow x \cdot b \leq y \cdot b$

proposition
  compact_attains_Sup_componentwise:
  fixes $b::'a::ordered\_euclidean\_space$
  assumes $b \in Basis$ **assumes** $X \neq \{\}$ *compact X*
  obtains x **where** $x \in X\ x \cdot b = Sup\ X \cdot b\ \bigwedge y.\ y \in X \Longrightarrow y \cdot b \leq x \cdot b$
proposition
  fixes $a :: 'a::ordered\_euclidean\_space$
  shows *cbox_interval*: $cbox\ a\ b = \{a..b\}$
    and *interval_cbox*: $\{a..b\} = cbox\ a\ b$
    and *eucl_le_atMost*: $\{x.\ \forall i{\in}Basis.\ x \cdot i <= a \cdot i\} = \{..a\}$
    and *eucl_le_atLeast*: $\{x.\ \forall i{\in}Basis.\ a \cdot i <= x \cdot i\} = \{a..\}$

instantiation *vec* :: (*ordered_euclidean_space*, *finite*) *ordered_euclidean_space*
begin

definition $inf\ x\ y = (\chi\ i.\ inf\ (x\ \$\ i)\ (y\ \$\ i))$
definition $sup\ x\ y = (\chi\ i.\ sup\ (x\ \$\ i)\ (y\ \$\ i))$
definition $Inf\ X = (\chi\ i.\ (INF\ x{\in}X.\ x\ \$\ i))$
definition $Sup\ X = (\chi\ i.\ (SUP\ x{\in}X.\ x\ \$\ i))$
definition $|x| = (\chi\ i.\ |x\ \$\ i|)$

end

7.10 Arcwise-Connected Sets

theory *Arcwise_Connected*
imports *Path_Connected Ordered_Euclidean_Space HOL−Computational_Algebra.Primes*
begin

7.10.1 The Brouwer reduction theorem

theorem *Brouwer_reduction_theorem_gen*:
fixes S :: $'a$::*euclidean_space set*
assumes *closed* S φ S
and φ: $\bigwedge F.\ [\![\bigwedge n.\ closed(F\ n); \bigwedge n.\ \varphi(F\ n); \bigwedge n.\ F(Suc\ n) \subseteq F\ n]\!] \Longrightarrow \varphi(\bigcap(range\ F))$
obtains T **where** $T \subseteq S$ *closed* T φ T $\bigwedge U.\ [\![U \subseteq S; closed\ U; \varphi\ U]\!] \Longrightarrow \neg\ (U \subset T)$

corollary *Brouwer_reduction_theorem*:
fixes S :: $'a$::*euclidean_space set*
assumes *compact* S φ S $S \neq \{\}$
and φ: $\bigwedge F.\ [\![\bigwedge n.\ compact(F\ n); \bigwedge n.\ F\ n \neq \{\}; \bigwedge n.\ \varphi(F\ n); \bigwedge n.\ F(Suc\ n) \subseteq F\ n]\!] \Longrightarrow \varphi(\bigcap(range\ F))$
obtains T **where** $T \subseteq S$ *compact* T $T \neq \{\}$ φ T
$\bigwedge U.\ [\![U \subseteq S; closed\ U; U \neq \{\}; \varphi\ U]\!] \Longrightarrow \neg\ (U \subset T)$

7.10.2 Density of points with dyadic rational coordinates

proposition *closure_dyadic_rationals*:
$closure\ (\bigcup k.\ \bigcup f \in Basis \to \mathbb{Z}.\ \{\ \sum i :: 'a :: euclidean\_space \in Basis.\ (f\ i\ /\ 2\hat{}k) *_R\ i\ \}) = UNIV$

corollary *closure_rational_coordinates*:
$closure\ (\bigcup f \in Basis \to \mathbb{Q}.\ \{\ \sum i :: 'a :: euclidean\_space \in Basis.\ f\ i *_R\ i\ \}) = UNIV$

theorem *homeomorphic_monotone_image_interval*:
fixes f :: *real* $\Rightarrow$ $'a$::{*real_normed_vector*,*complete_space*}
assumes *cont_f*: *continuous_on* $\{0..1\}$ f

and *conn*: $\bigwedge y.$ *connected* $(\{0..1\} \cap f -` \{y\})$
and *f_1not0*: $f\ 1 \neq f\ 0$
shows $(f ` \{0..1\})$ *homeomorphic* $\{0..1::real\}$

theorem *path_contains_arc*:
fixes p :: *real* ⇒ *'a*::{*complete_space*,*real_normed_vector*}
assumes *path p* **and** *a*: *pathstart p* = *a* **and** *b*: *pathfinish p* = *b* **and** $a \neq b$
obtains *q* **where** *arc q path_image q* ⊆ *path_image p pathstart q* = *a pathfinish q* = *b*

corollary *path_connected_arcwise*:
fixes *S* :: *'a*::{*complete_space*,*real_normed_vector*} *set*
shows *path_connected S* ⟷
(∀ *x* ∈ *S*. ∀ *y* ∈ *S*. *x* ≠ *y* ⟶ (∃ *g*. *arc g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*))
(**is** *?lhs* = *?rhs*)

corollary *arc_connected_trans*:
fixes *g* :: *real* ⇒ *'a*::{*complete_space*,*real_normed_vector*}
assumes *arc g arc h pathfinish g* = *pathstart h pathstart g* ≠ *pathfinish h*
obtains *i* **where** *arc i path_image i* ⊆ *path_image g* ∪ *path_image h*
pathstart i = *pathstart g pathfinish i* = *pathfinish h*

7.10.3 Accessibility of frontier points

end

7.11 The Urysohn lemma, its consequences and other advanced material about metric spaces

theory *Urysohn*
imports *Abstract_Topological_Spaces Abstract_Metric_Spaces Infinite_Sum Arcwise_Connected*
begin

7.11.1 Urysohn lemma and Tietze's theorem

proposition *Urysohn_lemma*:
fixes *a b* :: *real*
assumes *normal_space X closedin X S closedin X T disjnt S T* $a \leq b$
obtains *f* **where** *continuous_map X* (*top_of_set* $\{a..b\}$) *f* $f ` S \subseteq \{a\}$ $f ` T \subseteq \{b\}$

theorem *Tietze_extension_closed_real_interval*:
assumes *normal_space X* **and** *closedin X S*
and *contf*: *continuous_map* (*subtopology X S*) *euclideanreal f*
and *fim*: f ‘ $S \subseteq \{a..b\}$ **and** $a \leq b$
obtains *g*
where *continuous_map X euclideanreal g*
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$ g ‘ *topspace* $X \subseteq \{a..b\}$

theorem *Tietze_extension_realinterval*:
assumes *XS*: *normal_space X closedin X S* **and** *T*: *is_interval T* $T \neq \{\}$
and *contf*: *continuous_map* (*subtopology X S*) *euclideanreal f*
and f ‘ $S \subseteq T$
obtains *g* **where** *continuous_map X euclideanreal g* g ‘ *topspace* $X \subseteq T$ $\bigwedge x.$ $x \in S \Longrightarrow g\ x = f\ x$

7.11.2 Random metric space stuff

7.11.3 Hereditarily normal spaces

7.11.4 Completely regular spaces

proposition *locally_compact_regular_imp_completely_regular_space*:
assumes *locally_compact_space X Hausdorff_space X* $\vee$ *regular_space X*
shows *completely_regular_space X*

proposition *completely_regular_space_product_topology*:
completely_regular_space (*product_topology X I*) $\longleftrightarrow$
($\exists\, i{\in}I.\ X\ i =$ *trivial_topology*) $\vee$ ($\forall\, i \in I.$ *completely_regular_space* ($X\ i$))
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)

7.11.5 More generally, the k-ification functor

7.11.6 One-point compactifications and the Alexandroff extension construction

proposition *kc_space_one_point_compactification_gen*:

assumes *compact_space X*
shows *kc_space X* ⟷
openin X (*topspace X* − {*a*}) ∧ (∀ *K*. *compactin X K* ∧ *a*∉*K* ⟶ *closedin X K*) ∧
k_space (*subtopology X* (*topspace X* − {*a*})) ∧ *kc_space* (*subtopology X* (*topspace X* − {*a*}))
(**is** *?lhs* ⟷ *?rhs*)

proposition *istopology_Alexandroff_open*: *istopology* (*Alexandroff_open X*)

proposition *regular_space_one_point_compactification*:
assumes *compact_space X* **and** *ope*: *openin X* (*topspace X* − {*a*})
and §: ⋀*K*. ⟦*compactin* (*subtopology X* (*topspace X* − {*a*})) *K*; *closedin* (*subtopology X* (*topspace X* − {*a*})) *K*⟧ ⟹ *closedin X K*
shows *regular_space X* ⟷
regular_space (*subtopology X* (*topspace X* − {*a*})) ∧ *locally_compact_space* (*subtopology X* (*topspace X* − {*a*}))
(**is** *?lhs* ⟷ *?rhs*)

proposition *Hausdorff_space_one_point_compactification_asymmetric_prod*:
assumes *compact_space X*
shows *Hausdorff_space X* ⟷
kc_space (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) ∧
k_space (*prod_topology X* (*subtopology X* (*topspace X* − {*a*}))) (**is** *?lhs* ⟷ *?rhs*)

7.11.7 Extending continuous maps "pointwise" in a regular space

7.11.8 Extending Cauchy continuous functions to the closure

7.11.9 Metric space of bounded functions

7.11.10 Metric space of continuous bounded functions

7.11.11 Existence of completion for any metric space M as a subspace of M=>R

7.11.12 Contractions

7.11.13 The Baire Category Theorem

7.11.14 Sierpinski-Hausdorff type results about countable closed unions

7.11.15 The Tychonoff embedding

7.11.16 Urysohn and Tietze analogs for completely regular spaces

7.11.17 Size bounds on connected or path-connected spaces

7.11.18 Lavrentiev extension etc

7.11.19 Embedding in products and hence more about completely metrizable spaces

7.11.20 Theorems from Kuratowski

7.11.21 A perfect set in common cases must have at least the cardinality of the continuum

proposition *Kuratowski_component_number_invariance_aux*:
assumes *compact_space X* **and** *HsX*: *Hausdorff_space X*
and *lcX*: *locally_connected_space X* **and** *hnX*: *hereditarily normal_space X*
and *hom*: (*subtopology X S*) *homeomorphic_space* (*subtopology X T*)
and *leXS*: {..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *S*))
assumes §: ⋀*S T*.
⟦*closedin X S*; *closedin X T*; (*subtopology X S*) *homeomorphic_space* (*subtopology X T*);
{..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *S*))⟧
⟹ {..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *T*))
shows {..<*n*::*nat*} ≲ *connected_components_of* (*subtopology X* (*topspace X* − *T*))

```
theorem Kuratowski_component_number_invariance:
  assumes compact_space X Hausdorff_space X locally_connected_space X hered-
itarily normal_space X
  shows ((∀ S T n.
            closedin X S ∧ closedin X T ∧
            (subtopology X S) homeomorphic_space (subtopology X T)
            ⟶ (connected_components_of
                  (subtopology X (topspace X − S)) ≈ {..<n::nat} ⟷
                 connected_components_of
                  (subtopology X (topspace X − T)) ≈ {..<n::nat})) ⟷
          (∀ S T n.
            (subtopology X S) homeomorphic_space (subtopology X T)
            ⟶ (connected_components_of
                  (subtopology X (topspace X − S)) ≈ {..<n::nat} ⟷
                 connected_components_of
                  (subtopology X (topspace X − T)) ≈ {..<n::nat})))
        (is ?lhs = ?rhs)

end
theory Sparse_In
  imports Homotopy

begin
```

7.11.22 A set of points sparse in another set

7.11.23 Co-sparseness filter

```
end
theory Isolated
  imports Elementary_Metric_Spaces Sparse_In

begin
```

7.11.24 Isolate and discrete

```
end
```

7.12 Operator Norm

theory *Operator_Norm*
imports *Complex_Main*
begin

definition
onorm :: (*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*) ⇒ *real* **where**
onorm f = (*SUP x*. *norm* (*f x*) / *norm x*)

proposition *onorm_bound*:
  assumes *0* ≤ *b* **and** ⋀*x*. *norm* (*f x*) ≤ *b* ∗ *norm x*
  shows *onorm f* ≤ *b*

end

7.13 Limits on the Extended Real Number Line

theory *Extended_Real_Limits*
imports
  Topology_Euclidean_Space
  HOL−Library.Extended_Real
  HOL−Library.Extended_Nonnegative_Real
  HOL−Library.Indicator_Function
begin

7.13.1 Extended-Real.thy

Continuity of addition

Continuity of multiplication

Continuity of division

7.13.2 Extended-Nonnegative-Real.thy

7.13.3 monoset

7.13.4 Relate extended reals and the indicator function

end

7.14 Radius of Convergence and Summation Tests

theory *Summation_Tests*
imports
Complex_Main
HOL−Library.Discrete_Functions
HOL−Library.Extended_Real
HOL−Library.Liminf_Limsup
Extended_Real_Limits
begin

7.14.1 Convergence tests for infinite sums

theorem *root_test_convergence'*:
fixes $f :: nat \Rightarrow 'a :: banach$
defines $l \equiv limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n))))$
assumes l: $l < 1$
shows *summable f*

theorem *root_test_divergence*:
fixes $f :: nat \Rightarrow 'a :: banach$
defines $l \equiv limsup\ (\lambda n.\ ereal\ (root\ n\ (norm\ (f\ n))))$
assumes l: $l > 1$
shows $\neg summable\ f$

theorem *condensation_test*:
assumes *mono*: $\bigwedge m.\ 0 < m \Longrightarrow f\ (Suc\ m) \le f\ m$
assumes *nonneg*: $\bigwedge n.\ f\ n \ge 0$
shows $summable\ f \longleftrightarrow summable\ (\lambda n.\ 2\hat{}n * f\ (2\hat{}n))$

theorem *summable_complex_powr_iff*:
assumes $Re\ s < -1$
shows $summable\ (\lambda n.\ exp\ (of\_real\ (ln\ (of\_nat\ n)) * s))$
theorem *kummers_test_convergence*:
fixes $f\ p :: nat \Rightarrow real$
assumes *pos_f*: $eventually\ (\lambda n.\ f\ n > 0)\ sequentially$
assumes *nonneg_p*: $eventually\ (\lambda n.\ p\ n \ge 0)\ sequentially$
defines $l \equiv liminf\ (\lambda n.\ ereal\ (p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n)))$
assumes l: $l > 0$
shows *summable f*

theorem *kummers_test_divergence*:
fixes $f\ p :: nat \Rightarrow real$
assumes *pos_f*: $eventually\ (\lambda n.\ f\ n > 0)\ sequentially$
assumes *pos_p*: $eventually\ (\lambda n.\ p\ n > 0)\ sequentially$
assumes *divergent_p*: $\neg summable\ (\lambda n.\ inverse\ (p\ n))$
defines $l \equiv limsup\ (\lambda n.\ ereal\ (p\ n * f\ n\ /\ f\ (Suc\ n) - p\ (Suc\ n)))$
assumes l: $l < 0$

shows $\neg$*summable f*
theorem *ratio_test_convergence*:
fixes f :: *nat* $\Rightarrow$ *real*
assumes *pos_f*: *eventually* (λn. *f n* > *0*) *sequentially*
defines $l \equiv$ *liminf* (λn. *ereal* (*f n* / *f* (*Suc n*)))
assumes *l*: *l* > *1*
shows *summable f*

theorem *ratio_test_divergence*:
fixes f :: *nat* $\Rightarrow$ *real*
assumes *pos_f*: *eventually* (λn. *f n* > *0*) *sequentially*
defines $l \equiv$ *limsup* (λn. *ereal* (*f n* / *f* (*Suc n*)))
assumes *l*: *l* < *1*
shows $\neg$*summable f*
theorem *raabes_test_convergence*:
fixes f :: *nat* $\Rightarrow$ *real*
assumes *pos*: *eventually* (λn. *f n* > *0*) *sequentially*
defines $l \equiv$ *liminf* (λn. *ereal* (*of_nat n* * (*f n* / *f* (*Suc n*) − *1*)))
assumes *l*: *l* > *1*
shows *summable f*

theorem *raabes_test_divergence*:
fixes f :: *nat* $\Rightarrow$ *real*
assumes *pos*: *eventually* (λn. *f n* > *0*) *sequentially*
defines $l \equiv$ *limsup* (λn. *ereal* (*of_nat n* * (*f n* / *f* (*Suc n*) − *1*)))
assumes *l*: *l* < *1*
shows $\neg$*summable f*

7.14.2 Radius of convergence

definition *conv_radius* :: (*nat* $\Rightarrow$ *'a* :: *banach*) $\Rightarrow$ *ereal* **where**
conv_radius f = *inverse* (*limsup* (λn. *ereal* (*root n* (*norm* (*f n*)))))

theorem *abs_summable_in_conv_radius*:
fixes f :: *nat* $\Rightarrow$ *'a* :: {*banach*, *real_normed_div_algebra*}
assumes *ereal* (*norm z*) < *conv_radius f*
shows *summable* (λn. *norm* (*f n* * *z* ^ *n*))

theorem *not_summable_outside_conv_radius*:
fixes f :: *nat* $\Rightarrow$ *'a* :: {*banach*, *real_normed_div_algebra*}
assumes *ereal* (*norm z*) > *conv_radius f*
shows $\neg$*summable* (λn. *f n* * *z* ^ *n*)

end

7.15 Uniform Limit and Uniform Convergence

theory *Uniform_Limit*
imports *Connected Summation_Tests Infinite_Sum*
begin

7.15.1 Definition

definition *uniformly_on* :: *'a set* ⇒ (*'a* ⇒ *'b*::*metric_space*) ⇒ (*'a* ⇒ *'b*) *filter*
where *uniformly_on S l* = (*INF e*∈{*0* <..}. *principal* {*f*. ∀ *x*∈*S*. *dist* (*f x*) (*l x*) < *e*})

abbreviation
uniform_limit S f l ≡ *filterlim f* (*uniformly_on S l*)

proposition *uniform_limit_iff*:
uniform_limit S f l F ⟷ (∀ *e*>*0*. ∀$_F$ *n in F*. ∀ *x*∈*S*. *dist* (*f n x*) (*l x*) < *e*)

7.15.2 Exchange limits

proposition *swap_uniform_limit'*:
assumes *f*: ∀$_F$ *n in F*. (*f n* ⟶ *g n*) *G*
assumes *g*: (*g* ⟶ *l*) *F*
assumes *uc*: *uniform_limit S f h F*
assumes *ev*: ∀$_F$ *x in G*. *x* ∈ *S*
assumes ¬*trivial_limit F*
shows (*h* ⟶ *l*) *G*

corollary *swap_uniform_limit*:
assumes ∀$_F$ *n in F*. (*f n* ⟶ *g n*) (*at x within S*)
assumes (*g* ⟶ *l*) *F uniform_limit S f h F* ¬*trivial_limit F*
shows (*h* ⟶ *l*) (*at x within S*)

7.15.3 Uniform limit theorem

theorem *uniform_limit_theorem*:
assumes *c*: ∀$_F$ *n in F*. *continuous_on A* (*f n*)
assumes *ul*: *uniform_limit A f l F*
assumes ¬ *trivial_limit F*
shows *continuous_on A l*

7.15.4 Comparison Test

7.15.5 Weierstrass M-Test

proposition *Weierstrass_m_test_ev*:

fixes f :: _ ⇒ _ ⇒ _ :: *banach*
assumes *eventually* (λn. ∀x∈A. *norm* (f n x) ≤ M n) *sequentially*
assumes *summable M*
shows *uniform_limit A* (λn x. ∑i<n. f i x) (λx. *suminf* (λi. f i x)) *sequentially*

7.15.6 Power series and uniform convergence

proposition *powser_uniformly_convergent*:
fixes *a* :: *nat* ⇒ 'a::{*real_normed_div_algebra*,*banach*}
assumes *r* < *conv_radius a*
shows *uniformly_convergent_on* (*cball* ξ *r*) (λn x. ∑i<n. a i ∗ (x − ξ) ^ i)

7.15.7 Tannery's Theorem

end

7.16 Bounded Linear Function

theory *Bounded_Linear_Function*
imports
Topology_Euclidean_Space
Operator_Norm
Uniform_Limit
Function_Topology

begin

7.16.1 Type of bounded linear functions

typedef (**overloaded**) ('a, 'b) *blinfun* (‹(‹notation=‹infix ⇒$_L$››_ ⇒$_L$ /_)› [22, 21] 21) =
{f::'a::*real_normed_vector*⇒'b::*real_normed_vector*. *bounded_linear f*}
morphisms *blinfun_apply Blinfun*

7.16.2 Type class instantiations

instantiation *blinfun* :: (*real_normed_vector*, *real_normed_vector*) *real_normed_vector*
begin

lift_definition *norm_blinfun* :: 'a ⇒$_L$ 'b ⇒ *real* **is** *onorm*
lift_definition *zero_blinfun* :: 'a ⇒$_L$ 'b **is** λx. 0

lift_definition *plus_blinfun* :: 'a ⇒$_L$ 'b ⇒ 'a ⇒$_L$ 'b ⇒ 'a ⇒$_L$ 'b

is $\lambda f\ g\ x.\ f\ x + g\ x$

lift_definition *scaleR_blinfun*::*real* $\Rightarrow$ $'a \Rightarrow_L\ 'b \Rightarrow\ 'a \Rightarrow_L\ 'b$ **is** $\lambda r\ f\ x.\ r *_R f\ x$

7.16.3 The strong operator topology on continuous linear operators

definition *strong_operator_topology*::(*'a*::*real_normed_vector* $\Rightarrow_L$*'b*::*real_normed_vector*) *topology*
where *strong_operator_topology* = *pullback_topology UNIV blinfun_apply euclidean*

end

7.17 Derivative

theory *Derivative*
imports
Bounded_Linear_Function
Line_Segment
Convex_Euclidean_Space
begin

7.17.1 Derivatives

proposition *has_derivative_within'*:
(*f has_derivative f'*)(*at x within s*) $\longleftrightarrow$
bounded_linear f' $\wedge$
($\forall$ *e*>*0*. $\exists$ *d*>*0*. $\forall$ *x'*$\in$*s*. *0* < *norm* (*x'* − *x*) $\wedge$ *norm* (*x'* − *x*) < *d* $\longrightarrow$
norm (*f x'* − *f x* − *f'*(*x'* − *x*)) / *norm* (*x'* − *x*) < *e*)

7.17.2 Differentiability

definition
differentiable_on :: (*'a*::*real_normed_vector* $\Rightarrow$ *'b*::*real_normed_vector*) $\Rightarrow$ *'a set* $\Rightarrow$ *bool*
(**infix** ‹*differentiable'_on*› *50*)
where *f differentiable_on s* $\longleftrightarrow$ ($\forall$ *x*$\in$*s*. *f differentiable* (*at x within s*))

7.17.3 Frechet derivative and Jacobian matrix

proposition *frechet_derivative_works*:
f differentiable net ⟷ (*f has_derivative* (*frechet_derivative f net*)) *net*

7.17.4 Differentiability implies continuity

proposition *differentiable_imp_continuous_within*:
f differentiable (*at x within s*) ⟹ *continuous* (*at x within s*) *f*

7.17.5 The chain rule

proposition *diff_chain_within*[*derivative_intros*]:
assumes (*f has_derivative f′*) (*at x within s*)
and (*g has_derivative g′*) (*at* (*f x*) *within* (*f* ‘ *s*))
shows ((*g* ∘ *f*) *has_derivative* (*g′* ∘ *f′*))(*at x within s*)

7.17.6 Uniqueness of derivative

The general result is a bit messy because we need approachability of the limit point from any direction. But OK for nontrivial intervals etc.

proposition *frechet_derivative_unique_within*:
fixes *f* :: ′*a*::*euclidean_space* ⇒ ′*b*::*real_normed_vector*
assumes *1*: (*f has_derivative f′*) (*at x within S*)
and *2*: (*f has_derivative f″*) (*at x within S*)
and *S*: ⋀*i e*. ⟦*i*∈*Basis*; *e*>*0*⟧ ⟹ ∃ *d*. *0* < |*d*| ∧ |*d*| < *e* ∧ (*x* + *d* $*_R$ *i*) ∈ *S*
shows *f′* = *f″*

proposition *frechet_derivative_unique_within_closed_interval*:
fixes *f*::′*a*::*euclidean_space* ⇒ ′*b*::*real_normed_vector*
assumes *ab*: ⋀*i*. *i*∈*Basis* ⟹ *a*•*i* < *b*•*i*
and *x*: *x* ∈ *cbox a b*
and (*f has_derivative f′*) (*at x within cbox a b*)
and (*f has_derivative f″*) (*at x within cbox a b*)
shows *f′* = *f″*

7.17.7 Derivatives of local minima and maxima are zero

7.17.8 One-dimensional mean value theorem

7.17.9 More general bound theorems

proposition *differentiable_bound_general*:
fixes *f* :: *real* ⇒ ′*a*::*real_normed_vector*
assumes *a* < *b*

and *f_cont*: *continuous_on* $\{a..b\}$ *f*
and *phi_cont*: *continuous_on* $\{a..b\}$ φ
and f': $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow$ (f *has_vector_derivative* f' x) (*at* x)
and *phi'*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow$ (φ *has_vector_derivative* φ' x) (*at* x)
and *bnd*: $\bigwedge x.\ a < x \Longrightarrow x < b \Longrightarrow$ *norm* $(f'\ x) \le \varphi'\ x$
shows *norm* $(f\ b - f\ a) \le \varphi\ b - \varphi\ a$

7.17.10 Differentiability of inverse function (most basic form)

proposition *has_derivative_inverse*:
fixes f :: *'a::real_normed_vector* $\Rightarrow$ *'b::real_normed_vector*
assumes *compact S*
and $x \in S$
and *fx*: $f\ x \in$ *interior* (f ‘ S)
and *continuous_on S f*
and *gf*: $\bigwedge y.\ y \in S \Longrightarrow g\ (f\ y) = y$
and *B*: (f *has_derivative* f') (*at* x) *bounded_linear* g' $g' \circ f' = id$
shows (g *has_derivative* g') (*at* ($f\ x$))
proposition *has_derivative_locally_injective*:
fixes f :: *'n::euclidean_space* $\Rightarrow$ *'m::euclidean_space*
assumes $a \in S$
and *open S*
and *bling*: *bounded_linear* g'
and $g' \circ f'\ a = id$
and *derf*: $\bigwedge x.\ x \in S \Longrightarrow$ (f *has_derivative* f' x) (*at* x)
and $\bigwedge e.\ e > 0 \Longrightarrow \exists\, d{>}0.\ \forall\, x.$ *dist* $a\ x < d \longrightarrow$ *onorm* $(\lambda v.\ f'\ x\ v - f'\ a\ v)$ $< e$
obtains r **where** $r > 0$ *ball* $a\ r \subseteq S$ *inj_on* f (*ball* $a\ r$)

7.17.11 Uniformly convergent sequence of derivatives

proposition *has_derivative_sequence*:
fixes f :: *nat* $\Rightarrow$ *'a::real_normed_vector* $\Rightarrow$ *'b::banach*
assumes *convex S*
and *derf*: $\bigwedge n\ x.\ x \in S \Longrightarrow$ (($f\ n$) *has_derivative* ($f'\ n\ x$)) (*at* x *within* S)
and *nle*: $\bigwedge e.\ e > 0 \Longrightarrow \forall_F\ n$ *in sequentially*. $\forall\, x{\in}S.\ \forall\, h.$ *norm* $(f'\ n\ x\ h - g'\ x\ h) \le e * $ *norm* h
and $x0 \in S$
and *lim*: $((\lambda n.\ f\ n\ x0) \longrightarrow l)$ *sequentially*
shows $\exists\, g.\ \forall\, x{\in}S.\ (\lambda n.\ f\ n\ x) \longrightarrow g\ x \wedge$ (g *has_derivative* $g'(x)$) (*at* x *within* S)

7.17.12 Differentiation of a series

proposition *has_derivative_series*:
fixes f :: *nat* $\Rightarrow$ *'a::real_normed_vector* $\Rightarrow$ *'b::banach*

assumes *convex S*
and $\bigwedge n\ x.\ x \in S \Longrightarrow ((f\ n)\ has\_derivative\ (f'\ n\ x))\ (at\ x\ within\ S)$
and $\bigwedge e.\ e>0 \Longrightarrow \forall_F\ n\ in\ sequentially.\ \forall x{\in}S.\ \forall h.\ norm\ (sum\ (\lambda i.\ f'\ i\ x\ h)\ \{..<n\} - g'\ x\ h) \le e * norm\ h$
and $x \in S$
and $(\lambda n.\ f\ n\ x)\ sums\ l$
shows $\exists g.\ \forall x{\in}S.\ (\lambda n.\ f\ n\ x)\ sums\ (g\ x) \wedge (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$

7.17.13 Derivative as a vector

proposition *vector_derivative_works*:
$f\ differentiable\ net \longleftrightarrow (f\ has\_vector\_derivative\ (vector\_derivative\ f\ net))\ net$
(**is** $?l = ?r$)

7.17.14 Field differentiability

definition $field\_differentiable :: ['a \Rightarrow 'a{::}real\_normed\_field, 'a\ filter] \Rightarrow bool$
(**infixr** ‹(field'_differentiable)› *50*)
where $f\ field\_differentiable\ F \equiv \exists f'.\ (f\ has\_field\_derivative\ f')\ F$

7.17.15 Field derivative

definition $deriv :: ('a \Rightarrow 'a{::}real\_normed\_field) \Rightarrow 'a \Rightarrow 'a$ **where**
$deriv\ f\ x \equiv SOME\ D.\ DERIV\ f\ x :> D$

proposition *field_differentiable_derivI*:
$f\ field\_differentiable\ (at\ x) \Longrightarrow (f\ has\_field\_derivative\ deriv\ f\ x)\ (at\ x)$

7.17.16 Relation between convexity and derivative

proposition *convex_on_imp_above_tangent*:
assumes *convex*: *convex_on A f* **and** *connected*: *connected A*
assumes *c*: $c \in interior\ A$ **and** $x : x \in A$
assumes *deriv*: $(f\ has\_field\_derivative\ f')\ (at\ c\ within\ A)$
shows $f\ x - f\ c \ge f' * (x - c)$

7.17.17 Partial derivatives

proposition *has_derivative_partialsI*:
fixes $f{::}'a{::}real\_normed\_vector \Rightarrow 'b{::}real\_normed\_vector \Rightarrow 'c{::}real\_normed\_vector$
assumes *fx*: $((\lambda x.\ f\ x\ y)\ has\_derivative\ fx)\ (at\ x\ within\ X)$

assumes *fy*: $\bigwedge x\ y.\ x \in X \Longrightarrow y \in Y \Longrightarrow ((\lambda y.\ f\ x\ y)$ *has_derivative blinfun_apply* $(fy\ x\ y))$ $(at\ y\ within\ Y)$
assumes *fy_cont*[*unfolded continuous_within*]: *continuous* $(at\ (x,\ y)\ within\ X \times Y)$ $(\lambda(x,\ y).\ fy\ x\ y)$
assumes $y \in Y$ *convex* Y
shows $((\lambda(x,\ y).\ f\ x\ y)$ *has_derivative* $(\lambda(tx,\ ty).\ fx\ tx + fy\ x\ y\ ty))$ $(at\ (x,\ y)$ *within* $X \times Y)$

7.17.18 The Inverse Function Theorem

theorem *inverse_function_theorem*:
fixes f::$'a$::*euclidean_space* $\Rightarrow$ $'a$
and f'::$'a \Rightarrow ('a \Rightarrow_L\ 'a)$
assumes *open* U
and *derf*: $\bigwedge x.\ x \in U \Longrightarrow (f$ *has_derivative* $($*blinfun_apply* $(f'\ x)))$ $(at\ x)$
and *contf*: *continuous_on* $U\ f'$
and $x0 \in U$
and *invf*: *invf* o_L $f'\ x0 = $ *id_blinfun*
obtains $U'\ V\ g\ g'$ **where** *open* U' $U' \subseteq U$ $x0 \in U'$ *open* V $f\ x0 \in V$ *homeomorphism* $U'\ V\ f\ g$
$\bigwedge y.\ y \in V \Longrightarrow (g$ *has_derivative* $(g'\ y))$ $(at\ y)$
$\bigwedge y.\ y \in V \Longrightarrow g'\ y = $ *inv* $($*blinfun_apply* $(f'(g\ y)))$
$\bigwedge y.\ y \in V \Longrightarrow$ *bij* $($*blinfun_apply* $(f'(g\ y)))$

7.17.19 The concept of continuously differentiable

definition *C1_differentiable_on* :: $($*real* $\Rightarrow$ $'a$::*real_normed_vector*$)$ $\Rightarrow$ *real set* $\Rightarrow$ *bool*
(**infix** ‹*C1'_differentiable'_on*› *50*)
where
f C1_differentiable_on S $\longleftrightarrow$
$(\exists D.\ (\forall x \in S.\ (f$ *has_vector_derivative* $(D\ x))$ $(at\ x))$ $\wedge$ *continuous_on* $S\ D)$

definition *piecewise_C1_differentiable_on*
(**infixr** ‹*piecewise'_C1'_differentiable'_on*› *50*)
where *f piecewise_C1_differentiable_on i* $\equiv$
continuous_on i f $\wedge$
$(\exists S.$ *finite* $S \wedge (f$ *C1_differentiable_on* $(i - S)))$

end

7.18 Finite Cartesian Products of Euclidean Spaces

theory *Cartesian_Euclidean_Space*
imports *Derivative*
begin

7.18.1 Closures and interiors of halfspaces

7.18.2 Bounds on components etc. relative to operator norm

7.18.3 Convex Euclidean Space

7.18.4 Arbitrarily good rational approximations

proposition *matrix_rational_approximation*:
  fixes *A* :: *real^'n^'m*
  assumes *e* > *0*
  obtains *B* **where** ⋀*i j*. *Bij* ∈ ℚ *onorm*(λ*x*. (*A* − *B*) ∗*v x*) < *e*

7.18.5 Derivative

definition *jacobian f net* = *matrix*(*frechet_derivative f net*)

proposition *jacobian_works*:
  (*f*::(*real^'a*) ⇒ (*real^'b*)) *differentiable net* ⟷
    (*f has_derivative* (λ*h*. (*jacobian f net*) ∗*v h*)) *net* (**is** *?lhs* = *?rhs*)
proposition *differential_zero_maxmin_cart*:
  fixes *f*::*real^'a* ⇒ *real^'b*
  assumes *0* < *e* ((∀ *y* ∈ *ball x e*. (*f y*)$*k* ≤ (*f x*)$*k*) ∨ (∀ *y*∈*ball x e*. (*f x*)$*k* ≤ (*f y*)$*k*))
    f differentiable (*at x*)
  shows *jacobian f* (*at x*) $ *k* = *0*

end

7.19 Complex Analysis Basics

theory *Complex_Analysis_Basics*
  imports *Derivative HOL−Library.Nonpos_Ints Uncountable_Sets*
begin

7.19.1 Holomorphic functions

definition *holomorphic_on* :: [*complex* ⇒ *complex*, *complex set*] ⇒ *bool*
    (**infixl** ‹(*holomorphic′_on*)› *50*)
  where *f holomorphic_on s* ≡ ∀ *x*∈*s*. *f field_differentiable* (*at x within s*)

named_theorems *holomorphic_intros structural introduction rules for holomorphic_on*

7.19.2 Analyticity on a set

definition *analytic_on* (**infixl** ‹(*analytic′_on*)› *50*)
  where f *analytic_on* $S \equiv \forall x \in S.\ \exists\varepsilon.\ 0 < \varepsilon \wedge f$ *holomorphic_on* (*ball* $x\ \varepsilon$)

named_theorems *analytic_intros introduction rules for proving analyticity*

end

7.20 Complex Transcendental Functions

theory *Complex_Transcendental*
imports
  Complex_Analysis_Basics Summation_Tests HOL−Library.Periodic_Fun
begin

7.20.1 Möbius transformations

definition *moebius* $a\ b\ c\ d \equiv (\lambda z.\ (a*z+b)\ /\ (c*z+d :: {}'a :: \mathit{field}))$

theorem *moebius_inverse*:
  assumes $a * d \neq b * c\ \ c * z + d \neq 0$
  shows *moebius* $d\ (-b)\ (-c)\ a$ (*moebius* $a\ b\ c\ d\ z) = z$

7.20.2 Euler and de Moivre formulas

theorem *exp_Euler*: $exp(\mathrm{i} * z) = cos(z) + \mathrm{i} * sin(z)$

theorem *Euler*: $exp(z) = of\_real(exp(Re\ z)) *$
  $(of\_real(cos(Im\ z)) + \mathrm{i} * of\_real(sin(Im\ z)))$

7.20.3 The argument of a complex number (HOL Light version)

definition *is_Arg* :: [*complex*,*real*] ⇒ *bool*
  where *is_Arg* $z\ r \equiv z = of\_real(norm\ z) * exp(\mathrm{i} * of\_real\ r)$

definition *Arg2pi* :: *complex* ⇒ *real*
  where *Arg2pi* $z \equiv$ *if* $z = 0$ *then* 0 *else THE* $t.\ 0 \leq t \wedge t < 2*pi \wedge is\_Arg\ z\ t$

7.20.4 The principal branch of the Complex logarithm

instantiation *complex* :: *ln*
begin

definition *ln_complex* :: *complex* ⇒ *complex*
 where *ln_complex* ≡ λ*z*. *THE w*. *exp w* = *z* & −*pi* < *Im*(*w*) & *Im*(*w*) ≤ *pi*
theorem *Ln_series*:
 fixes *z* :: *complex*
 assumes *norm z* < *1*
 shows (λ*n*. (−*1*)^*Suc n* / *of_nat n* ∗ *z*^*n*) *sums ln* (*1* + *z*) (**is** (λ*n*. *?f n* ∗ *z*^*n*) *sums* _)

corollary *norm_Ln_prod_le*:
 fixes *f* :: ′*a* ⇒ *complex*
 assumes ⋀*x*. *x* ∈ *A* ⟹ *f x* ≠ *0*
 shows *cmod* (*Ln* (*prod f A*)) ≤ (∑*x* ∈ *A*. *cmod* (*Ln* (*f x*)))

7.20.5 The Argument of a Complex Number

lemma *Arg_def*:
 shows *Arg z* = (*if z* = *0 then 0 else Im* (*Ln z*))

7.20.6 The Unwinding Number and the Ln product Formula

definition *unwinding* :: *complex* ⇒ *int* **where**
 unwinding z ≡ *THE k*. *of_int k* = (*z* − *Ln*(*exp z*)) / (*of_real*(*2*∗*pi*) ∗ i)

7.20.7 Characterisation of *Im* (*Ln z*) (Wenda Li)

7.20.8 Complex arctangent

definition *Arctan* :: *complex* ⇒ *complex* **where**
 Arctan ≡ λ*z*. (i/*2*) ∗ *Ln*((*1* − i∗*z*) / (*1* + i∗*z*))

theorem *Arctan_series*:
 assumes *z*: *norm* (*z* :: *complex*) < *1*
 defines *g* ≡ λ*n*. *if odd n then* −i∗i^*n* / *n else 0*
 defines *h* ≡ λ*z n*. (−*1*)^*n* / *of_nat* (*2*∗*n*+*1*) ∗ (*z*::*complex*)^(*2*∗*n*+*1*)
 shows (λ*n*. *g n* ∗ *z*^*n*) *sums Arctan z*
 and *h z sums Arctan z*
theorem *ln_series_quadratic*:
 assumes *x*: *x* > (*0*::*real*)

shows $(\lambda n.\ (2*((x - 1) / (x + 1))\ \hat{}\ (2*n+1) / of\_nat\ (2*n+1)))\ sums\ ln\ x$

7.20.9 Inverse Sine

definition *Arcsin* :: *complex* ⇒ *complex* **where**
$Arcsin \equiv \lambda z.\ -\mathrm{i} * Ln(\mathrm{i} * z + csqrt(1 - z^2))$

7.20.10 Inverse Cosine

definition *Arccos* :: *complex* ⇒ *complex* **where**
$Arccos \equiv \lambda z.\ -\mathrm{i} * Ln(z + \mathrm{i} * csqrt(1 - z^2))$

7.20.11 Roots of unity

theorem *complex_root_unity*:
fixes *j::nat*
assumes $n \neq 0$
shows $exp(2 * of\_real\ pi * \mathrm{i} * of\_nat\ j / of\_nat\ n)\hat{}n = 1$

corollary *bij_betw_roots_unity*:
$bij\_betw\ (\lambda j.\ exp(2 * of\_real\ pi * \mathrm{i} * of\_nat\ j / of\_nat\ n))$
$\{..<n\}\ \ \{exp(2 * of\_real\ pi * \mathrm{i} * of\_nat\ j / of\_nat\ n) \mid j.\ j < n\}$

7.20.12 Normalisation of angles

7.20.13 Convexity of circular sectors in the complex plane

7.20.14 Complex cones

end

Chapter 8

Measure and Integration Theory

theory *Sigma_Algebra*
imports
  Complex_Main
  HOL−Library.Countable_Set
  HOL−Library.FuncSet
  HOL−Library.Indicator_Function
  HOL−Library.Extended_Nonnegative_Real
  HOL−Library.Disjoint_Sets
begin

8.1 Sigma Algebra

8.1.1 Families of sets

locale *subset_class* =
  fixes Ω :: *'a set* **and** M :: *'a set set*
  assumes *space_closed*: $M \subseteq Pow\ \Omega$
locale *semiring_of_sets* = *subset_class* +
  assumes *empty_sets*[*iff*]: $\{\} \in M$
  assumes *Int*[*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cap b \in M$
  assumes *Diff_cover*:
    $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow \exists\, C \subseteq M.\ finite\ C \wedge disjoint\ C \wedge a - b = \bigcup C$
locale *ring_of_sets* = *semiring_of_sets* +
  assumes *Un* [*intro*]: $\bigwedge a\ b.\ a \in M \Longrightarrow b \in M \Longrightarrow a \cup b \in M$
locale *algebra* = *ring_of_sets* +
  assumes *top* [*iff*]: $\Omega \in M$

proposition *algebra_iff_Un*:
  algebra $\Omega\ M \longleftrightarrow$
    $M \subseteq Pow\ \Omega\ \wedge$
    $\{\} \in M\ \wedge$
    $(\forall\, a \in M.\ \Omega - a \in M)\ \wedge$

$(\forall a \in M.\ \forall\ b \in M.\ a \cup b \in M)$ (**is** _ $\longleftrightarrow$ *?Un*)

proposition *algebra_iff_Int*:
algebra Ω $M \longleftrightarrow$
$M \subseteq Pow\ \Omega\ \&\ \{\} \in M\ \&$
$(\forall a \in M.\ \Omega - a \in M)\ \&$
$(\forall a \in M.\ \forall\ b \in M.\ a \cap b \in M)$ (**is** _ $\longleftrightarrow$ *?Int*)
locale *sigma_algebra* = *algebra* +
assumes *countable_nat_UN* [*intro*]: $\bigwedge A.\ range\ A \subseteq M \Longrightarrow (\bigcup i{::}nat.\ A\ i) \in M$

Sigma algebras can naturally be created as the closure of any set of M with regard to the properties just postulated.

inductive_set *sigma_sets* :: *'a set* ⇒ *'a set set* ⇒ *'a set set*
for *sp* :: *'a set* **and** *A* :: *'a set set*
where
Basic[*intro*, *simp*]: $a \in A \Longrightarrow a \in sigma\_sets\ sp\ A$
| *Empty*: $\{\} \in sigma\_sets\ sp\ A$
| *Compl*: $a \in sigma\_sets\ sp\ A \Longrightarrow sp - a \in sigma\_sets\ sp\ A$
| *Union*: $(\bigwedge i{::}nat.\ a\ i \in sigma\_sets\ sp\ A) \Longrightarrow (\bigcup i.\ a\ i) \in sigma\_sets\ sp\ A$
definition *closed_cdi* :: *'a set* ⇒ *'a set set* ⇒ *bool* **where**
closed_cdi Ω $M \longleftrightarrow$
$M \subseteq Pow\ \Omega\ \&$
$(\forall s \in M.\ \Omega - s \in M)\ \&$
$(\forall A.\ (range\ A \subseteq M)\ \&\ (A\ 0 = \{\})\ \&\ (\forall n.\ A\ n \subseteq A\ (Suc\ n)) \longrightarrow$
$(\bigcup i.\ A\ i) \in M)\ \&$
$(\forall A.\ (range\ A \subseteq M)\ \&\ disjoint\_family\ A \longrightarrow (\bigcup i{::}nat.\ A\ i) \in M)$

locale *Dynkin_system* = *subset_class* +
assumes *space*: $\Omega \in M$
and *compl*[*intro*!]: $\bigwedge A.\ A \in M \Longrightarrow \Omega - A \in M$
and *UN*[*intro*!]: $\bigwedge A.\ disjoint\_family\ A \Longrightarrow range\ A \subseteq M$
$\Longrightarrow (\bigcup i{::}nat.\ A\ i) \in M$
definition *Int_stable* :: *'a set set* ⇒ *bool* **where**
Int_stable $M \longleftrightarrow (\forall\ a \in M.\ \forall\ b \in M.\ a \cap b \in M)$
definition *Dynkin* :: *'a set* ⇒ *'a set set* ⇒ *'a set set* **where**
Dynkin Ω $M = (\bigcap \{D.\ Dynkin\_system\ \Omega\ D \wedge M \subseteq D\})$

The reason to introduce Dynkin-systems is the following induction rules for σ-algebras generated by a generator closed under intersection.

proposition *sigma_sets_induct_disjoint*[*consumes 3*, *case_names basic empty compl union*]:
assumes *Int_stable G*
and *closed*: $G \subseteq Pow\ \Omega$
and *A*: $A \in sigma\_sets\ \Omega\ G$
assumes *basic*: $\bigwedge A.\ A \in G \Longrightarrow P\ A$
and *empty*: $P\ \{\}$
and *compl*: $\bigwedge A.\ A \in sigma\_sets\ \Omega\ G \Longrightarrow P\ A \Longrightarrow P\ (\Omega - A)$

and *union*: $\bigwedge A.$ *disjoint_family* $A \Longrightarrow$ *range* $A \subseteq$ *sigma_sets* Ω $G \Longrightarrow (\bigwedge i.$ P $(A\ i)) \Longrightarrow P$ $(\bigcup i{::}nat.\ A\ i)$
shows $P\ A$

8.1.2 Measure type

definition *positive* :: *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *bool* **where**
positive M $\mu \longleftrightarrow \mu$ {} = *0*

definition *countably_additive* :: *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *bool* **where**
countably_additive M f $\longleftrightarrow$
($\forall A.$ *range* $A \subseteq M \longrightarrow$ *disjoint_family* $A \longrightarrow (\bigcup i.\ A\ i) \in M \longrightarrow$
$(\sum i.\ f\ (A\ i)) = f\ (\bigcup i.\ A\ i))$

definition *measure_space* :: *'a set* ⇒ *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *bool*
where
measure_space Ω A $\mu \longleftrightarrow$
sigma_algebra Ω $A \wedge$ *positive* A $\mu \wedge$ *countably_additive* A μ

typedef *'a measure* =
{(Ω::*'a set*, A, μ). ($\forall a \in -A.\ \mu\ a = 0$) $\wedge$ *measure_space* Ω A μ }

definition *space* :: *'a measure* ⇒ *'a set* **where**
space M = *fst* (*Rep_measure M*)

definition *sets* :: *'a measure* ⇒ *'a set set* **where**
sets M = *fst* (*snd* (*Rep_measure M*))

definition *emeasure* :: *'a measure* ⇒ *'a set* ⇒ *ennreal* **where**
emeasure M = *snd* (*snd* (*Rep_measure M*))

definition *measure* :: *'a measure* ⇒ *'a set* ⇒ *real* **where**
measure M A = *enn2real* (*emeasure M A*)

definition *measure_of* :: *'a set* ⇒ *'a set set* ⇒ (*'a set* ⇒ *ennreal*) ⇒ *'a measure*
where
measure_of Ω A $\mu \equiv$
Abs_measure (Ω, *if* $A \subseteq$ *Pow* Ω *then sigma_sets* Ω A *else* {{}, Ω},
$\lambda a.$ *if* $a \in$ *sigma_sets* Ω $A \wedge$ *measure_space* Ω (*sigma_sets* Ω A) μ *then* μ a *else 0*)
proposition *emeasure_measure_of*:
assumes *M*: M = *measure_of* Ω A μ
assumes *ms*: $A \subseteq$ *Pow* Ω *positive* (*sets M*) μ *countably_additive* (*sets M*) μ
assumes *X*: $X \in$ *sets M*
shows *emeasure M X* = μ X
definition *measurable* :: *'a measure* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) *set*
(**infixr** ‹$\rightarrow_M$› *60*) **where**
measurable A B = {$f \in$ *space A* → *space B*. $\forall y \in$ *sets B*. f −‘ y ∩ *space A* ∈ *sets*

A}
definition *count_space* :: *'a set* ⇒ *'a measure* **where**
count_space Ω = *measure_of* Ω (*Pow* Ω) (λ*A*. *if finite A then of_nat* (*card A*) *else* ∞)

8.1.3 The smallest σ-algebra regarding a function

definition *vimage_algebra* :: *'a set* ⇒ (*'a* ⇒ *'b*) ⇒ *'b measure* ⇒ *'a measure* **where**
vimage_algebra X f M = *sigma X* {*f* −‘ *A* ∩ *X* | *A*. *A* ∈ *sets M*}

end

8.2 Measurability Prover

theory *Measurable*
imports
Sigma_Algebra
HOL−Library.Order_Continuity
begin

method_setup *measurable* = ‹ *Scan.lift* (*Scan.succeed* (*METHOD o Measurable.measurable_tac*)) ›
measurability prover

simproc_setup *measurable* (*A* ∈ *sets M* | *f* ∈ *measurable M N*) =
‹*K Measurable.proc*›

end

8.3 Measure Spaces

theory *Measure_Space*
imports
Measurable HOL−Library.Extended_Nonnegative_Real
begin

8.3.1 μ-null sets

definition *null_sets* :: *'a measure* ⇒ *'a set set* **where**
null_sets M = {*N*∈*sets M*. *emeasure M N* = *0*}

8.3.2 The almost everywhere filter (i.e. quantifier)

definition *ae_filter* :: *'a measure* ⇒ *'a filter* **where**
ae_filter M = (*INF N*∈*null_sets M*. *principal* (*space M* − *N*))

8.3.3 σ-finite Measures

locale *sigma_finite_measure* =
fixes M :: *'a measure*
assumes *sigma_finite_countable*:
$\exists A{::}'a\ set\ set.\ countable\ A \wedge A \subseteq sets\ M \wedge (\bigcup A) = space\ M \wedge (\forall a{\in}A.\ emeasure\ M\ a \neq \infty)$

8.3.4 Measure space induced by distribution of $(\rightarrow_M)$-functions

definition *distr* :: *'a measure* ⇒ *'b measure* ⇒ (*'a* ⇒ *'b*) ⇒ *'b measure* **where**
$distr\ M\ N\ f =$
$measure\_of\ (space\ N)\ (sets\ N)\ (\lambda A.\ emeasure\ M\ (f - \text{`}\ A \cap space\ M))$

proposition *distr_distr*:
$g \in measurable\ N\ L \Longrightarrow f \in measurable\ M\ N \Longrightarrow distr\ (distr\ M\ N\ f)\ L\ g = distr\ M\ L\ (g \circ f)$

8.3.5 Set of measurable sets with finite measure

definition *fmeasurable* :: *'a measure* ⇒ *'a set set* **where**
$fmeasurable\ M = \{A{\in}sets\ M.\ emeasure\ M\ A < \infty\}$

8.3.6 Measure spaces with $emeasure\ M\ (space\ M) < \infty$

locale *finite_measure* = *sigma_finite_measure* M **for** M +
assumes *finite_emeasure_space*: $emeasure\ M\ (space\ M) \neq top$

8.3.7 Scaling a measure

definition *scale_measure* :: *ennreal* ⇒ *'a measure* ⇒ *'a measure* **where**
$scale\_measure\ r\ M = measure\_of\ (space\ M)\ (sets\ M)\ (\lambda A.\ r * emeasure\ M\ A)$

8.3.8 Complete lattice structure on measures

proposition *unsigned_Hahn_decomposition*:
assumes [*simp*]: $sets\ N = sets\ M$ **and** [*measurable*]: $A \in sets\ M$
and [*simp*]: $emeasure\ M\ A \neq top\ emeasure\ N\ A \neq top$
shows $\exists Y{\in}sets\ M.\ Y \subseteq A \wedge (\forall X{\in}sets\ M.\ X \subseteq Y \longrightarrow N\ X \leq M\ X) \wedge (\forall X{\in}sets\ M.\ X \subseteq A \longrightarrow X \cap Y = \{\} \longrightarrow M\ X \leq N\ X)$

Define a lexicographical order on *measure*, in the order space, sets and measure. The parts of the lexicographical order are point-wise ordered.

instantiation *measure* :: (*type*) *order_bot*
begin

definition *less_measure* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
less_measure M N ⟷ (*M* ≤ *N* ∧ ¬ *N* ≤ *M*)

definition *bot_measure* :: *'a measure* **where**
bot_measure = *sigma* {} {}

proposition *le_measure*: *sets M* = *sets N* ⟹ *M* ≤ *N* ⟷ (∀ *A*∈*sets M*. *emeasure M A* ≤ *emeasure N A*)

definition *sup_measure'* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
sup_measure' A B =
measure_of (*space A*) (*sets A*)
(λ*X*. *SUP Y*∈*sets A*. *emeasure A* (*X* ∩ *Y*) + *emeasure B* (*X* ∩ − *Y*))

definition *sup_lexord* :: *'a* ⇒ *'a* ⇒ (*'a* ⇒ *'b*::*order*) ⇒ *'a* ⇒ *'a* ⇒ *'a* **where**
sup_lexord A B k s c =
(*if k A* = *k B then c else*
if ¬ *k A* ≤ *k B* ∧ ¬ *k B* ≤ *k A then s else*
if k B ≤ *k A then A else B*)

instantiation *measure* :: (*type*) *semilattice_sup*
begin

definition *sup_measure* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
sup_measure A B =
sup_lexord A B space (*sigma* (*space A* ∪ *space B*) {})
(*sup_lexord A B sets* (*sigma* (*space A*) (*sets A* ∪ *sets B*)) (*sup_measure' A B*))

definition
Sup_lexord :: (*'a* ⇒ *'b*::*complete_lattice*) ⇒ (*'a set* ⇒ *'a*) ⇒ (*'a set* ⇒ *'a*) ⇒ *'a set* ⇒ *'a*
where
Sup_lexord k c s A =
(*let U* = (*SUP a*∈*A*. *k a*)
in if ∃ *a*∈*A*. *k a* = *U then c* {*a*∈*A*. *k a* = *U*} *else s A*)

instantiation *measure* :: (*type*) *complete_lattice*
begin

definition *Sup_measure'* :: *'a measure set* ⇒ *'a measure* **where**
Sup_measure' M =
measure_of (⋃ *a*∈*M*. *space a*) (⋃ *a*∈*M*. *sets a*)
(λ*X*. (*SUP P*∈{*P*. *finite P* ∧ *P* ⊆ *M* }. *sup_measure.F id P X*))

definition *Sup_measure* :: *'a measure set* ⇒ *'a measure* **where**
Sup_measure =
  Sup_lexord space
    (*Sup_lexord sets Sup_measure'*
      (λU. *sigma* ($\bigcup u \in U$. *space u*) ($\bigcup u \in U$. *sets u*)))
    (λU. *sigma* ($\bigcup u \in U$. *space u*) {})

definition *Inf_measure* :: *'a measure set* ⇒ *'a measure* **where**
  Inf_measure A = *Sup* {*x*. ∀ *a*∈*A*. *x* ≤ *a*}

definition *inf_measure* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure* **where**
  inf_measure a b = *Inf* {*a*, *b*}

definition *top_measure* :: *'a measure* **where**
  top_measure = *Inf* {}

end

8.4 Borel Space

theory *Borel_Space*
imports
  Measurable Derivative Ordered_Euclidean_Space Extended_Real_Limits
begin

proposition *open_prod_generated*: *open* = *generate_topology* {*A* × *B* | *A B*. *open A* ∧ *open B*}

proposition *mono_on_imp_deriv_nonneg*:
  assumes *mono*: *mono_on A f* **and** *deriv*: (*f has_real_derivative D*) (*at x*)
  assumes *x* ∈ *interior A*
  shows *D* ≥ *0*

proposition *mono_on_ctble_discont*:
  fixes *f* :: *real* ⇒ *real*
  fixes *A* :: *real set*
  assumes *mono_on A f*
  shows *countable* {*a*∈*A*. ¬ *continuous* (*at a within A*) *f*}

8.4.1 Generic Borel spaces

definition (**in** *topological_space*) *borel* :: *'a measure* **where**
  borel = *sigma UNIV* {*S*. *open S*}

theorem *second_countable_borel_measurable*:
  fixes X :: *'a::second_countable_topology set set*
  assumes *eq*: *open = generate_topology X*
  shows *borel = sigma UNIV X*

proposition *borel_eq_countable_basis*:
  fixes *B::'a::topological_space set set*
  assumes *countable B*
  assumes *topological_basis B*
  shows *borel = sigma UNIV B*

8.4.2 Borel spaces on order topologies

8.4.3 Borel spaces on topological monoids

8.4.4 Borel spaces on Euclidean spaces

8.4.5 Borel measurable operators

lemma *borel_measurable_complex_iff*:
  $f \in$ *borel_measurable M* $\longleftrightarrow$
    $(\lambda x.$ *Re* $(f\ x)) \in$ *borel_measurable M* $\wedge$ $(\lambda x.$ *Im* $(f\ x)) \in$ *borel_measurable M*
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)

8.4.6 Borel space on the extended reals

theorem *borel_measurable_ereal_iff_real*:
  fixes f :: *'a* $\Rightarrow$ *ereal*
  shows $f \in$ *borel_measurable M* $\longleftrightarrow$
    $((\lambda x.$ *real_of_ereal* $(f\ x)) \in$ *borel_measurable M* $\wedge f -` \{\infty\} \cap$ *space M* $\in$ *sets M* $\wedge f -` \{-\infty\} \cap$ *space M* $\in$ *sets M*)

8.4.7 Borel space on the extended non-negative reals

definition [*simp*]: *is_borel f M* $\longleftrightarrow f \in$ *borel_measurable M*

8.4.8 LIMSEQ is borel measurable

proposition *measurable_limit* [*measurable*]:
  fixes *f::nat* $\Rightarrow$ *'a* $\Rightarrow$ *'b::first_countable_topology*
  assumes [*measurable*]: $\bigwedge n$*::nat*. $f\ n \in$ *borel_measurable M*
  shows *Measurable.pred M* $(\lambda x. (\lambda n. f\ n\ x) \longrightarrow c)$

end

8.5 Lebesgue Integration for Nonnegative Functions

theory *Nonnegative_Lebesgue_Integration*
imports *Measure_Space Borel_Space*
begin

8.5.1 Simple function

definition *simple_function M g* $\longleftrightarrow$
$finite\ (g\ `\ space\ M) \wedge$
$(\forall x \in g\ `\ space\ M.\ g -` \{x\} \cap space\ M \in sets\ M)$

lemma *borel_measurable_implies_simple_function_sequence*:
fixes $u :: 'a \Rightarrow ennreal$
assumes $u[measurable]$: $u \in borel\_measurable\ M$
shows $\exists f.\ incseq\ f \wedge (\forall i.\ (\forall x.\ f\ i\ x < top) \wedge simple\_function\ M\ (f\ i)) \wedge u = (SUP\ i.\ f\ i)$

lemma *simple_function_induct*
$[consumes\ 1,\ case\_names\ cong\ set\ mult\ add,\ induct\ set:\ simple\_function]$:
fixes $u :: 'a \Rightarrow ennreal$
assumes u: $simple\_function\ M\ u$
assumes $cong$: $\bigwedge f\ g.\ simple\_function\ M\ f \Longrightarrow simple\_function\ M\ g \Longrightarrow (AE\ x\ in\ M.\ f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
assumes set: $\bigwedge A.\ A \in sets\ M \Longrightarrow P\ (indicator\ A)$
assumes $mult$: $\bigwedge u\ c.\ P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$
assumes add: $\bigwedge u\ v.\ P\ u \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ v\ x + u\ x)$
shows $P\ u$

lemma *borel_measurable_induct*
$[consumes\ 1,\ case\_names\ cong\ set\ mult\ add\ seq,\ induct\ set:\ borel\_measurable]$:
fixes $u :: 'a \Rightarrow ennreal$
assumes u: $u \in borel\_measurable\ M$
assumes $cong$: $\bigwedge f\ g.\ f \in borel\_measurable\ M \Longrightarrow g \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x) \Longrightarrow P\ g \Longrightarrow P\ f$
assumes set: $\bigwedge A.\ A \in sets\ M \Longrightarrow P\ (indicator\ A)$
assumes $mult'$: $\bigwedge u\ c.\ c < top \Longrightarrow u \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow u\ x < top) \Longrightarrow P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$
assumes add: $\bigwedge u\ v.\ u \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow u\ x < top) \Longrightarrow P\ u \Longrightarrow v \in borel\_measurable\ M \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow v\ x < top) \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow u\ x = 0 \vee v\ x = 0) \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ v\ x + u\ x)$
assumes seq: $\bigwedge U.\ (\bigwedge i.\ U\ i \in borel\_measurable\ M) \Longrightarrow (\bigwedge i\ x.\ x \in space\ M \Longrightarrow U\ i\ x < top) \Longrightarrow (\bigwedge i.\ P\ (U\ i)) \Longrightarrow incseq\ U \Longrightarrow u = (SUP\ i.\ U\ i) \Longrightarrow P\ (SUP\ i.\ U\ i)$

shows $P\ u$

8.5.2 Simple integral

definition $simple\_integral :: 'a\ measure \Rightarrow ('a \Rightarrow ennreal) \Rightarrow ennreal$ (‹$integral^S$›)
where
$integral^S\ M\, f = (\sum x \in f\ `\ space\ M.\ x * emeasure\ M\ (f\ -`\ \{x\} \cap space\ M))$

8.5.3 Integral on nonnegative functions

definition $nn\_integral :: 'a\ measure \Rightarrow ('a \Rightarrow ennreal) \Rightarrow ennreal$ (‹$integral^N$›)
where
$integral^N\ M\, f = (SUP\ g \in \{g.\ simple\_function\ M\ g \wedge g \le f\}.\ integral^S\ M\ g)$

theorem *nn_integral_monotone_convergence_SUP_AE*:
assumes f: $\bigwedge i.\ AE\ x\ in\ M.\ f\ i\ x \le f\ (Suc\ i)\ x\ \bigwedge i.\ f\ i \in borel\_measurable\ M$
shows $(\int^+ x.\ (SUP\ i.\ f\ i\ x)\ \partial M) = (SUP\ i.\ integral^N\ M\ (f\ i))$

theorem *nn_integral_suminf*:
assumes f: $\bigwedge i.\ f\ i \in borel\_measurable\ M$
shows $(\int^+ x.\ (\sum i.\ f\ i\ x)\ \partial M) = (\sum i.\ integral^N\ M\ (f\ i))$

theorem *nn_integral_Markov_inequality*:
assumes u: $(\lambda x.\ u\ x * indicator\ A\ x) \in borel\_measurable\ M$ **and** $A \in sets\ M$
shows $(emeasure\ M)\ (\{x{\in}A.\ 1 \le c * u\ x\}) \le c * (\int^+ x.\ u\ x * indicator\ A\ x$
$\partial M)$
(**is** $(emeasure\ M)\ ?A \le \_ * ?PI$)

theorem *nn_integral_monotone_convergence_INF_AE*:
fixes $f :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes f: $\bigwedge i.\ AE\ x\ in\ M.\ f\ (Suc\ i)\ x \le f\ i\ x$
and [*measurable*]: $\bigwedge i.\ f\ i \in borel\_measurable\ M$
and *fin*: $(\int^+ x.\ f\ i\ x\ \partial M) < \infty$
shows $(\int^+ x.\ (INF\ i.\ f\ i\ x)\ \partial M) = (INF\ i.\ integral^N\ M\ (f\ i))$

theorem *nn_integral_liminf*:
fixes $u :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes u: $\bigwedge i.\ u\ i \in borel\_measurable\ M$
shows $(\int^+ x.\ liminf\ (\lambda n.\ u\ n\ x)\ \partial M) \le liminf\ (\lambda n.\ integral^N\ M\ (u\ n))$

theorem *nn_integral_limsup*:
fixes $u :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes [*measurable*]: $\bigwedge i.\ u\ i \in borel\_measurable\ M\ w \in borel\_measurable\ M$
assumes *bounds*: $\bigwedge i.\ AE\ x\ in\ M.\ u\ i\ x \le w\ x$ **and** w: $(\int^+ x.\ w\ x\ \partial M) < \infty$
shows $limsup\ (\lambda n.\ integral^N\ M\ (u\ n)) \le (\int^+ x.\ limsup\ (\lambda n.\ u\ n\ x)\ \partial M)$

theorem *nn_integral_dominated_convergence*:
assumes [*measurable*]:

$\bigwedge i.\ u\ i \in borel\_measurable\ M\ u' \in borel\_measurable\ M\ w \in borel\_measurable\ M$
and *bound*: $\bigwedge j.\ AE\ x\ in\ M.\ u\ j\ x \leq w\ x$
and *w*: $(\int^+ x.\ w\ x\ \partial M) < \infty$
and *u'*: $AE\ x\ in\ M.\ (\lambda i.\ u\ i\ x) \longrightarrow u'\ x$
shows $(\lambda i.\ (\int^+ x.\ u\ i\ x\ \partial M)) \longrightarrow (\int^+ x.\ u'\ x\ \partial M)$

theorem *nn_integral_lfp*:
assumes *sets*[*simp*]: $\bigwedge s.\ sets\ (M\ s) = sets\ N$
assumes *f*: *sup_continuous f*
assumes *g*: *sup_continuous g*
assumes *meas*: $\bigwedge F.\ F \in borel\_measurable\ N \Longrightarrow f\ F \in borel\_measurable\ N$
assumes *step*: $\bigwedge F\ s.\ F \in borel\_measurable\ N \Longrightarrow integral^N\ (M\ s)\ (f\ F) = g\ (\lambda s.\ integral^N\ (M\ s)\ F)\ s$
shows $(\int^+ \omega.\ lfp\ f\ \omega\ \partial M\ s) = lfp\ g\ s$

theorem *nn_integral_gfp*:
assumes *sets*[*simp*]: $\bigwedge s.\ sets\ (M\ s) = sets\ N$
assumes *f*: *inf_continuous f* **and** *g*: *inf_continuous g*
assumes *meas*: $\bigwedge F.\ F \in borel\_measurable\ N \Longrightarrow f\ F \in borel\_measurable\ N$
assumes *bound*: $\bigwedge F\ s.\ F \in borel\_measurable\ N \Longrightarrow (\int^+ x.\ f\ F\ x\ \partial M\ s) < \infty$
assumes *non_zero*: $\bigwedge s.\ emeasure\ (M\ s)\ (space\ (M\ s)) \neq 0$
assumes *step*: $\bigwedge F\ s.\ F \in borel\_measurable\ N \Longrightarrow integral^N\ (M\ s)\ (f\ F) = g\ (\lambda s.\ integral^N\ (M\ s)\ F)\ s$
shows $(\int^+ \omega.\ gfp\ f\ \omega\ \partial M\ s) = gfp\ g\ s$

8.5.4 Integral under concrete measures

definition *density* :: $'a\ measure \Rightarrow ('a \Rightarrow ennreal) \Rightarrow 'a\ measure$ **where**
$density\ M\ f = measure\_of\ (space\ M)\ (sets\ M)\ (\lambda A.\ \int^+ x.\ f\ x * indicator\ A\ x\ \partial M)$

lemma *nn_integral_density*:
assumes *f*: $f \in borel\_measurable\ M$
assumes *g*: $g \in borel\_measurable\ M$
shows $integral^N\ (density\ M\ f)\ g = (\int^+ x.\ f\ x * g\ x\ \partial M)$
definition *point_measure* :: $'a\ set \Rightarrow ('a \Rightarrow ennreal) \Rightarrow 'a\ measure$ **where**
$point\_measure\ A\ f = density\ (count\_space\ A)\ f$
definition $uniform\_measure\ M\ A = density\ M\ (\lambda x.\ indicator\ A\ x\ /\ emeasure\ M\ A)$
definition $uniform\_count\_measure\ A = point\_measure\ A\ (\lambda x.\ 1\ /\ card\ A)$

end

8.6 Binary Product Measure

theory *Binary_Product_Measure*

imports *Nonnegative_Lebesgue_Integration*
begin

8.6.1 Binary products

definition *pair_measure* (**infixr** ‹$\bigotimes_M$› *80*) **where**
$A \bigotimes_M B$ = *measure_of* (*space* $A \times$ *space* B)
{$a \times b$ | a b. $a \in$ *sets* $A \wedge b \in$ *sets* B}
($\lambda X. \int^+ x. (\int^+ y.$ *indicator* X (x,y) $\partial B)$ ∂A)

proposition (**in** *sigma_finite_measure*) *emeasure_pair_measure_Times*:
assumes A: $A \in$ *sets* N **and** B: $B \in$ *sets* M
shows *emeasure* $(N \bigotimes_M M)$ $(A \times B)$ = *emeasure* N A $*$ *emeasure* M B

8.6.2 Binary products of σ-finite emeasure spaces

proposition (**in** *pair_sigma_finite*) *sigma_finite_up_in_pair_measure_generator*:
defines $E \equiv$ {$A \times B$ | A B. $A \in$ *sets M1* $\wedge$ $B \in$ *sets M2*}
shows $\exists F$::*nat* $\Rightarrow$ ($'a \times 'b$) *set*. *range* $F \subseteq E \wedge$ *incseq* $F \wedge (\bigcup i. F\ i)$ = *space M1* $\times$ *space M2* $\wedge$
($\forall i.$ *emeasure* (*M1* $\bigotimes_M$ *M2*) $(F\ i) \neq \infty$)

8.6.3 Fubinis theorem

proposition (**in** *pair_sigma_finite*) *nn_integral_snd*:
assumes f[*measurable*]: $f \in$ *borel_measurable* (*M1* $\bigotimes_M$ *M2*)
shows ($\int^+ y. (\int^+ x. f\ (x, y)\ \partial M1)\ \partial M2$) = *integral*^{N} (*M1* $\bigotimes_M$ *M2*) f

theorem (**in** *pair_sigma_finite*) *Fubini*:
assumes f: $f \in$ *borel_measurable* (*M1* $\bigotimes_M$ *M2*)
shows ($\int^+ y. (\int^+ x. f\ (x, y)\ \partial M1)\ \partial M2$) = ($\int^+ x. (\int^+ y. f\ (x, y)\ \partial M2)$ $\partial M1$)

theorem (**in** *pair_sigma_finite*) *Fubini'*:
assumes f: *case_prod* $f \in$ *borel_measurable* (*M1* $\bigotimes_M$ *M2*)
shows ($\int^+ y. (\int^+ x. f\ x\ y\ \partial M1)\ \partial M2$) = ($\int^+ x. (\int^+ y. f\ x\ y\ \partial M2)\ \partial M1$)

8.6.4 Products on counting spaces, densities and distributions

proposition *sigma_prod*:
assumes *X_cover*: $\exists E \subseteq A.$ *countable* $E \wedge X = \bigcup E$ **and** A: $A \subseteq$ *Pow* X
assumes *Y_cover*: $\exists E \subseteq B.$ *countable* $E \wedge Y = \bigcup E$ **and** B: $B \subseteq$ *Pow* Y

shows *sigma X A* $\bigotimes_M$ *sigma Y B* = *sigma* (*X* × *Y*) {*a* × *b* | *a b*. *a* ∈ *A* ∧ *b* ∈ *B*}
(**is** *?P* = *?S*)

proposition *sets_pair_eq*:
assumes *Ea*: *Ea* ⊆ *Pow* (*space A*) *sets A* = *sigma_sets* (*space A*) *Ea*
and *Ca*: *countable Ca Ca* ⊆ *Ea* ⋃ *Ca* = *space A*
and *Eb*: *Eb* ⊆ *Pow* (*space B*) *sets B* = *sigma_sets* (*space B*) *Eb*
and *Cb*: *countable Cb Cb* ⊆ *Eb* ⋃ *Cb* = *space B*
shows *sets* (*A* $\bigotimes_M$ *B*) = *sets* (*sigma* (*space A* × *space B*) { *a* × *b* | *a b*. *a* ∈ *Ea* ∧ *b* ∈ *Eb* })
(**is** _ = *sets* (*sigma ?Ω ?E*))

proposition *borel_prod*:
(*borel* $\bigotimes_M$ *borel*) = (*borel* :: ('*a*::*second_countable_topology* × '*b*::*second_countable_topology*) *measure*)
(**is** *?P* = *?B*)

proposition *pair_measure_count_space*:
assumes *A*: *finite A* **and** *B*: *finite B*
shows *count_space A* $\bigotimes_M$ *count_space B* = *count_space* (*A* × *B*) (**is** *?P* = *?C*)

theorem *pair_measure_density*:
assumes *f*: *f* ∈ *borel_measurable M1*
assumes *g*: *g* ∈ *borel_measurable M2*
assumes *sigma_finite_measure M2 sigma_finite_measure* (*density M2 g*)
shows *density M1 f* $\bigotimes_M$ *density M2 g* = *density* (*M1* $\bigotimes_M$ *M2*) (λ(*x*,*y*). *f x* ∗ *g y*) (**is** *?L* = *?R*)

proposition *nn_integral_fst_count_space*:
($\int^+$ *x*. $\int^+$ *y*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = $integral^N$ (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)

proposition *nn_integral_snd_count_space*:
($\int^+$ *y*. $\int^+$ *x*. *f* (*x*, *y*) ∂*count_space UNIV* ∂*count_space UNIV*) = $integral^N$ (*count_space UNIV*) *f*
(**is** *?lhs* = *?rhs*)

8.6.5 Product of Borel spaces

theorem *borel_Times*:
fixes *A* :: '*a*::*topological_space set* **and** *B* :: '*b*::*topological_space set*
assumes *A*: *A* ∈ *sets borel* **and** *B*: *B* ∈ *sets borel*
shows *A* × *B* ∈ *sets borel*

end

8.7 Finite Product Measure

theory *Finite_Product_Measure*
imports *Binary_Product_Measure Function_Topology*
begin

8.7.1 Finite product spaces

definition *prod_emb* **where**
$prod\_emb\ I\ M\ K\ X = (\lambda x.\ restrict\ x\ K) -` X \cap (\Pi_E\ i{\in}I.\ space\ (M\ i))$

definition *PiM* :: $'i\ set \Rightarrow ('i \Rightarrow 'a\ measure) \Rightarrow ('i \Rightarrow 'a)\ measure$ **where**
$PiM\ I\ M = extend\_measure\ (\Pi_E\ i{\in}I.\ space\ (M\ i))$
$\{(J,\ X).\ (J \neq \{\} \vee I = \{\}) \wedge finite\ J \wedge J \subseteq I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$
$(\lambda(J,\ X).\ prod\_emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ X\ j))$
$(\lambda(J,\ X).\ \prod j{\in}J \cup \{i{\in}I.\ emeasure\ (M\ i)\ (space\ (M\ i)) \neq 1\}.\ if\ j \in J\ then\ emeasure\ (M\ j)\ (X\ j)\ else\ emeasure\ (M\ j)\ (space\ (M\ j)))$

definition *prod_algebra* :: $'i\ set \Rightarrow ('i \Rightarrow 'a\ measure) \Rightarrow ('i \Rightarrow 'a)\ set\ set$ **where**
$prod\_algebra\ I\ M = (\lambda(J,\ X).\ prod\_emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ X\ j))$ `
$\{(J,\ X).\ (J \neq \{\} \vee I = \{\}) \wedge finite\ J \wedge J \subseteq I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$

proposition *prod_algebra_mono*:
assumes *space*: $\bigwedge i.\ i \in I \Longrightarrow space\ (E\ i) = space\ (F\ i)$
assumes *sets*: $\bigwedge i.\ i \in I \Longrightarrow sets\ (E\ i) \subseteq sets\ (F\ i)$
shows $prod\_algebra\ I\ E \subseteq prod\_algebra\ I\ F$
proposition *prod_algebra_cong*:
assumes $I = J$ **and** $(\bigwedge i.\ i \in I \Longrightarrow sets\ (M\ i) = sets\ (N\ i))$
shows $prod\_algebra\ I\ M = prod\_algebra\ J\ N$

proposition *sets_PiM_single*: $sets\ (PiM\ I\ M) =$
$sigma\_sets\ (\Pi_E\ i{\in}I.\ space\ (M\ i))\ \{\{f{\in}\Pi_E\ i{\in}I.\ space\ (M\ i).\ f\ i \in A\} \mid i\ A.\ i \in I \wedge A \in sets\ (M\ i)\}$
(**is** $\_ = sigma\_sets\ ?\Omega\ ?R$)

proposition *sets_PiM_sigma*:
assumes *Ω_cover*: $\bigwedge i.\ i \in I \Longrightarrow \exists S{\subseteq}E\ i.\ countable\ S \wedge \Omega\ i = \bigcup S$
assumes *E*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq Pow\ (\Omega\ i)$
assumes *J*: $\bigwedge j.\ j \in J \Longrightarrow finite\ j\ \bigcup J = I$
defines $P \equiv \{\{f{\in}(\Pi_E\ i{\in}I.\ \Omega\ i).\ \forall i{\in}j.\ f\ i \in A\ i\} \mid A\ j.\ j \in J \wedge A \in Pi\ j\ E\}$
shows $sets\ (\Pi_M\ i{\in}I.\ sigma\ (\Omega\ i)\ (E\ i)) = sets\ (sigma\ (\Pi_E\ i{\in}I.\ \Omega\ i)\ P)$

proposition *measurable_PiM*:
assumes *space*: $f \in space\ N \to (\Pi_E\ i{\in}I.\ space\ (M\ i))$
assumes *sets*: $\bigwedge X\ J.\ J \neq \{\} \vee I = \{\} \Longrightarrow finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J$

$\Longrightarrow X\ i \in sets\ (M\ i)) \Longrightarrow$
$f\ -`\ prod\_emb\ I\ M\ J\ (Pi_E\ J\ X) \cap space\ N \in sets\ N$
shows $f \in measurable\ N\ (PiM\ I\ M)$

proposition *measurable_fun_upd*:
assumes I: $I = J \cup \{i\}$
assumes $f[measurable]$: $f \in measurable\ N\ (PiM\ J\ M)$
assumes $h[measurable]$: $h \in measurable\ N\ (M\ i)$
shows $(\lambda x.\ (f\ x)\ (i := h\ x)) \in measurable\ N\ (PiM\ I\ M)$

proposition *measure_eqI_PiM_finite*:
assumes $[simp]$: $finite\ I\ sets\ P = PiM\ I\ M\ sets\ Q = PiM\ I\ M$
assumes eq: $\bigwedge A.\ (\bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow P\ (Pi_E\ I\ A) = Q\ (Pi_E\ I\ A)$
assumes A: $range\ A \subseteq prod\_algebra\ I\ M\ (\bigcup i.\ A\ i) = space\ (PiM\ I\ M)\ \bigwedge i{::}nat.\ P\ (A\ i) \neq \infty$
shows $P = Q$

proposition *measure_eqI_PiM_infinite*:
assumes $[simp]$: $sets\ P = PiM\ I\ M\ sets\ Q = PiM\ I\ M$
assumes eq: $\bigwedge A\ J.\ finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow$
$P\ (prod\_emb\ I\ M\ J\ (Pi_E\ J\ A)) = Q\ (prod\_emb\ I\ M\ J\ (Pi_E\ J\ A))$
assumes A: $finite\_measure\ P$
shows $P = Q$

proposition (**in** *finite_product_sigma_finite*) *sigma_finite_pairs*:
$\exists F{::}'i \Rightarrow nat \Rightarrow 'a\ set.$
$(\forall i{\in}I.\ range\ (F\ i) \subseteq sets\ (M\ i)) \wedge$
$(\forall k.\ \forall i{\in}I.\ emeasure\ (M\ i)\ (F\ i\ k) \neq \infty) \wedge incseq\ (\lambda k.\ \Pi_E\ i{\in}I.\ F\ i\ k) \wedge$
$(\bigcup k.\ \Pi_E\ i{\in}I.\ F\ i\ k) = space\ (PiM\ I\ M)$

lemma (**in** *product_sigma_finite*) *distr_merge*:
assumes $IJ[simp]$: $I \cap J = \{\}$ **and** fin: $finite\ I\ finite\ J$
shows $distr\ (Pi_M\ I\ M \bigotimes_M Pi_M\ J\ M)\ (Pi_M\ (I \cup J)\ M)\ (merge\ I\ J) = Pi_M\ (I \cup J)\ M$
(**is** $?D = ?P$)

proposition (**in** *product_sigma_finite*) *product_nn_integral_fold*:
assumes IJ: $I \cap J = \{\}\ finite\ I\ finite\ J$
and $f[measurable]$: $f \in borel\_measurable\ (Pi_M\ (I \cup J)\ M)$
shows $integral^N\ (Pi_M\ (I \cup J)\ M)\ f = (\int^+ x.\ (\int^+ y.\ f\ (merge\ I\ J\ (x,\ y))\ \partial(Pi_M\ J\ M))\ \partial(Pi_M\ I\ M))$
(**is** $?lhs = ?rhs$)

proposition (**in** *product_sigma_finite*) *product_nn_integral_insert*:
assumes $I[simp]$: $finite\ I\ i \notin I$
and f: $f \in borel\_measurable\ (Pi_M\ (insert\ i\ I)\ M)$
shows $integral^N\ (Pi_M\ (insert\ i\ I)\ M)\ f = (\int^+ x.\ (\int^+ y.\ f\ (x(i := y))\ \partial(M\ i))$

$\partial(Pi_M\ I\ M))$

proposition (**in** *product_sigma_finite*) *product_nn_integral_pair*:
assumes [*measurable*]: *case_prod f* $\in$ *borel_measurable* $(M\ x \bigotimes_M M\ y)$
assumes *xy*: $x \neq y$
shows $(\int^+ \sigma.\ f\ (\sigma\ x)\ (\sigma\ y)\ \partial PiM\ \{x,\ y\}\ M) = (\int^+ z.\ f\ (fst\ z)\ (snd\ z)\ \partial(M\ x \bigotimes_M M\ y))$

8.7.2 Measurability

proposition *sets_PiM_equal_borel*:
sets $(Pi_M\ UNIV\ (\lambda i{::}('a{::}countable).\ borel{::}('b{::}second\_countable\_topology\ measure))) = sets\ borel$

end

8.8 Caratheodory Extension Theorem

theory *Caratheodory*
imports *Measure_Space*
begin

8.8.1 Characterizations of Measures

definition *outer_measure_space* **where**
outer_measure_space M f $\longleftrightarrow$ *positive M f* $\wedge$ *increasing M f* $\wedge$ *countably_subadditive M f*

Lambda Systems

definition *lambda_system* :: $'a\ set \Rightarrow 'a\ set\ set \Rightarrow ('a\ set \Rightarrow ennreal) \Rightarrow 'a\ set\ set$
where
lambda_system $\Omega\ M\ f = \{l \in M.\ \forall x \in M.\ f\ (l \cap x) + f\ ((\Omega - l) \cap x) = f\ x\}$

proposition (**in** *sigma_algebra*) *lambda_system_caratheodory*:
assumes *oms*: *outer_measure_space M f*
and *A*: *range A* $\subseteq$ *lambda_system* Ω *M f*
and *disj*: *disjoint_family A*
shows $(\bigcup i.\ A\ i) \in lambda\_system\ \Omega\ M\ f \wedge (\sum i.\ f\ (A\ i)) = f\ (\bigcup i.\ A\ i)$

proposition (**in** *sigma_algebra*) *caratheodory_lemma*:
assumes *oms*: *outer_measure_space M f*
defines $L \equiv lambda\_system\ \Omega\ M\ f$
shows *measure_space* Ω *L f*

definition *outer_measure* :: $'a$ *set set* ⇒ ($'a$ *set* ⇒ *ennreal*) ⇒ $'a$ *set* ⇒ *ennreal*
where
outer_measure M f X =
(*INF A*∈{*A*. *range A* ⊆ *M* ∧ *disjoint_family A* ∧ *X* ⊆ (⋃*i*. *A i*)}. ∑*i*. *f* (*A i*))

8.8.2 Caratheodory's theorem

theorem (**in** *ring_of_sets*) *caratheodory'*:
assumes *posf*: *positive M f* **and** *ca*: *countably_additive M f*
shows ∃ *μ* :: $'a$ *set* ⇒ *ennreal*. (∀ *s* ∈ *M*. *μ s* = *f s*) ∧ *measure_space* Ω (*sigma_sets* Ω *M*) *μ*

8.8.3 Volumes

definition *volume* :: $'a$ *set set* ⇒ ($'a$ *set* ⇒ *ennreal*) ⇒ *bool* **where**
volume M f ⟷
(*f* {} = *0*) ∧ (∀ *a*∈*M*. *0* ≤ *f a*) ∧
(∀ *C*⊆*M*. *disjoint C* ⟶ *finite C* ⟶ ⋃ *C* ∈ *M* ⟶ *f* (⋃ *C*) = (∑ *c*∈*C*. *f c*))

proposition *volume_finite_additive*:
assumes *volume M f*
assumes *A*: ⋀*i*. *i* ∈ *I* ⟹ *A i* ∈ *M disjoint_family_on A I finite I* ⋃(*A* ‘ *I*) ∈ *M*
shows *f* (⋃(*A* ‘ *I*)) = (∑ *i*∈*I*. *f* (*A i*))

proposition (**in** *semiring_of_sets*) *extend_volume*:
assumes *volume M μ*
shows ∃ *μ'*. *volume generated_ring μ'* ∧ (∀ *a*∈*M*. *μ' a* = *μ a*)

Caratheodory on semirings

theorem (**in** *semiring_of_sets*) *caratheodory*:
assumes *pos*: *positive M μ* **and** *ca*: *countably_additive M μ*
shows ∃ *μ'* :: $'a$ *set* ⇒ *ennreal*. (∀ *s* ∈ *M*. *μ' s* = *μ s*) ∧ *measure_space* Ω (*sigma_sets* Ω *M*) *μ'*

proposition *extend_measure_caratheodory_pair*:
fixes *G* :: $'i$ ⇒ $'j$ ⇒ $'a$ *set*
assumes *M*: *M* = *extend_measure* Ω {(*a*, *b*). *P a b*} (λ(*a*, *b*). *G a b*) (λ(*a*, *b*). *μ a b*)
assumes *P i j*
assumes *semiring*: *semiring_of_sets* Ω {*G a b* | *a b*. *P a b*}
assumes *empty*: ⋀*i j*. *P i j* ⟹ *G i j* = {} ⟹ *μ i j* = *0*
assumes *inj*: ⋀*i j k l*. *P i j* ⟹ *P k l* ⟹ *G i j* = *G k l* ⟹ *μ i j* = *μ k l*
assumes *nonneg*: ⋀*i j*. *P i j* ⟹ *0* ≤ *μ i j*
assumes *add*: ⋀*A*::*nat* ⇒ $'i$. ⋀*B*::*nat* ⇒ $'j$. ⋀*j k*.

$(\bigwedge n.\ P\ (A\ n)\ (B\ n)) \Longrightarrow P\ j\ k \Longrightarrow$ *disjoint_family* $(\lambda n.\ G\ (A\ n)\ (B\ n)) \Longrightarrow$
$(\bigcup i.\ G\ (A\ i)\ (B\ i)) = G\ j\ k \Longrightarrow (\sum n.\ \mu\ (A\ n)\ (B\ n)) = \mu\ j\ k$
shows *emeasure M* $(G\ i\ j) = \mu\ i\ j$

end

8.9 Bochner Integration for Vector-Valued Functions

theory *Bochner_Integration*
imports *Finite_Product_Measure*
beginproposition *borel_measurable_implies_sequence_metric*:
fixes *f* :: *'a* ⇒ *'b* :: {*metric_space*, *second_countable_topology*}
assumes [*measurable*]: *f* ∈ *borel_measurable M*
shows ∃ *F*. (∀ *i*. *simple_function M* (*F i*)) ∧ (∀ *x*∈*space M*. (λ*i*. *F i x*) ⟶ *f x*) ∧
(∀ *i*. ∀ *x*∈*space M*. *dist* (*F i x*) *z* ≤ *2* ∗ *dist* (*f x*) *z*)

definition *simple_bochner_integral* :: *'a measure* ⇒ (*'a* ⇒ *'b*::*real_vector*) ⇒ *'b*
where
simple_bochner_integral M f = (∑ *y*∈*f*‘*space M*. *measure M* {*x*∈*space M*. *f x* = *y*} $*_R$ *y*)

proposition *simple_bochner_integral_partition*:
assumes *f*: *simple_bochner_integrable M f* **and** *g*: *simple_function M g*
assumes *sub*: ⋀*x y*. *x* ∈ *space M* ⟹ *y* ∈ *space M* ⟹ *g x* = *g y* ⟹ *f x* = *f y*
assumes *v*: ⋀*x*. *x* ∈ *space M* ⟹ *f x* = *v* (*g x*)
shows *simple_bochner_integral M f* = (∑ *y*∈*g* ‘ *space M*. *measure M* {*x*∈*space M*. *g x* = *y*} $*_R$ *v y*)
(**is** _ = *?r*)

proposition *has_bochner_integral_implies_finite_norm*:
has_bochner_integral M f x ⟹ (∫⁺*x*. *norm* (*f x*) ∂*M*) < ∞

proposition *has_bochner_integral_norm_bound*:
assumes *i*: *has_bochner_integral M f x*
shows *norm x* ≤ (∫⁺*x*. *norm* (*f x*) ∂*M*)

definition *lebesgue_integral* (‹$integral^L$›) **where**
$integral^L$ *M f* = (*if* ∃ *x*. *has_bochner_integral M f x then THE x*. *has_bochner_integral M f x else 0*)

proposition *nn_integral_dominated_convergence_norm*:
fixes *u'* :: _ ⇒ _::{*real_normed_vector*, *second_countable_topology*}
assumes [*measurable*]:
⋀*i*. *u i* ∈ *borel_measurable M u'* ∈ *borel_measurable M w* ∈ *borel_measurable M*
and *bound*: ⋀*j*. *AE x in M*. *norm* (*u j x*) ≤ *w x*

and w: $(\int^{+}x.\ w\ x\ \partial M) < \infty$
and u': $AE\ x\ in\ M.\ (\lambda i.\ u\ i\ x) \longrightarrow u'\ x$
shows $(\lambda i.\ (\int^{+}x.\ norm\ (u'\ x - u\ i\ x)\ \partial M)) \longrightarrow 0$

proposition *integrableI_bounded*:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\_countable\_topology\}$
assumes $f[measurable]$: $f \in borel\_measurable\ M$ **and** *fin*: $(\int^{+}x.\ norm\ (f\ x)\ \partial M) < \infty$
shows *integrable M f*

proposition *nn_integral_eq_integral*:
assumes f: *integrable M f*
assumes *nonneg*: $AE\ x\ in\ M.\ 0 \leq f\ x$
shows $(\int^{+}\ x.\ f\ x\ \partial M) = integral^{L}\ M\ f$

proposition *integral_norm_bound* [*simp*]:
fixes $f :: \_ \Rightarrow 'a :: \{banach,\ second\_countable\_topology\}$
shows $norm\ (integral^{L}\ M\ f) \leq (\int x.\ norm\ (f\ x)\ \partial M)$

proposition *integral_abs_bound* [*simp*]:
fixes $f :: 'a \Rightarrow real$ **shows** $abs\ (\int x.\ f\ x\ \partial M) \leq (\int x.\ |f\ x|\ \partial M)$

proposition *integrable_induct*[*consumes 1*, *case_names base add lim*, *induct pred*: *integrable*]:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\_countable\_topology\}$
assumes *integrable M f*
assumes *base*: $\bigwedge A\ c.\ A \in sets\ M \Longrightarrow emeasure\ M\ A < \infty \Longrightarrow P\ (\lambda x.\ indicator\ A\ x *_R c)$
assumes *add*: $\bigwedge f\ g.\ integrable\ M\ f \Longrightarrow P\ f \Longrightarrow integrable\ M\ g \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
assumes *lim*: $\bigwedge f\ s.\ (\bigwedge i.\ integrable\ M\ (s\ i)) \Longrightarrow (\bigwedge i.\ P\ (s\ i)) \Longrightarrow$
$(\bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ s\ i\ x) \longrightarrow f\ x) \Longrightarrow$
$(\bigwedge i\ x.\ x \in space\ M \Longrightarrow norm\ (s\ i\ x) \leq 2 * norm\ (f\ x)) \Longrightarrow integrable\ M\ f \Longrightarrow P\ f$
shows *P f*

theorem *integral_Markov_inequality*:
assumes [*measurable*]: *integrable M u* **and** $AE\ x\ in\ M.\ 0 \leq u\ x\ 0 < (c::real)$
shows $(emeasure\ M)\ \{x \in space\ M.\ u\ x \geq c\} \leq (1/c) * (\int x.\ u\ x\ \partial M)$

theorem *integral_Markov_inequality_measure*:
assumes [*measurable*]: *integrable M u* **and** $A \in sets\ M$ **and** $AE\ x\ in\ M.\ 0 \leq u\ x\ 0 < (c::real)$
shows $measure\ M\ \{x \in space\ M.\ u\ x \geq c\} \leq (\int x.\ u\ x\ \partial M)\ /\ c$

theorem (**in** *finite_measure*) *second_moment_method*:
assumes [*measurable*]: $f \in M \rightarrow_M borel$
assumes $integrable\ M\ (\lambda x.\ f\ x\ \hat{}\ 2)$

defines $\mu \equiv$ *lebesgue_integral M f*
assumes $a > 0$
shows *measure M* $\{x\in space\ M.\ |f\ x| \geq a\} \leq$ *lebesgue_integral M* $(\lambda x.\ f\ x\ \hat{}\ 2)$ / a^2
proof −
have *integrable*: *integrable M f*
using *assms* **by** (*blast dest*: *square_integrable_imp_integrable*)
have $\{x\in space\ M.\ |f\ x| \geq a\} = \{x\in space\ M.\ f\ x\ \hat{}\ 2 \geq a^2\}$
using ‹$a > 0$› **by** (*simp flip*: *abs_le_square_iff*)
hence *measure M* $\{x\in space\ M.\ |f\ x| \geq a\}$ = *measure M* $\{x\in space\ M.\ f\ x\ \hat{}\ 2 \geq a^2\}$
by *simp*
also have ... $\leq$ *lebesgue_integral M* $(\lambda x.\ f\ x\ \hat{}\ 2)$ / a^2
using *assms* **by** (*intro integral_Markov_inequality_measure*) *auto*
finally show *?thesis* **.**
qed

proposition *tendsto_L1_int*:
fixes *u* :: _ ⇒ _ ⇒ ′*b*::{*banach*, *second_countable_topology*}
assumes [*measurable*]: $\bigwedge n.$ *integrable M* (*u n*) *integrable M f*
and $((\lambda n.\ (\int^+ x.\ norm(u\ n\ x - f\ x)\ \partial M)) \longrightarrow 0)$ *F*
shows $((\lambda n.\ (\int x.\ u\ n\ x\ \partial M)) \longrightarrow (\int x.\ f\ x\ \partial M))$ *F*
proposition *tendsto_L1_AE_subseq*:
fixes *u* :: *nat* ⇒ ′*a* ⇒ ′*b*::{*banach*, *second_countable_topology*}
assumes [*measurable*]: $\bigwedge n.$ *integrable M* (*u n*)
and $(\lambda n.\ (\int x.\ norm(u\ n\ x)\ \partial M)) \longrightarrow 0$
shows ∃ *r*::*nat*⇒*nat*. *strict_mono r* ∧ (*AE x in M*. $(\lambda n.\ u\ (r\ n)\ x) \longrightarrow 0$)

8.9.1 Restricted measure spaces

8.9.2 Measure spaces with an associated density

8.9.3 Distributions

8.9.4 Lebesgue integration on *count_space*

8.9.5 Point measure

proposition *integrable_point_measure_finite*:
fixes *g* :: ′*a* ⇒ ′*b*::{*banach*, *second_countable_topology*} **and** *f* :: ′*a* ⇒ *real*
assumes *finite A*
shows *integrable* (*point_measure A f*) *g*

8.9.6 Lebesgue integration on *null_measure*

8.9.7 Legacy lemmas for the real-valued Lebesgue integral

theorem *real_lebesgue_integral_def*:

assumes *f*[*measurable*]: *integrable M f*
shows *integral*^{L} *M f* = *enn2real* ($\int^+$*x. f x* ∂*M*) − *enn2real* ($\int^+$*x. ennreal* (− *f x*) ∂*M*)

theorem *real_integrable_def*:
integrable M f ⟷ *f* ∈ *borel_measurable M* ∧
($\int^+$ *x. ennreal* (*f x*) ∂*M*) ≠ ∞ ∧ ($\int^+$ *x. ennreal* (− *f x*) ∂*M*) ≠ ∞

8.9.8 Product measure

proposition (**in** *sigma_finite_measure*) *borel_measurable_lebesgue_integral*[*measurable* (*raw*)]:
fixes *f* :: _ ⇒ _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *case_prod f* ∈ *borel_measurable* (*N* ⨂$_M$ *M*)
shows (λ*x*. ∫ *y. f x y* ∂*M*) ∈ *borel_measurable N*

theorem (**in** *pair_sigma_finite*) *Fubini_integrable*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*[*measurable*]: *f* ∈ *borel_measurable* (*M1* ⨂$_M$ *M2*)
and *integ1*: *integrable M1* (λ*x*. ∫ *y. norm* (*f* (*x*, *y*)) ∂*M2*)
and *integ2*: *AE x in M1*. *integrable M2* (λ*y. f* (*x*, *y*))
shows *integrable* (*M1* ⨂$_M$ *M2*) *f*

proposition (**in** *pair_sigma_finite*) *integral_fst'*:
fixes *f* :: _ ⇒ _::{*banach*, *second_countable_topology*}
assumes *f*: *integrable* (*M1* ⨂$_M$ *M2*) *f*
shows (∫ *x*. (∫ *y. f* (*x*, *y*) ∂*M2*) ∂*M1*) = *integral*^{L} (*M1* ⨂$_M$ *M2*) *f*

proposition (**in** *pair_sigma_finite*) *Fubini_integral*:
fixes *f* :: _ ⇒ _ ⇒ _ :: {*banach*, *second_countable_topology*}
assumes *f*: *integrable* (*M1* ⨂$_M$ *M2*) (*case_prod f*)
shows (∫ *y*. (∫ *x. f x y* ∂*M1*) ∂*M2*) = (∫ *x*. (∫ *y. f x y* ∂*M2*) ∂*M1*)

end

8.10 Complete Measures

theory *Complete_Measure*
imports *Bochner_Integration*
begin

locale *complete_measure* =
fixes *M* :: *'a measure*
assumes *complete*: ⋀*A B*. *B* ⊆ *A* ⟹ *A* ∈ *null_sets M* ⟹ *B* ∈ *sets M*

definition
split_completion M A p = (*if A* ∈ *sets M then p* = (*A*, {}) *else*

$\exists N'.\ A = fst\ p \cup snd\ p \wedge fst\ p \cap snd\ p = \{\} \wedge fst\ p \in sets\ M \wedge snd\ p \subseteq N' \wedge N' \in null\_sets\ M)$

definition
$main\_part\ M\ A = fst\ (Eps\ (split\_completion\ M\ A))$

definition
$null\_part\ M\ A = snd\ (Eps\ (split\_completion\ M\ A))$

definition $completion :: 'a\ measure \Rightarrow 'a\ measure$ **where**
$completion\ M = measure\_of\ (space\ M)\ \{\ S \cup N\ |S\ N\ N'.\ S \in sets\ M \wedge N' \in null\_sets\ M \wedge N \subseteq N'\ \}$
$(emeasure\ M \circ main\_part\ M)$

lemma $sets\_completion$:
$sets\ (completion\ M) = \{\ S \cup N\ |S\ N\ N'.\ S \in sets\ M \wedge N' \in null\_sets\ M \wedge N \subseteq N'\ \}$

lemma $measurable\_completion$: $f \in M \rightarrow_M N \Longrightarrow f \in completion\ M \rightarrow_M N$

lemma $split\_completion$:
assumes $A \in sets\ (completion\ M)$
shows $split\_completion\ M\ A\ (main\_part\ M\ A,\ null\_part\ M\ A)$

lemma $emeasure\_completion[simp]$:
assumes S: $S \in sets\ (completion\ M)$
shows $emeasure\ (completion\ M)\ S = emeasure\ M\ (main\_part\ M\ S)$

lemma $completion\_ex\_borel\_measurable$:
fixes $g :: 'a \Rightarrow ennreal$
assumes g: $g \in borel\_measurable\ (completion\ M)$
shows $\exists g' \in borel\_measurable\ M.\ (AE\ x\ in\ M.\ g\ x = g'\ x)$

locale $semifinite\_measure =$
fixes $M :: 'a\ measure$
assumes $semifinite$:
$\bigwedge A.\ A \in sets\ M \Longrightarrow emeasure\ M\ A = \infty \Longrightarrow \exists B \in sets\ M.\ B \subseteq A \wedge emeasure\ M\ B < \infty$

locale $locally\_determined\_measure = semifinite\_measure +$
assumes $locally\_determined$:
$\bigwedge A.\ A \subseteq space\ M \Longrightarrow (\bigwedge B.\ B \in sets\ M \Longrightarrow emeasure\ M\ B < \infty \Longrightarrow A \cap B \in sets\ M) \Longrightarrow A \in sets\ M$

locale $cld\_measure =$
$complete\_measure\ M + locally\_determined\_measure\ M$ **for** $M :: 'a\ measure$

definition $outer\_measure\_of :: 'a\ measure \Rightarrow 'a\ set \Rightarrow ennreal$
where $outer\_measure\_of\ M\ A = (INF\ B \in \{B \in sets\ M.\ A \subseteq B\}.\ emeasure\ M$

B)

definition *measurable_envelope* :: *'a measure* ⇒ *'a set* ⇒ *'a set* ⇒ *bool*
where *measurable_envelope M A E* ⟷
(*A* ⊆ *E* ∧ *E* ∈ *sets M* ∧ (∀ *F*∈*sets M*. *emeasure M* (*F* ∩ *E*) = *outer_measure_of M* (*F* ∩ *A*)))

lemma *measurable_envelope_eq2*:
assumes *A* ⊆ *E E* ∈ *sets M emeasure M E* < ∞
shows *measurable_envelope M A E* ⟷ (*emeasure M E* = *outer_measure_of M A*)

proposition (**in** *complete_measure*) *fmeasurable_inner_outer*:
S ∈ *fmeasurable M* ⟷
(∀ *e*>*0*. ∃ *T*∈*fmeasurable M*. ∃ *U*∈*fmeasurable M*. *T* ⊆ *S* ∧ *S* ⊆ *U* ∧ |*measure M T* − *measure M U*| < *e*)
(**is** _ ⟷ *?approx*)

end

8.11 Regularity of Measures

theory *Regularity*
imports *Measure_Space Borel_Space*
begin

theorem
fixes *M*::*'a*::{*second_countable_topology*, *complete_space*} *measure*
assumes *sb*: *sets M* = *sets borel*
assumes *emeasure M* (*space M*) ≠ ∞
assumes *B* ∈ *sets borel*
shows *inner_regular*: *emeasure M B* =
(*SUP K* ∈ {*K*. *K* ⊆ *B* ∧ *compact K*}. *emeasure M K*) (**is** *?inner B*)
and *outer_regular*: *emeasure M B* =
(*INF U* ∈ {*U*. *B* ⊆ *U* ∧ *open U*}. *emeasure M U*) (**is** *?outer B*)

end

8.12 Lebesgue Measure

theory *Lebesgue_Measure*
imports
Finite_Product_Measure
Caratheodory
Complete_Measure
Summation_Tests
Regularity
begin

8.12.1 Measures defined by monotonous functions

definition *interval_measure* :: (*real* ⇒ *real*) ⇒ *real measure* **where**
interval_measure F =
extend_measure UNIV {(*a*, *b*). *a* ≤ *b*} (λ(*a*, *b*). {*a*<..*b*}) (λ(*a*, *b*). *ennreal* (*F b* − *F a*))

lemma *emeasure_interval_measure_Ioc*:
assumes *a* ≤ *b*
assumes *mono_F*: ⋀*x y*. *x* ≤ *y* ⟹ *F x* ≤ *F y*
assumes *right_cont_F* : ⋀*a*. *continuous* (*at_right a*) *F*
shows *emeasure* (*interval_measure F*) {*a*<..*b*} = *F b* − *F a*

lemma *sets_interval_measure* [*simp*, *measurable_cong*]:
sets (*interval_measure F*) = *sets borel*

lemma *sigma_finite_interval_measure*:
assumes *mono_F*: ⋀*x y*. *x* ≤ *y* ⟹ *F x* ≤ *F y*
assumes *right_cont_F* : ⋀*a*. *continuous* (*at_right a*) *F*
shows *sigma_finite_measure* (*interval_measure F*)

8.12.2 Lebesgue-Borel measure

definition *lborel* :: (′*a* :: *euclidean_space*) *measure* **where**
lborel = *distr* (Π_M *b*∈*Basis*. *interval_measure* (λ*x*. *x*)) *borel* (λ*f*. ∑*b*∈*Basis*. *f b* $*_R$ *b*)

abbreviation *lebesgue* :: ′*a*::*euclidean_space measure*
where *lebesgue* ≡ *completion lborel*

abbreviation *lebesgue_on* :: ′*a set* ⇒ ′*a*::*euclidean_space measure*
where *lebesgue_on* Ω ≡ *restrict_space* (*completion lborel*) Ω

8.12.3 Borel measurability

lemma *emeasure_lborel_cbox*[*simp*]:
assumes [*simp*]: ⋀*b*. *b* ∈ *Basis* ⟹ *l* · *b* ≤ *u* · *b*
shows *emeasure lborel* (*cbox l u*) = (∏*b*∈*Basis*. (*u* − *l*) · *b*)

8.12.4 Affine transformation on the Lebesgue-Borel

lemma *lborel_eqI*:
fixes *M* :: ′*a*::*euclidean_space measure*

assumes *emeasure_eq*: $\bigwedge l\ u.\ (\bigwedge b.\ b \in Basis \Longrightarrow l \cdot b \leq u \cdot b) \Longrightarrow emeasure\ M\ (box\ l\ u) = (\prod b{\in}Basis.\ (u - l) \cdot b)$
assumes *sets_eq*: $sets\ M = sets\ borel$
shows $lborel = M$

lemma *lborel_affine_euclidean*:
fixes $c :: 'a::euclidean\_space \Rightarrow real$ **and** t
defines $T\ x \equiv t + (\sum j{\in}Basis.\ (c\ j * (x \cdot j)) *_R j)$
assumes *c*: $\bigwedge j.\ j \in Basis \Longrightarrow c\ j \neq 0$
shows $lborel = density\ (distr\ lborel\ borel\ T)\ (\lambda\_.\ (\prod j{\in}Basis.\ |c\ j|))$ (**is** $\_ = ?D$)

lemma *lborel_integral_real_affine*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\_countable\_topology\}$ **and** $c :: real$
assumes *c*: $c \neq 0$ **shows** $(\int x.\ f\ x\ \partial\ lborel) = |c| *_R (\int x.\ f\ (t + c * x)\ \partial lborel)$

corollary *lebesgue_real_affine*:
$c \neq 0 \Longrightarrow lebesgue = density\ (distr\ lebesgue\ lebesgue\ (\lambda x.\ t + c * x))\ (\lambda\_.\ ennreal\ (abs\ c))$

lemma *lborel_prod*:
$lborel \bigotimes_M lborel = (lborel :: ('a::euclidean\_space \times 'b::euclidean\_space)\ measure)$

8.12.5 Lebesgue measurable sets

abbreviation $lmeasurable :: 'a::euclidean\_space\ set\ set$
where
$lmeasurable \equiv fmeasurable\ lebesgue$

lemma *lmeasurable_iff_integrable*:
$S \in lmeasurable \longleftrightarrow integrable\ lebesgue\ (indicator\ S :: 'a::euclidean\_space \Rightarrow real)$

8.12.6 A nice lemma for negligibility proofs

proposition *starlike_negligible_bounded_gmeasurable*:
fixes $S :: 'a :: euclidean\_space\ set$
assumes *S*: $S \in sets\ lebesgue$ **and** $bounded\ S$
and *eq1*: $\bigwedge c\ x.\ [\![(c *_R x) \in S;\ 0 \leq c;\ x \in S]\!] \Longrightarrow c = 1$
shows $S \in null\_sets\ lebesgue$

corollary *starlike_negligible_compact*:
$compact\ S \Longrightarrow (\bigwedge c\ x.\ [\![(c *_R x) \in S;\ 0 \leq c;\ x \in S]\!] \Longrightarrow c = 1) \Longrightarrow S \in null\_sets\ lebesgue$

proposition *outer_regular_lborel_le*:

assumes $B[measurable]$: $B \in sets\ borel$ **and** $0 < (e::real)$
obtains U **where** $open\ U\ B \subseteq U$ **and** $emeasure\ lborel\ (U - B) \leq e$

lemma *outer_regular_lborel*:
assumes B: $B \in sets\ borel$ **and** $0 < (e::real)$
obtains U **where** $open\ U\ B \subseteq U\ emeasure\ lborel\ (U - B) < e$

8.12.7 *F_sigma* and *G_delta* sets.

inductive $fsigma :: 'a::topological\_space\ set \Rightarrow bool$ **where**
$(\bigwedge n::nat.\ closed\ (F\ n)) \Longrightarrow fsigma\ (\bigcup (F\ `\ UNIV))$

inductive $gdelta :: 'a::topological\_space\ set \Rightarrow bool$ **where**
$(\bigwedge n::nat.\ open\ (F\ n)) \Longrightarrow gdelta\ (\bigcap (F\ `\ UNIV))$

end

8.13 Tagged Divisions for Henstock-Kurzweil Integration

theory *Tagged_Division*
imports *Topology_Euclidean_Space*
begin

8.13.1 Some useful lemmas about intervals

8.13.2 Bounds on intervals where they exist

definition $interval\_upperbound :: ('a::euclidean\_space)\ set \Rightarrow 'a$
where $interval\_upperbound\ s = (\sum i\in Basis.\ (SUP\ x\in s.\ x\cdot i) *_R\ i)$

definition $interval\_lowerbound :: ('a::euclidean\_space)\ set \Rightarrow 'a$
where $interval\_lowerbound\ s = (\sum i\in Basis.\ (INF\ x\in s.\ x\cdot i) *_R\ i)$

8.13.3 The notion of a gauge — simply an open set containing the point

definition $gauge\ \gamma \longleftrightarrow (\forall x.\ x \in \gamma\ x \wedge open\ (\gamma\ x))$

8.13.4 Attempt a systematic general set of "offset" results for components

8.13.5 Divisions

definition *division_of* (**infixl** ‹*division'_of*› *40*)

where
  s division_of i ⟷
    finite s ∧
    (∀ *K*∈*s*. *K* ⊆ *i* ∧ *K* ≠ {} ∧ (∃ *a b*. *K* = *cbox a b*)) ∧
    (∀ *K1*∈*s*. ∀ *K2*∈*s*. *K1* ≠ *K2* ⟶ *interior*(*K1*) ∩ *interior*(*K2*) = {}) ∧
    (⋃ *s* = *i*)

proposition *partial_division_extend_interval*:
  assumes *p division_of* (⋃ *p*) (⋃ *p*) ⊆ *cbox a b*
  obtains *q* **where** *p* ⊆ *q q division_of cbox a* (*b*::′*a*::*euclidean_space*)

proposition *division_union_intervals_exists*:
  assumes *cbox a b* ≠ {}
  obtains *p* **where** (*insert* (*cbox a b*) *p*) *division_of* (*cbox a b* ∪ *cbox c d*)

8.13.6 Tagged (partial) divisions

definition *tagged_partial_division_of* (**infixr** ‹*tagged′_partial′_division′_of*› *40*)
  where *s tagged_partial_division_of i* ⟷
    finite s ∧
    (∀ *x K*. (*x*, *K*) ∈ *s* ⟶ *x* ∈ *K* ∧ *K* ⊆ *i* ∧ (∃ *a b*. *K* = *cbox a b*)) ∧
    (∀ *x1 K1 x2 K2*. (*x1*, *K1*) ∈ *s* ∧ (*x2*, *K2*) ∈ *s* ∧ (*x1*, *K1*) ≠ (*x2*, *K2*) ⟶
      interior K1 ∩ *interior K2* = {})

definition *tagged_division_of* (**infixr** ‹*tagged′_division′_of*› *40*)
  where *s tagged_division_of i* ⟷ *s tagged_partial_division_of i* ∧ (⋃ {*K*. ∃ *x*. (*x*,*K*) ∈ *s*} = *i*)

8.13.7 Functions closed on boxes: morphisms from boxes to monoids

Using additivity of lifted function to encode definedness. **definition** *lift_option* :: (′*a* ⇒ ′*b* ⇒ ′*c*) ⇒ ′*a option* ⇒ ′*b option* ⇒ ′*c option*
where
  lift_option f a′ b′ = *Option.bind a′* (λ*a*. *Option.bind b′* (λ*b*. *Some* (*f a b*)))

lemma *comm_monoid_lift_option*:
  assumes *comm_monoid f z*
  shows *comm_monoid* (*lift_option f*) (*Some z*)

Misc

Division points **definition** *division_points* (*k*::(′*a*::*euclidean_space*) *set*) *d* =

$\{(j,x).\ j \in Basis \wedge (interval\_lowerbound\ k)\cdot j < x \wedge x < (interval\_upperbound\ k)\cdot j \wedge$
$(\exists\, i{\in}d.\ (interval\_lowerbound\ i)\cdot j = x \vee (interval\_upperbound\ i)\cdot j = x)\}$

Operative

proposition *tagged_division*:
assumes d *tagged_division_of* $(cbox\ a\ b)$
shows $F\ (\lambda(\_,\ l).\ g\ l)\ d = g\ (cbox\ a\ b)$

8.13.8 Special case of additivity we need for the FTC

8.13.9 Fine-ness of a partition w.r.t. a gauge

definition *fine* (**infixr** ‹*fine*› *46*)
where d *fine* $s \longleftrightarrow (\forall\,(x,k) \in s.\ k \subseteq d\ x)$

8.13.10 Some basic combining lemmas

8.13.11 General bisection principle for intervals; might be useful elsewhere

8.13.12 Cousin's lemma

8.13.13 A technical lemma about "refinement" of division

Covering lemma

proposition *covering_lemma*:
assumes $S \subseteq cbox\ a\ b\ \ box\ a\ b \neq \{\}\ \ gauge\ g$
obtains $\mathcal{D}$ **where**
$countable\ \mathcal{D}\ \ \bigcup\mathcal{D} \subseteq cbox\ a\ b$
$\bigwedge K.\ K \in \mathcal{D} \Longrightarrow interior\ K \neq \{\} \wedge (\exists\, c\ d.\ K = cbox\ c\ d)$
$pairwise\ (\lambda A\ B.\ interior\ A \cap interior\ B = \{\})\ \mathcal{D}$
$\bigwedge K.\ K \in \mathcal{D} \Longrightarrow \exists\, x \in S \cap K.\ K \subseteq g\ x$
$\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists\, n.\ \forall\, i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i)\ /\ 2\hat{\ }n$
$S \subseteq \bigcup\mathcal{D}$

8.13.14 Division filter

definition $division\_filter :: 'a::euclidean\_space\ set \Rightarrow ('a \times 'a\ set)\ set\ filter$
where $division\_filter\ s = (INF\ g{\in}\{g.\ gauge\ g\}.\ principal\ \{p.\ p\ tagged\_division\_of\ s \wedge g\ fine\ p\})$

proposition *eventually_division_filter*:
$(\forall_F\ p\ in\ division\_filter\ s.\ P\ p) \longleftrightarrow$
$(\exists\, g.\ gauge\ g \wedge (\forall\, p.\ p\ tagged\_division\_of\ s \wedge g\ fine\ p \longrightarrow P\ p))$

end

8.14 Henstock-Kurzweil Gauge Integration in Many Dimensions

theory *Henstock_Kurzweil_Integration*
imports
Lebesgue_Measure Tagged_Division HOL−Real_Asymp.Real_Asymp

begin

8.14.1 Content (length, area, volume, etc.) of an interval

8.14.2 Gauge integral

8.14.3 Basic theorems about integrals

corollary *integral_mult_left* [*simp*]:
fixes *c*:: *'a*::{*real_normed_algebra*,*division_ring*}
shows *integral S* (*λx. f x* ∗ *c*) = *integral S f* ∗ *c*

corollary *integral_mult_right* [*simp*]:
fixes *c*:: *'a*::{*real_normed_field*}
shows *integral S* (*λx. c* ∗ *f x*) = *c* ∗ *integral S f*

corollary *integral_divide* [*simp*]:
fixes *z* :: *'a*::*real_normed_field*
shows *integral S* (*λx. f x* / *z*) = *integral S* (*λx. f x*) / *z*

8.14.4 Cauchy-type criterion for integrability

proposition *integrable_Cauchy*:
fixes *f* :: *'n*::*euclidean_space* ⇒ *'a*::{*real_normed_vector*,*complete_space*}
shows *f integrable_on cbox a b* ⟷
(∀ *e*>*0*. ∃ *γ*. *gauge γ* ∧
(∀ *𝒟1 𝒟2*. *𝒟1 tagged_division_of* (*cbox a b*) ∧ *γ fine 𝒟1* ∧
𝒟2 tagged_division_of (*cbox a b*) ∧ *γ fine 𝒟2* ⟶
norm ((∑ (*x*,*K*)∈*𝒟1*. *content K* $*_R$ *f x*) − (∑ (*x*,*K*)∈*𝒟2*. *content K* $*_R$ *f x*)) < *e*))
(**is** *?l* = (∀ *e*>*0*. ∃ *γ*. *?P e γ*))

8.14.5 Additivity of integral on abutting intervals

proposition *has_integral_split*:

fixes *f* :: *'a::euclidean_space* ⇒ *'b::real_normed_vector*
assumes *fi*: (*f has_integral i*) (*cbox a b* ∩ {*x*. *x*•*k* ≤ *c*})
and *fj*: (*f has_integral j*) (*cbox a b* ∩ {*x*. *x*•*k* ≥ *c*})
and *k*: *k* ∈ *Basis*
shows (*f has_integral* (*i* + *j*)) (*cbox a b*)

8.14.6 A sort of converse, integrability on subintervals

8.14.7 Bounds on the norm of Riemann sums and the integral itself

corollary *integrable_bound*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::real_normed_vector*
assumes *0* ≤ *B*
and *f integrable_on* (*cbox a b*)
and ⋀*x*. *x*∈*cbox a b* ⟹ *norm* (*f x*) ≤ *B*
shows *norm* (*integral* (*cbox a b*) *f*) ≤ *B* ∗ *content* (*cbox a b*)

8.14.8 Similar theorems about relationship among components

8.14.9 Uniform limit of integrable functions is integrable

8.14.10 Negligible sets

proposition *negligible_standard_hyperplane*[*intro*]:
fixes *k* :: *'a::euclidean_space*
assumes *k*: *k* ∈ *Basis*
shows *negligible* {*x*. *x*•*k* = *c*}

corollary *negligible_standard_hyperplane_cart*:
fixes *k* :: *'a::finite*
shows *negligible* {*x*. *x*$*k* = (*0::real*)}

proposition *has_integral_negligible*:
fixes *f* :: *'b::euclidean_space* ⇒ *'a::real_normed_vector*
assumes *negs*: *negligible S*
and ⋀*x*. *x* ∈ (*T* − *S*) ⟹ *f x* = *0*
shows (*f has_integral 0*) *T*

8.14.11 Some other trivialities about negligible sets

8.14.12 Finite case of the spike theorem is quite commonly needed

corollary *has_integral_bound_real*:
fixes f :: *real* ⇒ *'b::real_normed_vector*
assumes $0 \le B$ *finite* S
and (f *has_integral* i) $\{a..b\}$
and $\bigwedge x.\ x \in \{a..b\} - S \Longrightarrow$ *norm* $(f\ x) \le B$
shows *norm* $i \le B *$ *content* $\{a..b\}$

8.14.13 In particular, the boundary of an interval is negligible

8.14.14 Integrability of continuous functions

8.14.15 Specialization of additivity to one dimension

8.14.16 A useful lemma allowing us to factor out the content size

8.14.17 Fundamental theorem of calculus

theorem *fundamental_theorem_of_calculus*:
fixes f :: *real* ⇒ *'a::banach*
assumes $a \le b$
and *vecd*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ (f *has_vector_derivative* $f'\ x$) (*at* x *within* $\{a..b\}$)
shows (f' *has_integral* $(f\ b - f\ a)$) $\{a..b\}$

8.14.18 Taylor series expansion

8.14.19 Only need trivial subintervals if the interval itself is trivial

proposition *division_of_nontrivial*:
fixes $\mathcal{D}$ :: *'a::euclidean_space set set*
assumes *sdiv*: $\mathcal{D}$ *division_of* (*cbox* a b)
and *cont0*: *content* (*cbox* a b) $\neq 0$
shows $\{k.\ k \in \mathcal{D} \wedge$ *content* $k \neq 0\}$ *division_of* (*cbox* a b)

8.14.20 Integrability on subintervals

8.14.21 Combining adjacent intervals in 1 dimension

8.14.22 Reduce integrability to "local" integrability

8.14.23 Second FTC or existence of antiderivative

8.14.24 Combined fundamental theorem of calculus

8.14.25 General "twiddling" for interval-to-interval function image

8.14.26 Special case of a basic affine transformation

8.14.27 Special case of stretching coordinate axes separately

8.14.28 even more special cases

8.14.29 Stronger form of FCT; quite a tedious proof

theorem *fundamental_theorem_of_calculus_interior*:
  fixes f :: *real* ⇒ *'a::real_normed_vector*
  assumes $a \leq b$
    and *contf*: *continuous_on* $\{a..b\}$ f
    and *derf*: $\bigwedge x.\ x \in \{a <..< b\} \Longrightarrow$ (f *has_vector_derivative* f' x) (*at* x)
  shows (f' *has_integral* ($f\ b - f\ a$)) $\{a..b\}$

8.14.30 Stronger form with finite number of exceptional points

corollary *fundamental_theorem_of_calculus_strong*:
  fixes f :: *real* ⇒ *'a::banach*
  assumes *finite S*
    and $a \leq b$
    and *vec*: $\bigwedge x.\ x \in \{a..b\} - S \Longrightarrow$ (f *has_vector_derivative* $f'(x)$) (*at* x)
    and *continuous_on* $\{a..b\}$ f
  shows (f' *has_integral* ($f\ b - f\ a$)) $\{a..b\}$

proposition *indefinite_integral_continuous_left*:
  fixes f:: *real* ⇒ *'a::banach*
  assumes *intf*: f *integrable_on* $\{a..b\}$ **and** $a < c$ $c \leq b$ $e > 0$
  obtains d **where** $d > 0$
    and $\forall t.\ c - d < t \wedge t \leq c \longrightarrow$ *norm* (*integral* $\{a..c\}$ f − *integral* $\{a..t\}$ f) $<$ e

theorem *integral_has_vector_derivative'*:

fixes f :: *real* ⇒ *'b*::*banach*
assumes *continuous_on* {*a*..*b*} *f*
and $x \in$ {*a*..*b*}
shows ((λ*u*. *integral* {*u*..*b*} *f*) *has_vector_derivative* − *f x*) (*at x within* {*a*..*b*})

8.14.31 This doesn't directly involve integration, but that gives an easy proof

8.14.32 Generalize a bit to any convex set

8.14.33 Integrating characteristic function of an interval

corollary *has_integral_restrict_UNIV*:
fixes f :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
shows ((λ*x*. *if* $x \in s$ *then f x else 0*) *has_integral i*) *UNIV* ⟷ (*f has_integral i*) *s*

8.14.34 Integrals on set differences

corollary *integral_spike_set*:
fixes f :: *'n*::*euclidean_space* ⇒ *'a*::*banach*
assumes *negligible* {$x \in S - T$. *f x* ≠ *0*} *negligible* {$x \in T - S$. *f x* ≠ *0*}
shows *integral S f* = *integral T f*

8.14.35 More lemmas that are useful later

8.14.36 Continuity of the integral (for a 1-dimensional interval)

8.14.37 A straddling criterion for integrability

8.14.38 Adding integrals over several sets

8.14.39 Also tagged divisions

8.14.40 Henstock's lemma

8.14.41 Monotone convergence (bounded interval first)

8.14.42 differentiation under the integral sign

8.14.43 Exchange uniform limit and integral

8.14.44 Integration by parts

8.14.45 Integration by substitution

8.14.46 Compute a double integral using iterated integrals and switching the order of integration

theorem *integral_swap_continuous*:
  fixes *f* :: [*'a*::*euclidean_space*, *'b*::*euclidean_space*] ⇒ *'c*::*banach*
  assumes *continuous_on* (*cbox* (*a*,*c*) (*b*,*d*)) (λ(*x*,*y*). *f x y*)
    shows *integral* (*cbox a b*) (λ*x*. *integral* (*cbox c d*) (*f x*)) =
          integral (*cbox c d*) (λ*y*. *integral* (*cbox a b*) (λ*x*. *f x y*))

8.14.47 Definite integrals for exponential and power function

8.14.48 Adaption to ordered Euclidean spaces and the Cartesian Euclidean space

end

Chapter 9

Kronecker's Theorem with Applications

theory *Kronecker_Approximation_Theorem*

imports *Complex_Transcendental Henstock_Kurzweil_Integration HOL−Real_Asymp.Real_Asymp*

begin

9.1 Dirichlet's Approximation Theorem

theorem *Dirichlet_approx_simult*:
fixes $\vartheta :: nat \Rightarrow real$ **and** $N\ n :: nat$
assumes $N > 0$
obtains $q\ p$ **where** $0<q\ q \leq int\ (N\hat{\ }n)$
and $\bigwedge i.\ i<n \Longrightarrow |of\_int\ q * \vartheta\ i - of\_int(p\ i)| < 1/N$
corollary *Dirichlet_approx*:
fixes $\vartheta :: real$ **and** $N :: nat$
assumes $N > 0$
obtains $h\ k$ **where** $0 < k\ k \leq int\ N\ |of\_int\ k * \vartheta - of\_int\ h| < 1/N$
corollary *Dirichlet_approx_coprime*:
fixes $\vartheta :: real$ **and** $N :: nat$
assumes $N > 0$
obtains $h\ k$ **where** $coprime\ h\ k\ 0 < k\ k \leq int\ N\ |of\_int\ k * \vartheta - of\_int\ h| < 1/N$
theorem *infinite_approx_set*:
assumes $infinite\ (approx\_set\ \vartheta)$
shows $\exists h\ k.\ (h,k) \in approx\_set\ \vartheta \land k > K$
theorem *rational_iff_finite_approx_set*:
shows $\vartheta \in \mathbb{Q} \longleftrightarrow finite\ (approx\_set\ \vartheta)$

9.2 Kronecker's Approximation Theorem: the One-dimensional Case

theorem *Kronecker_approx_1_explicit*:
  fixes ϑ :: *real*
  assumes $\vartheta \notin \mathbb{Q}$ **and** α: $0 \leq \alpha$ $\alpha \leq 1$ **and** $\varepsilon > 0$
  obtains k **where** $k{>}0$ $|frac(real\ k * \vartheta) - \alpha| < \varepsilon$
corollary *Kronecker_approx_1*:
  fixes ϑ :: *real*
  assumes $\vartheta \notin \mathbb{Q}$
  shows $closure\ (range\ (\lambda n.\ frac\ (real\ n * \vartheta))) = \{0..1\}$ (**is** $?C = \_$)
corollary *sequence_of_fractional_parts_is_dense*:
  fixes ϑ :: *real*
  assumes $\vartheta \notin \mathbb{Q}$ $\varepsilon > 0$
  obtains h k **where** $k > 0$ $|of\_int\ k * \vartheta - of\_int\ h - \alpha| < \varepsilon$

9.3 Extension of Kronecker's Theorem to Simultaneous Approximation

9.3.1 Towards Lemma 1

9.3.2 Towards Lemma 2

9.3.3 Towards lemma 3

9.3.4 And finally Kroncker's theorem itself

theorem *Kronecker_thm_1*:
  fixes α ϑ:: *nat* $\Rightarrow$ *real* **and** n:: *nat*
  assumes *indp*: $module.independent\ (\lambda r.\ (*)\ (real\_of\_int\ r))\ (\vartheta\ `\ \{..{<}n\})$
    and *injϑ*: $inj\_on\ \vartheta\ \{..{<}n\}$ **and** $\varepsilon > 0$
  obtains t h **where** $\bigwedge i.\ i < n \Longrightarrow |t * \vartheta\ i - of\_int\ (h\ i) - \alpha\ i| < \varepsilon$
corollary *Kronecker_thm_2*:
  fixes α ϑ :: *nat* $\Rightarrow$ *real* **and** n :: *nat*
  assumes *indp*: $module.independent\ (\lambda r\ x.\ of\_int\ r * x)\ (\vartheta\ `\ \{..n\})$
    and *injϑ*: $inj\_on\ \vartheta\ \{..n\}$ **and** [*simp*]: $\vartheta\ n = 1$ **and** $\varepsilon > 0$
  obtains k m **where** $\bigwedge i.\ i < n \Longrightarrow |of\_int\ k * \vartheta\ i - of\_int\ (m\ i) - \alpha\ i| < \varepsilon$

end

9.4 Bernstein-Weierstrass and Stone-Weierstrass

theory *Weierstrass_Theorems*
imports *Uniform_Limit Path_Connected Derivative*
begin

9.4.1 Bernstein polynomials

definition *Bernstein* :: [*nat*,*nat*,*real*] ⇒ *real* **where**
Bernstein n k x ≡ *of_nat* (*n choose k*) ∗ *x*^*k* ∗ (*1* − *x*)^(*n* − *k*)

9.4.2 Explicit Bernstein version of the 1D Weierstrass approximation theorem

theorem *Bernstein_Weierstrass*:
fixes *f* :: *real* ⇒ *real*
assumes *contf*: *continuous_on* {*0*..*1*} *f* **and** *e*: *0* < *e*
shows ∃ *N*. ∀ *n x*. *N* ≤ *n* ∧ *x* ∈ {*0*..*1*}
⟶ |*f x* − (∑ *k*≤*n*. *f*(*k*/*n*) ∗ *Bernstein n k x*)| < *e*

9.4.3 General Stone-Weierstrass theorem

definition *normf* :: ('*a*::*t2_space* ⇒ *real*) ⇒ *real*
where *normf f* ≡ *SUP x*∈*S*. |*f x*|
proposition (**in** *function_ring_on*) *Stone_Weierstrass_basic*:
assumes *f*: *continuous_on S f* **and** *e*: *e* > *0*
shows ∃ *g* ∈ *R*. ∀ *x*∈*S*. |*f x* − *g x*| < *e*

theorem (**in** *function_ring_on*) *Stone_Weierstrass*:
assumes *f*: *continuous_on S f*
shows ∃ *F*∈*UNIV* → *R*. *LIM n sequentially*. *F n* :> *uniformly_on S f*
corollary *Stone_Weierstrass_HOL*:
fixes *R* :: ('*a*::*t2_space* ⇒ *real*) *set* **and** *S* :: '*a set*
assumes *compact S* ⋀*c*. *P*(λ*x*. *c*::*real*)
⋀*f*. *P f* ⟹ *continuous_on S f*
⋀*f g*. *P*(*f*) ∧ *P*(*g*) ⟹ *P*(λ*x*. *f x* + *g x*) ⋀*f g*. *P*(*f*) ∧ *P*(*g*) ⟹ *P*(λ*x*. *f x* ∗ *g x*)
⋀*x y*. *x* ∈ *S* ∧ *y* ∈ *S* ∧ *x* ≠ *y* ⟹ ∃ *f*. *P*(*f*) ∧ *f x* ≠ *f y*
continuous_on S f
0 < *e*
shows ∃ *g*. *P*(*g*) ∧ (∀ *x* ∈ *S*. |*f x* − *g x*| < *e*)

9.4.4 Polynomial functions

definition *polynomial_function* :: (*'a*::*real_normed_vector* ⇒ *'b*::*real_normed_vector*) ⇒ *bool*
where
polynomial_function p ≡ (∀ *f*. *bounded_linear f* ⟶ *real_polynomial_function* (*f o p*))

9.4.5 Stone-Weierstrass theorem for polynomial functions

theorem *Stone_Weierstrass_polynomial_function*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *S*: *compact S*
and *f*: *continuous_on S f*
and *e*: *0* < *e*
shows ∃ *g*. *polynomial_function g* ∧ (∀ *x* ∈ *S*. *norm*(*f x* − *g x*) < *e*)

proposition *Stone_Weierstrass_uniform_limit*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *S*: *compact S*
and *f*: *continuous_on S f*
obtains *g* **where** *uniform_limit S g f sequentially* ⋀*n*. *polynomial_function* (*g n*)

9.4.6 Polynomial functions as paths

proposition *connected_open_polynomial_connected*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *S*: *open S connected S*
and *x* ∈ *S y* ∈ *S*
shows ∃ *g*. *polynomial_function g* ∧ *path_image g* ⊆ *S* ∧ *pathstart g* = *x* ∧ *pathfinish g* = *y*

theorem *Stone_Weierstrass_polynomial_function_subspace*:
fixes *f* :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes *compact S*
and *contf*: *continuous_on S f*
and *0* < *e*
and *subspace T f ‘ S* ⊆ *T*
obtains *g* **where** *polynomial_function g g ‘ S* ⊆ *T*
⋀*x*. *x* ∈ *S* ⟹ *norm*(*f x* − *g x*) < *e*

end

9.5 Radon-Nikodým Derivative

theory *Radon_Nikodym*
imports *Bochner_Integration*
begin

definition *diff_measure* :: *'a measure* ⇒ *'a measure* ⇒ *'a measure*
where
diff_measure M N = *measure_of* (*space M*) (*sets M*) (*λA. emeasure M A* − *emeasure N A*)
proposition (**in** *sigma_finite_measure*) *obtain_positive_integrable_function*:
obtains *f*::*'a* ⇒ *real* **where**
f ∈ *borel_measurable M*
⋀*x. f x* > *0*
⋀*x. f x* ≤ *1*
integrable M f

9.5.1 Absolutely continuous

definition *absolutely_continuous* :: *'a measure* ⇒ *'a measure* ⇒ *bool* **where**
absolutely_continuous M N ⟷ *null_sets M* ⊆ *null_sets N*

9.5.2 Existence of the Radon-Nikodym derivative

proposition
(**in** *finite_measure*) *Radon_Nikodym_finite_measure*:
assumes *finite_measure N* **and** *sets_eq*[*simp*]: *sets N* = *sets M*
assumes *absolutely_continuous M N*
shows ∃*f* ∈ *borel_measurable M*. *density M f* = *N*

proposition (**in** *finite_measure*) *Radon_Nikodym_finite_measure_infinite*:
assumes *absolutely_continuous M N* **and** *sets_eq*: *sets N* = *sets M*
shows ∃*f*∈*borel_measurable M*. *density M f* = *N*

theorem (**in** *sigma_finite_measure*) *Radon_Nikodym*:
assumes *ac*: *absolutely_continuous M N* **assumes** *sets_eq*: *sets N* = *sets M*
shows ∃*f* ∈ *borel_measurable M*. *density M f* = *N*

9.5.3 Uniqueness of densities

proposition (**in** *sigma_finite_measure*) *density_unique*:
assumes *f*: *f* ∈ *borel_measurable M*
assumes *f'*: *f'* ∈ *borel_measurable M*
assumes *density_eq*: *density M f* = *density M f'*
shows *AE x in M. f x* = *f' x*

9.5.4 Radon-Nikodym derivative

definition *RN_deriv* :: *'a measure* ⇒ *'a measure* ⇒ *'a* ⇒ *ennreal* **where**
RN_deriv M N =
(*if* ∃*f*. *f* ∈ *borel_measurable M* ∧ *density M f* = *N*
then SOME f. *f* ∈ *borel_measurable M* ∧ *density M f* = *N*
else (λ_. *0*))

proposition (**in** *sigma_finite_measure*) *real_RN_deriv*:
assumes *finite_measure N*
assumes *ac*: *absolutely_continuous M N sets N* = *sets M*
obtains *D* **where** *D* ∈ *borel_measurable M*
and *AE x in M*. *RN_deriv M N x* = *ennreal* (*D x*)
and *AE x in N*. *0* < *D x*
and ⋀*x*. *0* ≤ *D x*

end

Chapter 10

Integrals over a Set

theory *Set_Integral*
  imports *Radon_Nikodym*
begin

10.1 Notation

definition *set_borel_measurable M A f* $\equiv$ ($\lambda x.$ *indicator A x* $*_R$ *f x*) $\in$ *borel_measurable M*

definition *set_integrable M A f* $\equiv$ *integrable M* ($\lambda x.$ *indicator A x* $*_R$ *f x*)

definition *set_lebesgue_integral M A f* $\equiv$ *lebesgue_integral M* ($\lambda x.$ *indicator A x* $*_R$ *f x*)

10.2 Basic properties

proposition *set_borel_measurable_subset*:
  fixes *f* :: _ $\Rightarrow$ _ :: {*banach*, *second_countable_topology*}
  assumes [*measurable*]: *set_borel_measurable M A f B* $\in$ *sets M* **and** $B \subseteq A$
  shows *set_borel_measurable M B f*

10.3 Complex integrals

10.4 NN Set Integrals

proposition *nn_integral_disjoint_family*:
assumes [*measurable*]: $f \in$ *borel_measurable M* $\bigwedge(n::nat).\ B\ n \in$ *sets M*
and *disjoint_family B*
shows $(\int^{+} x \in (\bigcup n.\ B\ n).\ f\ x\ \partial M) = (\sum n.\ (\int^{+} x \in B\ n.\ f\ x\ \partial M))$

10.5 Scheffé's lemma

proposition *Scheffe_lemma1*:
assumes $\bigwedge n.$ *integrable M* $(F\ n)$ *integrable M f*
AE x in M. $(\lambda n.\ F\ n\ x) \longrightarrow f\ x$
limsup $(\lambda n.\ \int^{+} x.\ norm(F\ n\ x)\ \partial M) \leq (\int^{+} x.\ norm(f\ x)\ \partial M)$
shows $(\lambda n.\ \int^{+} x.\ norm(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$

proposition *Scheffe_lemma2*:
fixes F::*nat* $\Rightarrow$ *'a* $\Rightarrow$ *'b*::{*banach*, *second_countable_topology*}
assumes $\bigwedge$ *n::nat*. $F\ n \in$ *borel_measurable M integrable M f*
AE x in M. $(\lambda n.\ F\ n\ x) \longrightarrow f\ x$
$\bigwedge n.\ (\int^{+} x.\ norm(F\ n\ x)\ \partial M) \leq (\int^{+} x.\ norm(f\ x)\ \partial M)$
shows $(\lambda n.\ \int^{+} x.\ norm(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$

10.6 Convergence of integrals over an interval

proposition *tendsto_set_lebesgue_integral_at_top*:
fixes *f* :: *real* $\Rightarrow$ *'a*::{*banach*, *second_countable_topology*}
assumes *sets*: $\bigwedge b.\ b \geq a \Longrightarrow \{a..b\} \in$ *sets M*
and *int*: *set_integrable M* $\{a..\}$ *f*
shows $((\lambda b.$ *set_lebesgue_integral M* $\{a..b\}\ f) \longrightarrow$ *set_lebesgue_integral M* $\{a..\}\ f)$ *at_top*

proposition *tendsto_set_lebesgue_integral_at_bot*:
fixes *f* :: *real* $\Rightarrow$ *'a*::{*banach*, *second_countable_topology*}
assumes *sets*: $\bigwedge a.\ a \leq b \Longrightarrow \{a..b\} \in$ *sets M*
and *int*: *set_integrable M* $\{..b\}$ *f*
shows $((\lambda a.$ *set_lebesgue_integral M* $\{a..b\}\ f) \longrightarrow$ *set_lebesgue_integral M* $\{..b\}\ f)$ *at_bot*

theorem *integral_Markov_inequality'*:
fixes *u* :: *'a* $\Rightarrow$ *real*
assumes [*measurable*]: *set_integrable M A u* **and** $A \in$ *sets M*
assumes *AE x in M*. $x \in A \longrightarrow u\ x \geq 0$ **and** $0 < (c::real)$

shows *emeasure M* $\{x{\in}A.\ u\ x \geq c\} \leq (1/c{::}real) * (\int x{\in}A.\ u\ x\ \partial M)$

theorem *integral_Markov_inequality'_measure*:
assumes [*measurable*]: *set_integrable M A u* **and** $A \in$ *sets M*
and *AE x in M*. $x \in A \longrightarrow 0 \leq u\ x\ 0 < (c{::}real)$
shows *measure M* $\{x{\in}A.\ u\ x \geq c\} \leq (\int x{\in}A.\ u\ x\ \partial M) / c$

theorem (**in** *finite_measure*) *Chernoff_ineq_ge*:
assumes *s*: $s > 0$
assumes *integrable*: *set_integrable M A* ($\lambda x.$ *exp* $(s * f\ x)$) **and** $A \in$ *sets M*
shows *measure M* $\{x{\in}A.\ f\ x \geq a\} \leq exp\ (-s * a) * (\int x{\in}A.\ exp\ (s * f\ x)\ \partial M)$
proof –
have $\{x{\in}A.\ f\ x \geq a\} = \{x{\in}A.\ exp\ (s * f\ x) \geq exp\ (s * a)\}$
using *s* **by** *auto*
also have *measure M* ... $\leq$ *set_lebesgue_integral M A* ($\lambda x.$ *exp* $(s * f\ x)$) / *exp* $(s * a)$
by (*intro integral_Markov_inequality'_measure assms*) *auto*
finally show *?thesis*
by (*simp add*: *exp_minus field_simps*)
qed

theorem (**in** *finite_measure*) *Chernoff_ineq_le*:
assumes *s*: $s > 0$
assumes *integrable*: *set_integrable M A* ($\lambda x.$ *exp* $(-s * f\ x)$) **and** $A \in$ *sets M*
shows *measure M* $\{x{\in}A.\ f\ x \leq a\} \leq exp\ (s * a) * (\int x{\in}A.\ exp\ (-s * f\ x)\ \partial M)$
proof –
have $\{x{\in}A.\ f\ x \leq a\} = \{x{\in}A.\ exp\ (-s * f\ x) \geq exp\ (-s * a)\}$
using *s* **by** *auto*
also have *measure M* ... $\leq$ *set_lebesgue_integral M A* ($\lambda x.$ *exp* $(-s * f\ x)$) / *exp* $(-s * a)$
by (*intro integral_Markov_inequality'_measure assms*) *auto*
finally show *?thesis*
by (*simp add*: *exp_minus field_simps*)
qed

10.7 Integrable Simple Functions

lemma *integrable_simple_function_induct*[*consumes 2*, *case_names cong indicator add*, *induct set*: *simple_function*]:
fixes $f :: 'a \Rightarrow 'b ::$ {*second_countable_topology*, *banach*}
assumes *f*: *simple_function M f emeasure M* $\{y \in space\ M.\ f\ y \neq 0\} \neq \infty$
assumes *cong*: $\bigwedge f\ g.$ *simple_function M f* $\Longrightarrow$ *emeasure M* $\{y \in space\ M.\ f\ y \neq 0\} \neq \infty$
$\Longrightarrow$ *simple_function M g* $\Longrightarrow$ *emeasure M* $\{y \in space\ M.\ g\ y \neq 0\} \neq \infty$
$\Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
assumes *indicator*: $\bigwedge A\ y.\ A \in sets\ M \Longrightarrow emeasure\ M\ A < \infty \Longrightarrow P\ (\lambda x.\ indicator\ A\ x *_R y)$
assumes *add*: $\bigwedge f\ g.$ *simple_function M f* $\Longrightarrow$ *emeasure M* $\{y \in space\ M.\ f\ y \neq$

0} $\neq \infty \Longrightarrow$
simple_function M g $\Longrightarrow$ *emeasure M* $\{y \in$ *space M*. *g y* $\neq$ *0*$\}$ $\neq \infty \Longrightarrow$
$(\bigwedge z.\ z \in$ *space M* $\Longrightarrow$ *norm* $(f\ z + g\ z) =$ *norm* $(f\ z) +$ *norm* $(g\ z)) \Longrightarrow$
$P\ f \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
shows *P f*

lemma *integrable_simple_function_induct_nn*[*consumes 3*, *case_names cong indicator add*, *induct set*: *simple_function*]:
fixes *f* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *f*: *simple_function M f emeasure M* $\{y \in$ *space M*. $f\ y \neq$ *0*$\} \neq \infty$ $\bigwedge x.\ x \in$ *space M* $\longrightarrow f\ x \geq$ *0*
assumes *cong*: $\bigwedge f\ g.$ *simple_function M f* $\Longrightarrow$ *emeasure M* $\{y \in$ *space M*. $f\ y \neq$ *0*$\} \neq \infty \Longrightarrow (\bigwedge x.\ x \in$ *space M* $\Longrightarrow f\ x \geq$ *0*$) \Longrightarrow$ *simple_function M g* $\Longrightarrow$ *emeasure M* $\{y \in$ *space M*. $g\ y \neq$ *0*$\} \neq \infty \Longrightarrow (\bigwedge x.\ x \in$ *space M* $\Longrightarrow g\ x \geq$ *0*$) \Longrightarrow (\bigwedge x.\ x \in$ *space M* $\Longrightarrow f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
assumes *indicator*: $\bigwedge A\ y.\ y \geq$ *0* $\Longrightarrow A \in$ *sets M* $\Longrightarrow$ *emeasure M A* $< \infty \Longrightarrow P\ (\lambda x.$ *indicator A x* $*_R\ y)$
assumes *add*: $\bigwedge f\ g.\ (\bigwedge x.\ x \in$ *space M* $\Longrightarrow f\ x \geq$ *0*$) \Longrightarrow$ *simple_function M f* $\Longrightarrow$ *emeasure M* $\{y \in$ *space M*. $f\ y \neq$ *0*$\} \neq \infty \Longrightarrow$
$(\bigwedge x.\ x \in$ *space M* $\Longrightarrow g\ x \geq$ *0*$) \Longrightarrow$ *simple_function M g* $\Longrightarrow$ *emeasure M* $\{y \in$ *space M*. $g\ y \neq$ *0*$\} \neq \infty \Longrightarrow$
$(\bigwedge z.\ z \in$ *space M* $\Longrightarrow$ *norm* $(f\ z + g\ z) =$ *norm* $(f\ z) +$ *norm* $(g\ z)) \Longrightarrow$
$P\ f \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
shows *P f*

10.7.1 Totally Ordered Banach Spaces

10.7.2 Auxiliary Lemmas for Set Integrals

10.7.3 Integrability and Measurability of the Diameter

10.7.4 Averaging Theorem

corollary *integral_nonneg_eq_0_iff_AE_banach*:
fixes *f* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *f*[*measurable*]: *integrable M f* **and** *nonneg*: *AE x in M*. *0* $\leq f\ x$
shows $integral^L$ *M f* = *0* $\longleftrightarrow$ (*AE x in M*. $f\ x =$ *0*)

corollary *integral_eq_mono_AE_eq_AE*:
fixes *f g* :: *'a* ⇒ *'b* :: {*second_countable_topology*, *banach*, *linorder_topology*, *ordered_real_vector*}
assumes *integrable M f integrable M g* $integral^L$ *M f* = $integral^L$ *M g AE x in M*. $f\ x \leq g\ x$
shows *AE x in M*. $f\ x = g\ x$

end

10.8 Homeomorphism Theorems

theory *Homeomorphism*
imports *Homotopy*
begin

10.8.1 Homeomorphism of all convex compact sets with nonempty interior

proposition
fixes *S* :: *'a::euclidean_space set*
assumes *compact S* **and** *0*: *0 ∈ rel_interior S*
and *star*: $\bigwedge x.\ x \in S \Longrightarrow$ *open_segment 0 x ⊆ rel_interior S*
shows *starlike_compact_projective1_0*:
S − rel_interior S homeomorphic sphere 0 1 ∩ affine hull S
(**is** *?SMINUS homeomorphic ?SPHER*)
and *starlike_compact_projective2_0*:
S homeomorphic cball 0 1 ∩ affine hull S
(**is** *S homeomorphic ?CBALL*)

corollary
fixes *S* :: *'a::euclidean_space set*
assumes *compact S* **and** *a*: *a ∈ rel_interior S*
and *star*: $\bigwedge x.\ x \in S \Longrightarrow$ *open_segment a x ⊆ rel_interior S*
shows *starlike_compact_projective1*:
S − rel_interior S homeomorphic sphere a 1 ∩ affine hull S
and *starlike_compact_projective2*:
S homeomorphic cball a 1 ∩ affine hull S

corollary *starlike_compact_projective_special*:
assumes *compact S*
and *cb01*: *cball (0::'a::euclidean_space) 1 ⊆ S*
and *scale*: $\bigwedge x\ u.\ [\![x \in S;\ 0 \le u;\ u < 1]\!] \Longrightarrow u *_R x \in S - \mathit{frontier}\ S$
shows *S homeomorphic (cball (0::'a::euclidean_space) 1)*

10.8.2 Homeomorphisms between punctured spheres and affine sets

theorem *homeomorphic_punctured_affine_sphere_affine*:
fixes *a* :: *'a :: euclidean_space*
assumes *0 < r b ∈ sphere a r affine T a ∈ T b ∈ T affine p*
and *aff*: *aff_dim T = aff_dim p + 1*
shows *(sphere a r ∩ T) − {b} homeomorphic p*

corollary *homeomorphic_punctured_sphere_affine*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *0* < *r* **and** *b*: *b* ∈ *sphere a r*
and *affine T* **and** *affS*: *aff_dim T* + *1* = *DIM*(*'a*)
shows (*sphere a r* − {*b*}) *homeomorphic T*

corollary *homeomorphic_punctured_sphere_hyperplane*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *0* < *r* **and** *b*: *b* ∈ *sphere a r*
and *c* ≠ *0*
shows (*sphere a r* − {*b*}) *homeomorphic* {*x*::*'a*. *c* · *x* = *d*}

proposition *homeomorphic_punctured_sphere_affine_gen*:
fixes *a* :: *'a* :: *euclidean_space*
assumes *convex S bounded S* **and** *a*: *a* ∈ *rel_frontier S*
and *affine T* **and** *affS*: *aff_dim S* = *aff_dim T* + *1*
shows *rel_frontier S* − {*a*} *homeomorphic T*
proposition *homeomorphic_closedin_convex*:
fixes *S* :: *'m*::*euclidean_space set*
assumes *aff_dim S* < *DIM*(*'n*)
obtains *U* **and** *T* :: *'n*::*euclidean_space set*
where *convex U U* ≠ {} *closedin* (*top_of_set U*) *T*
S homeomorphic T

10.8.3 Locally compact sets in an open set

proposition *locally_compact_homeomorphic_closed*:
fixes *S* :: *'a*::*euclidean_space set*
assumes *locally compact S* **and** *dimlt*: *DIM*(*'a*) < *DIM*(*'b*)
obtains *T* :: *'b*::*euclidean_space set* **where** *closed T S homeomorphic T*

proposition *homeomorphic_convex_compact_cball*:
fixes *e* :: *real*
and *S* :: *'a*::*euclidean_space set*
assumes *S*: *convex S compact S interior S* ≠ {} **and** *e* > *0*
shows *S homeomorphic* (*cball* (*b*::*'a*) *e*)

corollary *homeomorphic_convex_compact*:
fixes *S* :: *'a*::*euclidean_space set*
and *T* :: *'a set*
assumes *convex S compact S interior S* ≠ {}
and *convex T compact T interior T* ≠ {}
shows *S homeomorphic T*

10.8.4 Covering spaces and lifting results for them

definition *covering_space*
:: $'a::topological\_space\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b::topological\_space\ set \Rightarrow bool$
where
$covering\_space\ c\ p\ S \equiv$
$continuous\_on\ c\ p \wedge p\ `\ c = S \wedge$
$(\forall x \in S.\ \exists T.\ x \in T \wedge openin\ (top\_of\_set\ S)\ T \wedge$
$(\exists v.\ \bigcup v = c \cap p\ -`\ T \wedge$
$(\forall u \in v.\ openin\ (top\_of\_set\ c)\ u) \wedge$
$pairwise\ disjnt\ v \wedge$
$(\forall u \in v.\ \exists q.\ homeomorphism\ u\ T\ p\ q)))$

proposition *covering_space_open_map*:
fixes $S :: 'a :: metric\_space\ set$ **and** $T :: 'b :: metric\_space\ set$
assumes p: $covering\_space\ c\ p\ S$ **and** T: $openin\ (top\_of\_set\ c)\ T$
shows $openin\ (top\_of\_set\ S)\ (p\ `\ T)$

proposition *covering_space_lift_unique*:
fixes $f :: 'a::topological\_space \Rightarrow 'b::topological\_space$
fixes $g1 :: 'a \Rightarrow 'c::real\_normed\_vector$
assumes $covering\_space\ c\ p\ S$
$g1\ a = g2\ a$
$continuous\_on\ T\ f \quad f \in T \to S$
$continuous\_on\ T\ g1 \quad g1 \in T \to c \quad \bigwedge x.\ x \in T \Longrightarrow f\ x = p(g1\ x)$
$continuous\_on\ T\ g2 \quad g2 \in T \to c \quad \bigwedge x.\ x \in T \Longrightarrow f\ x = p(g2\ x)$
$connected\ T \quad a \in T \quad x \in T$
shows $g1\ x = g2\ x$

proposition *covering_space_locally_eq*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
assumes cov: $covering\_space\ C\ p\ S$
and pim: $\bigwedge T.\ [\![T \subseteq C;\ \varphi\ T]\!] \Longrightarrow \psi(p\ `\ T)$
and qim: $\bigwedge q\ U.\ [\![U \subseteq S;\ continuous\_on\ U\ q;\ \psi\ U]\!] \Longrightarrow \varphi(q\ `\ U)$
shows $locally\ \psi\ S \longleftrightarrow locally\ \varphi\ C$
(**is** $?lhs = ?rhs$)

proposition *covering_space_lift_homotopy*:
fixes $p :: 'a::real\_normed\_vector \Rightarrow 'b::real\_normed\_vector$
and $h :: real \times 'c::real\_normed\_vector \Rightarrow 'b$
assumes cov: $covering\_space\ C\ p\ S$
and $conth$: $continuous\_on\ (\{0..1\} \times U)\ h$
and him: $h \in (\{0..1\} \times U) \to S$

and *heq*: $\bigwedge y.\ y \in U \Longrightarrow h\ (0,y) = p(f\ y)$
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \rightarrow C$
obtains *k* **where** *continuous_on* $(\{0..1\} \times U)$ *k*
$k \in (\{0..1\} \times U) \rightarrow C$
$\bigwedge y.\ y \in U \Longrightarrow k(0,\ y) = f\ y$
$\bigwedge z.\ z \in \{0..1\} \times U \Longrightarrow h\ z = p(k\ z)$

corollary *covering_space_lift_homotopy_alt*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
and *h* :: *'c::real_normed_vector* × *real* ⇒ *'b*
assumes *cov*: *covering_space C p S*
and *conth*: *continuous_on* $(U \times \{0..1\})$ *h*
and *him*: $h \in (U \times \{0..1\}) \rightarrow S$
and *heq*: $\bigwedge y.\ y \in U \Longrightarrow h\ (y,0) = p(f\ y)$
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \rightarrow C$
obtains *k* **where** *continuous_on* $(U \times \{0..1\})$ *k*
$k \in (U \times \{0..1\}) \rightarrow C$
$\bigwedge y.\ y \in U \Longrightarrow k(y,\ 0) = f\ y$
$\bigwedge z.\ z \in U \times \{0..1\} \Longrightarrow h\ z = p(k\ z)$

corollary *covering_space_lift_homotopic_function*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector* **and** *g*:: *'c::real_normed_vector* ⇒ *'a*
assumes *cov*: *covering_space C p S*
and *contg*: *continuous_on U g*
and *gim*: $g \in U \rightarrow C$
and *pgeq*: $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$
and *hom*: *homotopic_with_canon* (λ*x. True*) *U S f f'*
obtains *g'* **where** *continuous_on U g' image g' U* ⊆ *C* $\bigwedge y.\ y \in U \Longrightarrow p(g'\ y) = f'\ y$

corollary *covering_space_lift_inessential_function*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector* **and** *U* :: *'c::real_normed_vector set*
assumes *cov*: *covering_space C p S*
and *hom*: *homotopic_with_canon* (λ*x. True*) *U S f* (λ*x. a*)
obtains *g* **where** *continuous_on U g g ‘ U* ⊆ *C* $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

10.8.5 Lifting of general functions to covering space

proposition *covering_space_lift_path_strong*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
and *f* :: *'c::real_normed_vector* ⇒ *'b*
assumes *cov*: *covering_space C p S* **and** $a \in C$
and *path g* **and** *pag*: *path_image g* ⊆ *S* **and** *pas*: *pathstart g* = *p a*
obtains *h* **where** *path h path_image h* ⊆ *C pathstart h* = *a*
and $\bigwedge t.\ t \in \{0..1\} \Longrightarrow p(h\ t) = g\ t$

corollary *covering_space_lift_path*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *cov*: *covering_space C p S* **and** *path g* **and** *pig*: *path_image g* ⊆ *S*
obtains *h* **where** *path h path_image h* ⊆ *C* ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h t*) = *g t*

proposition *covering_space_lift_homotopic_paths*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *cov*: *covering_space C p S*
and *path g1* **and** *pig1*: *path_image g1* ⊆ *S*
and *path g2* **and** *pig2*: *path_image g2* ⊆ *S*
and *hom*: *homotopic_paths S g1 g2*
and *path h1* **and** *pih1*: *path_image h1* ⊆ *C* **and** *ph1*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h1 t*) = *g1 t*
and *path h2* **and** *pih2*: *path_image h2* ⊆ *C* **and** *ph2*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h2 t*) = *g2 t*
and *h1h2*: *pathstart h1* = *pathstart h2*
shows *homotopic_paths C h1 h2*

corollary *covering_space_monodromy*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *cov*: *covering_space C p S*
and *path g1* **and** *pig1*: *path_image g1* ⊆ *S*
and *path g2* **and** *pig2*: *path_image g2* ⊆ *S*
and *hom*: *homotopic_paths S g1 g2*
and *path h1* **and** *pih1*: *path_image h1* ⊆ *C* **and** *ph1*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h1 t*) = *g1 t*
and *path h2* **and** *pih2*: *path_image h2* ⊆ *C* **and** *ph2*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*h2 t*) = *g2 t*
and *h1h2*: *pathstart h1* = *pathstart h2*
shows *pathfinish h1* = *pathfinish h2*

corollary *covering_space_lift_homotopic_path*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
assumes *cov*: *covering_space C p S*
and *hom*: *homotopic_paths S f f'*
and *path g* **and** *pig*: *path_image g* ⊆ *C*
and *a*: *pathstart g* = *a* **and** *b*: *pathfinish g* = *b*
and *pgeq*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*g t*) = *f t*
obtains *g'* **where** *path g' path_image g'* ⊆ *C*
pathstart g' = *a pathfinish g'* = *b* ⋀*t*. *t* ∈ {*0..1*} ⟹ *p*(*g' t*) = *f' t*

proposition *covering_space_lift_general*:
fixes p :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
and *f* :: *'c::real_normed_vector* ⇒ *'b*
assumes *cov*: *covering_space C p S* **and** *a* ∈ *C z* ∈ *U*

and *U*: *path_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \rightarrow S$
and *feq*: $f\ z = p\ a$
and *hom*: $\bigwedge r.$ ⟦*path r*; *path_image r* $\subseteq U$; *pathstart r* $= z$; *pathfinish r* $= z$⟧
$\Longrightarrow \exists\, q.$ *path q* $\wedge$ *path_image q* $\subseteq C$ $\wedge$
pathstart q $= a \wedge$ *pathfinish q* $= a \wedge$
homotopic_paths S $(f \circ r)$ $(p \circ q)$
obtains *g* **where** *continuous_on U g* $g \in U \rightarrow C$ $g\ z = a$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

corollary *covering_space_lift_stronger*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
and *f* :: *'c::real_normed_vector* ⇒ *'b*
assumes *cov*: *covering_space C p S* $a \in C$ $z \in U$
and *U*: *path_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \rightarrow S$
and *feq*: $f\ z = p\ a$
and *hom*: $\bigwedge r.$ ⟦*path r*; *path_image r* $\subseteq U$; *pathstart r* $= z$; *pathfinish r* $= z$⟧
$\Longrightarrow \exists\, b.$ *homotopic_paths S* $(f \circ r)$ (*linepath b b*)
obtains *g* **where** *continuous_on U g* $g \in U \rightarrow C$ $g\ z = a$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

corollary *covering_space_lift_strong*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
and *f* :: *'c::real_normed_vector* ⇒ *'b*
assumes *cov*: *covering_space C p S* $a \in C$ $z \in U$
and *scU*: *simply_connected U* **and** *lpcU*: *locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \rightarrow S$
and *feq*: $f\ z = p\ a$
obtains *g* **where** *continuous_on U g* $g \in U \rightarrow C$ $g\ z = a$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

corollary *covering_space_lift*:
fixes *p* :: *'a::real_normed_vector* ⇒ *'b::real_normed_vector*
and *f* :: *'c::real_normed_vector* ⇒ *'b*
assumes *cov*: *covering_space C p S*
and *U*: *simply_connected U locally path_connected U*
and *contf*: *continuous_on U f* **and** *fim*: $f \in U \rightarrow S$
obtains *g* **where** *continuous_on U g* $g \in U \rightarrow C$ $\bigwedge y.\ y \in U \Longrightarrow p(g\ y) = f\ y$

end

theory *Equivalence_Lebesgue_Henstock_Integration*
imports
Lebesgue_Measure
Henstock_Kurzweil_Integration
Complete_Measure
Set_Integral

Homeomorphism
Cartesian_Euclidean_Space
begin

10.8.6 Equivalence Lebesgue integral on *lborel* and HK-integral

10.8.7 Absolute integrability (this is the same as Lebesgue integrability)

corollary *absolutely_integrable_spike_set*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes f: f *absolutely_integrable_on* S **and** *neg*: *negligible* $\{x \in S - T. f\ x \neq 0\}$ *negligible* $\{x \in T - S. f\ x \neq 0\}$
shows f *absolutely_integrable_on* T

10.8.8 Applications to Negligibility

corollary *eventually_ae_filter_negligible*:
eventually P (*ae_filter lebesgue*) $\longleftrightarrow$ ($\exists N.$ *negligible* $N \wedge \{x. \neg P\ x\} \subseteq N$)

proposition *negligible_convex_frontier*:
fixes $S :: 'N :: euclidean\_space\ set$
assumes *convex* S
shows *negligible*(*frontier* S)

corollary *negligible_sphere*: *negligible* (*sphere* a e)

proposition *open_not_negligible*:
assumes *open* S $S \neq \{\}$
shows $\neg$ *negligible* S

10.8.9 Negligibility of image under non-injective linear map

10.8.10 Negligibility of a Lipschitz image of a negligible set

proposition *negligible_locally_Lipschitz_image*:
fixes $f :: 'M::euclidean\_space \Rightarrow 'N::euclidean\_space$
assumes *MleN*: $DIM('M) \leq DIM('N)$ *negligible* S
and *lips*: $\bigwedge x.\ x \in S$
$\Longrightarrow \exists T\ B.$ *open* $T \wedge x \in T \wedge$

$(\forall y \in S \cap T.\ norm(f\ y - f\ x) \leq B * norm(y - x))$
shows *negligible* $(f\ `\ S)$

corollary *negligible_differentiable_image_negligible*:
fixes f :: *'M::euclidean_space* ⇒ *'N::euclidean_space*
assumes *MleN*: $DIM('M) \leq DIM('N)$ *negligible S*
and *diff_f*: *f differentiable_on S*
shows *negligible* $(f\ `\ S)$

corollary *negligible_differentiable_image_lowdim*:
fixes f :: *'M::euclidean_space* ⇒ *'N::euclidean_space*
assumes *MlessN*: $DIM('M) < DIM('N)$ **and** *diff_f*: *f differentiable_on S*
shows *negligible* $(f\ `\ S)$

10.8.11 Measurability of countable unions and intersections of various kinds.

10.8.12 Negligibility is a local property

10.8.13 Integral bounds

proposition *bounded_variation_absolutely_integrable_interval*:
fixes f :: *'n::euclidean_space* ⇒ *'m::euclidean_space*
assumes *f*: *f integrable_on cbox a b*
and ∗: $\bigwedge d.\ d$ *division_of* $(cbox\ a\ b) \Longrightarrow sum\ (\lambda K.\ norm(integral\ K\ f))\ d \leq B$
shows *f absolutely_integrable_on cbox a b*

10.8.14 Outer and inner approximation of measurable sets by well-behaved sets.

proposition *measurable_outer_intervals_bounded*:
assumes $S \in lmeasurable\ S \subseteq cbox\ a\ b\ e > 0$
obtains $\mathcal{D}$
where *countable* $\mathcal{D}$
$\bigwedge K.\ K \in \mathcal{D} \Longrightarrow K \subseteq cbox\ a\ b \wedge K \neq \{\} \wedge (\exists c\ d.\ K = cbox\ c\ d)$
$pairwise\ (\lambda A\ B.\ interior\ A \cap interior\ B = \{\})\ \mathcal{D}$
$\bigwedge u\ v.\ cbox\ u\ v \in \mathcal{D} \Longrightarrow \exists n.\ \forall i \in Basis.\ v \cdot i - u \cdot i = (b \cdot i - a \cdot i)/2\hat{\ }n$
$\bigwedge K.\ [\![K \in \mathcal{D};\ box\ a\ b \neq \{\}]\!] \Longrightarrow interior\ K \neq \{\}$
$S \subseteq \bigcup\mathcal{D}\ \bigcup\mathcal{D} \in lmeasurable\ measure\ lebesgue\ (\bigcup\mathcal{D}) \leq measure\ lebesgue\ S + e$

10.8.15 Transformation of measure by linear maps

proposition *measure_linear_sufficient*:

fixes *f* :: *'n::euclidean_space* ⇒ *'n*
assumes *linear f* **and** *S*: *S* ∈ *lmeasurable*
and *im*: ⋀*a b. measure lebesgue* (*f* ' (*cbox a b*)) = *m* ∗ *measure lebesgue* (*cbox a b*)
shows *f* ' *S* ∈ *lmeasurable* ∧ *m* ∗ *measure lebesgue S* = *measure lebesgue* (*f* ' *S*)

10.8.16 Lemmas about absolute integrability

corollary *absolutely_integrable_on_const* [*simp*]:
fixes *c* :: *'a::euclidean_space*
assumes *S* ∈ *lmeasurable*
shows (λ*x. c*) *absolutely_integrable_on S*

10.8.17 Componentwise

proposition *absolutely_integrable_componentwise_iff*:
shows *f absolutely_integrable_on A* ⟷ (∀ *b*∈*Basis.* (λ*x. f x* · *b*) *absolutely_integrable_on A*)

corollary *absolutely_integrable_max_1*:
fixes *f* :: *'n::euclidean_space* ⇒ *real*
assumes *f absolutely_integrable_on S g absolutely_integrable_on S*
shows (λ*x. max* (*f x*) (*g x*)) *absolutely_integrable_on S*

corollary *absolutely_integrable_min_1*:
fixes *f* :: *'n::euclidean_space* ⇒ *real*
assumes *f absolutely_integrable_on S g absolutely_integrable_on S*
shows (λ*x. min* (*f x*) (*g x*)) *absolutely_integrable_on S*

10.8.18 Dominated convergence

proposition *integral_countable_UN*:
fixes *f* :: *real^'m* ⇒ *real^'n*
assumes *f*: *f absolutely_integrable_on* (⋃(*range s*))
and *s*: ⋀*m. s m* ∈ *sets lebesgue*
shows ⋀*n. f absolutely_integrable_on* (⋃*m*≤*n. s m*)
and (λ*n. integral* (⋃*m*≤*n. s m*) *f*) ⟶ *integral* (⋃(*s* ' *UNIV*)) *f* (**is** *?F* ⟶ *?I*)

10.8.19 Fundamental Theorem of Calculus for the Lebesgue integral

10.8.20 Integration by parts

10.8.21 A non-negative continuous function whose integral is zero must be zero

corollary *integral_cbox_eq_0_iff*:
  fixes f :: $'a$::*euclidean_space* $\Rightarrow$ *real*
  assumes *continuous_on* (*cbox a b*) f **and** *box a b* $\neq$ {}
    and $\bigwedge x.\ x \in$ *cbox a b* $\Longrightarrow f\, x \geq 0$
  shows *integral* (*cbox a b*) $f = 0 \longleftrightarrow (\forall x \in$ *cbox a b*. $f\, x = 0)$ (**is** *?lhs* = *?rhs*)

10.8.22 Various common equivalent forms of function measurability

10.8.23 Lebesgue sets and continuous images

proposition *lebesgue_regular_inner*:
 assumes $S \in$ *sets lebesgue*
 obtains K C **where** *negligible* K $\bigwedge n$::*nat*. *compact*($C\, n$) $S = (\bigcup n.\ C\, n) \cup K$

10.8.24 Affine lemmas

lemma *lebesgue_integral_real_affine*:
  fixes f :: *real* $\Rightarrow$ $'a$ :: *euclidean_space* **and** c :: *real*
  assumes c: $c \neq 0$ **shows** $(\int x.\ f\ x\ \partial\ lebesgue) = |c| *_R (\int x.\ f(t + c * x)\ \partial lebesgue)$

10.8.25 More results on integrability

proposition *measurable_bounded_by_integrable_imp_integrable*:
  fixes f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
  assumes f: $f \in$ *borel_measurable* (*lebesgue_on* S) **and** g: g *integrable_on* S
    and *normf*: $\bigwedge x.\ x \in S \Longrightarrow$ *norm*($f\, x$) $\leq g\ x$ **and** S: $S \in$ *sets lebesgue*
  shows f *integrable_on* S

corollary *measurable_bounded_by_integrable_imp_lebesgue_integrable*:
  fixes f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
 assumes f: $f \in$ *borel_measurable* (*lebesgue_on* S) **and** g: *integrable* (*lebesgue_on* S) g

and *normf*: $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \leq g\ x$ **and** *S*: $S \in sets\ lebesgue$
shows *integrable* (*lebesgue_on* *S*) *f*

corollary *measurable_bounded_by_integrable_imp_integrable_real*:
fixes f :: *'a::euclidean_space* ⇒ *real*
assumes $f \in$ *borel_measurable* (*lebesgue_on* *S*) *g integrable_on S* $\bigwedge x.\ x \in S \Longrightarrow abs(f\ x) \leq g\ x$ $S \in$ *sets lebesgue*
shows *f integrable_on S*

10.8.26 Relation between Borel measurability and integrability.

proposition *negligible_differentiable_vimage*:
fixes f :: *'a* ⇒ *'a::euclidean_space*
assumes *negligible T*
and f': $\bigwedge x.\ x \in S \Longrightarrow inj(f'\ x)$
and *derf*: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
shows *negligible* $\{x \in S.\ f\ x \in T\}$
proposition *has_derivative_inverse_within*:
fixes f :: *'a::real_normed_vector* ⇒ *'b::euclidean_space*
assumes *der_f*: (*f has_derivative f'*) (*at a within S*)
and *cont_g*: *continuous* (*at* (*f a*) *within f ‘ S*) *g*
and $a \in S$ *linear g'* **and** *id*: $g' \circ f' = id$
and *gf*: $\bigwedge x.\ x \in S \Longrightarrow g(f\ x) = x$
shows (*g has_derivative g'*) (*at* (*f a*) *within f ‘ S*)

end

10.9 Harmonic Numbers

theory *Harmonic_Numbers*
imports
Complex_Transcendental
Summation_Tests
begin

10.9.1 The Harmonic numbers

definition *harm* :: *nat* ⇒ *'a* :: *real_normed_field* **where**
harm n = ($\sum$*k=1..n. inverse* (*of_nat k*))

theorem *not_convergent_harm*: ¬*convergent* (*harm* :: *nat* ⇒ *'a* :: *real_normed_field*)

10.9.2 The Euler-Mascheroni constant

lemma *euler_mascheroni_LIMSEQ*:
(*λn. harm n* − *ln* (*of_nat n*) :: *real*) ⟶ *euler_mascheroni*

theorem *alternating_harmonic_series_sums*: (*λk.* (−*1*) ^*k* / *real_of_nat* (*Suc k*)) *sums ln 2*

end

10.10 The Gamma Function

theory *Gamma_Function*
imports
Equivalence_Lebesgue_Henstock_Integration
Summation_Tests
Harmonic_Numbers
HOL−Library.Nonpos_Ints
HOL−Library.Periodic_Fun
begin

10.10.1 The Euler form and the logarithmic Gamma function

definition *Gamma_series* :: (*'a* :: {*banach*,*real_normed_field*}) ⇒ *nat* ⇒ *'a* **where**
Gamma_series z n = *fact n* ∗ *exp* (*z* ∗ *of_real* (*ln* (*of_nat n*))) / *pochhammer z* (*n*+*1*)
definition *ln_Gamma_series* :: (*'a* :: {*banach*,*real_normed_field*,*ln*}) ⇒ *nat* ⇒ *'a* **where**
ln_Gamma_series z n = *z* ∗ *ln* (*of_nat n*) − *ln z* − (∑*k*=*1*..*n*. *ln* (*z* / *of_nat k* + *1*))

theorem *ln_Gamma_complex_LIMSEQ*: (*z* :: *complex*) ∉ $\mathbb{Z}_{\le 0}$ ⟹ *ln_Gamma_series z* ⟶ *ln_Gamma z*

10.10.2 The Polygamma functions

definition *Polygamma* :: *nat* ⇒ (*'a* :: {*real_normed_field*,*banach*}) ⇒ *'a* **where**
Polygamma n z = (*if n* = *0 then*
(∑*k*. *inverse* (*of_nat* (*Suc k*)) − *inverse* (*z* + *of_nat k*)) − *euler_mascheroni else*
(−*1*) ^*Suc n* ∗ *fact n* ∗ (∑*k*. *inverse* ((*z* + *of_nat k*) ^*Suc n*)))

abbreviation *Digamma* :: (*'a* :: {*real_normed_field*,*banach*}) ⇒ *'a* **where**
Digamma ≡ *Polygamma 0*

theorem *Digamma_LIMSEQ*:
fixes z :: $'a$ :: {*banach*,*real_normed_field*}
assumes z: $z \neq 0$
shows $(\lambda m.\ of\_real\ (ln\ (real\ m)) - (\sum n<m.\ inverse\ (z + of\_nat\ n))) \longrightarrow Digamma\ z$

theorem *Polygamma_LIMSEQ*:
fixes z :: $'a$ :: {*banach*,*real_normed_field*}
assumes $z \neq 0$ **and** $n > 0$
shows $(\lambda k.\ inverse\ ((z + of\_nat\ k)\hat{}Suc\ n))\ sums\ ((-1)\ \hat{}\ Suc\ n * Polygamma\ n\ z\ /\ fact\ n)$

theorem *has_field_derivative_ln_Gamma_complex* [*derivative_intros*]:
fixes z :: *complex*
assumes z: $z \notin \mathbb{R}_{\leq 0}$
shows $(ln\_Gamma\ has\_field\_derivative\ Digamma\ z)\ (at\ z)$

theorem *Polygamma_plus1*:
assumes $z \neq 0$
shows $Polygamma\ n\ (z + 1) = Polygamma\ n\ z + (-1)\hat{}n * fact\ n\ /\ (z\ \hat{}\ Suc\ n)$

theorem *Digamma_of_nat*:
$Digamma\ (of\_nat\ (Suc\ n) :: 'a :: \{real\_normed\_field,banach\}) = harm\ n - euler\_mascheroni$

theorem *has_field_derivative_Polygamma* [*derivative_intros*]:
fixes z :: $'a$ :: {*real_normed_field*,*euclidean_space*}
assumes z: $z \notin \mathbb{Z}_{\leq 0}$
shows $(Polygamma\ n\ has\_field\_derivative\ Polygamma\ (Suc\ n)\ z)\ (at\ z\ within\ A)$

10.10.3 Basic properties

theorem *Gamma_series_LIMSEQ* [*tendsto_intros*]:
$Gamma\_series\ z \longrightarrow Gamma\ z$

theorem *Gamma_plus1*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow Gamma\ (z + 1) = z * Gamma\ z$

theorem *pochhammer_Gamma*: $z \notin \mathbb{Z}_{\leq 0} \Longrightarrow pochhammer\ z\ n = Gamma\ (z + of\_nat\ n)\ /\ Gamma\ z$

theorem *Gamma_fact*: $Gamma\ (1 + of\_nat\ n) = fact\ n$

10.10.4 Differentiability

theorem *has_field_derivative_Gamma* [*derivative_intros*]:
$z \notin \mathbb{Z}_{\leq 0} \Longrightarrow$ (*Gamma has_field_derivative Gamma z* ∗ *Digamma z*) (*at z within A*)

theorem *log_convex_Gamma_real*: *convex_on* {*0*<..} (*ln* ∘ *Gamma* :: *real* ⇒ *real*)

10.10.5 The uniqueness of the real Gamma function

theorem *Gamma_pos_real_unique*:
assumes *x*: $x > 0$
shows *G x* = *Gamma x*

10.10.6 The Beta function

theorem *Beta_plus1_plus1*:
assumes $x \notin \mathbb{Z}_{\leq 0}$ $y \notin \mathbb{Z}_{\leq 0}$
shows *Beta* (*x* + *1*) *y* + *Beta x* (*y* + *1*) = *Beta x y*

theorem *Beta_plus1_left*:
assumes $x \notin \mathbb{Z}_{\leq 0}$
shows (*x* + *y*) ∗ *Beta* (*x* + *1*) *y* = *x* ∗ *Beta x y*

theorem *Beta_plus1_right*:
assumes $y \notin \mathbb{Z}_{\leq 0}$
shows (*x* + *y*) ∗ *Beta x* (*y* + *1*) = *y* ∗ *Beta x y*

10.10.7 Legendre duplication theorem

theorem *Gamma_legendre_duplication*:
fixes *z* :: *complex*
assumes $z \notin \mathbb{Z}_{\leq 0}$ $z + 1/2 \notin \mathbb{Z}_{\leq 0}$
shows *Gamma z* ∗ *Gamma* (*z* + *1*/*2*) =
exp ((*1* − *2*∗*z*) ∗ *of_real* (*ln 2*)) ∗ *of_real* (*sqrt pi*) ∗ *Gamma* (*2*∗*z*)

10.10.8 Alternative definitions

theorem *Gamma_series_euler'*:
assumes *z*: $(z :: 'a :: Gamma) \notin \mathbb{Z}_{\leq 0}$
shows $(\lambda n.\ Gamma\_series\_euler'\ z\ n) \longrightarrow Gamma\ z$

theorem *Gamma_Weierstrass_complex*: $Gamma\_series\_Weierstrass\ z \longrightarrow Gamma\ (z :: complex)$

theorem *gbinomial_Gamma*:
assumes $z + 1 \notin \mathbb{Z}_{\leq 0}$
shows $(z\ gchoose\ n) = Gamma\ (z + 1)\ /\ (fact\ n * Gamma\ (z - of\_nat\ n + 1))$

theorem *Gamma_integral_complex*:
assumes *z*: $Re\ z > 0$
shows $((\lambda t.\ of\_real\ t\ powr\ (z - 1)\ /\ of\_real\ (exp\ t))\ has\_integral\ Gamma\ z)\ \{0..\}$

theorem *has_integral_Beta_real*:
assumes *a*: $a > 0$ **and** *b*: $b > (0 :: real)$
shows $((\lambda t.\ t\ powr\ (a - 1) * (1 - t)\ powr\ (b - 1))\ has\_integral\ Beta\ a\ b)\ \{0..1\}$

10.10.9 The Weierstraß product formula for the sine

theorem *sin_product_formula_complex*:
fixes $z :: complex$
shows $(\lambda n.\ of\_real\ pi * z * (\prod k{=}1..n.\ 1 - z\hat{}2\ /\ of\_nat\ k\hat{}2)) \longrightarrow sin\ (of\_real\ pi * z)$

theorem *wallis*: $(\lambda n.\ \prod k{=}1..n.\ (4{*}real\ k\hat{}2)\ /\ (4{*}real\ k\hat{}2 - 1)) \longrightarrow pi\ /\ 2$

10.10.10 The Solution to the Basel problem

theorem *inverse_squares_sums*: $(\lambda n.\ 1\ /\ (n + 1)^2)\ sums\ (pi^2\ /\ 6)$

end

theory *Interval_Integral*
imports *Equivalence_Lebesgue_Henstock_Integration*
begin

10.10.11 Approximating a (possibly infinite) interval

proposition *einterval_Icc_approximation*:
fixes $a\ b$ :: *ereal*
assumes $a < b$
obtains $u\ l$ :: $nat \Rightarrow real$ **where**
$einterval\ a\ b = (\bigcup i.\ \{l\ i\ ..\ u\ i\})$
$incseq\ u\ decseq\ l\ \bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longrightarrow a\ u \longrightarrow b$

definition *interval_lebesgue_integral* :: *real measure* $\Rightarrow$ *ereal* $\Rightarrow$ *ereal* $\Rightarrow$ (*real* $\Rightarrow$ *'a*) $\Rightarrow$ *'a*::{*banach*, *second_countable_topology*} **where**
interval_lebesgue_integral M a b f =
(*if* $a \le b$ *then* (*LINT x*:*einterval a b*|*M*. *f x*) *else* $-$ (*LINT x*:*einterval b a*|*M*. *f x*))

definition *interval_lebesgue_integrable* :: *real measure* $\Rightarrow$ *ereal* $\Rightarrow$ *ereal* $\Rightarrow$ (*real* $\Rightarrow$ *'a*::{*banach*, *second_countable_topology*}) $\Rightarrow$ *bool* **where**
interval_lebesgue_integrable M a b f =
(*if* $a \le b$ *then set_integrable M* (*einterval a b*) *f else set_integrable M* (*einterval b a*) *f*)

10.10.12 Basic properties of integration over an interval

proposition *interval_integrable_to_infinity_eq*: (*interval_lebesgue_integrable M a* ∞ *f*) =
(*set_integrable M* {*a*<..} *f*)

10.10.13 Basic properties of integration over an interval wrt lebesgue measure

10.10.14 General limit approximation arguments

proposition *interval_integral_Icc_approx_nonneg*:
fixes $a\ b$ :: *ereal*
assumes $a < b$
fixes $u\ l$ :: $nat \Rightarrow real$
assumes *approx*: $einterval\ a\ b = (\bigcup i.\ \{l\ i\ ..\ u\ i\})$
$incseq\ u\ decseq\ l\ \bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longrightarrow a\ u \longrightarrow b$
fixes f :: $real \Rightarrow real$

assumes *f_integrable*: $\bigwedge i.$ *set_integrable lborel* $\{l\ i..u\ i\}$ *f*
assumes *f_nonneg*: *AE x in lborel.* $a <$ *ereal* $x \longrightarrow$ *ereal* $x < b \longrightarrow 0 \leq f\ x$
assumes *f_measurable*: *set_borel_measurable lborel* (*einterval a b*) *f*
assumes *lbint_lim*: $(\lambda i.$ *LBINT* $x{=}l\ i..\ u\ i.\ f\ x) \longlongrightarrow C$
shows
set_integrable lborel (*einterval a b*) *f*
(*LBINT* $x{=}a..b.\ f\ x) = C$

proposition *interval_integral_Icc_approx_integrable*:
fixes $u\ l$ :: *nat* $\Rightarrow$ *real* **and** $a\ b$ :: *ereal*
fixes f :: *real* $\Rightarrow$ *'a*::{*banach*, *second_countable_topology*}
assumes $a < b$
assumes *approx*: *einterval a b* $= (\bigcup i.\ \{l\ i\ ..\ u\ i\})$
incseq u decseq l $\bigwedge i.\ l\ i < u\ i\ \bigwedge i.\ a < l\ i\ \bigwedge i.\ u\ i < b$
$l \longlongrightarrow a\ u \longlongrightarrow b$
assumes *f_integrable*: *set_integrable lborel* (*einterval a b*) *f*
shows $(\lambda i.$ *LBINT* $x{=}l\ i..\ u\ i.\ f\ x) \longlongrightarrow$ (*LBINT* $x{=}a..b.\ f\ x$)

10.10.15 A slightly stronger Fundamental Theorem of Calculus

theorem *interval_integral_FTC_integrable*:
fixes $f\ F$ :: *real* $\Rightarrow$ *'a*::*euclidean_space* **and** $a\ b$ :: *ereal*
assumes $a < b$
assumes *F*: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow$ (*F has_vector_derivative f x*) (*at x*)
assumes *f*: $\bigwedge x.\ a <$ *ereal* $x \Longrightarrow$ *ereal* $x < b \Longrightarrow$ *isCont f x*
assumes *f_integrable*: *set_integrable lborel* (*einterval a b*) *f*
assumes *A*: $((F \circ$ *real_of_ereal*$) \longlongrightarrow A)$ (*at_right a*)
assumes *B*: $((F \circ$ *real_of_ereal*$) \longlongrightarrow B)$ (*at_left b*)
shows (*LBINT* $x{=}a..b.\ f\ x) = B - A$

theorem *interval_integral_FTC2*:
fixes $a\ b\ c$ :: *real* **and** f :: *real* $\Rightarrow$ *'a*::*euclidean_space*
assumes $a \leq c\ c \leq b$
and *contf*: *continuous_on* $\{a..b\}$ *f*
fixes x :: *real*
assumes $a \leq x$ **and** $x \leq b$
shows $((\lambda u.$ *LBINT* $y{=}c..u.\ f\ y)$ *has_vector_derivative* $(f\ x))$ (*at x within* $\{a..b\}$)

proposition *einterval_antiderivative*:
fixes $a\ b$ :: *ereal* **and** f :: *real* $\Rightarrow$ *'a*::*euclidean_space*
assumes $a < b$ **and** *contf*: $\bigwedge x$ :: *real*. $a < x \Longrightarrow x < b \Longrightarrow$ *isCont f x*
shows $\exists F.\ \forall x$ :: *real*. $a < x \longrightarrow x < b \longrightarrow$ (*F has_vector_derivative f x*) (*at x*)

10.10.16 The substitution theorem

theorem *interval_integral_substitution_finite*:
  fixes $a\ b :: real$ **and** $f :: real \Rightarrow 'a::euclidean\_space$
  assumes $a \leq b$
  and *derivg*: $\bigwedge x.\ a \leq x \Longrightarrow x \leq b \Longrightarrow (g\ has\_real\_derivative\ (g'\ x))\ (at\ x\ within\ \{a..b\})$
  and *contf* : $continuous\_on\ (g\ `\ \{a..b\})\ f$
  and *contg'*: $continuous\_on\ \{a..b\}\ g'$
  shows $(LBINT\ x{=}a..b.\ g'\ x *_R f\ (g\ x)) = (LBINT\ y{=}g\ a..g\ b.\ f\ y)$

theorem *interval_integral_substitution_integrable*:
  fixes $f :: real \Rightarrow 'a::euclidean\_space$ **and** $a\ b\ u\ v :: ereal$
  assumes $a < b$
  and *deriv_g*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow DERIV\ g\ x :> g'\ x$
  and *contf*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ f\ (g\ x)$
  and *contg'*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ g'\ x$
  and *g'_nonneg*: $\bigwedge x.\ a \leq ereal\ x \Longrightarrow ereal\ x \leq b \Longrightarrow 0 \leq g'\ x$
  and *A*: $((ereal \circ g \circ real\_of\_ereal) \longrightarrow A)\ (at\_right\ a)$
  and *B*: $((ereal \circ g \circ real\_of\_ereal) \longrightarrow B)\ (at\_left\ b)$
  and *integrable*: $set\_integrable\ lborel\ (einterval\ a\ b)\ (\lambda x.\ g'\ x *_R f\ (g\ x))$
  and *integrable2*: $set\_integrable\ lborel\ (einterval\ A\ B)\ (\lambda x.\ f\ x)$
  shows $(LBINT\ x{=}A..B.\ f\ x) = (LBINT\ x{=}a..b.\ g'\ x *_R f\ (g\ x))$

theorem *interval_integral_substitution_nonneg*:
  fixes $f\ g\ g' :: real \Rightarrow real$ **and** $a\ b\ u\ v :: ereal$
  assumes $a < b$
  and *deriv_g*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow DERIV\ g\ x :> g'\ x$
  and *contf*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ f\ (g\ x)$
  and *contg'*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow isCont\ g'\ x$
  and *f_nonneg*: $\bigwedge x.\ a < ereal\ x \Longrightarrow ereal\ x < b \Longrightarrow 0 \leq f\ (g\ x)$
  and *g'_nonneg*: $\bigwedge x.\ a \leq ereal\ x \Longrightarrow ereal\ x \leq b \Longrightarrow 0 \leq g'\ x$
  and *A*: $((ereal \circ g \circ real\_of\_ereal) \longrightarrow A)\ (at\_right\ a)$
  and *B*: $((ereal \circ g \circ real\_of\_ereal) \longrightarrow B)\ (at\_left\ b)$
  and *integrable_fg*: $set\_integrable\ lborel\ (einterval\ a\ b)\ (\lambda x.\ f\ (g\ x) * g'\ x)$
  shows
    $set\_integrable\ lborel\ (einterval\ A\ B)\ f$
    $(LBINT\ x{=}A..B.\ f\ x) = (LBINT\ x{=}a..b.\ (f\ (g\ x) * g'\ x))$

proposition *interval_integral_norm*:
  fixes $f :: real \Rightarrow 'a :: \{banach,\ second\_countable\_topology\}$
  shows $interval\_lebesgue\_integrable\ lborel\ a\ b\ f \Longrightarrow a \leq b \Longrightarrow$

norm (*LBINT t=a..b. f t*) $\leq$ *LBINT t=a..b. norm* (*f t*)

proposition *interval_integral_norm2*:
interval_lebesgue_integrable lborel a b f $\Longrightarrow$
norm (*LBINT t=a..b. f t*) $\leq$ |*LBINT t=a..b. norm* (*f t*)|

end

10.11 Integration by Substition for the Lebesgue Integral

theory *Lebesgue_Integral_Substitution*
imports *Interval_Integral*
begin

theorem *nn_integral_substitution*:
fixes *f* :: *real* $\Rightarrow$ *real*
assumes *Mf*[*measurable*]: *set_borel_measurable borel* {*g a..g b*} *f*
assumes *derivg*: $\bigwedge$*x. x* $\in$ {*a..b*} $\Longrightarrow$ (*g has_real_derivative g′ x*) (*at x*)
assumes *contg′*: *continuous_on* {*a..b*} *g′*
assumes *derivg_nonneg*: $\bigwedge$*x. x* $\in$ {*a..b*} $\Longrightarrow$ *g′ x* $\geq$ *0*
assumes *a* $\leq$ *b*
shows ($\int^+$*x. f x* $*$ *indicator* {*g a..g b*} *x* ∂*lborel*) =
($\int^+$*x. f* (*g x*) $*$ *g′ x* $*$ *indicator* {*a..b*} *x* ∂*lborel*)

theorem *integral_substitution*:
assumes *integrable*: *set_integrable lborel* {*g a..g b*} *f*
assumes *derivg*: $\bigwedge$*x. x* $\in$ {*a..b*} $\Longrightarrow$ (*g has_real_derivative g′ x*) (*at x*)
assumes *contg′*: *continuous_on* {*a..b*} *g′*
assumes *derivg_nonneg*: $\bigwedge$*x. x* $\in$ {*a..b*} $\Longrightarrow$ *g′ x* $\geq$ *0*
assumes *a* $\leq$ *b*
shows *set_integrable lborel* {*a..b*} (λ*x. f* (*g x*) $*$ *g′ x*)
and (*LBINT x. f x* $*$ *indicator* {*g a..g b*} *x*) = (*LBINT x. f* (*g x*) $*$ *g′ x* $*$ *indicator* {*a..b*} *x*)

theorem *interval_integral_substitution*:
assumes *integrable*: *set_integrable lborel* {*g a..g b*} *f*
assumes *derivg*: $\bigwedge$*x. x* $\in$ {*a..b*} $\Longrightarrow$ (*g has_real_derivative g′ x*) (*at x*)
assumes *contg′*: *continuous_on* {*a..b*} *g′*
assumes *derivg_nonneg*: $\bigwedge$*x. x* $\in$ {*a..b*} $\Longrightarrow$ *g′ x* $\geq$ *0*
assumes *a* $\leq$ *b*
shows *set_integrable lborel* {*a..b*} (λ*x. f* (*g x*) $*$ *g′ x*)
and (*LBINT x=g a..g b. f x*) = (*LBINT x=a..b. f* (*g x*) $*$ *g′ x*)

end

10.12 The Volume of an n-Dimensional Ball

theory *Ball_Volume*
imports *Gamma_Function Lebesgue_Integral_Substitution*
begindefinition *unit_ball_vol* :: *real* ⇒ *real* **where**
unit_ball_vol n = *pi powr* (*n* / *2*) / *Gamma* (*n* / *2* + *1*)

corollary *content_ball*:
content (*ball c r*) = *unit_ball_vol* (*DIM*('*a*)) ∗ *r* ^ *DIM*('*a*)

end

10.13 Integral Test for Summability

theory *Integral_Test*
imports *Henstock_Kurzweil_Integration*
beginlocale *antimono_fun_sum_integral_diff* =
fixes *f* :: *real* ⇒ *real*
assumes *dec*: $\bigwedge x\ y.\ x \geq 0 \Longrightarrow x \leq y \Longrightarrow f\ x \geq f\ y$
assumes *nonneg*: $\bigwedge x.\ x \geq 0 \Longrightarrow f\ x \geq 0$
assumes *cont*: *continuous_on* {*0*..} *f*
begin

theorem *integral_test*:
summable (λ*n*. *f* (*of_nat n*)) ⟷ *convergent* (λ*n*. *integral* {*0*..*of_nat n*} *f*)

end

10.14 Continuity of the indefinite integral; improper integral theorem

theory *Improper_Integral*
imports *Equivalence_Lebesgue_Henstock_Integration*
begin

10.14.1 Equiintegrability

definition *equiintegrable_on* (**infixr** ‹*equiintegrable'_on*› *46*)
where *F equiintegrable_on I* ≡
(∀*f* ∈ *F*. *f integrable_on I*) ∧
(∀ *e* > *0*. ∃γ. *gauge* γ ∧
(∀*f* 𝒟. *f* ∈ *F* ∧ 𝒟 *tagged_division_of I* ∧ γ *fine* 𝒟
⟶ *norm* (($\sum$(*x*,*K*) ∈ 𝒟. *content K* $*_R$ *f x*) − *integral I f*)
< *e*))

corollary *equiintegrable_sum_real*:
fixes F :: (*real* ⇒ *'b*::*euclidean_space*) *set*
assumes F *equiintegrable_on* $\{a..b\}$
shows $(\bigcup I \in \text{Collect finite}. \bigcup c \in \{c. (\forall i \in I. c\ i \geq 0) \wedge \text{sum } c\ I = 1\}.$
$\bigcup f \in I \to F. \{(\lambda x. \text{sum } (\lambda i. c\ i *_R f\ i\ x)\ I)\})$
equiintegrable_on $\{a..b\}$

theorem *equiintegrable_limit*:
fixes g :: *'a* :: *euclidean_space* ⇒ *'b* :: *banach*
assumes *feq*: *range f equiintegrable_on cbox a b*
and *to_g*: $\bigwedge x. x \in \text{cbox } a\ b \Longrightarrow (\lambda n. f\ n\ x) \longrightarrow g\ x$
shows g *integrable_on cbox a b* $\wedge$ $(\lambda n. \text{integral } (\text{cbox } a\ b)\ (f\ n)) \longrightarrow \text{integral}$ $(\text{cbox } a\ b)\ g$

10.14.2 Subinterval restrictions for equiintegrable families

proposition *sum_content_area_over_thin_division*:
assumes *div*: $\mathcal{D}$ *division_of* S **and** S: $S \subseteq \text{cbox } a\ b$ **and** i: $i \in \text{Basis}$
and $a \cdot i \leq c\ c \leq b \cdot i$
and *nonmt*: $\bigwedge K. K \in \mathcal{D} \Longrightarrow K \cap \{x. x \cdot i = c\} \neq \{\}$
shows $(b \cdot i - a \cdot i) * (\sum K \in \mathcal{D}. \text{content } K / (\text{interval\_upperbound } K \cdot i - \text{interval\_lowerbound } K \cdot i))$
$\leq 2 * \text{content}(\text{cbox } a\ b)$

proposition *bounded_equiintegral_over_thin_tagged_partial_division*:
fixes f :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes F: F *equiintegrable_on cbox a b* **and** f: $f \in F$ **and** $0 < \varepsilon$
and *norm_f*: $\bigwedge h\ x. [\![h \in F; x \in \text{cbox } a\ b]\!] \Longrightarrow \text{norm}(h\ x) \leq \text{norm}(f\ x)$
obtains γ **where** *gauge* γ
$\bigwedge c\ i\ S\ h. [\![c \in \text{cbox } a\ b; i \in \text{Basis}; S\ \text{tagged\_partial\_division\_of cbox } a\ b;$
$\gamma \text{ fine } S; h \in F; \bigwedge x\ K. (x,K) \in S \Longrightarrow (K \cap \{x. x \cdot i = c \cdot i\} \neq \{\})]\!]$
$\Longrightarrow (\sum (x,K) \in S. \text{norm } (\text{integral } K\ h)) < \varepsilon$

proposition *equiintegrable_halfspace_restrictions_le*:
fixes f :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes F: F *equiintegrable_on cbox a b* **and** f: $f \in F$
and *norm_f*: $\bigwedge h\ x. [\![h \in F; x \in \text{cbox } a\ b]\!] \Longrightarrow \text{norm}(h\ x) \leq \text{norm}(f\ x)$
shows $(\bigcup i \in \text{Basis}. \bigcup c. \bigcup h \in F. \{(\lambda x. \text{if } x \cdot i \leq c \text{ then } h\ x \text{ else } 0)\})$
equiintegrable_on cbox a b

corollary *equiintegrable_halfspace_restrictions_ge*:
fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
assumes F: F *equiintegrable_on cbox a b* **and** f: $f \in F$
and *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \leq norm(f\ x)$
shows $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i \geq c\ then\ h\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*

corollary *equiintegrable_halfspace_restrictions_lt*:
fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
assumes F: F *equiintegrable_on cbox a b* **and** f: $f \in F$
and *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \leq norm(f\ x)$
shows $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i < c\ then\ h\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on cbox a b*)

corollary *equiintegrable_halfspace_restrictions_gt*:
fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
assumes F: F *equiintegrable_on cbox a b* **and** f: $f \in F$
and *norm_f*: $\bigwedge h\ x.\ [\![h \in F;\ x \in cbox\ a\ b]\!] \Longrightarrow norm(h\ x) \leq norm(f\ x)$
shows $(\bigcup i \in Basis.\ \bigcup c.\ \bigcup h \in F.\ \{(\lambda x.\ if\ x \cdot i > c\ then\ h\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*
(**is** *?G equiintegrable_on cbox a b*)

proposition *equiintegrable_closed_interval_restrictions*:
fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
assumes f: *f integrable_on cbox a b*
shows $(\bigcup c\ d.\ \{(\lambda x.\ if\ x \in cbox\ c\ d\ then\ f\ x\ else\ 0)\})$ *equiintegrable_on cbox a b*

10.14.3 Continuity of the indefinite integral

proposition *indefinite_integral_continuous*:
fixes f :: $'a$ :: *euclidean_space* ⇒ $'b$ :: *euclidean_space*
assumes *int_f*: *f integrable_on cbox a b*
and c: $c \in cbox\ a\ b$ **and** d: $d \in cbox\ a\ b$ $0 < \varepsilon$
obtains δ **where** $0 < \delta$
$\bigwedge c'\ d'.\ [\![c' \in cbox\ a\ b;\ d' \in cbox\ a\ b;\ norm(c' - c) \leq \delta;\ norm(d' - d) \leq \delta]\!]$
$\Longrightarrow norm(integral(cbox\ c'\ d')\ f - integral(cbox\ c\ d)\ f) < \varepsilon$

corollary *indefinite_integral_uniformly_continuous*:
fixes f :: $'a$ :: *euclidean_space* ⇒ $'b$ :: *euclidean_space*
assumes *f integrable_on cbox a b*
shows *uniformly_continuous_on* (*cbox* (*Pair a a*) (*Pair b b*)) (λy. *integral* (*cbox* (*fst y*) (*snd y*)) *f*)

corollary *bounded_integrals_over_subintervals*:
fixes f :: $'a$ :: *euclidean_space* ⇒ $'b$ :: *euclidean_space*
assumes f *integrable_on cbox a b*
shows *bounded* {*integral* (*cbox c d*) f |*c d*. *cbox c d* ⊆ *cbox a b*}
theorem *absolutely_integrable_improper*:
fixes f :: $'M$::*euclidean_space* ⇒ $'N$::*euclidean_space*
assumes *int_f*: ⋀*c d*. *cbox c d* ⊆ *box a b* ⟹ f *integrable_on cbox c d*
and *bo*: *bounded* {*integral* (*cbox c d*) f |*c d*. *cbox c d* ⊆ *box a b*}
and *absi*: ⋀*i*. *i* ∈ *Basis*
⟹ ∃ *g*. *g absolutely_integrable_on cbox a b* ∧
((∀ *x* ∈ *cbox a b*. f *x* · *i* ≤ *g x*) ∨ (∀ *x* ∈ *cbox a b*. f *x* · *i* ≥ *g x*))
shows f *absolutely_integrable_on cbox a b*

10.14.4 Second mean value theorem and corollaries

theorem *second_mean_value_theorem_full*:
fixes f :: *real* ⇒ *real*
assumes f: f *integrable_on* {*a..b*} **and** $a \le b$
and *g*: ⋀*x y*. ⟦$a \le x$; $x \le y$; $y \le b$⟧ ⟹ $g\ x \le g\ y$
obtains *c* **where** $c \in$ {*a..b*}
and ((λ*x*. *g x* ∗ *f x*) *has_integral* (*g a* ∗ *integral* {*a..c*} *f* + *g b* ∗ *integral* {*c..b*} *f*)) {*a..b*}

corollary *second_mean_value_theorem*:
fixes f :: *real* ⇒ *real*
assumes f: f *integrable_on* {*a..b*} **and** $a \le b$
and *g*: ⋀*x y*. ⟦$a \le x$; $x \le y$; $y \le b$⟧ ⟹ $g\ x \le g\ y$
obtains *c* **where** $c \in$ {*a..b*}
integral {*a..b*} (λ*x*. *g x* ∗ *f x*) = *g a* ∗ *integral* {*a..c*} *f* + *g b* ∗ *integral* {*c..b*} *f*

end

10.15 Continuous Extensions of Functions

theory *Continuous_Extension*
imports *Starlike*
begin

10.15.1 Partitions of unity subordinate to locally finite open coverings

proposition *subordinate_partition_of_unity*:

fixes S :: $'a$::*metric_space set*
assumes $S \subseteq \bigcup\mathcal{C}$ **and** *opC*: $\bigwedge T.\ T \in \mathcal{C} \Longrightarrow$ *open* T
and *fin*: $\bigwedge x.\ x \in S \Longrightarrow \exists\, V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U \in \mathcal{C}.\ U \cap V \neq \{\}\}$
obtains F :: $['a\ set,\ 'a] \Rightarrow$ *real*
where $\bigwedge U.\ U \in \mathcal{C} \Longrightarrow$ *continuous_on* S $(F\ U) \wedge (\forall\, x \in S.\ 0 \leq F\ U\ x)$
and $\bigwedge x\ U.\ [\![U \in \mathcal{C};\ x \in S;\ x \notin U]\!] \Longrightarrow F\ U\ x = 0$
and $\bigwedge x.\ x \in S \Longrightarrow$ *supp_sum* $(\lambda W.\ F\ W\ x)\ \mathcal{C} = 1$
and $\bigwedge x.\ x \in S \Longrightarrow \exists\, V.$ *open* $V \wedge x \in V \wedge$ *finite* $\{U \in \mathcal{C}.\ \exists x{\in}V.\ F\ U\ x \neq 0\}$

10.15.2 Urysohn's Lemma for Euclidean Spaces

proposition *Urysohn_local_strong*:
assumes *US*: *closedin* (*top_of_set* U) S
and *UT*: *closedin* (*top_of_set* U) T
and $S \cap T = \{\}$ $a \neq b$
obtains f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
where *continuous_on* U f
$\bigwedge x.\ x \in U \Longrightarrow f\,x \in$ *closed_segment* a b
$\bigwedge x.\ x \in U \Longrightarrow (f\,x = a \longleftrightarrow x \in S)$
$\bigwedge x.\ x \in U \Longrightarrow (f\,x = b \longleftrightarrow x \in T)$

proposition *Urysohn*:
assumes *US*: *closed* S
and *UT*: *closed* T
and $S \cap T = \{\}$
obtains f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
where *continuous_on* *UNIV* f
$\bigwedge x.\ f\,x \in$ *closed_segment* a b
$\bigwedge x.\ x \in S \Longrightarrow f\,x = a$
$\bigwedge x.\ x \in T \Longrightarrow f\,x = b$

10.15.3 Dugundji's Extension Theorem and Tietze Variants

theorem *Dugundji*:
fixes f :: $'a$::{*metric_space*,*second_countable_topology*} $\Rightarrow$ $'b$::*real_inner*
assumes *convex* C $C \neq \{\}$
and *cloin*: *closedin* (*top_of_set* U) S
and *contf*: *continuous_on* S f **and** f ‘ $S \subseteq C$
obtains g **where** *continuous_on* U g g ‘ $U \subseteq C$
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

corollary *Tietze*:
fixes f :: $'a$::{*metric_space*,*second_countable_topology*} $\Rightarrow$ $'b$::*real_inner*
assumes *continuous_on* S f
and *closedin* (*top_of_set* U) S

and $0 \leq B$
and $\bigwedge x.\ x \in S \Longrightarrow norm(f\ x) \leq B$
obtains g **where** $continuous\_on\ U\ g\ \bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$
$\bigwedge x.\ x \in U \Longrightarrow norm(g\ x) \leq B$

end

10.16 Equivalence Between Classical Borel Measurability and HOL Light's

theory *Equivalence_Measurable_On_Borel*
imports *Equivalence_Lebesgue_Henstock_Integration Improper_Integral Continuous_Extension*
begin

10.16.1 Austin's Lemma

10.16.2 A differentiability-like property of the indefinite integral.

proposition *integrable_ccontinuous_explicit*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $\bigwedge a\ b::'a.\ f\ integrable\_on\ cbox\ a\ b$
obtains N **where**
$negligible\ N$
$\bigwedge x\ e.\ [\![x \notin N;\ 0 < e]\!] \Longrightarrow$
$\exists d>0.\ \forall h.\ 0 < h \wedge h < d \longrightarrow$
$norm(integral\ (cbox\ x\ (x + h *_R One))\ f\ /_R\ h\ \hat{}\ DIM('a) - f\ x) < e$

10.16.3 HOL Light measurability

proposition *integrable_subintervals_imp_measurable*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $\bigwedge a\ b.\ f\ integrable\_on\ cbox\ a\ b$
shows $f\ measurable\_on\ UNIV$

10.16.4 Composing continuous and measurable functions; a few variants

proposition *indicator_measurable_on*:

assumes $S \in$ *sets lebesgue*
shows *indicat_real S measurable_on UNIV*

lemma *simple_function_induct_real*
[*consumes 1*, *case_names cong set mult add*, *induct set*: *simple_function*]:
fixes $u :: 'a \Rightarrow real$
assumes *u*: *simple_function M u*
assumes *cong*: $\bigwedge f\ g.$ *simple_function M f* $\Longrightarrow$ *simple_function M g* $\Longrightarrow$ (*AE x in M*. $f\ x = g\ x$) $\Longrightarrow P\ f \Longrightarrow P\ g$
assumes *set*: $\bigwedge A.\ A \in$ *sets M* $\Longrightarrow P$ (*indicator A*)
assumes *mult*: $\bigwedge u\ c.\ P\ u \Longrightarrow P\ (\lambda x.\ c * u\ x)$
assumes *add*: $\bigwedge u\ v.\ P\ u \Longrightarrow P\ v \Longrightarrow P\ (\lambda x.\ u\ x + v\ x)$
and *nn*: $\bigwedge x.\ u\ x \geq 0$
shows $P\ u$

proposition *simple_function_measurable_on_UNIV*:
fixes $f :: 'a::euclidean\_space \Rightarrow real$
assumes *f*: *simple_function lebesgue f* **and** *nn*: $\bigwedge x.\ f\ x \geq 0$
shows *f measurable_on UNIV*

corollary *simple_function_measurable_on*:
fixes $f :: 'a::euclidean\_space \Rightarrow real$
assumes *f*: *simple_function lebesgue f* **and** *nn*: $\bigwedge x.\ f\ x \geq 0$ **and** *S*: $S \in$ *sets lebesgue*
shows *f measurable_on S*

proposition *measurable_on_componentwise_UNIV*:
f measurable_on UNIV $\longleftrightarrow$ ($\forall\, i \in Basis.$ ($\lambda x.$ ($f\ x \cdot i$) $*_R\ i$) *measurable_on UNIV*)
(**is** *?lhs* = *?rhs*)

corollary *measurable_on_componentwise*:
f measurable_on S $\longleftrightarrow$ ($\forall\, i \in Basis.$ ($\lambda x.$ ($f\ x \cdot i$) $*_R\ i$) *measurable_on S*)

lemma *borel_measurable_implies_simple_function_sequence_real*:
fixes $u :: 'a \Rightarrow real$
assumes *u*[*measurable*]: $u \in$ *borel_measurable M* **and** *nn*: $\bigwedge x.\ u\ x \geq 0$
shows $\exists f.$ *incseq f* $\wedge$ ($\forall\, i.$ *simple_function M* ($f\ i$)) $\wedge$ ($\forall\, x.$ *bdd_above* (*range* ($\lambda i.\ f\ i\ x$))) $\wedge$
($\forall\, i\ x.\ 0 \leq f\ i\ x$) $\wedge$ $u = (SUP\ i.\ f\ i)$

proposition *homeomorphic_box_UNIV*:
fixes $a\ b :: 'a::euclidean\_space$
assumes *box a b* $\neq \{\}$
shows *box a b homeomorphic* ($UNIV::'a\ set$)

proposition *measurable_on_imp_borel_measurable_lebesgue_UNIV*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *f measurable_on UNIV*
shows $f \in$ *borel_measurable lebesgue*

corollary *measurable_on_imp_borel_measurable_lebesgue*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *f measurable_on S* **and** *S*: $S \in$ *sets lebesgue*
shows $f \in$ *borel_measurable* (*lebesgue_on S*)

proposition *measurable_on_limit*:
fixes f :: *nat* ⇒ *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *f*: $\bigwedge n.$ *f n measurable_on S* **and** *N*: *negligible N*
and *lim*: $\bigwedge x.\ x \in S - N \Longrightarrow (\lambda n.\ f\ n\ x) \longrightarrow g\ x$
shows *g measurable_on S*

proposition *lebesgue_measurable_imp_measurable_on*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *f*: $f \in$ *borel_measurable lebesgue* **and** *S*: $S \in$ *sets lebesgue*
shows *f measurable_on S*

proposition *measurable_on_iff_borel_measurable*:
fixes f :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes $S \in$ *sets lebesgue*
shows *f measurable_on S* ⟷ $f \in$ *borel_measurable* (*lebesgue_on S*) (**is** *?lhs* = *?rhs*)

10.16.5 Monotonic functions are Lebesgue integrable

10.16.6 Measurability on generalisations of the binary product

end

10.17 Embedding Measure Spaces with a Function

theory *Embed_Measure*
imports *Binary_Product_Measure*
begindefinition *embed_measure* :: *'a measure* ⇒ (*'a* ⇒ *'b*) ⇒ *'b measure* **where**

$embed\_measure\ M\ f = measure\_of\ (f\ `\ space\ M)\ \{f\ `\ A\ |A.\ A \in sets\ M\}$
$(\lambda A.\ emeasure\ M\ (f\ -`\ A \cap space\ M))$

end

10.18 Brouwer's Fixed Point Theorem

theory *Brouwer_Fixpoint*
imports *Homeomorphism Derivative*
begin

10.18.1 Retractions

10.18.2 Kuhn Simplices

10.18.3 Brouwer's fixed point theorem

theorem *brouwer*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'a$
assumes *S*: $compact\ S\ convex\ S\ S \neq \{\}$
and *contf*: $continuous\_on\ S\ f$
and *fim*: $f \in S \to S$
obtains x **where** $x \in S$ **and** $f\ x = x$

10.18.4 Applications

corollary *no_retraction_cball*:
fixes $a :: 'a::euclidean\_space$
assumes $e > 0$
shows $\neg\ (frontier\ (cball\ a\ e)\ retract\_of\ (cball\ a\ e))$

corollary *contractible_sphere*:
fixes $a :: 'a::euclidean\_space$
shows $contractible(sphere\ a\ r) \longleftrightarrow r \le 0$

corollary *connected_sphere_eq*:
fixes $a :: 'a :: euclidean\_space$
shows $connected(sphere\ a\ r) \longleftrightarrow 2 \le DIM('a) \vee r \le 0$
(**is** $?lhs = ?rhs$)

corollary *path_connected_sphere_eq*:
fixes $a :: 'a :: euclidean\_space$
shows $path\_connected(sphere\ a\ r) \longleftrightarrow 2 \le DIM('a) \vee r \le 0$
(**is** $?lhs = ?rhs$)

proposition *frontier_subset_retraction*:
  fixes S :: *'a::euclidean_space set*
  assumes *bounded* S **and** *fros*: *frontier* $S \subseteq T$
    and *contf*: *continuous_on* (*closure* S) f
    and *fim*: $f \in S \to T$
    and *fid*: $\bigwedge x.\ x \in T \Longrightarrow f\ x = x$
   shows $S \subseteq T$

corollary *rel_frontier_retract_of_punctured_affine_hull*:
  fixes S :: *'a::euclidean_space set*
  assumes *bounded* S *convex* S $a \in$ *rel_interior* S
   shows *rel_frontier* S *retract_of* (*affine hull* $S - \{a\}$)

corollary *rel_boundary_retract_of_punctured_affine_hull*:
  fixes S :: *'a::euclidean_space set*
  assumes *compact* S *convex* S $a \in$ *rel_interior* S
   shows ($S -$ *rel_interior* S) *retract_of* (*affine hull* $S - \{a\}$)
theorem *has_derivative_inverse_on*:
  fixes f :: *'n::euclidean_space* $\Rightarrow$ *'n*
  assumes *open* S
   and $\bigwedge x.\ x \in S \Longrightarrow$ (f *has_derivative* $f'(x)$) (*at* x)
   and $\bigwedge x.\ x \in S \Longrightarrow g\ (f\ x) = x$
   and $f'\ x \circ g'\ x = id$
   and $x \in S$
  shows (g *has_derivative* $g'(x)$) (*at* ($f\ x$))

end

10.19 Fashoda Meet Theorem

theory *Fashoda_Theorem*
imports *Brouwer_Fixpoint Path_Connected Cartesian_Euclidean_Space*
begin

10.19.1 Bijections between intervals

definition *interval_bij* :: $'a \times 'a \Rightarrow 'a \times 'a \Rightarrow 'a \Rightarrow$ *'a::euclidean_space*
  where *interval_bij* =
   $(\lambda(a,\ b)\ (u,\ v)\ x.\ (\sum i{\in}Basis.\ (u{\cdot}i + (x{\cdot}i - a{\cdot}i) / (b{\cdot}i - a{\cdot}i) * (v{\cdot}i - u{\cdot}i)) *_R\ i))$

10.19.2 Fashoda meet theorem

proposition *fashoda_unit*:
  fixes $f\ g$ :: *real* $\Rightarrow$ *real^2*

assumes f ‘ $\{-1\ ..\ 1\} \subseteq cbox\ (-1)\ 1$
and g ‘ $\{-1\ ..\ 1\} \subseteq cbox\ (-1)\ 1$
and *continuous_on* $\{-1\ ..\ 1\}\ f$
and *continuous_on* $\{-1\ ..\ 1\}\ g$
and $f\ (-\ 1)\$1 = -\ 1$
and $f\ 1\$1 = 1\ g\ (-\ 1)\ \$2 = -1$
and $g\ 1\ \$2 = 1$
shows $\exists\, s\in\{-1\ ..\ 1\}.\ \exists\, t\in\{-1\ ..\ 1\}.\ f\ s = g\ t$

proposition *fashoda_unit_path*:
fixes *f g* :: *real* ⇒ *real^2*
assumes *path f*
and *path g*
and *path_image f* ⊆ *cbox* (−*1*) *1*
and *path_image g* ⊆ *cbox* (−*1*) *1*
and (*pathstart f*)$*1* = −*1*
and (*pathfinish f*)$*1* = *1*
and (*pathstart g*)$*2* = −*1*
and (*pathfinish g*)$*2* = *1*
obtains *z* **where** *z* ∈ *path_image f* **and** *z* ∈ *path_image g*

theorem *fashoda*:
fixes *b* :: *real^2*
assumes *path f*
and *path g*
and *path_image f* ⊆ *cbox a b*
and *path_image g* ⊆ *cbox a b*
and (*pathstart f*)$*1* = *a*$*1*
and (*pathfinish f*)$*1* = *b*$*1*
and (*pathstart g*)$*2* = *a*$*2*
and (*pathfinish g*)$*2* = *b*$*2*
obtains *z* **where** *z* ∈ *path_image f* **and** *z* ∈ *path_image g*

10.19.3 Useful Fashoda corollary pointed out to me by Tom Hales

corollary *fashoda_interlace*:
fixes *a* :: *real^2*
assumes *path f*
and *path g*
and *paf*: *path_image f* ⊆ *cbox a b*
and *pag*: *path_image g* ⊆ *cbox a b*
and (*pathstart f*)$*2* = *a*$*2*
and (*pathfinish f*)$*2* = *a*$*2*
and (*pathstart g*)$*2* = *a*$*2*
and (*pathfinish g*)$*2* = *a*$*2*
and (*pathstart f*)$*1* < (*pathstart g*)$*1*
and (*pathstart g*)$*1* < (*pathfinish f*)$*1*

and (*pathfinish f*)\$*1* < (*pathfinish g*)\$*1*
obtains *z* **where** *z* ∈ *path_image f* **and** *z* ∈ *path_image g*

end

10.20 Vector Cross Products in 3 Dimensions

theory *Cross3*
imports *Determinants Cartesian_Euclidean_Space*
begin

definition *cross3* :: [*real^3*, *real^3*] ⇒ *real^3* (**infixr** ‹×› *80*)
where *a* × *b* ≡
vector [*a*\$*2* ∗ *b*\$*3* − *a*\$*3* ∗ *b*\$*2*,
a\$*3* ∗ *b*\$*1* − *a*\$*1* ∗ *b*\$*3*,
a\$*1* ∗ *b*\$*2* − *a*\$*2* ∗ *b*\$*1*]

10.20.1 Basic lemmas

proposition *Jacobi*: *x* × (*y* × *z*) + *y* × (*z* × *x*) + *z* × (*x* × *y*) = *0* **for** *x*::*real^3*

proposition *Lagrange*: $x \times (y \times z) = (x \cdot z) *_R y - (x \cdot y) *_R z$

proposition *cross_triple*: (*x* × *y*) · *z* = (*y* × *z*) · *x*

proposition *dot_cross*: (*w* × *x*) · (*y* × *z*) = (*w* · *y*) ∗ (*x* · *z*) − (*w* · *z*) ∗ (*x* · *y*)

proposition *norm_cross*: $(norm\ (x \times y))^2 = (norm\ x)^2 * (norm\ y)^2 - (x \cdot y)^2$

10.20.2 Preservation by rotation, or other orthogonal transformation up to sign

10.20.3 Continuity

end

10.21 Bounded Continuous Functions

theory *Bounded_Continuous_Function*
imports
Topology_Euclidean_Space
Uniform_Limit
begin

10.21.1 Definition

definition *bcontfun* = {*f*. *continuous_on UNIV f* ∧ *bounded* (*range f*)}

instantiation *bcontfun* :: (*topological_space*, *metric_space*) *metric_space*
begin

lift_definition *dist_bcontfun* :: $'a \Rightarrow_C 'b \Rightarrow 'a \Rightarrow_C 'b \Rightarrow real$
  is λ*f g*. (*SUP x*. *dist* (*f x*) (*g x*))

10.21.2 Complete Space

instance *bcontfun* :: (*metric_space*, *complete_space*) *complete_space*

end

10.22 Infinite Products

theory *Infinite_Products*
  imports *Topology_Euclidean_Space Complex_Transcendental*
begin

10.22.1 Definitions and basic properties

definition *raw_has_prod* :: [*nat* ⇒ '*a*::{*t2_space*, *comm_semiring_1*}, *nat*, '*a*] ⇒ *bool*
  where *raw_has_prod f M p* ≡ (λ*n*. ∏ *i*≤*n*. *f* (*i*+*M*)) ⟶ *p* ∧ *p* ≠ *0*

definition
  has_prod :: (*nat* ⇒ '*a*::{*t2_space*, *comm_semiring_1*}) ⇒ '*a* ⇒ *bool* (**infixr** ‹*has'_prod*› *80*)
  where *f has_prod p* ≡ *raw_has_prod f 0 p* ∨ (∃ *i q*. *p* = *0* ∧ *f i* = *0* ∧ *raw_has_prod f* (*Suc i*) *q*)

definition *convergent_prod* :: (*nat* ⇒ '*a* :: {*t2_space*,*comm_semiring_1*}) ⇒ *bool*
where
  convergent_prod f ≡ ∃ *M p*. *raw_has_prod f M p*

definition *prodinf* :: (*nat* ⇒ '*a*::{*t2_space*, *comm_semiring_1*}) ⇒ '*a*
  (**binder** ‹∏› *10*)
  where *prodinf f* = (*THE p*. *f has_prod p*)

10.22.2 Absolutely convergent products

definition *abs_convergent_prod* :: (*nat* ⇒ _) ⇒ *bool* **where**
  abs_convergent_prod f ⟷ *convergent_prod* (λ*i*. *1* + *norm* (*f i* − *1*))

lemma *convergent_prod_iff_convergent*:

fixes f :: $nat \Rightarrow 'a$ :: {$topological\_semigroup\_mult,t2\_space,idom$}
assumes $\bigwedge i.\ f\ i \neq 0$
shows $convergent\_prod\ f \longleftrightarrow convergent\ (\lambda n.\ \prod i{\le}n.\ f\ i) \wedge lim\ (\lambda n.\ \prod i{\le}n.\ f\ i) \neq 0$

theorem *abs_convergent_prod_conv_summable*:
fixes f :: $nat \Rightarrow 'a$ :: $real\_normed\_div\_algebra$
shows $abs\_convergent\_prod\ f \longleftrightarrow summable\ (\lambda i.\ norm\ (f\ i - 1))$

10.22.3 More elementary properties

theorem *abs_convergent_prod_imp_convergent_prod*:
fixes f :: $nat \Rightarrow 'a$ :: {$real\_normed\_div\_algebra,complete\_space,comm\_ring\_1$}
assumes $abs\_convergent\_prod\ f$
shows $convergent\_prod\ f$

corollary *convergent_prod_offset_0*:
fixes f :: $nat \Rightarrow 'a$ :: {$idom,topological\_semigroup\_mult,t2\_space$}
assumes $convergent\_prod\ f\ \bigwedge i.\ f\ i \neq 0$
shows $\exists\, p.\ raw\_has\_prod\ f\ 0\ p$

theorem *has_prod_iff*: $f\ has\_prod\ x \longleftrightarrow convergent\_prod\ f \wedge prodinf\ f = x$

10.22.4 Exponentials and logarithms

theorem *convergent_prod_iff_summable_real*:
fixes a :: $nat \Rightarrow real$
assumes $\bigwedge n.\ a\ n > 0$
shows $convergent\_prod\ (\lambda k.\ 1 + a\ k) \longleftrightarrow summable\ a$ (**is** $?lhs = ?rhs$)

theorem *Ln_prodinf_complex*:
fixes z :: $nat \Rightarrow complex$
assumes z: $\bigwedge j.\ z\ j \neq 0$ **and** ξ: $\xi \neq 0$
shows $((\lambda n.\ \prod j{\le}n.\ z\ j) \longrightarrow \xi) \longleftrightarrow (\exists\, k.\ (\lambda n.\ (\sum j{\le}n.\ Ln\ (z\ j))) \longrightarrow Ln\ \xi + of\_int\ k * (of\_real(2*pi) * \mathrm{i}))$ (**is** $?lhs = ?rhs$)
proposition *convergent_prod_iff_summable_complex*:
fixes z :: $nat \Rightarrow complex$
assumes $\bigwedge k.\ z\ k \neq 0$
shows $convergent\_prod\ (\lambda k.\ z\ k) \longleftrightarrow summable\ (\lambda k.\ Ln\ (z\ k))$ (**is** $?lhs = ?rhs$)
proposition *summable_imp_convergent_prod_complex*:
fixes z :: $nat \Rightarrow complex$
assumes z: $summable\ (\lambda k.\ norm\ (z\ k))$ **and** $non0$: $\bigwedge k.\ z\ k \neq -1$
shows $convergent\_prod\ (\lambda k.\ 1 + z\ k)$

corollary *summable_imp_convergent_prod_real*:
fixes z :: *nat* ⇒ *real*
assumes z: *summable* (λk. $|z\ k|$) **and** *non0*: $\bigwedge k.\ z\ k \neq -1$
shows *convergent_prod* (λk. *1* + $z\ k$)

10.22.5 Convergence criteria: especially uniform convergence of infinite products

end

10.23 Sums over Infinite Sets

theory *Infinite_Set_Sum*
imports *Set_Integral Infinite_Sum*
begin

definition *abs_summable_on* ::
('*a* ⇒ '*b* :: {*banach*, *second_countable_topology*}) ⇒ '*a set* ⇒ *bool*
(**infix** ‹*abs'_summable'_on*› *50*)
where
f abs_summable_on A ⟷ *integrable* (*count_space A*) *f*

definition *infsetsum* ::
('*a* ⇒ '*b* :: {*banach*, *second_countable_topology*}) ⇒ '*a set* ⇒ '*b*
where
infsetsum f A = *lebesgue_integral* (*count_space A*) *f*

theorem *infsetsum_reindex*:
assumes *inj_on g A*
shows *infsetsum f* (*g* ‘ *A*) = *infsetsum* (λx. *f* (*g x*)) *A*

theorem *infsetsum_Sigma*:
fixes *A* :: '*a set* **and** *B* :: '*a* ⇒ '*b set*
assumes [*simp*]: *countable A* **and** $\bigwedge i$. *countable* (*B i*)
assumes *summable*: *f abs_summable_on* (*Sigma A B*)
shows *infsetsum f* (*Sigma A B*) = *infsetsum* (λx. *infsetsum* (λy. *f* (*x*, *y*)) (*B x*)) *A*

theorem *abs_summable_on_Sigma_iff*:
assumes [*simp*]: *countable A* **and** $\bigwedge x$. $x \in A \Longrightarrow$ *countable* (*B x*)
shows *f abs_summable_on Sigma A B* ⟷
(∀ *x*∈*A*. (λy. *f* (*x*, *y*)) *abs_summable_on B x*) ∧
((λx. *infsetsum* (λy. *norm* (*f* (*x*, *y*))) (*B x*)) *abs_summable_on A*)

theorem *infsetsum_prod_PiE*:
fixes f :: $'a \Rightarrow 'b \Rightarrow 'c$ :: {*real_normed_field,banach,second_countable_topology*}
assumes *finite*: *finite A* **and** *countable*: $\bigwedge x.\ x \in A \Longrightarrow$ *countable* $(B\ x)$
assumes *summable*: $\bigwedge x.\ x \in A \Longrightarrow f\ x$ *abs_summable_on* $B\ x$
shows *infsetsum* $(\lambda g.\ \prod x{\in}A.\ f\ x\ (g\ x))$ (*PiE A B*) = $(\prod x{\in}A.$ *infsetsum* $(f\ x)$ $(B\ x))$

end

10.24 Faces, Extreme Points, Polytopes, Polyhedra etc

theory *Polytope*
imports *Cartesian_Euclidean_Space Path_Connected*
begin

10.24.1 Faces of a (usually convex) set

definition *face_of* :: [*'a::real_vector set*, *'a set*] ⇒ *bool* (**infixr** ‹(*face'_of*)› *50*)
where
T face_of S ⟷
$T \subseteq S \land$ *convex* $T \land$
$(\forall a \in S.\ \forall b \in S.\ \forall x \in T.\ x \in$ *open_segment* $a\ b \longrightarrow a \in T \land b \in T)$

proposition *face_of_imp_eq_affine_Int*:
fixes S :: *'a::euclidean_space set*
assumes S: *convex S* **and** T: *T face_of S*
shows T = (*affine hull T*) ∩ S

proposition *face_of_conic*:
assumes *conic S f face_of S*
shows *conic f*

proposition *face_of_convex_hulls*:
assumes S: *finite S* $T \subseteq S$ **and** *disj*: *affine hull T* ∩ *convex hull* $(S - T)$ = {}
shows (*convex hull T*) *face_of* (*convex hull S*)

proposition *face_of_convex_hull_insert*:
assumes *finite S* $a \notin$ *affine hull S* **and** T: *T face_of convex hull S*
shows *T face_of convex hull insert a S*

proposition *face_of_affine_trivial*:

assumes *affine S T face_of S*
shows $T = \{\} \vee T = S$

proposition *Inter_faces_finite_altbound*:
fixes T :: *'a::euclidean_space set set*
assumes *cfaI*: $\bigwedge c.\ c \in T \Longrightarrow c$ *face_of S*
shows $\exists F'.$ *finite* $F' \wedge F' \subseteq T \wedge$ *card* $F' \leq DIM('a) + 2 \wedge \bigcap F' = \bigcap T$

proposition *face_of_Times*:
assumes *F face_of S* **and** *F' face_of S'*
shows $(F \times F')$ *face_of* $(S \times S')$

corollary *face_of_Times_decomp*:
fixes S :: *'a::euclidean_space set* **and** S' :: *'b::euclidean_space set*
shows *C face_of* $(S \times S') \longleftrightarrow (\exists F\ F'.\ F$ *face_of* $S \wedge F'$ *face_of* $S' \wedge C = F \times F')$
(**is** *?lhs = ?rhs*)

10.24.2 Exposed faces

definition *exposed_face_of* :: *['a::euclidean_space set, 'a set] ⇒ bool*
(**infixr** ‹(*exposed'_face'_of*)› *50*)
where *T exposed_face_of S* $\longleftrightarrow$
T face_of S $\wedge (\exists a\ b.\ S \subseteq \{x.\ a \cdot x \leq b\} \wedge T = S \cap \{x.\ a \cdot x = b\})$

proposition *exposed_face_of_Int*:
assumes *T exposed_face_of S*
and *U exposed_face_of S*
shows $(T \cap U)$ *exposed_face_of S*

proposition *exposed_face_of_Inter*:
fixes P :: *'a::euclidean_space set set*
assumes $P \neq \{\}$
and $\bigwedge T.\ T \in P \Longrightarrow T$ *exposed_face_of S*
shows $\bigcap P$ *exposed_face_of S*

proposition *exposed_face_of_sums*:
assumes *convex S* **and** *convex T*
and *F exposed_face_of* $\{x + y \mid x\ y.\ x \in S \wedge y \in T\}$
(**is** *F exposed_face_of ?ST*)
obtains *k l*
where *k exposed_face_of S l exposed_face_of T*
$F = \{x + y \mid x\ y.\ x \in k \wedge y \in l\}$

proposition *exposed_face_of_parallel*:
T exposed_face_of S $\longleftrightarrow$

T *face_of* $S \wedge$
$(\exists\, a\ b.\ S \subseteq \{x.\ a \cdot x \leq b\} \wedge T = S \cap \{x.\ a \cdot x = b\} \wedge$
$(T \neq \{\} \longrightarrow T \neq S \longrightarrow a \neq 0) \wedge$
$(T \neq S \longrightarrow (\forall\, w \in \mathit{affine\ hull}\ S.\ (w + a) \in \mathit{affine\ hull}\ S)))$
(**is** *?lhs* = *?rhs*)

10.24.3 Extreme points of a set: its singleton faces

definition *extreme_point_of* :: [*'a::real_vector*, *'a set*] ⇒ *bool*
(**infixr** ‹(*extreme'_point'_of*)› *50*)
where x *extreme_point_of* $S \longleftrightarrow$
$x \in S \wedge (\forall\, a \in S.\ \forall\, b \in S.\ x \notin$ *open_segment* $a\ b)$

proposition *extreme_points_of_convex_hull*:
$\{x.\ x$ *extreme_point_of* (*convex hull* $S)\} \subseteq S$

10.24.4 Facets

definition *facet_of* :: [*'a::euclidean_space set*, *'a set*] ⇒ *bool*
(**infixr** ‹(*facet'_of*)› *50*)
where F *facet_of* $S \longleftrightarrow F$ *face_of* $S \wedge F \neq \{\} \wedge$ *aff_dim* F = *aff_dim* $S - 1$

10.24.5 Edges: faces of affine dimension 1

definition *edge_of* :: [*'a::euclidean_space set*, *'a set*] ⇒ *bool* (**infixr** ‹(*edge'_of*)› *50*)
where e *edge_of* $S \longleftrightarrow e$ *face_of* $S \wedge$ *aff_dim* $e = 1$

10.24.6 Existence of extreme points

proposition *different_norm_3_collinear_points*:
fixes a :: *'a::euclidean_space*
assumes $x \in$ *open_segment* $a\ b$ *norm*(a) = *norm*(b) *norm*(x) = *norm*(b)
shows *False*

proposition *extreme_point_exists_convex*:
fixes S :: *'a::euclidean_space set*
assumes *compact* S *convex* S $S \neq \{\}$
obtains x **where** x *extreme_point_of* S

10.24.7 Krein-Milman, the weaker form

proposition *Krein_Milman*:
fixes S :: *'a::euclidean_space set*
assumes *compact* S *convex* S

shows *S* = *closure*(*convex hull* {*x*. *x extreme_point_of S*})

theorem *Krein_Milman_Minkowski*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *compact S convex S*
shows *S* = *convex hull* {*x*. *x extreme_point_of S*}

10.24.8 Applying it to convex hulls of explicitly indicated finite sets

corollary *Krein_Milman_polytope*:
fixes *S* :: ′*a*::*euclidean_space set*
shows
finite S
$\Longrightarrow$ *convex hull S* =
convex hull {*x*. *x extreme_point_of* (*convex hull S*)}

proposition *face_of_convex_hull_insert_eq*:
fixes *a* :: ′*a* :: *euclidean_space*
assumes *finite S* **and** *a*: $a \notin$ *affine hull S*
shows (*F face_of* (*convex hull* (*insert a S*)) $\longleftrightarrow$
F face_of (*convex hull S*) $\vee$
($\exists$ *F*′. *F*′ *face_of* (*convex hull S*) $\wedge$ *F* = *convex hull* (*insert a F*′)))
(**is** *F face_of ?CAS* $\longleftrightarrow$ _)

proposition *face_of_convex_hull_affine_independent*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes $\neg$ *affine_dependent S*
shows (*T face_of* (*convex hull S*) $\longleftrightarrow$ ($\exists$ *c*. $c \subseteq S \wedge T$ = *convex hull c*))
(**is** *?lhs* = *?rhs*)

proposition *Krein_Milman_frontier*:
fixes *S* :: ′*a*::*euclidean_space set*
assumes *convex S compact S*
shows *S* = *convex hull* (*frontier S*)
(**is** *?lhs* = *?rhs*)

10.24.9 Polytopes

definition *polytope* **where**
polytope S $\equiv$ $\exists$ *v*. *finite v* $\wedge$ *S* = *convex hull v*

proposition *face_of_polytope_insert2*:

fixes $a :: 'a :: euclidean\_space$
assumes $polytope\ S\ a \notin affine\ hull\ S\ F\ face\_of\ S$
shows $convex\ hull\ (insert\ a\ F)\ face\_of\ convex\ hull\ (insert\ a\ S)$

10.24.10 Polyhedra

definition *polyhedron* **where**
$polyhedron\ S \equiv$
$\exists F.\ finite\ F \wedge$
$S = \bigcap\ F \wedge$
$(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\})$

10.24.11 Canonical polyhedron representation making facial structure explicit

proposition *polyhedron_Int_affine*:
fixes $S :: 'a :: euclidean\_space\ set$
shows $polyhedron\ S \longleftrightarrow$
$(\exists F.\ finite\ F \wedge S = (affine\ hull\ S) \cap \bigcap F \wedge$
$(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\}))$

proposition *rel_interior_polyhedron_explicit*:
assumes $finite\ F$
and seq: $S = affine\ hull\ S \cap \bigcap F$
and $faceq$: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
and $psub$: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset affine\ hull\ S \cap \bigcap F'$
shows $rel\_interior\ S = \{x \in S.\ \forall h \in F.\ a\ h \cdot x < b\ h\}$

proposition *polyhedron_Int_affine_parallel_minimal*:
fixes $S :: 'a :: euclidean\_space\ set$
shows $polyhedron\ S \longleftrightarrow$
$(\exists F.\ finite\ F \wedge$
$S = (affine\ hull\ S) \cap (\bigcap F) \wedge$
$(\forall h \in F.\ \exists a\ b.\ a \neq 0 \wedge h = \{x.\ a \cdot x \leq b\} \wedge$
$(\forall x \in affine\ hull\ S.\ (x + a) \in affine\ hull\ S)) \wedge$
$(\forall F'.\ F' \subset F \longrightarrow S \subset (affine\ hull\ S) \cap (\bigcap F')))$
(**is** $?lhs = ?rhs$)

proposition *facet_of_polyhedron_explicit*:
assumes $finite\ F$
and seq: $S = affine\ hull\ S \cap \bigcap F$
and $faceq$: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
and $psub$: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset affine\ hull\ S \cap \bigcap F'$

shows C *facet_of* $S \longleftrightarrow (\exists h.\ h \in F \wedge C = S \cap \{x.\ a\ h \cdot x = b\ h\})$

proposition *face_of_polyhedron_explicit*:
fixes S :: *'a* :: *euclidean_space set*
assumes *finite* F
and *seq*: $S =$ *affine hull* $S \cap \bigcap F$
and *faceq*: $\bigwedge h.\ h \in F \Longrightarrow a\ h \neq 0 \wedge h = \{x.\ a\ h \cdot x \leq b\ h\}$
and *psub*: $\bigwedge F'.\ F' \subset F \Longrightarrow S \subset$ *affine hull* $S \cap \bigcap F'$
and C: C *face_of* S **and** $C \neq \{\}$ $C \neq S$
shows $C = \bigcap\{S \cap \{x.\ a\ h \cdot x = b\ h\} \mid h.\ h \in F \wedge C \subseteq S \cap \{x.\ a\ h \cdot x = b\ h\}\}$

10.24.12 More general corollaries from the explicit representation

corollary *facet_of_polyhedron*:
assumes *polyhedron* S **and** C *facet_of* S
obtains $a\ b$ **where** $a \neq 0$ $S \subseteq \{x.\ a \cdot x \leq b\}$ $C = S \cap \{x.\ a \cdot x = b\}$

corollary *face_of_polyhedron*:
assumes *polyhedron* S **and** C *face_of* S **and** $C \neq \{\}$ **and** $C \neq S$
shows $C = \bigcap\{F.\ F$ *facet_of* $S \wedge C \subseteq F\}$

proposition *rel_interior_of_polyhedron*:
fixes S :: *'a* :: *euclidean_space set*
assumes *polyhedron* S
shows *rel_interior* $S = S - \bigcup\{F.\ F$ *facet_of* $S\}$

proposition *polyhedron_eq_finite_exposed_faces*:
fixes S :: *'a* :: *euclidean_space set*
shows *polyhedron* $S \longleftrightarrow$ *closed* $S \wedge$ *convex* $S \wedge$ *finite* $\{F.\ F$ *exposed_face_of* $S\}$
(**is** *?lhs* = *?rhs*)

corollary *polyhedron_eq_finite_faces*:
fixes S :: *'a* :: *euclidean_space set*
shows *polyhedron* $S \longleftrightarrow$ *closed* $S \wedge$ *convex* $S \wedge$ *finite* $\{F.\ F$ *face_of* $S\}$
(**is** *?lhs* = *?rhs*)

10.24.13 Relation between polytopes and polyhedra

proposition *polytope_eq_bounded_polyhedron*:
fixes S :: *'a* :: *euclidean_space set*
shows *polytope* $S \longleftrightarrow$ *polyhedron* $S \wedge$ *bounded* S
(**is** *?lhs* = *?rhs*)

10.24.14 Relative and absolute frontier of a polytope

proposition *frontier_of_convex_hull*:
fixes S :: $'a::euclidean\_space\ set$
assumes $card\ S = Suc\ (DIM('a))$
shows $frontier(convex\ hull\ S) = \bigcup\ \{convex\ hull\ (S - \{a\}) \mid a.\ a \in S\}$

10.24.15 Special case of a triangle

proposition *frontier_of_triangle*:
fixes a :: $'a::euclidean\_space$
assumes $DIM('a) = 2$
shows $frontier(convex\ hull\ \{a,b,c\}) = closed\_segment\ a\ b \cup closed\_segment\ b\ c \cup closed\_segment\ c\ a$
(**is** $?lhs = ?rhs$)

corollary *inside_of_triangle*:
fixes a :: $'a::euclidean\_space$
assumes $DIM('a) = 2$
shows $inside\ (closed\_segment\ a\ b \cup closed\_segment\ b\ c \cup closed\_segment\ c\ a) = interior(convex\ hull\ \{a,b,c\})$

corollary *interior_of_triangle*:
fixes a :: $'a::euclidean\_space$
assumes $DIM('a) = 2$
shows $interior(convex\ hull\ \{a,b,c\}) =$
$convex\ hull\ \{a,b,c\} - (closed\_segment\ a\ b \cup closed\_segment\ b\ c \cup closed\_segment\ c\ a)$

10.24.16 Subdividing a cell complex

proposition *cell_complex_subdivision_exists*:
fixes $\mathcal{F}$ :: $'a::euclidean\_space\ set\ set$
assumes $0 < e\ finite\ \mathcal{F}$
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow polytope\ X$
and *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow aff\_dim\ X \le d$
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow X \cap Y\ face\_of\ X$
obtains $\mathcal{F}'$ **where** $finite\ \mathcal{F}'\ \bigcup\mathcal{F}' = \bigcup\mathcal{F}\ \bigwedge X.\ X \in \mathcal{F}' \Longrightarrow diameter\ X < e$
$\bigwedge X.\ X \in \mathcal{F}' \Longrightarrow polytope\ X\ \bigwedge X.\ X \in \mathcal{F}' \Longrightarrow aff\_dim\ X \le d$
$\bigwedge X\ Y.\ [\![X \in \mathcal{F}';\ Y \in \mathcal{F}']\!] \Longrightarrow X \cap Y\ face\_of\ X$
$\bigwedge C.\ C \in \mathcal{F}' \Longrightarrow \exists D.\ D \in \mathcal{F} \land C \subseteq D$
$\bigwedge C\ x.\ C \in \mathcal{F} \land x \in C \Longrightarrow \exists D.\ D \in \mathcal{F}' \land x \in D \land D \subseteq C$

10.24.17 Simplexes

definition *simplex* :: *int* ⇒ *'a*::*euclidean_space set* ⇒ *bool* (**infix** ‹*simplex*› *50*)
where *n simplex S* ≡ ∃ *C*. ¬ *affine_dependent C* ∧ *int*(*card C*) = *n* + *1* ∧ *S* = *convex hull C*

10.24.18 Simplicial complexes and triangulations

definition *simplicial_complex* **where**
simplicial_complex $\mathcal{C}$ ≡
finite $\mathcal{C}$ ∧
(∀ *S* ∈ $\mathcal{C}$. ∃ *n*. *n simplex S*) ∧
(∀ *F S*. *S* ∈ $\mathcal{C}$ ∧ *F face_of S* ⟶ *F* ∈ $\mathcal{C}$) ∧
(∀ *S S'*. *S* ∈ $\mathcal{C}$ ∧ *S'* ∈ $\mathcal{C}$ ⟶ (*S* ∩ *S'*) *face_of S*)

definition *triangulation* **where**
triangulation $\mathcal{T}$ ≡
finite $\mathcal{T}$ ∧
(∀ *T* ∈ $\mathcal{T}$. ∃ *n*. *n simplex T*) ∧
(∀ *T T'*. *T* ∈ $\mathcal{T}$ ∧ *T'* ∈ $\mathcal{T}$ ⟶ (*T* ∩ *T'*) *face_of T*)

10.24.19 Refining a cell complex to a simplicial complex

proposition *convex_hull_insert_Int_eq*:
fixes *z* :: *'a* :: *euclidean_space*
assumes *z*: *z* ∈ *rel_interior S*
and *T*: *T* ⊆ *rel_frontier S*
and *U*: *U* ⊆ *rel_frontier S*
and *convex S convex T convex U*
shows *convex hull* (*insert z T*) ∩ *convex hull* (*insert z U*) = *convex hull* (*insert z* (*T* ∩ *U*))
(**is** *?lhs* = *?rhs*)

proposition *simplicial_subdivision_of_cell_complex*:
assumes *finite* $\mathcal{M}$
and *poly*: ⋀*C*. *C* ∈ $\mathcal{M}$ ⟹ *polytope C*
and *face*: ⋀*C1 C2*. ⟦*C1* ∈ $\mathcal{M}$; *C2* ∈ $\mathcal{M}$⟧ ⟹ *C1* ∩ *C2 face_of C1*
obtains $\mathcal{T}$ **where** *simplicial_complex* $\mathcal{T}$
⋃$\mathcal{T}$ = ⋃$\mathcal{M}$
⋀*C*. *C* ∈ $\mathcal{M}$ ⟹ ∃ *F*. *finite F* ∧ *F* ⊆ $\mathcal{T}$ ∧ *C* = ⋃*F*
⋀*K*. *K* ∈ $\mathcal{T}$ ⟹ ∃ *C*. *C* ∈ $\mathcal{M}$ ∧ *K* ⊆ *C*

corollary *fine_simplicial_subdivision_of_cell_complex*:
assumes *0* < *e finite* $\mathcal{M}$
and *poly*: ⋀*C*. *C* ∈ $\mathcal{M}$ ⟹ *polytope C*
and *face*: ⋀*C1 C2*. ⟦*C1* ∈ $\mathcal{M}$; *C2* ∈ $\mathcal{M}$⟧ ⟹ *C1* ∩ *C2 face_of C1*
obtains $\mathcal{T}$ **where** *simplicial_complex* $\mathcal{T}$

$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow$ *diameter* $K < e$
$\bigcup \mathcal{T} = \bigcup \mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists F.$ *finite* $F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$
$\bigwedge K.\ K \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge K \subseteq C$

10.24.20 Some results on cell division with full-dimensional cells only

proposition *fine_triangular_subdivision_of_cell_complex*:
assumes $0 < e$ *finite* $\mathcal{M}$
and *poly*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *polytope* C
and *aff*: $\bigwedge C.\ C \in \mathcal{M} \Longrightarrow$ *aff_dim* $C = d$
and *face*: $\bigwedge C1\ C2.\ [\![C1 \in \mathcal{M};\ C2 \in \mathcal{M}]\!] \Longrightarrow C1 \cap C2$ *face_of* $C1$
obtains $\mathcal{T}$ **where** *triangulation* $\mathcal{T}$ $\bigwedge k.\ k \in \mathcal{T} \Longrightarrow$ *diameter* $k < e$
$\bigwedge k.\ k \in \mathcal{T} \Longrightarrow$ *aff_dim* $k = d$ $\bigcup \mathcal{T} = \bigcup \mathcal{M}$
$\bigwedge C.\ C \in \mathcal{M} \Longrightarrow \exists f.$ *finite* $f \wedge f \subseteq \mathcal{T} \wedge C = \bigcup f$
$\bigwedge k.\ k \in \mathcal{T} \Longrightarrow \exists C.\ C \in \mathcal{M} \wedge k \subseteq C$

10.25 Finitely generated cone is polyhedral, and hence closed

proposition *polyhedron_convex_cone_hull*:
fixes S :: *'a::euclidean_space set*
assumes *finite* S
shows *polyhedron*(*convex_cone hull* S)

end

10.26 Absolute Retracts, Absolute Neighbourhood Retracts and Euclidean Neighbourhood Retracts

theory *Retracts*
imports
Brouwer_Fixpoint
Continuous_Extension
begindefinition *AR* :: *'a::topological_space set* $\Rightarrow$ *bool* **where**
AR $S \equiv \forall U.\ \forall S'$::(*'a* $*$ *real*) *set*.
S *homeomorphic* $S' \wedge$ *closedin* (*top_of_set* U) $S' \longrightarrow S'$ *retract_of* U

definition *ANR* :: *'a::topological_space set* $\Rightarrow$ *bool* **where**
ANR $S \equiv \forall U.\ \forall S'$::(*'a* $*$ *real*) *set*.
S *homeomorphic* $S' \wedge$ *closedin* (*top_of_set* U) S'

$\longrightarrow (\exists\, T.\ openin\ (top\_of\_set\ U)\ T \wedge S'\ retract\_of\ T)$

definition *ENR* :: *'a::topological_space set* ⇒ *bool* **where**
$ENR\ S \equiv \exists\, U.\ open\ U \wedge S\ retract\_of\ U$

corollary *ANR_imp_absolute_neighbourhood_retract*:
fixes *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
assumes *ANR S S homeomorphic S'*
and *clo*: *closedin (top_of_set U) S'*
obtains *V* **where** *openin (top_of_set U) V S' retract_of V*

corollary *ANR_imp_absolute_neighbourhood_retract_UNIV*:
fixes *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
assumes *ANR S* **and** *hom*: *S homeomorphic S'* **and** *clo*: *closed S'*
obtains *V* **where** *open V S' retract_of V*

corollary *neighbourhood_extension_into_ANR*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *contf*: *continuous_on S f* **and** *fim*: $f \in S \to T$ **and** *ANR T closed S*
obtains *V g* **where** $S \subseteq V$ *open V continuous_on V g*
$g \in V \to T\ \bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

10.26.1 Analogous properties of ENRs

corollary *ENR_imp_absolute_neighbourhood_retract_UNIV*:
fixes *S* :: *'a::euclidean_space set* **and** *S'* :: *'b::euclidean_space set*
assumes *ENR S S homeomorphic S'*
obtains *T'* **where** *open T' S' retract_of T'*

corollary *AR_closed_Un*:
fixes *S* :: *'a::euclidean_space set*
shows $[\![closed\ S;\ closed\ T;\ AR\ S;\ AR\ T;\ AR\ (S \cap T)]\!] \Longrightarrow AR\ (S \cup T)$

corollary *ANR_closed_Un*:
fixes *S* :: *'a::euclidean_space set*
shows $[\![closed\ S;\ closed\ T;\ ANR\ S;\ ANR\ T;\ ANR\ (S \cap T)]\!] \Longrightarrow ANR\ (S \cup T)$

10.26.2 More advanced properties of ANRs and ENRs

10.26.3 Original ANR material, now for ENRs

10.26.4 Finally, spheres are ANRs and ENRs

10.26.5 Spheres are connected, etc

10.26.6 Borsuk homotopy extension theorem

theorem *Borsuk_homotopy_extension_homotopic*:
fixes f :: $'a$::*euclidean_space* ⇒ $'b$::*euclidean_space*
assumes *cloTS*: *closedin* (*top_of_set* T) S
and *anr*: (*ANR* S ∧ *ANR* T) ∨ *ANR* U
and *contf*: *continuous_on* T f
and $f \in T \to U$
and *homotopic_with_canon* (λx. *True*) S U f g
obtains g' **where** *homotopic_with_canon* (λx. *True*) T U f g'
continuous_on T g' *image* g' $T \subseteq U$
$\bigwedge x.\ x \in S \Longrightarrow g'\ x = g\ x$

10.26.7 More extension theorems

10.26.8 The complement of a set and path-connectedness

theorem *connected_complement_homeomorphic_convex_compact*:
fixes S :: $'a$::*euclidean_space set* **and** T :: $'b$::*euclidean_space set*
assumes *hom*: S *homeomorphic* T **and** T: *convex* T *compact* T **and** *2*: $2 \leq$ *DIM*($'a$)
shows *connected*($-$ S)

corollary *path_connected_complement_homeomorphic_convex_compact*:
fixes S :: $'a$::*euclidean_space set* **and** T :: $'b$::*euclidean_space set*
assumes *hom*: S *homeomorphic* T *convex* T *compact* T $2 \leq$ *DIM*($'a$)
shows *path_connected*($-$ S)

end

10.27 Extending Continous Maps, Invariance of Domain, etc

theory *Further_Topology*
imports *Weierstrass_Theorems Polytope Complex_Transcendental Equivalence_Lebesgue_Henstock_I*
Retracts
begin

10.27.1 A map from a sphere to a higher dimensional sphere is nullhomotopic

proposition *inessential_spheremap_lowdim_gen*:
fixes f :: $'M$::*euclidean_space* $\Rightarrow$ $'a$::*euclidean_space*
assumes *convex S bounded S convex T bounded T*
and *affST*: *aff_dim S* $<$ *aff_dim T*
and *contf*: *continuous_on* (*rel_frontier S*) *f*
and *fim*: $f \in$ (*rel_frontier S*) $\to$ *rel_frontier T*
obtains *c* **where** *homotopic_with_canon* ($\lambda z.$ *True*) (*rel_frontier S*) (*rel_frontier T*) *f* ($\lambda x.$ *c*)

10.27.2 Some technical lemmas about extending maps from cell complexes

theorem *extend_map_cell_complex_to_sphere*:
assumes *finite* $\mathcal{F}$ **and** *S*: $S \subseteq \bigcup\mathcal{F}$ *closed S* **and** *T*: *convex T bounded T*
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope X*
and *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim X* $<$ *aff_dim T*
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow (X \cap Y)$ *face_of X*
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \to$ *rel_frontier T*
obtains *g* **where** *continuous_on* $(\bigcup\mathcal{F})$ *g*
$g \in (\bigcup\mathcal{F}) \to$ *rel_frontier T* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

theorem *extend_map_cell_complex_to_sphere_cofinite*:
assumes *finite* $\mathcal{F}$ **and** *S*: $S \subseteq \bigcup\mathcal{F}$ *closed S* **and** *T*: *convex T bounded T*
and *poly*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *polytope X*
and *aff*: $\bigwedge X.\ X \in \mathcal{F} \Longrightarrow$ *aff_dim X* $\leq$ *aff_dim T*
and *face*: $\bigwedge X\ Y.\ [\![X \in \mathcal{F};\ Y \in \mathcal{F}]\!] \Longrightarrow (X \cap Y)$ *face_of X*
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \to$ *rel_frontier T*
obtains *C g* **where** *finite C disjnt C S continuous_on* $(\bigcup\mathcal{F} - C)$ *g*
$g \in (\bigcup\mathcal{F} - C) \to$ *rel_frontier T* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

10.27.3 Special cases and corollaries involving spheres

proposition *extend_map_affine_to_sphere_cofinite_simple*:
fixes f :: $'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *compact S convex U bounded U*
and *aff*: $aff\_dim\ T \leq aff\_dim\ U$
and $S \subseteq T$ **and** *contf*: *continuous_on S f*
and *fim*: $f \in S \rightarrow rel\_frontier\ U$
obtains $K\ g$ **where** *finite K* $K \subseteq T$ *disjnt K S continuous_on* $(T - K)\ g$
$g \in (T - K) \rightarrow rel\_frontier\ U$
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

10.27.4 Extending maps to spheres

proposition *extend_map_affine_to_sphere_cofinite_gen*:
fixes f :: $'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *SUT*: *compact S convex U bounded U affine T* $S \subseteq T$
and *aff*: $aff\_dim\ T \leq aff\_dim\ U$
and *contf*: *continuous_on S f*
and *fim*: $f \in S \rightarrow rel\_frontier\ U$
and *dis*: $\bigwedge C.\ [\![C \in components(T - S);\ bounded\ C]\!] \Longrightarrow C \cap L \neq \{\}$
obtains $K\ g$ **where** *finite K* $K \subseteq L$ $K \subseteq T$ *disjnt K S continuous_on* $(T - K)$
g
$g \in (T - K) \rightarrow rel\_frontier\ U$
$\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

corollary *extend_map_affine_to_sphere_cofinite*:
fixes f :: $'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *SUT*: *compact S affine T* $S \subseteq T$
and *aff*: $aff\_dim\ T \leq DIM('b)$ **and** $0 \leq r$
and *contf*: *continuous_on S f*
and *fim*: $f \in S \rightarrow sphere\ a\ r$
and *dis*: $\bigwedge C.\ [\![C \in components(T - S);\ bounded\ C]\!] \Longrightarrow C \cap L \neq \{\}$
obtains $K\ g$ **where** *finite K* $K \subseteq L$ $K \subseteq T$ *disjnt K S continuous_on* $(T - K)$
g
$g \in (T - K) \rightarrow sphere\ a\ r$ $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

corollary *extend_map_UNIV_to_sphere_cofinite*:
fixes f :: $'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes $DIM('a) \leq DIM('b)$ **and** $0 \leq r$
and *compact S*
and *continuous_on S f*
and $f \in S \rightarrow sphere\ a\ r$
and $\bigwedge C.\ [\![C \in components(-\ S);\ bounded\ C]\!] \Longrightarrow C \cap L \neq \{\}$
obtains $K\ g$ **where** *finite K* $K \subseteq L$ *disjnt K S continuous_on* $(-\ K)\ g$

$g \in (-\ K) \to$ *sphere a r* $\bigwedge x.\ x \in S \Longrightarrow g\ x = f\ x$

corollary *extend_map_UNIV_to_sphere_no_bounded_component*:
fixes *f* :: *'a::euclidean_space* ⇒ *'b::euclidean_space*
assumes *aff*: *DIM('a)* ≤ *DIM('b)* **and** *0* ≤ *r*
and *SUT*: *compact S*
and *contf*: *continuous_on S f*
and *fim*: *f* ∈ *S* → *sphere a r*
and *dis*: ⋀*C*. *C* ∈ *components*(− *S*) ⟹ ¬ *bounded C*
obtains *g* **where** *continuous_on UNIV g g* ∈ *UNIV* → *sphere a r* ⋀*x*. *x* ∈ *S* ⟹ *g x* = *f x*

theorem *Borsuk_separation_theorem_gen*:
fixes *S* :: *'a::euclidean_space set*
assumes *compact S*
shows (∀ *c* ∈ *components*(− *S*). ¬*bounded c*) ⟷
(∀ *f*. *continuous_on S f* ∧ *f* ∈ *S* → *sphere* (*0::'a*) *1*
⟶ (∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *S* (*sphere 0 1*) *f* (λ*x*. *c*)))
(**is** *?lhs* = *?rhs*)

corollary *Borsuk_separation_theorem*:
fixes *S* :: *'a::euclidean_space set*
assumes *compact S* **and** *2*: *2* ≤ *DIM('a)*
shows *connected*(− *S*) ⟷
(∀ *f*. *continuous_on S f* ∧ *f* ∈ *S* → *sphere* (*0::'a*) *1*
⟶ (∃ *c*. *homotopic_with_canon* (λ*x*. *True*) *S* (*sphere 0 1*) *f* (λ*x*. *c*)))
(**is** *?lhs* = *?rhs*)

proposition *Jordan_Brouwer_separation*:
fixes *S* :: *'a::euclidean_space set* **and** *a::'a*
assumes *hom*: *S homeomorphic sphere a r* **and** *0* < *r*
shows ¬ *connected*(− *S*)

proposition *Jordan_Brouwer_frontier*:
fixes *S* :: *'a::euclidean_space set* **and** *a::'a*
assumes *S*: *S homeomorphic sphere a r* **and** *T*: *T* ∈ *components*(− *S*) **and** *2*: *2* ≤ *DIM('a)*
shows *frontier T* = *S*

proposition *Jordan_Brouwer_nonseparation*:
fixes *S* :: *'a::euclidean_space set* **and** *a::'a*
assumes *S*: *S homeomorphic sphere a r* **and** *T* ⊂ *S* **and** *2*: *2* ≤ *DIM('a)*
shows *connected*(− *T*)

10.27.5 Invariance of domain and corollaries

theorem *invariance_of_domain*:
fixes $f :: 'a \Rightarrow 'a::euclidean\_space$
assumes *continuous_on S f open S inj_on f S*
shows *open*(f ‘ S)

corollary *invariance_of_domain_subspaces*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *ope*: *openin* (*top_of_set U*) *S*
and *subspace U subspace V* **and** *VU*: $dim\ V \leq dim\ U$
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \rightarrow V$
and *injf*: *inj_on f S*
shows *openin* (*top_of_set V*) (f ‘ S)

corollary *invariance_of_dimension_subspaces*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *ope*: *openin* (*top_of_set U*) *S*
and *subspace U subspace V*
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \rightarrow V$
and *injf*: *inj_on f S* **and** $S \neq \{\}$
shows $dim\ U \leq dim\ V$

corollary *invariance_of_domain_affine_sets*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V* $aff\_dim\ V \leq aff\_dim\ U$
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \rightarrow V$
and *injf*: *inj_on f S*
shows *openin* (*top_of_set V*) (f ‘ S)

corollary *invariance_of_dimension_affine_sets*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *ope*: *openin* (*top_of_set U*) *S*
and *aff*: *affine U affine V*
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \rightarrow V$
and *injf*: *inj_on f S* **and** $S \neq \{\}$
shows $aff\_dim\ U \leq aff\_dim\ V$

corollary *invariance_of_dimension*:
fixes $f :: 'a::euclidean\_space \Rightarrow 'b::euclidean\_space$
assumes *contf*: *continuous_on S f* **and** *open S*
and *injf*: *inj_on f S* **and** $S \neq \{\}$
shows $DIM('a) \leq DIM('b)$

corollary *continuous_injective_image_subspace_dim_le*:
fixes f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
assumes *subspace S subspace T*
and *contf*: *continuous_on S f* **and** *fim*: $f \in S \to T$
and *injf*: *inj_on f S*
shows $dim\ S \leq dim\ T$

corollary *invariance_of_domain_homeomorphic*:
fixes f :: $'a$::*euclidean_space* $\Rightarrow$ $'b$::*euclidean_space*
assumes *open S continuous_on S f* $DIM('b) \leq DIM('a)$ *inj_on f S*
shows *S homeomorphic* (*f* ‘ *S*)

proposition *homeomorphic_interiors*:
fixes S :: $'a$::*euclidean_space set* **and** T :: $'b$::*euclidean_space set*
assumes *S homeomorphic T interior* $S = \{\} \longleftrightarrow$ *interior* $T = \{\}$
shows (*interior S*) *homeomorphic* (*interior T*)

proposition *uniformly_continuous_homeomorphism_UNIV_trivial*:
fixes f :: $'a$::*euclidean_space* $\Rightarrow$ $'a$
assumes *contf*: *uniformly_continuous_on S f* **and** *hom*: *homeomorphism S UNIV f g*
shows $S = UNIV$

10.27.6 Formulation of loop homotopy in terms of maps out of type complex

proposition *simply_connected_eq_homotopic_circlemaps*:
fixes S :: $'a$::*real_normed_vector set*
shows *simply_connected S* $\longleftrightarrow$
($\forall$ *f g*::*complex* $\Rightarrow$ $'a$.
continuous_on (*sphere 0 1*) *f* $\wedge$ $f \in$ (*sphere 0 1*) $\to S \wedge$
continuous_on (*sphere 0 1*) *g* $\wedge$ $g \in$ (*sphere 0 1*) $\to S$
$\longrightarrow$ *homotopic_with_canon* ($\lambda h.$ *True*) (*sphere 0 1*) *S f g*)

proposition *simply_connected_eq_contractible_circlemap*:
fixes S :: $'a$::*real_normed_vector set*
shows *simply_connected S* $\longleftrightarrow$
path_connected S $\wedge$
($\forall$ *f*::*complex* $\Rightarrow$ $'a$.
continuous_on (*sphere 0 1*) *f* $\wedge$ *f* ‘(*sphere 0 1*) $\subseteq S$
$\longrightarrow$ ($\exists$ *a*. *homotopic_with_canon* ($\lambda h.$ *True*) (*sphere 0 1*) *S f* ($\lambda x.$ *a*)))

corollary *homotopy_eqv_simple_connectedness*:
fixes *S* :: *'a::real_normed_vector set* **and** *T* :: *'b::real_normed_vector set*
shows *S homotopy_eqv T* $\Longrightarrow$ *simply_connected S* $\longleftrightarrow$ *simply_connected T*

10.27.7 Homeomorphism of simple closed curves to circles

proposition *homeomorphic_simple_path_image_circle*:
fixes *a* :: *complex* **and** γ :: *real* $\Rightarrow$ *'a::t2_space*
assumes *simple_path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ **and** $0 < r$
shows (*path_image* γ) *homeomorphic sphere a r*

10.27.8 Dimension-based conditions for various homeomorphisms

10.27.9 more invariance of domain

proposition *invariance_of_domain_sphere_affine_set_gen*:
fixes *f* :: *'a::euclidean_space* $\Rightarrow$ *'b::euclidean_space*
assumes *contf*: *continuous_on S f* **and** *injf*: *inj_on f S* **and** *fim*: $f \in S \to T$
and *U*: *bounded U convex U*
and *affine T* **and** *affTU*: *aff_dim T* < *aff_dim U*
and *ope*: *openin* (*top_of_set* (*rel_frontier U*)) *S*
shows *openin* (*top_of_set T*) (*f ‘ S*)

proposition *simply_connected_punctured_convex*:
fixes *a* :: *'a::euclidean_space*
assumes *convex S* **and** *3*: $3 \leq$ *aff_dim S*
shows *simply_connected*($S - \{a\}$)

corollary *simply_connected_punctured_universe*:
fixes *a* :: *'a::euclidean_space*
assumes $3 \leq$ *DIM*(*'a*)
shows *simply_connected*($- \{a\}$)

10.27.10 The power, squaring and exponential functions as covering maps

proposition *covering_space_power_punctured_plane*:
assumes $0 < n$
shows *covering_space* ($- \{0\}$) (λz::*complex*. *z*^*n*) ($- \{0\}$)

corollary *covering_space_square_punctured_plane*:
covering_space ($- \{0\}$) (λz::*complex*. *z*^*2*) ($- \{0\}$)

proposition *covering_space_exp_punctured_plane*:
covering_space UNIV (λz::*complex*. *exp z*) ($-$ {*0*})

10.27.11 Hence the Borsukian results about mappings into circles

corollary *inessential_imp_continuous_logarithm_circle*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *complex*
assumes *homotopic_with_canon* (λh. *True*) *S* (*sphere 0 1*) *f* (λt. *a*)
obtains *g* **where** *continuous_on S g* **and** $\bigwedge x.\ x \in S \Longrightarrow f\ x = exp(g\ x)$

proposition *homotopic_with_sphere_times*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *complex*
assumes *hom*: *homotopic_with_canon* (λx. *True*) *S* (*sphere 0 1*) *f g* **and** *conth*: *continuous_on S h*
and *hin*: $\bigwedge x.\ x \in S \Longrightarrow h\ x \in sphere\ 0\ 1$
shows *homotopic_with_canon* (λx. *True*) *S* (*sphere 0 1*) (λx. *f x* ∗ *h x*) (λx. *g x* ∗ *h x*)

proposition *homotopic_circlemaps_divide*:
fixes *f* :: *'a*::*real_normed_vector* ⇒ *complex*
shows *homotopic_with_canon* (λx. *True*) *S* (*sphere 0 1*) *f g* ⟷
continuous_on S f ∧ *f* ∈ *S* → *sphere 0 1* ∧
continuous_on S g ∧ *g* ∈ *S* → *sphere 0 1* ∧
(∃ *c*. *homotopic_with_canon* (λx. *True*) *S* (*sphere 0 1*) (λx. *f x* / *g x*) (λx. *c*))

10.27.12 Upper and lower hemicontinuous functions

proposition *upper_lower_hemicontinuous_explicit*:
fixes *T* :: (*'b*::{*real_normed_vector*,*heine_borel*}) *set*
assumes *fST*: $\bigwedge x.\ x \in S \Longrightarrow f\ x \subseteq T$
and *ope*: $\bigwedge U$. *openin* (*top_of_set T*) *U*
⟹ *openin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *U*}
and *clo*: $\bigwedge U$. *closedin* (*top_of_set T*) *U*
⟹ *closedin* (*top_of_set S*) {*x* ∈ *S*. *f x* ⊆ *U*}
and *x* ∈ *S* *0* < *e* **and** *bofx*: *bounded*(*f x*) **and** *fx_ne*: *f x* ≠ {}
obtains *d* **where** *0* < *d*
$\bigwedge x'$. ⟦*x'* ∈ *S*; *dist x x'* < *d*⟧
⟹ (∀ *y* ∈ *f x*. ∃ *y'*. *y'* ∈ *f x'* ∧ *dist y y'* < *e*) ∧
(∀ *y'* ∈ *f x'*. ∃ *y*. *y* ∈ *f x* ∧ *dist y' y* < *e*)

10.27.13 Complex logs exist on various "well-behaved" sets

10.27.14 Another simple case where sphere maps are nullhomotopic

10.27.15 Holomorphic logarithms and square roots

10.27.16 The "Borsukian" property of sets

definition *Borsukian* **where**
Borsukian S ≡
∀*f*. *continuous_on S f* ∧ *f* ∈ *S* → (− {*0*::*complex*})
⟶ (∃ *a*. *homotopic_with_canon* (λ*h*. *True*) *S* (− {*0*}) *f* (λ*x*. *a*))

proposition *Borsukian_sphere*:
fixes *a* :: *'a*::*euclidean_space*
shows *3* ≤ *DIM*(*'a*) ⟹ *Borsukian* (*sphere a r*)

proposition *Borsukian_open_Un*:
fixes *S* :: *'a*::*real_normed_vector set*
assumes *opeS*: *openin* (*top_of_set* (*S* ∪ *T*)) *S*
and *opeT*: *openin* (*top_of_set* (*S* ∪ *T*)) *T*
and *BS*: *Borsukian S* **and** *BT*: *Borsukian T* **and** *ST*: *connected*(*S* ∩ *T*)
shows *Borsukian*(*S* ∪ *T*)

proposition *closed_irreducible_separator*:
fixes *a* :: *'a*::*real_normed_vector*
assumes *closed S* **and** *ab*: ¬ *connected_component* (− *S*) *a b*
obtains *T* **where** *T* ⊆ *S closed T T* ≠ {} ¬ *connected_component* (− *T*) *a b*
⋀*U*. *U* ⊂ *T* ⟹ *connected_component* (− *U*) *a b*

10.27.17 Unicoherence (closed)

definition *unicoherent* **where**
unicoherent U ≡
∀ *S T*. *connected S* ∧ *connected T* ∧ *S* ∪ *T* = *U* ∧
closedin (*top_of_set U*) *S* ∧ *closedin* (*top_of_set U*) *T*
⟶ *connected* (*S* ∩ *T*)

proposition *homeomorphic_unicoherent*:
assumes *ST*: *S homeomorphic T* **and** *S*: *unicoherent S*
shows *unicoherent T*

corollary *contractible_imp_unicoherent*:
fixes *U* :: *'a::euclidean_space set*
assumes *contractible U* **shows** *unicoherent U*

corollary *convex_imp_unicoherent*:
fixes *U* :: *'a::euclidean_space set*
assumes *convex U* **shows** *unicoherent U*
corollary *unicoherent_UNIV*: *unicoherent* (*UNIV* :: *'a* :: *euclidean_space set*)

10.27.18 Several common variants of unicoherence

10.27.19 Some separation results

proposition *separation_by_component_open*:
fixes *S* :: *'a* :: *euclidean_space set*
assumes *open S* **and** *non*: ¬ *connected*(− *S*)
obtains *C* **where** *C* ∈ *components S* ¬ *connected*(− *C*)

proposition *inessential_eq_extensible*:
fixes *f* :: *'a::euclidean_space* ⇒ *complex*
assumes *closed S*
shows (∃ *a. homotopic_with_canon* (λ*h. True*) *S* (−{*0*}) *f* (λ*t. a*)) ⟷
(∃ *g. continuous_on UNIV g* ∧ (∀ *x* ∈ *S. g x* = *f x*) ∧ (∀ *x. g x* ≠ *0*))
(**is** *?lhs* = *?rhs*)

proposition *Janiszewski_dual*:
fixes *S* :: *complex set*
assumes *compact S compact T connected S connected T connected*(− (*S* ∪ *T*))
shows *connected*(*S* ∩ *T*)

end

10.28 The Jordan Curve Theorem and Applications

theory *Jordan_Curve*
imports *Arcwise_Connected Further_Topology*
begin

10.28.1 Janiszewski's theorem

theorem *Janiszewski*:
  fixes *a b* :: *complex*
  assumes *compact S closed T* **and** *conST*: *connected* (*S* ∩ *T*)
      and *ccS*: *connected_component* (− *S*) *a b* **and** *ccT*: *connected_component* (− *T*) *a b*
    shows *connected_component* (− (*S* ∪ *T*)) *a b*

10.28.2 The Jordan Curve theorem

corollary *Jordan_inside_outside*:
  fixes *c* :: *real* ⇒ *complex*
  assumes *simple_path c pathfinish c* = *pathstart c*
    shows *inside*(*path_image c*) ≠ {} ∧
          open(*inside*(*path_image c*)) ∧
          connected(*inside*(*path_image c*)) ∧
          outside(*path_image c*) ≠ {} ∧
          open(*outside*(*path_image c*)) ∧
          connected(*outside*(*path_image c*)) ∧
          bounded(*inside*(*path_image c*)) ∧
          ¬ *bounded*(*outside*(*path_image c*)) ∧
          inside(*path_image c*) ∩ *outside*(*path_image c*) = {} ∧
          inside(*path_image c*) ∪ *outside*(*path_image c*) =
          − *path_image c* ∧
          frontier(*inside*(*path_image c*)) = *path_image c* ∧
          frontier(*outside*(*path_image c*)) = *path_image c*

theorem *split_inside_simple_closed_curve*:
  fixes *c* :: *real* ⇒ *complex*
  assumes *simple_path c1* **and** *c1*: *pathstart c1* = *a pathfinish c1* = *b*
      and *simple_path c2* **and** *c2*: *pathstart c2* = *a pathfinish c2* = *b*
      and *simple_path c* **and** *c*: *pathstart c* = *a pathfinish c* = *b*
      and *a* ≠ *b*
      and *c1c2*: *path_image c1* ∩ *path_image c2* = {*a*,*b*}
      and *c1c*: *path_image c1* ∩ *path_image c* = {*a*,*b*}
      and *c2c*: *path_image c2* ∩ *path_image c* = {*a*,*b*}
      and *ne_12*: *path_image c* ∩ *inside*(*path_image c1* ∪ *path_image c2*) ≠ {}
   obtains *inside*(*path_image c1* ∪ *path_image c*) ∩ *inside*(*path_image c2* ∪ *path_image c*) = {}
             inside(*path_image c1* ∪ *path_image c*) ∪ *inside*(*path_image c2* ∪ *path_image c*) ∪
         (*path_image c* − {*a*,*b*}) = *inside*(*path_image c1* ∪ *path_image c2*)

end

10.29 Polynomial Functions: Extremal Behaviour and Root Counts

theory *Poly_Roots*
imports *Complex_Main*
begin

10.29.1 Basics about polynomial functions: extremal behaviour and root counts

proposition *polyfun_extremal_lemma*:
fixes $c :: nat \Rightarrow 'a::real\_normed\_div\_algebra$
assumes $e > 0$
shows $\exists M. \forall z. M \leq norm\ z \longrightarrow norm(\sum i{\leq}n.\ c\ i * z\widehat{\ }i) \leq e * norm(z)\ \widehat{\ }$ $Suc\ n$

proposition *polyfun_extremal*:
fixes $c :: nat \Rightarrow 'a::real\_normed\_div\_algebra$
assumes $\exists k. k \neq 0 \wedge k \leq n \wedge c\ k \neq 0$
shows $eventually\ (\lambda z.\ norm(\sum i{\leq}n.\ c\ i * z\widehat{\ }i) \geq B)\ at\_infinity$

proposition *polyfun_rootbound*:
fixes $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
assumes $\exists k. k \leq n \wedge c\ k \neq 0$
shows $finite\ \{z.\ (\sum i{\leq}n.\ c\ i * z\widehat{\ }i) = 0\} \wedge card\ \{z.\ (\sum i{\leq}n.\ c\ i * z\widehat{\ }i) = 0\}$ $\leq n$

corollary
fixes $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
assumes $\exists k. k \leq n \wedge c\ k \neq 0$
shows *polyfun_rootbound_finite*: $finite\ \{z.\ (\sum i{\leq}n.\ c\ i * z\widehat{\ }i) = 0\}$
and *polyfun_rootbound_card*: $card\ \{z.\ (\sum i{\leq}n.\ c\ i * z\widehat{\ }i) = 0\} \leq n$

proposition *polyfun_finite_roots*:
fixes $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
shows $finite\ \{z.\ (\sum i{\leq}n.\ c\ i * z\widehat{\ }i) = 0\} \longleftrightarrow (\exists k. k \leq n \wedge c\ k \neq 0)$

theorem *polyfun_eq_const*:
fixes $c :: nat \Rightarrow 'a::\{comm\_ring,real\_normed\_div\_algebra\}$
shows $(\forall z.\ (\sum i{\leq}n.\ c\ i * z\widehat{\ }i) = k) \longleftrightarrow c\ 0 = k \wedge (\forall k. k \neq 0 \wedge k \leq n \longrightarrow$ $c\ k = 0)$

end

10.30 Generalised Binomial Theorem

theory *Generalised_Binomial_Theorem*

imports
Complex_Main
Complex_Transcendental
Summation_Tests
begin

theorem *gen_binomial_complex*:
fixes z :: *complex*
assumes *norm* $z < 1$
shows $(\lambda n.\ (a\ gchoose\ n) * z\hat{\ }n)$ *sums* $(1 + z)$ *powr* a

end

10.31 Vitali Covering Theorem and an Application to Negligibility

theory *Vitali_Covering_Theorem*
imports
HOL−Combinatorics.Permutations
Equivalence_Lebesgue_Henstock_Integration
begin

10.31.1 Vitali covering theorem

theorem *Vitali_covering_theorem_cballs*:
fixes $a :: 'a \Rightarrow 'n::euclidean\_space$
assumes r: $\bigwedge i.\ i \in K \Longrightarrow 0 < r\ i$
and S: $\bigwedge x\ d.\ [\![x \in S;\ 0 < d]\!]$
$\Longrightarrow \exists i.\ i \in K \wedge x \in cball\ (a\ i)\ (r\ i) \wedge r\ i < d$
obtains C **where** *countable* C $C \subseteq K$
pairwise $(\lambda i\ j.\ disjnt\ (cball\ (a\ i)\ (r\ i))\ (cball\ (a\ j)\ (r\ j)))\ C$
negligible$(S - (\bigcup i \in C.\ cball\ (a\ i)\ (r\ i)))$

theorem *Vitali_covering_theorem_balls*:
fixes $a :: 'a \Rightarrow 'b::euclidean\_space$
assumes S: $\bigwedge x\ d.\ [\![x \in S;\ 0 < d]\!] \Longrightarrow \exists i.\ i \in K \wedge x \in ball\ (a\ i)\ (r\ i) \wedge r\ i < d$
obtains C **where** *countable* C $C \subseteq K$
pairwise $(\lambda i\ j.\ disjnt\ (ball\ (a\ i)\ (r\ i))\ (ball\ (a\ j)\ (r\ j)))\ C$
negligible$(S - (\bigcup i \in C.\ ball\ (a\ i)\ (r\ i)))$

proposition *negligible_eq_zero_density*:
negligible $S \longleftrightarrow$

(∀ *x*∈*S*. ∀ *r*>*0*. ∀ *e*>*0*. ∃ *d*. *0* < *d* ∧ *d* ≤ *r* ∧
(∃ *U*. *S* ∩ *ball x d* ⊆ *U* ∧ *U* ∈ *lmeasurable* ∧ *measure lebesgue U* < *e* ∗ *measure lebesgue* (*ball x d*)))

end

10.32 Change of Variables Theorems

theory *Change_Of_Vars*
imports *Vitali_Covering_Theorem Determinants*

begin

10.32.1 Measurable Shear and Stretch

proposition
fixes *a* :: *real^'n*
assumes *m* ≠ *n* **and** *ab_ne*: *cbox a b* ≠ {} **and** *an*: *0* ≤ *a$n*
shows *measurable_shear_interval*: (λ*x*. χ *i*. *if i* = *m then x$m* + *x$n else x$i*) ‘ (*cbox a b*) ∈ *lmeasurable*
(**is** *?f* ‘ _ ∈ _)
and *measure_shear_interval*: *measure lebesgue* ((λ*x*. χ *i*. *if i* = *m then x$m* + *x$n else x$i*) ‘ *cbox a b*)
= *measure lebesgue* (*cbox a b*) (**is** *?Q*)

proposition
fixes *S* :: (*real^'n*) *set*
assumes *S* ∈ *lmeasurable*
shows *measurable_stretch*: ((λ*x*. χ *k*. *m k* ∗ *x$k*) ‘ *S*) ∈ *lmeasurable* (**is** *?f* ‘ *S* ∈ _)
and *measure_stretch*: *measure lebesgue* ((λ*x*. χ *k*. *m k* ∗ *x$k*) ‘ *S*) = |*prod m UNIV*| ∗ *measure lebesgue S*
(**is** *?MEQ*)

proposition
fixes *f* :: *real^'n*::{*finite*,*wellorder*} ⇒ *real^'n*::_
assumes *linear f S* ∈ *lmeasurable*
shows *measurable_linear_image*: (*f* ‘ *S*) ∈ *lmeasurable*
and *measure_linear_image*: *measure lebesgue* (*f* ‘ *S*) = |*det* (*matrix f*)| ∗ *measure lebesgue S* (**is** *?Q f S*)

proposition *measure_semicontinuous_with_hausdist_explicit*:
assumes *bounded S* **and** *neg*: *negligible*(*frontier S*) **and** *e* > *0*
obtains *d* **where** *d* > *0*
⋀*T*. ⟦*T* ∈ *lmeasurable*; ⋀*y*. *y* ∈ *T* ⟹ ∃ *x*. *x* ∈ *S* ∧ *dist x y* < *d*⟧
⟹ *measure lebesgue T* < *measure lebesgue S* + *e*

proposition
fixes f :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real*^'*n*::_
assumes S: $S \in$ *lmeasurable*
and *deriv*: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
and *int*: ($\lambda x.$ |*det* (*matrix* (*f' x*))|) *integrable_on S*
and *bounded*: $\bigwedge x.\ x \in S \Longrightarrow$ |*det* (*matrix* (*f' x*))| $\leq B$
shows *measurable_bounded_differentiable_image*:
f ' S $\in$ *lmeasurable*
and *measure_bounded_differentiable_image*:
measure lebesgue (*f ' S*) $\leq B *$ *measure lebesgue S* (**is** *?M*)

theorem
fixes f :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real*^'*n*::_
assumes S: $S \in$ *sets lebesgue*
and *deriv*: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
and *int*: ($\lambda x.$ |*det* (*matrix* (*f' x*))|) *integrable_on S*
shows *measurable_differentiable_image*: *f ' S* $\in$ *lmeasurable*
and *measure_differentiable_image*:
measure lebesgue (*f ' S*) $\leq$ *integral S* ($\lambda x.$ |*det* (*matrix* (*f' x*))|) (**is** *?M*)

10.32.2 Borel measurable Jacobian determinant

proposition *borel_measurable_partial_derivatives*:
fixes f :: *real*^'*m*::{*finite*,*wellorder*} ⇒ *real*^'*n*
assumes S: $S \in$ *sets lebesgue*
and f: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
shows ($\lambda x.$ (*matrix*(*f' x*)\$*m*\$*n*)) $\in$ *borel_measurable* (*lebesgue_on S*)

theorem *borel_measurable_det_Jacobian*:
fixes f :: *real*^'*n*::{*finite*,*wellorder*} ⇒ *real*^'*n*::_
assumes S: $S \in$ *sets lebesgue* **and** f: $\bigwedge x.\ x \in S \Longrightarrow$ (*f has_derivative f' x*) (*at x within S*)
shows ($\lambda x.$ *det*(*matrix*(*f' x*))) $\in$ *borel_measurable* (*lebesgue_on S*)
theorem *borel_measurable_lebesgue_on_preimage_borel*:
fixes f :: *'a*::*euclidean_space* ⇒ *'b*::*euclidean_space*
assumes $S \in$ *sets lebesgue*
shows $f \in$ *borel_measurable* (*lebesgue_on S*) $\longleftrightarrow$
($\forall T.\ T \in$ *sets borel* $\longrightarrow \{x \in S.\ f\ x \in T\} \in$ *sets lebesgue*)

10.32.3 Simplest case of Sard's theorem (we don't need continuity of derivative)

theorem *baby_Sard*:
  fixes f :: *real*$\widehat{\ }$'*m*::{*finite*,*wellorder*} ⇒ *real*$\widehat{\ }$'*n*::{*finite*,*wellorder*}
  assumes *mlen*: $CARD('m) \leq CARD('n)$
    and *der*: $\bigwedge x.\ x \in S \Longrightarrow (f\ has\_derivative\ f'\ x)\ (at\ x\ within\ S)$
    and *rank*: $\bigwedge x.\ x \in S \Longrightarrow rank(matrix(f'\ x)) < CARD('n)$
  shows *negligible*(f ‘ S)

10.32.4 A one-way version of change-of-variables not assuming injectivity.

proposition *absolutely_integrable_on_image*:
  fixes f :: *real*$\widehat{\ }$'*m*::{*finite*,*wellorder*} ⇒ *real*$\widehat{\ }$'*n* **and** g :: *real*$\widehat{\ }$'*m*::_ ⇒ *real*$\widehat{\ }$'*m*::_
  assumes *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$
    and *intS*: $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x))$ *absolutely_integrable_on* S
  shows f *absolutely_integrable_on* (g ‘ S)

proposition *integral_on_image_ubound*:
  fixes f :: *real*$\widehat{\ }$'*n*::{*finite*,*wellorder*} ⇒ *real* **and** g :: *real*$\widehat{\ }$'*n*::_ ⇒ *real*$\widehat{\ }$'*n*::_
  assumes $\bigwedge x.\ x \in S \Longrightarrow 0 \leq f(g\ x)$
    and $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$
    and $(\lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x))$ *integrable_on* S
  shows *integral* (g ‘ S) $f \leq$ *integral* S $(\lambda x.\ |det\ (matrix\ (g'\ x))| * f(g\ x))$

10.32.5 Change-of-variables theorem

theorem *has_absolute_integral_change_of_variables_invertible*:
  fixes f :: *real*$\widehat{\ }$'*m*::{*finite*,*wellorder*} ⇒ *real*$\widehat{\ }$'*n* **and** g :: *real*$\widehat{\ }$'*m*::_ ⇒ *real*$\widehat{\ }$'*m*::_
  assumes *der_g*: $\bigwedge x.\ x \in S \Longrightarrow (g\ has\_derivative\ g'\ x)\ (at\ x\ within\ S)$
      and *hg*: $\bigwedge x.\ x \in S \Longrightarrow h(g\ x) = x$
      and *conth*: *continuous_on* (g ‘ S) h
  shows $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x))$ *absolutely_integrable_on* S ∧ *integral* S $(\lambda x.\ |det\ (matrix\ (g'\ x))| *_R f(g\ x)) = b \longleftrightarrow$
        f *absolutely_integrable_on* (g ‘ S) ∧ *integral* (g ‘ S) $f = b$

(**is** *?lhs* = *?rhs*)

theorem *has_absolute_integral_change_of_variables_compact*:
fixes f :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** g :: *real^'m::_* ⇒ *real^'m::_*
assumes *compact S*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
and *inj*: *inj_on g S*
shows ((λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S* ∧
integral S (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) = *b*)
⟷ *f absolutely_integrable_on* (*g* ' *S*) ∧ *integral* (*g* ' *S*) *f* = *b*)

theorem *has_absolute_integral_change_of_variables*:
fixes f :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** g :: *real^'m::_* ⇒ *real^'m::_*
assumes *S*: *S* ∈ *sets lebesgue*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
and *inj*: *inj_on g S*
shows (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S* ∧
integral S (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) = *b*
⟷ *f absolutely_integrable_on* (*g* ' *S*) ∧ *integral* (*g* ' *S*) *f* = *b*

corollary *absolutely_integrable_change_of_variables*:
fixes f :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** g :: *real^'m::_* ⇒ *real^'m::_*
assumes *S* ∈ *sets lebesgue*
and $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
and *inj_on g S*
shows *f absolutely_integrable_on* (*g* ' *S*)
⟷ (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*

corollary *integral_change_of_variables*:
fixes f :: *real^'m::{finite,wellorder}* ⇒ *real^'n* **and** g :: *real^'m::_* ⇒ *real^'m::_*
assumes *S*: *S* ∈ *sets lebesgue*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_derivative g' x*) (*at x within S*)
and *inj*: *inj_on g S*
and *disj*: (*f absolutely_integrable_on* (*g* ' *S*) ∨
(λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*)
shows *integral* (*g* ' *S*) *f* = *integral S* (λ*x*. |*det* (*matrix* (*g' x*))| $*_R$ *f*(*g x*))

corollary *absolutely_integrable_change_of_variables_1*:
fixes f :: *real* ⇒ *real^'n::{finite,wellorder}* **and** g :: *real* ⇒ *real*
assumes *S*: *S* ∈ *sets lebesgue*
and *der_g*: $\bigwedge x.\ x \in S \Longrightarrow$ (*g has_vector_derivative g' x*) (*at x within S*)
and *inj*: *inj_on g S*
shows (*f absolutely_integrable_on g* ' *S* ⟷
(λ*x*. |*g' x*| $*_R$ *f*(*g x*)) *absolutely_integrable_on S*)

10.32.6 Change of variables for integrals: special case of linear function

10.32.7 Change of variable for measure

end

10.33 Lipschitz Continuity

theory *Lipschitz*
  imports
    Derivative Abstract_Metric_Spaces
begin

definition *lipschitz_on*
  where *lipschitz_on C U f* $\longleftrightarrow$ ($0 \leq C \wedge (\forall x \in U.\ \forall y{\in}U.\ dist\ (f\ x)\ (f\ y) \leq C * dist\ x\ y)$)
notation
  lipschitz_on (‹(‹open_block notation=‹postfix lipschitz_on››_−lipschitz′_on)› [*1000*])
proposition *lipschitz_on_uniformly_continuous*:
  assumes *L−lipschitz_on X f*
  shows *uniformly_continuous_on X f*

proposition *lipschitz_on_continuous_on*:
  continuous_on X f **if** *L−lipschitz_on X f*
proposition *bounded_derivative_imp_lipschitz*:
  assumes $\bigwedge x.\ x \in X \Longrightarrow$ (*f has_derivative f′ x*) (*at x within X*)
  assumes *convex*: *convex X*
  assumes $\bigwedge x.\ x \in X \Longrightarrow$ *onorm* ($f'\ x$) $\leq C$ $0 \leq C$
  shows *C−lipschitz_on X f*

10.33.1 Local Lipschitz continuity

proposition *lipschitz_on_closed_Union*:
  assumes $\bigwedge i.\ i \in I \Longrightarrow$ *lipschitz_on M* (*U i*) *f*
    $\bigwedge i.\ i \in I \Longrightarrow$ *closed* (*U i*)
    finite I
    $M \geq 0$
    $\{u..(v{::}real)\} \subseteq (\bigcup i{\in}I.\ U\ i)$
  shows *lipschitz_on M* $\{u..v\}$ *f*

10.33.2 Local Lipschitz continuity (uniform for a family of functions)

definition *local_lipschitz*::
$'a$::*metric_space set* ⇒ $'b$::*metric_space set* ⇒ ($'a$ ⇒ $'b$ ⇒ $'c$::*metric_space*) ⇒ *bool*
where
local_lipschitz T X f ≡ ∀ *x* ∈ *X*. ∀ *t* ∈ *T*.
∃ *u*>*0*. ∃ *L*. ∀ *t* ∈ *cball t u* ∩ *T*. *L*−*lipschitz_on* (*cball x u* ∩ *X*) (*f t*)

proposition *c1_implies_local_lipschitz*:
fixes *T*::*real set* **and** *X*::$'a$::{*banach*,*heine_borel*} *set*
and *f*::*real* ⇒ $'a$ ⇒ $'a$
assumes f': ⋀*t x*. *t* ∈ *T* ⟹ *x* ∈ *X* ⟹ (*f t has_derivative blinfun_apply* (f' (*t*, *x*))) (*at x*)
assumes *cont_*f': *continuous_on* (*T* × *X*) f'
assumes *open T*
assumes *open X*
shows *local_lipschitz T X f*

end
theory
Multivariate_Analysis
imports
Ordered_Euclidean_Space
Determinants
Cross3
Lipschitz
Starlike
beginend

10.34 Volume of a Simplex

theory *Simplex_Content*
imports *Change_Of_Vars*
begin

theorem *content_std_simplex*:
measure lborel (*convex hull* (*insert 0 Basis* :: $'a$ :: *euclidean_space set*)) =
1 / *fact DIM*($'a$)

proposition *measure_lebesgue_linear_transformation*:
fixes *A* :: (*real* ^ $'n$ :: {*finite*, *wellorder*}) *set*
fixes *f* :: _ ⇒ *real* ^ $'n$ :: {*finite*, *wellorder*}
assumes *bounded A A* ∈ *sets lebesgue linear f*
shows *measure lebesgue* (*f* ‘ *A*) = |*det* (*matrix f*)| ∗ *measure lebesgue A*

theorem *content_simplex*:

fixes *X* :: (*real* ^ *'n* :: {*finite*, *wellorder*}) *set* **and** *f* :: *'n* :: _ ⇒ *real* ^(*'n* :: _)
assumes *finite X card X* = *Suc CARD*(*'n*) **and** *x0*: *x0* ∈ *X* **and** *bij*: *bij_betw f UNIV* (*X* − {*x0*})
defines *M* ≡ (χ *i*. χ *j*. *f j* $ *i* − *x0* $ *i*)
shows *content* (*convex hull X*) = |*det M*| / *fact* (*CARD*(*'n*))

theorem *content_triangle*:
fixes *A B C* :: *real* ^ *2*
shows *content* (*convex hull* {*A*, *B*, *C*}) =
|(*C* $ *1* − *A* $ *1*) ∗ (*B* $ *2* − *A* $ *2*) − (*B* $ *1* − *A* $ *1*) ∗ (*C* $ *2* − *A* $ *2*)| / *2*

theorem *heron*:
fixes *A B C* :: *real* ^ *2*
defines *a* ≡ *dist B C* **and** *b* ≡ *dist A C* **and** *c* ≡ *dist A B*
defines *s* ≡ (*a* + *b* + *c*) / *2*
shows *content* (*convex hull* {*A*, *B*, *C*}) = *sqrt* (*s* ∗ (*s* − *a*) ∗ (*s* − *b*) ∗ (*s* − *c*))

end

10.35 Convergence of Formal Power Series

theory *FPS_Convergence*
imports
Generalised_Binomial_Theorem
HOL−Computational_Algebra.Formal_Power_Series
HOL−Computational_Algebra.Polynomial_FPS

begin

10.35.1 Basic properties of convergent power series

definition *fps_conv_radius* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps* ⇒ *ereal* **where**
fps_conv_radius f = *conv_radius* (*fps_nth f*)

definition *eval_fps* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps* ⇒ *'a* ⇒ *'a* **where**
eval_fps f z = (∑ *n*. *fps_nth f n* ∗ *z* ^ *n*)

theorem *sums_eval_fps*:
fixes *f* :: *'a* :: {*banach*, *real_normed_div_algebra*} *fps*
assumes *norm z* < *fps_conv_radius f*
shows (λ*n*. *fps_nth f n* ∗ *z* ^ *n*) *sums eval_fps f z*

10.35.2 Evaluating power series

theorem *eval_fps_deriv*:
assumes *norm z < fps_conv_radius f*
shows *eval_fps (fps_deriv f) z = deriv (eval_fps f) z*

theorem *fps_nth_conv_deriv*:
fixes *f :: complex fps*
assumes *fps_conv_radius f > 0*
shows *fps_nth f n = (deriv ^^ n) (eval_fps f) 0 / fact n*

theorem *eval_fps_eqD*:
fixes *f g :: complex fps*
assumes *fps_conv_radius f > 0 fps_conv_radius g > 0*
assumes *eventually (λz. eval_fps f z = eval_fps g z) (nhds 0)*
shows *f = g*

10.35.3 FPS of a polynomial

10.35.4 Power series expansions of analytic functions

definition
has_fps_expansion :: ('a :: {banach,real_normed_div_algebra} ⇒ 'a) ⇒ 'a fps ⇒ bool
(**infixl** ‹*has'_fps'_expansion*› *60*)
where *(f has_fps_expansion F) ⟷*
fps_conv_radius F > 0 ∧ eventually (λz. eval_fps F z = f z) (nhds 0)

end
theory *Smooth_Paths*
imports *Retracts*
begin

10.35.5 Piecewise differentiability of paths

10.35.6 Valid paths, and their start and finish

definition *valid_path :: (real ⇒ 'a :: real_normed_vector) ⇒ bool*
where *valid_path f ≡ f piecewise_C1_differentiable_on {0..1::real}*

end

10.36 Metrics on product spaces

theory *Function_Metric*
imports
Function_Topology

Elementary_Metric_Spaces
begininstantiation *fun* :: (*countable*, *metric_space*) *metric_space*
begin

definition *dist_fun_def*:
dist x y = ($\sum$ *n*. (*1/2*)^*n* ∗ *min* (*dist* (*x* (*from_nat n*)) (*y* (*from_nat n*))) *1*)

definition *uniformity_fun_def*:
(*uniformity*::((′*a* ⇒ ′*b*) × (′*a* ⇒ ′*b*)) *filter*) = (*INF e*∈{*0*<..}. *principal* {(*x*, *y*). *dist* (*x*::(′*a*⇒′*b*)) *y* < *e*})
end
theory *Analysis*
imports

Convex
Determinants

FSigma
Sum_Topology
Abstract_Topological_Spaces
Abstract_Metric_Spaces
Urysohn
Connected
Abstract_Limits
Isolated
Sparse_In

Elementary_Normed_Spaces
Norm_Arith

Convex_Euclidean_Space
Operator_Norm

Line_Segment
Derivative
Cartesian_Euclidean_Space
Kronecker_Approximation_Theorem
Weierstrass_Theorems

Ball_Volume
Integral_Test
Improper_Integral
Equivalence_Measurable_On_Borel
Lebesgue_Integral_Substitution
Embed_Measure
Complete_Measure
Radon_Nikodym
Fashoda_Theorem
Cross3

Homeomorphism
Bounded_Continuous_Function
Abstract_Topology
Product_Topology
Lindelof_Spaces
Infinite_Products
Infinite_Sum
Infinite_Set_Sum
Polytope
Jordan_Curve
Poly_Roots
Generalised_Binomial_Theorem
Gamma_Function
Change_Of_Vars
Multivariate_Analysis
Simplex_Content
FPS_Convergence
Smooth_Paths
Abstract_Euclidean_Space
Function_Metric
begin

end

10.37 Poly Mappings as a Real Normed Vector

theory *Finite_Function_Topology*
imports *Function_Topology HOL−Library.Poly_Mapping*

begin

instantiation *poly_mapping* :: (*type*, *real_vector*) *real_vector*
begin

instantiation *poly_mapping* :: (*type*, *real_normed_vector*) *metric_space*
begin

instantiation *poly_mapping* :: (*type*, *real_normed_vector*) *real_normed_vector*
begin

end

Bibliography

[1]