# Equivalents of the Axiom of Choice

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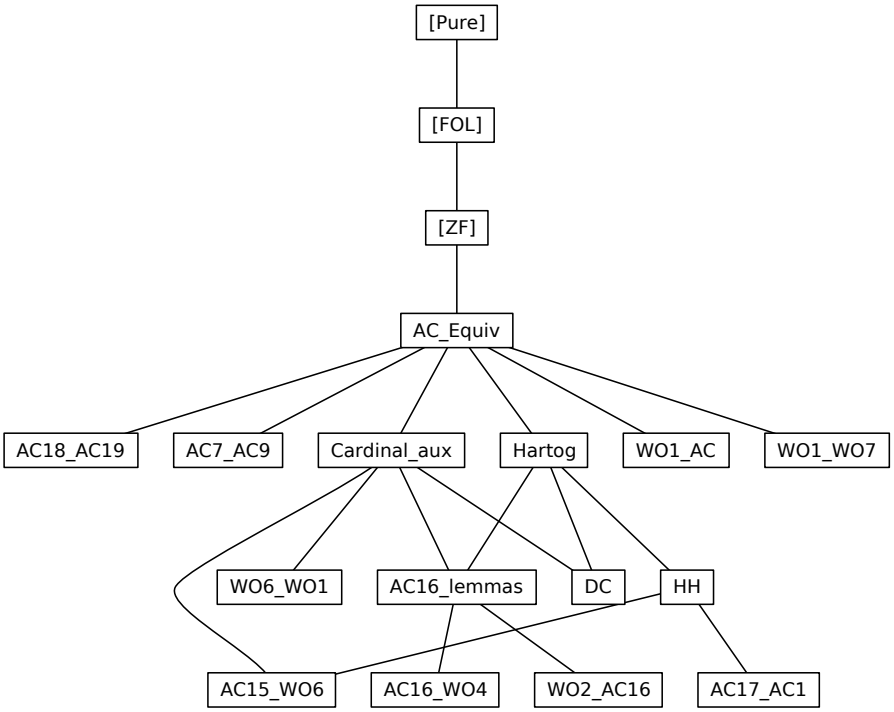
**Abstract**

This development [1] proves the equivalence of seven formulations of the well-ordering theorem and twenty formulations of the axiom of choice. It formalizes the first two chapters of the monograph *Equivalents of the Axiom of Choice* by Rubin and Rubin [2]. Some of this mmaterial involves extremely complex techniques.

## Contents





**theory** *AC_Equiv*
**imports** *ZF*
**begin**

**definition**
"*WO1* ≡ ∀*A*. ∃*R*. *well_ord(A,R)*"

**definition**
"*WO2* ≡ ∀*A*. ∃*a*. *Ord(a)* ∧ *A*≈*a*"

**definition**
"*WO3* ≡ ∀*A*. ∃*a*. *Ord(a)* ∧ (∃*b*. *b* ⊆ *a* ∧ *A*≈*b*)"

**definition**
"*WO4(m)* ≡ ∀*A*. ∃*a f*. *Ord(a)* ∧ *domain(f)*=*a* ∧
(⋃*b*<*a*. *f*‘*b*) = *A* ∧ (∀*b*<*a*. *f*‘*b* ≲ *m*)"

**definition**
"*WO5* ≡ ∃*m* ∈ *nat*. *1*≤*m* ∧ *WO4(m)*"

**definition**
"*WO6* ≡ ∀*A*. ∃*m* ∈ *nat*. *1*≤*m* ∧ (∃*a f*. *Ord(a)* ∧ *domain(f)*=*a*
∧ (⋃*b*<*a*. *f*‘*b*) = *A* ∧ (∀*b*<*a*. *f*‘*b* ≲ *m*))"

**definition**
"*WO7* ≡ ∀*A*. *Finite(A)* ⟷ (∀*R*. *well_ord(A,R)* ⟶ *well_ord(A,converse(R))*)"

**definition**
"*WO8* ≡ ∀*A*. (∃*f*. *f* ∈ (∏*X* ∈ *A*. *X*)) ⟶ (∃*R*. *well_ord(A,R)*)"

**definition**

*pairwise_disjoint* :: "*i* ⇒ *o*" **where**
"*pairwise_disjoint(A)* ≡ ∀*A1* ∈ *A*. ∀*A2* ∈ *A*. *A1* ∩ *A2* ≠ *0* ⟶ *A1*=*A2*"

**definition**
*sets_of_size_between* :: "[*i*, *i*, *i*] ⇒ *o*" **where**
"*sets_of_size_between(A,m,n)* ≡ ∀*B* ∈ *A*. *m* ≲ *B* ∧ *B* ≲ *n*"

**definition**
"*AC0* ≡ ∀*A*. ∃*f*. *f* ∈ (∏*X* ∈ *Pow(A)*-{*0*}. *X*)"

**definition**
"AC1 ≡ ∀A. 0∉A ⟶ (∃f. f ∈ (∏X ∈ A. X))"

**definition**
"AC2 ≡ ∀A. 0∉A ∧ pairwise_disjoint(A)
⟶ (∃C. ∀B ∈ A. ∃y. B ∩ C = {y})"
**definition**
"AC3 ≡ ∀A B. ∀f ∈ A->B. ∃g. g ∈ (∏x ∈ {a ∈ A. f‘a≠0}. f‘x)"

**definition**
"AC4 ≡ ∀R A B. (R ⊆ A*B ⟶ (∃f. f ∈ (∏x ∈ domain(R). R‘‘{x})))"

**definition**
"AC5 ≡ ∀A B. ∀f ∈ A->B. ∃g ∈ range(f)->A. ∀x ∈ domain(g). f‘(g‘x)
= x"

**definition**
"AC6 ≡ ∀A. 0∉A ⟶ (∏B ∈ A. B)≠0"

**definition**
"AC7 ≡ ∀A. 0∉A ∧ (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) ⟶ (∏B ∈ A. B) ≠
0"

**definition**
"AC8 ≡ ∀A. (∀B ∈ A. ∃B1 B2. B=⟨B1,B2⟩ ∧ B1≈B2)
⟶ (∃f. ∀B ∈ A. f‘B ∈ bij(fst(B),snd(B)))"

**definition**
"AC9 ≡ ∀A. (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) ⟶
(∃f. ∀B1 ∈ A. ∀B2 ∈ A. f‘⟨B1,B2⟩ ∈ bij(B1,B2))"

**definition**
"AC10(n) ≡ ∀A. (∀B ∈ A. ¬Finite(B)) ⟶
(∃f. ∀B ∈ A. (pairwise_disjoint(f‘B) ∧
sets_of_size_between(f‘B, 2, succ(n)) ∧ ⋃(f‘B)=B))"

**definition**
"AC11 ≡ ∃n ∈ nat. 1≤n ∧ AC10(n)"

**definition**
"AC12 ≡ ∀A. (∀B ∈ A. ¬Finite(B)) ⟶
(∃n ∈ nat. 1≤n ∧ (∃f. ∀B ∈ A. (pairwise_disjoint(f‘B)
∧
sets_of_size_between(f‘B, 2, succ(n)) ∧ ⋃(f‘B)=B)))"

**definition**
"AC13(m) ≡ ∀A. 0∉A ⟶ (∃f. ∀B ∈ A. f‘B≠0 ∧ f‘B ⊆ B ∧ f‘B ≲
m)"

**definition**
```
    "AC14 ≡ ∃m ∈ nat. 1≤m ∧ AC13(m)"
```

**definition**
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    "AC15 ≡ ∀A. 0∉A ⟶
                (∃m ∈ nat. 1≤m ∧ (∃f. ∀B ∈ A. f‘B≠0 ∧ f‘B ⊆ B ∧
f‘B ≲ m))"
```

**definition**
```
    "AC16(n, k)  ≡
       ∀A. ¬Finite(A) ⟶
           (∃T. T ⊆ {X ∈ Pow(A). X≈succ(n)} ∧
           (∀X ∈ {X ∈ Pow(A). X≈succ(k)}. ∃! Y. Y ∈ T ∧ X ⊆ Y))"
```

**definition**
```
    "AC17 ≡ ∀A. ∀g ∈ (Pow(A)-{0} -> A) -> Pow(A)-{0}.
                  ∃f ∈ Pow(A)-{0} -> A. f‘(g‘f) ∈ g‘f"
```

**locale** *AC18 =*
**assumes** *AC18: "A≠0 ∧ (∀a ∈ A. B(a) ≠ 0) ⟶*
*((⋂a ∈ A. ⋃b ∈ B(a). X(a,b)) =*
*(⋃f ∈ ∏a ∈ A. B(a). ⋂a ∈ A. X(a, f‘a)))"*
— AC18 cannot be expressed within the object-logic

**definition**
```
    "AC19 ≡ ∀A. A≠0 ∧ 0∉A ⟶ ((⋂a ∈ A. ⋃b ∈ a. b) =
                  (⋃f ∈ (∏B ∈ A. B). ⋂a ∈ A. f‘a))"
```

**lemma** *rvimage_id: "rvimage(A,id(A),r) = r ∩ A*A"*
**unfolding** *rvimage_def*
**apply** *(rule equalityI, safe)*
**apply** *(drule_tac P = "λa. <id (A) ‘xb,a>:r" in id_conv [THEN subst],*
*assumption)*
**apply** *(drule_tac P = "λa. ⟨a,ya⟩:r" in id_conv [THEN subst], (assumption+))*
**apply** *(fast intro: id_conv [THEN ssubst])*
**done**

**lemma** *ordertype_Int:*
*"well_ord(A,r) ⟹ ordertype(A, r ∩ A*A) = ordertype(A,r)"*
**apply** *(rule_tac P = "λa. ordertype (A,a) =ordertype (A,r) " in rvimage_id*
*[THEN subst])*

**apply** *(erule id_bij [THEN bij_ordertype_vimage])*
**done**

**lemma** *lam_sing_bij: "(λx ∈ A. {x}) ∈ bij(A, {{x}. x ∈ A})"*
**apply** *(rule_tac d = "λz. THE x. z={x}"* **in** *lam_bijective)*
**apply** *(auto simp add: the_equality)*
**done**

**lemma** *inj_strengthen_type:*
*"⟦f ∈ inj(A, B); ⋀a. a ∈ A ⟹ f‘a ∈ C⟧ ⟹ f ∈ inj(A,C)"*
**by** *(unfold inj_def, blast intro: Pi_type)*

**lemma** *ex1_two_eq: "⟦∃! x. P(x); P(x); P(y)⟧ ⟹ x=y"*
**by** *blast*

**lemma** *first_in_B:*
*"⟦well_ord(⋃(A),r); 0 ∉ A; B ∈ A⟧ ⟹ (THE b. first(b,B,r)) ∈ B"*
**by** *(blast dest!: well_ord_imp_ex1_first*
*[THEN theI, THEN first_def [THEN def_imp_iff, THEN*
*iffD1]])*

**lemma** *ex_choice_fun: "⟦well_ord(⋃(A), R); 0 ∉ A⟧ ⟹ ∃f. f ∈ (∏X*
*∈ A. X)"*
**by** *(fast elim!: first_in_B intro!: lam_type)*

**lemma** *ex_choice_fun_Pow: "well_ord(A, R) ⟹ ∃f. f ∈ (∏X ∈ Pow(A)-{0}.*
*X)"*
**by** *(fast elim!: well_ord_subset [THEN ex_choice_fun])*

**lemma** *lepoll_m_imp_domain_lepoll_m:*
*"⟦m ∈ nat; u ≲ m⟧ ⟹ domain(u) ≲ m"*
**unfolding** *lepoll_def*
**apply** *(erule exE)*

```
apply (rule_tac x = "λx ∈ domain(u). μ i. ∃y. ⟨x,y⟩ ∈ u ∧ f‘⟨x,y⟩ =
i"
       in exI)
apply (rule_tac d = "λy. fst (converse(f) ‘ y) " in lam_injective)
apply (fast intro: LeastI2 nat_into_Ord [THEN Ord_in_Ord]
                           inj_is_fun [THEN apply_type])
apply (erule domainE)
apply (frule inj_is_fun [THEN apply_type], assumption)
apply (rule LeastI2)
apply (auto elim!: nat_into_Ord [THEN Ord_in_Ord])
done

lemma rel_domain_ex1:
    "⟦succ(m) ≲ domain(r); r ≲ succ(m); m ∈ nat⟧ ⟹ function(r)"
apply (unfold function_def, safe)
apply (rule ccontr)
apply (fast elim!: lepoll_trans [THEN succ_lepoll_natE]
                   lepoll_m_imp_domain_lepoll_m [OF _ Diff_sing_lepoll]
            elim: domain_Diff_eq [OF _ not_sym, THEN subst])
done

lemma rel_is_fun:
     "⟦succ(m) ≲ domain(r);  r ≲ succ(m);  m ∈ nat;
        r ⊆ A*B; A=domain(r)⟧ ⟹ r ∈ A->B"
by (simp add: Pi_iff rel_domain_ex1)

end


theory Cardinal_aux imports AC_Equiv begin

lemma Diff_lepoll: "⟦A ≲ succ(m); B ⊆ A; B≠0⟧ ⟹ A-B ≲ m"
apply (rule not_emptyE, assumption)
apply (blast intro: lepoll_trans [OF subset_imp_lepoll Diff_sing_lepoll])
done

lemma lepoll_imp_ex_le_eqpoll:
     "⟦A ≲ i; Ord(i)⟧ ⟹ ∃j. j ≤ i ∧ A ≈ j"
by (blast intro!: lepoll_cardinal_le well_ord_Memrel
                  well_ord_cardinal_eqpoll [THEN eqpoll_sym]
          dest: lepoll_well_ord)
```

```
lemma lesspoll_imp_ex_lt_eqpoll:
     "⟦A ≺ i; Ord(i)⟧ ⟹ ∃j. j<i ∧ A ≈ j"
by (unfold lesspoll_def, blast dest!: lepoll_imp_ex_le_eqpoll elim!: leE)

lemma Un_eqpoll_Inf_Ord:
  assumes A: "A ≈ i" and B: "B ≈ i" and NFI: "¬ Finite(i)" and i:
"Ord(i)"
  shows "A ∪ B ≈ i"
proof (rule eqpollI)
  have AB: "A ≈ B" using A B by (blast intro: eqpoll_sym eqpoll_trans)

  have "2 ≲ nat"
    by (rule subset_imp_lepoll) (rule OrdmemD [OF nat_2I Ord_nat])
  also have "... ≲ i"
    by (simp add: nat_le_infinite_Ord le_imp_lepoll NFI i)+
  also have "... ≈ A" by (blast intro: eqpoll_sym A)
  finally have "2 ≲ A" .
  have ICI: "InfCard(|i|)"
    by (simp add: Inf_Card_is_InfCard Finite_cardinal_iff NFI i)
  have "A ∪ B ≲ A + B" by (rule Un_lepoll_sum)
  also have "... ≲ A × B"
    by (rule lepoll_imp_sum_lepoll_prod [OF AB [THEN eqpoll_imp_lepoll]
<2 ≲ A>])
  also have "... ≈ i × i"
    by (blast intro: prod_eqpoll_cong eqpoll_imp_lepoll A B)
  also have "... ≈ i"
    by (blast intro: well_ord_InfCard_square_eq well_ord_Memrel ICI i)

  finally show "A ∪ B ≲ i" .
next
  have "i ≈ A" by (blast intro: A eqpoll_sym)
  also have "... ≲ A ∪ B" by (blast intro: subset_imp_lepoll)
  finally show "i ≲ A ∪ B" .
qed

schematic_goal paired_bij: "?f ∈ bij({{y,z}. y ∈ x}, x)"
apply (rule RepFun_bijective)
apply (simp add: doubleton_eq_iff, blast)
done

lemma paired_eqpoll: "{{y,z}. y ∈ x} ≈ x"
by (unfold eqpoll_def, fast intro!: paired_bij)

lemma ex_eqpoll_disjoint: "∃B. B ≈ A ∧ B ∩ C = 0"
by (fast intro!: paired_eqpoll equals0I elim: mem_asym)
```

**lemma** *Un_lepoll_Inf_Ord:*
"⟦A ≲ i; B ≲ i; ¬Finite(i); Ord(i)⟧ ⟹ A ∪ B ≲ i"
**apply** (*rule_tac A1 = i* **and** *C1 = i* **in** *ex_eqpoll_disjoint [THEN exE]*)
**apply** (*erule conjE*)
**apply** (*drule lepoll_trans*)
**apply** (*erule eqpoll_sym [THEN eqpoll_imp_lepoll]*)
**apply** (*rule Un_lepoll_Un [THEN lepoll_trans], (assumption+)*)
**apply** (*blast intro: eqpoll_refl Un_eqpoll_Inf_Ord eqpoll_imp_lepoll*)
**done**

**lemma** *Least_in_Ord:* "⟦P(i); i ∈ j; Ord(j)⟧ ⟹ (μ i. P(i)) ∈ j"
**apply** (*erule Least_le [THEN leE]*)
**apply** (*erule Ord_in_Ord, assumption*)
**apply** (*erule ltE*)
**apply** (*fast dest: OrdmemD*)
**apply** (*erule subst_elem, assumption*)
**done**

**lemma** *Diff_first_lepoll:*
"⟦well_ord(x,r); y ⊆ x; y ≲ succ(n); n ∈ nat⟧
⟹ y - {THE b. first(b,y,r)} ≲ n"
**apply** (*case_tac "y=0", simp add: empty_lepollI*)
**apply** (*fast intro!: Diff_sing_lepoll the_first_in*)
**done**

**lemma** *UN_subset_split:*
"(⋃x ∈ X. P(x)) ⊆ (⋃x ∈ X. P(x)-Q(x)) ∪ (⋃x ∈ X. Q(x))"
**by** *blast*

**lemma** *UN_sing_lepoll:* "Ord(a) ⟹ (⋃x ∈ a. {P(x)}) ≲ a"
**unfolding** *lepoll_def*
**apply** (*rule_tac x = "λz ∈ (⋃x ∈ a. {P (x) }) . (μ i. P (i) =z) "* **in** *exI*)
**apply** (*rule_tac d = "λz. P (z) "* **in** *lam_injective*)
**apply** (*fast intro!: Least_in_Ord*)
**apply** (*fast intro: LeastI elim!: Ord_in_Ord*)
**done**

**lemma** *UN_fun_lepoll_lemma [rule_format]:*
"⟦well_ord(T, R); ¬Finite(a); Ord(a); n ∈ nat⟧
⟹ ∀f. (∀b ∈ a. f‘b ≲ n ∧ f‘b ⊆ T) ⟶ (⋃b ∈ a. f‘b) ≲ a"
**apply** (*induct_tac "n"*)
**apply** (*rule allI*)
**apply** (*rule impI*)
**apply** (*rule_tac b = "⋃b ∈ a. f‘b"* **in** *subst*)
**apply** (*rule_tac [2] empty_lepollI*)
**apply** (*rule equals0I [symmetric], clarify*)
**apply** (*fast dest: lepoll_0_is_0 [THEN subst]*)
**apply** (*rule allI*)

```
apply (rule impI)
apply (erule_tac x = "λx ∈ a. f‘x - {THE b. first (b,f‘x,R) }" in allE)
apply (erule impE, simp)
apply (fast intro!: Diff_first_lepoll, simp)
apply (rule UN_subset_split [THEN subset_imp_lepoll, THEN lepoll_trans])
apply (fast intro: Un_lepoll_Inf_Ord UN_sing_lepoll)
done

lemma UN_fun_lepoll:
     "⟦∀b ∈ a. f‘b ≲ n ∧ f‘b ⊆ T; well_ord(T, R);
        ¬Finite(a); Ord(a); n ∈ nat⟧ ⟹ (⋃b ∈ a. f‘b) ≲ a"
by (blast intro: UN_fun_lepoll_lemma)

lemma UN_lepoll:
     "⟦∀b ∈ a. F(b) ≲ n ∧ F(b) ⊆ T; well_ord(T, R);
        ¬Finite(a); Ord(a); n ∈ nat⟧
      ⟹ (⋃b ∈ a. F(b)) ≲ a"
apply (rule rev_mp)
apply (rule_tac f="λb ∈ a. F (b)" in UN_fun_lepoll)
apply auto
done

lemma UN_eq_UN_Diffs:
     "Ord(a) ⟹ (⋃b ∈ a. F(b)) = (⋃b ∈ a. F(b) - (⋃c ∈ b. F(c)))"
apply (rule equalityI)
 prefer 2 apply fast
apply (rule subsetI)
apply (erule UN_E)
apply (rule UN_I)
 apply (rule_tac P = "λz. x ∈ F (z) " in Least_in_Ord, (assumption+))
apply (rule DiffI, best intro: Ord_in_Ord LeastI, clarify)
apply (erule_tac P = "λz. x ∈ F (z) " and i = c in less_LeastE)
apply (blast intro: Ord_Least ltI)
done

lemma lepoll_imp_eqpoll_subset:
     "a ≲ X ⟹ ∃Y. Y ⊆ X ∧ a ≈ Y"
apply (unfold lepoll_def eqpoll_def, clarify)
apply (blast intro: restrict_bij
            dest: inj_is_fun [THEN fun_is_rel, THEN image_subset])
done




lemma Diff_lesspoll_eqpoll_Card_lemma:
     "⟦A≈a; ¬Finite(a); Card(a); B ≺ a; A-B ≺ a⟧ ⟹ P"
apply (elim lesspoll_imp_ex_lt_eqpoll [THEN exE] Card_is_Ord conjE)
```

```
apply (frule_tac j=xa in Un_upper1_le [OF lt_Ord lt_Ord], assumption)
apply (frule_tac j=xa in Un_upper2_le [OF lt_Ord lt_Ord], assumption)
apply (drule Un_least_lt, assumption)
apply (drule eqpoll_imp_lepoll [THEN lepoll_trans],
       rule le_imp_lepoll, assumption)+
apply (case_tac "Finite(x ∪ xa)")
```

finite case

```
 apply (drule Finite_Un [OF lepoll_Finite lepoll_Finite], assumption+)
 apply (drule subset_Un_Diff [THEN subset_imp_lepoll, THEN lepoll_Finite])
 apply (fast dest: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_Finite])
```

infinite case

```
apply (drule Un_lepoll_Inf_Ord, (assumption+))
apply (blast intro: le_Ord2)
apply (drule lesspoll_trans1
             [OF subset_Un_Diff [THEN subset_imp_lepoll, THEN lepoll_trans]
                 lt_Card_imp_lesspoll], assumption+)
apply (simp add: lesspoll_def)
done

lemma Diff_lesspoll_eqpoll_Card:
     "⟦A ≈ a; ¬Finite(a); Card(a); B ≺ a⟧ ⟹ A - B ≈ a"
apply (rule ccontr)
apply (rule Diff_lesspoll_eqpoll_Card_lemma, (assumption+))
apply (blast intro: lesspoll_def [THEN def_imp_iff, THEN iffD2]
                    subset_imp_lepoll eqpoll_imp_lepoll lepoll_trans)
done

end


theory WO6_WO1
imports Cardinal_aux
begin


definition
  NN  :: "i ⇒ i"  where
    "NN(y) ≡ {m ∈ nat. ∃a. ∃f. Ord(a) ∧ domain(f)=a  ∧
                        (⋃b<a. f‘b) = y ∧ (∀b<a. f‘b ≲ m)}"

definition
  uu  :: "[i, i, i, i] ⇒ i"  where
    "uu(f, beta, gamma, delta) ≡ (f‘beta * f‘gamma) ∩ f‘delta"



definition
```

```
  vv1 :: "[i, i, i] ⇒ i"  where
    "vv1(f,m,b) ≡
          let g = μ g. (∃d. Ord(d) ∧ (domain(uu(f,b,g,d)) ≠ 0 ∧
                                domain(uu(f,b,g,d)) ≲ m));
              d = μ d. domain(uu(f,b,g,d)) ≠ 0 ∧
                           domain(uu(f,b,g,d)) ≲ m
          in  if f`b ≠ 0 then domain(uu(f,b,g,d)) else 0"

definition
  ww1 :: "[i, i, i] ⇒ i"  where
    "ww1(f,m,b) ≡ f`b - vv1(f,m,b)"

definition
  gg1 :: "[i, i, i] ⇒ i"  where
    "gg1(f,a,m) ≡ λb ∈ a++a. if b<a then vv1(f,m,b) else ww1(f,m,b--a)"



definition
  vv2 :: "[i, i, i, i] ⇒ i"  where
    "vv2(f,b,g,s) ≡
          if f`g ≠ 0 then {uu(f, b, g, μ d. uu(f,b,g,d) ≠ 0)`s} else
0"

definition
  ww2 :: "[i, i, i, i] ⇒ i"  where
    "ww2(f,b,g,s) ≡ f`g - vv2(f,b,g,s)"

definition
  gg2 :: "[i, i, i, i] ⇒ i"  where
    "gg2(f,a,b,s) ≡
             λg ∈ a++a. if g<a then vv2(f,b,g,s) else ww2(f,b,g--a,s)"


lemma WO2_WO3: "WO2 ⟹ WO3"
by (unfold WO2_def WO3_def, fast)



lemma WO3_WO1: "WO3 ⟹ WO1"
  unfolding eqpoll_def WO1_def WO3_def
apply (intro allI)
apply (drule_tac x=A in spec)
apply (blast intro: bij_is_inj well_ord_rvimage
                    well_ord_Memrel [THEN well_ord_subset])
done
```

**lemma** *WO1_WO2: "WO1 ⟹ WO2"*
  **unfolding** *eqpoll_def WO1_def WO2_def*
**apply** *(blast intro!: Ord_ordertype ordermap_bij)*
**done**

**lemma** *lam_sets: "f ∈ A->B ⟹ (λx ∈ A. {f‘x}): A -> {{b}. b ∈ B}"*
**by** *(fast intro!: lam_type apply_type)*

**lemma** *surj_imp_eq': "f ∈ surj(A,B) ⟹ (⋃a ∈ A. {f‘a}) = B"*
  **unfolding** *surj_def*
**apply** *(fast elim!: apply_type)*
**done**

**lemma** *surj_imp_eq: "⟦f ∈ surj(A,B); Ord(A)⟧ ⟹ (⋃a<A. {f‘a}) = B"*
**by** *(fast dest!: surj_imp_eq' intro!: ltI elim!: ltE)*

**lemma** *WO1_WO4: "WO1 ⟹ WO4(1)"*
  **unfolding** *WO1_def WO4_def*
**apply** *(rule allI)*
**apply** *(erule_tac x = A* **in** *allE)*
**apply** *(erule exE)*
**apply** *(intro exI conjI)*
**apply** *(erule Ord_ordertype)*
**apply** *(erule ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun, THEN lam_sets, THEN domain_of_fun])*
**apply** *(simp_all add: singleton_eqpoll_1 eqpoll_imp_lepoll Ord_ordertype ordermap_bij [THEN bij_converse_bij, THEN bij_is_surj, THEN surj_imp_eq] ltD)*
**done**

**lemma** *WO4_mono: "⟦m≤n; WO4(m)⟧ ⟹ WO4(n)"*
  **unfolding** *WO4_def*
**apply** *(blast dest!: spec intro: lepoll_trans [OF _ le_imp_lepoll])*
**done**

**lemma** *WO4_WO5: "⟦m ∈ nat; 1≤m; WO4(m)⟧ ⟹ WO5"*
**by** *(unfold WO4_def WO5_def, blast)*

**lemma** *WO5_WO6: "WO5 ⟹ WO6"*
**by** *(unfold WO4_def WO5_def WO6_def, blast)*

**lemma** *lt_oadd_odiff_disj:*
"⟦k < i++j;  Ord(i);  Ord(j)⟧
⟹ k < i  |  (¬ k<i ∧ k = i ++ (k--i) ∧ (k--i)<j)"
**apply** *(rule_tac i = k* **and** *j = i* **in** *Ord_linear2)*
**prefer** *4*
**apply** *(drule odiff_lt_mono2, assumption)*
**apply** *(simp add: oadd_odiff_inverse odiff_oadd_inverse)*
**apply** *(auto elim!: lt_Ord)*
**done**

**lemma** *domain_uu_subset: "domain(uu(f,b,g,d)) ⊆ f‘b"*
**by** *(unfold uu_def, blast)*

**lemma** *quant_domain_uu_lepoll_m:*
"∀b<a. f‘b ≲ m ⟹ ∀b<a. ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≲ m"
**by** *(blast intro: domain_uu_subset [THEN subset_imp_lepoll] lepoll_trans)*

**lemma** *uu_subset1: "uu(f,b,g,d) ⊆ f‘b * f‘g"*
**by** *(unfold uu_def, blast)*

**lemma** *uu_subset2: "uu(f,b,g,d) ⊆ f‘d"*
**by** *(unfold uu_def, blast)*

**lemma** *uu_lepoll_m: "⟦∀b<a. f‘b ≲ m;  d<a⟧ ⟹ uu(f,b,g,d) ≲ m"*
**by** *(blast intro: uu_subset2 [THEN subset_imp_lepoll] lepoll_trans)*

**lemma** *cases:*
"∀b<a. ∀g<a. ∀d<a. u(f,b,g,d) ≲ m
⟹ (∀b<a. f‘b ≠ 0 ⟶
(∃g<a. ∃d<a. u(f,b,g,d) ≠ 0 ∧ u(f,b,g,d) ≺ m))
| (∃b<a. f‘b ≠ 0 ∧ (∀g<a. ∀d<a. u(f,b,g,d) ≠ 0 ⟶
u(f,b,g,d) ≈ m))"
**unfolding** *lesspoll_def*
**apply** *(blast del: equalityI)*

**done**

**lemma** *UN_oadd: "Ord(a) ⟹ (⋃b<a++a. C(b)) = (⋃b<a. C(b) ∪ C(a++b))"*
**by** *(blast intro: ltI lt_oadd1 oadd_lt_mono2 dest!: lt_oadd_disj)*

**lemma** *vv1_subset: "vv1(f,m,b) ⊆ f‘b"*
**by** *(simp add: vv1_def Let_def domain_uu_subset)*

**lemma** *UN_gg1_eq:*
*"⟦Ord(a); m ∈ nat⟧ ⟹ (⋃b<a++a. gg1(f,a,m)‘b) = (⋃b<a. f‘b)"*
**by** *(simp add: gg1_def UN_oadd lt_oadd1 oadd_le_self [THEN le_imp_not_lt]*

*lt_Ord odiff_oadd_inverse ltD vv1_subset [THEN Diff_partition]*
*ww1_def)*

**lemma** *domain_gg1: "domain(gg1(f,a,m)) = a++a"*
**by** *(simp add: lam_funtype [THEN domain_of_fun] gg1_def)*

**lemma** *nested_LeastI:*
*"⟦P(a, b); Ord(a); Ord(b);*
*Least_a = (μ a. ∃x. Ord(x) ∧ P(a, x))⟧*
*⟹ P(Least_a, μ b. P(Least_a, b))"*
**apply** *(erule ssubst)*
**apply** *(rule_tac Q = "λz. P (z, μ b. P (z, b))" in LeastI2)*
**apply** *(fast elim!: LeastI)+*
**done**

**lemmas** *nested_Least_instance =*
*nested_LeastI [of "λg d. domain(uu(f,b,g,d)) ≠ 0 ∧*
*domain(uu(f,b,g,d)) ≲ m"] for f b m*

**lemma** *gg1_lepoll_m:*
*"⟦Ord(a); m ∈ nat;*
*∀b<a. f‘b ≠0 ⟶*
*(∃g<a. ∃d<a. domain(uu(f,b,g,d)) ≠ 0 ∧*

```
                        domain(uu(f,b,g,d)) ≲ m);
         ∀b<a. f‘b ≲ succ(m);  b<a++a⟧
      ⟹ gg1(f,a,m)‘b ≲ m"
apply (simp add: gg1_def empty_lepollI)
apply (safe dest!: lt_oadd_odiff_disj)

 apply (simp add: vv1_def Let_def empty_lepollI)
 apply (fast intro: nested_Least_instance [THEN conjunct2]
             elim!: lt_Ord)

apply (simp add: ww1_def empty_lepollI)
apply (case_tac "f‘ (b--a) = 0", simp add: empty_lepollI)
apply (rule Diff_lepoll, blast)
apply (rule vv1_subset)
apply (drule ospec [THEN mp], assumption+)
apply (elim oexE conjE)
apply (simp add: vv1_def Let_def lt_Ord nested_Least_instance [THEN conjunct1])
done
```

```
lemma ex_d_uu_not_empty:
     "⟦b<a;  g<a;  f‘b≠0;  f‘g≠0;
        y*y ⊆ y;  (⋃b<a. f‘b)=y⟧
      ⟹ ∃d<a. uu(f,b,g,d) ≠ 0"
by (unfold uu_def, blast)

lemma uu_not_empty:
     "⟦b<a; g<a; f‘b≠0; f‘g≠0;  y*y ⊆ y; (⋃b<a. f‘b)=y⟧
      ⟹ uu(f,b,g,μ d. (uu(f,b,g,d) ≠ 0)) ≠ 0"
apply (drule ex_d_uu_not_empty, assumption+)
apply (fast elim!: LeastI lt_Ord)
done

lemma not_empty_rel_imp_domain: "⟦r ⊆ A*B; r≠0⟧ ⟹ domain(r)≠0"
by blast

lemma Least_uu_not_empty_lt_a:
     "⟦b<a; g<a; f‘b≠0; f‘g≠0; y*y ⊆ y; (⋃b<a. f‘b)=y⟧
      ⟹ (μ d. uu(f,b,g,d) ≠ 0) < a"
apply (erule ex_d_uu_not_empty [THEN oexE], assumption+)
apply (blast intro: Least_le [THEN lt_trans1] lt_Ord)
done
```

```
lemma subset_Diff_sing: "⟦B ⊆ A; a∉B⟧ ⟹ B ⊆ A-{a}"
by blast


lemma supset_lepoll_imp_eq:
     "⟦A ≲ m; m ≲ B; B ⊆ A; m ∈ nat⟧ ⟹ A=B"
apply (erule natE)
apply (fast dest!: lepoll_0_is_0 intro!: equalityI)
apply (safe intro!: equalityI)
apply (rule ccontr)
apply (rule succ_lepoll_natE)
 apply (erule lepoll_trans)
 apply (rule lepoll_trans)
  apply (erule subset_Diff_sing [THEN subset_imp_lepoll], assumption)
 apply (rule Diff_sing_lepoll, assumption+)
done

lemma uu_Least_is_fun:
     "⟦∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 ⟶
         domain(uu(f, b, g, d)) ≈ succ(m);
         ∀b<a. f‘b ≲ succ(m);  y*y ⊆ y;
         (⋃b<a. f‘b)=y;  b<a;  g<a;  d<a;
         f‘b≠0;  f‘g≠0;  m ∈ nat;  s ∈ f‘b⟧
      ⟹ uu(f, b, g, μ d. uu(f,b,g,d)≠0) ∈ f‘b -> f‘g"
apply (drule_tac x2=g in ospec [THEN ospec, THEN mp])
   apply (rule_tac [3] not_empty_rel_imp_domain [OF uu_subset1 uu_not_empty])
        apply (rule_tac [2] Least_uu_not_empty_lt_a, assumption+)
apply (rule rel_is_fun)
    apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll])
   apply (erule uu_lepoll_m)
   apply (rule Least_uu_not_empty_lt_a, assumption+)
apply (rule uu_subset1)
apply (rule supset_lepoll_imp_eq [OF _ eqpoll_sym [THEN eqpoll_imp_lepoll]])
apply (fast intro!: domain_uu_subset)+
done

lemma vv2_subset:
     "⟦∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 ⟶
                     domain(uu(f, b, g, d)) ≈ succ(m);
        ∀b<a. f‘b ≲ succ(m); y*y ⊆ y;
        (⋃b<a. f‘b)=y;  b<a;  g<a;  m ∈ nat;  s ∈ f‘b⟧
      ⟹ vv2(f,b,g,s) ⊆ f‘g"
apply (simp add: vv2_def)
apply (blast intro: uu_Least_is_fun [THEN apply_type])
done
```

```
lemma UN_gg2_eq:
     "⟦∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 ⟶
                   domain(uu(f,b,g,d)) ≈ succ(m);
        ∀b<a. f‘b ≲ succ(m); y*y ⊆ y;
        (⋃b<a. f‘b)=y;  Ord(a);  m ∈ nat;  s ∈ f‘b;  b<a⟧
      ⟹ (⋃g<a++a. gg2(f,a,b,s) ‘ g) = y"
  unfolding gg2_def
apply (drule sym)
apply (simp add: ltD UN_oadd  oadd_le_self [THEN le_imp_not_lt]
                 lt_Ord odiff_oadd_inverse ww2_def
                 vv2_subset [THEN Diff_partition])
done

lemma domain_gg2: "domain(gg2(f,a,b,s)) = a++a"
by (simp add: lam_funtype [THEN domain_of_fun] gg2_def)




lemma vv2_lepoll: "⟦m ∈ nat; m≠0⟧ ⟹ vv2(f,b,g,s) ≲ m"
  unfolding vv2_def
apply (simp add: empty_lepollI)
apply (fast dest!: le_imp_subset [THEN subset_imp_lepoll, THEN lepoll_0_is_0]

       intro!: singleton_eqpoll_1 [THEN eqpoll_imp_lepoll, THEN lepoll_trans]
               not_lt_imp_le [THEN le_imp_subset, THEN subset_imp_lepoll]
               nat_into_Ord nat_1I)
done

lemma ww2_lepoll:
    "⟦∀b<a. f‘b ≲ succ(m);  g<a;  m ∈ nat;  vv2(f,b,g,d) ⊆ f‘g⟧
     ⟹ ww2(f,b,g,d) ≲ m"
  unfolding ww2_def
apply (case_tac "f‘g = 0")
apply (simp add: empty_lepollI)
apply (drule ospec, assumption)
apply (rule Diff_lepoll, assumption+)
apply (simp add: vv2_def not_emptyI)
done

lemma gg2_lepoll_m:
     "⟦∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 ⟶
                   domain(uu(f,b,g,d)) ≈ succ(m);
        ∀b<a. f‘b ≲ succ(m);  y*y ⊆ y;
        (⋃b<a. f‘b)=y;  b<a;  s ∈ f‘b;  m ∈ nat;  m≠ 0;  g<a++a⟧
      ⟹ gg2(f,a,b,s) ‘ g ≲ m"
```

```
apply (simp add: gg2_def empty_lepollI)
apply (safe elim!: lt_Ord2 dest!: lt_oadd_odiff_disj)
 apply (simp add: vv2_lepoll)
apply (simp add: ww2_lepoll vv2_subset)
done
```

```
lemma lemma_ii: "⟦succ(m) ∈ NN(y); y*y ⊆ y; m ∈ nat; m≠0⟧ ⟹ m ∈
NN(y)"
  unfolding NN_def
apply (elim CollectE exE conjE)
apply (rule quant_domain_uu_lepoll_m [THEN cases, THEN disjE], assumption)

apply (simp add: lesspoll_succ_iff)
apply (rule_tac x = "a++a" in exI)
apply (fast intro!: Ord_oadd domain_gg1 UN_gg1_eq gg1_lepoll_m)

apply (elim oexE conjE)
apply (rule_tac A = "f`B" for B in not_emptyE, assumption)
apply (rule CollectI)
apply (erule succ_natD)
apply (rule_tac x = "a++a" in exI)
apply (rule_tac x = "gg2 (f,a,b,x) " in exI)
apply (simp add: Ord_oadd domain_gg2 UN_gg2_eq gg2_lepoll_m)
done
```

```
lemma z_n_subset_z_succ_n:
     "∀n ∈ nat. rec(n, x, λk r. r ∪ r*r) ⊆ rec(succ(n), x, λk r. r ∪
r*r)"
by (fast intro: rec_succ [THEN ssubst])

lemma le_subsets:
     "⟦∀n ∈ nat. f(n)<=f(succ(n)); n≤m; n ∈ nat; m ∈ nat⟧
      ⟹ f(n)<=f(m)"
apply (erule_tac P = "n≤m" in rev_mp)
apply (rule_tac P = "λz. n≤z ⟶ f (n) ⊆ f (z) " in nat_induct)
apply (auto simp add: le_iff)
done
```

**lemma** *le_imp_rec_subset:*
"⟦n≤m; m ∈ nat⟧
⟹ rec(n, x, λk r. r ∪ r*r) ⊆ rec(m, x, λk r. r ∪ r*r)"
**apply** *(rule z_n_subset_z_succ_n [THEN le_subsets])*
**apply** *(blast intro: lt_nat_in_nat)+*
**done**

**lemma** *lemma_iv: "∃y. x ∪ y*y ⊆ y"*
**apply** *(rule_tac x = "⋃n ∈ nat. rec (n, x, λk r. r ∪ r*r) " in exI)*
**apply** *safe*
**apply** *(rule nat_0I [THEN UN_I], simp)*
**apply** *(rule_tac a = "succ (n ∪ na) " in UN_I)*
**apply** *(erule Un_nat_type [THEN nat_succI], assumption)*
**apply** *(auto intro: le_imp_rec_subset [THEN subsetD]*
*intro!: Un_upper1_le Un_upper2_le Un_nat_type*
*elim!: nat_into_Ord)*
**done**

**lemma** *WO6_imp_NN_not_empty: "WO6 ⟹ NN(y) ≠ 0"*
**by** *(unfold WO6_def NN_def, clarify, blast)*

**lemma** *lemma1:*
"⟦(⋃b<a. f‘b)=y; x ∈ y; ∀b<a. f‘b ≲ 1; Ord(a)⟧ ⟹ ∃c<a. f‘c = {x}"
**by** *(fast elim!: lepoll_1_is_sing)*

**lemma** *lemma2:*
"⟦(⋃b<a. f‘b)=y; x ∈ y; ∀b<a. f‘b ≲ 1; Ord(a)⟧
⟹ f‘ (μ i. f‘i = {x}) = {x}"

```
apply (drule lemma1, assumption+)
apply (fast elim!: lt_Ord intro: LeastI)
done

lemma NN_imp_ex_inj: "1 ∈ NN(y) ⟹ ∃a f. Ord(a) ∧ f ∈ inj(y, a)"
  unfolding NN_def
apply (elim CollectE exE conjE)
apply (rule_tac x = a in exI)
apply (rule_tac x = "λx ∈ y. μ i. f`i = {x}" in exI)
apply (rule conjI, assumption)
apply (rule_tac d = "λi. THE x. x ∈ f`i" in lam_injective)
apply (drule lemma1, assumption+)
apply (fast elim!: Least_le [THEN lt_trans1, THEN ltD] lt_Ord)
apply (rule lemma2 [THEN ssubst], assumption+, blast)
done

lemma y_well_ord: "⟦y*y ⊆ y; 1 ∈ NN(y)⟧ ⟹ ∃r. well_ord(y, r)"
apply (drule NN_imp_ex_inj)
apply (fast elim!: well_ord_rvimage [OF _ well_ord_Memrel])
done




lemma rev_induct_lemma [rule_format]:
     "⟦n ∈ nat; ⋀m. ⟦m ∈ nat; m≠0; P(succ(m))⟧ ⟹ P(m)⟧
      ⟹ n≠0 ⟶ P(n) ⟶ P(1)"
by (erule nat_induct, blast+)

lemma rev_induct:
     "⟦n ∈ nat;  P(n);  n≠0;
        ⋀m. ⟦m ∈ nat; m≠0; P(succ(m))⟧ ⟹ P(m)⟧
      ⟹ P(1)"
by (rule rev_induct_lemma, blast+)

lemma NN_into_nat: "n ∈ NN(y) ⟹ n ∈ nat"
by (simp add: NN_def)

lemma lemma3: "⟦n ∈ NN(y); y*y ⊆ y; n≠0⟧ ⟹ 1 ∈ NN(y)"
apply (rule rev_induct [OF NN_into_nat], assumption+)
apply (rule lemma_ii, assumption+)
done
```

```
lemma NN_y_0: "0 ∈ NN(y) ⟹ y=0"
  unfolding NN_def
apply (fast intro!: equalityI dest!: lepoll_0_is_0 elim: subst)
done

lemma WO6_imp_WO1: "WO6 ⟹ WO1"
  unfolding WO1_def
apply (rule allI)
apply (case_tac "A=0")
apply (fast intro!: well_ord_Memrel nat_0I [THEN nat_into_Ord])
apply (rule_tac x = A in lemma_iv [elim_format])
apply (erule exE)
apply (drule WO6_imp_NN_not_empty)
apply (erule Un_subset_iff [THEN iffD1, THEN conjE])
apply (erule_tac A = "NN (y) " in not_emptyE)
apply (frule y_well_ord)
 apply (fast intro!: lemma3 dest!: NN_y_0 elim!: not_emptyE)
apply (fast elim: well_ord_subset)
done

end


theory WO1_WO7
imports AC_Equiv
begin

definition
    "LEMMA ≡
     ∀X. ¬Finite(X) ⟶ (∃R. well_ord(X,R) ∧ ¬well_ord(X,converse(R)))"




lemma WO7_iff_LEMMA: "WO7 ⟷ LEMMA"
  unfolding WO7_def LEMMA_def
apply (blast intro: Finite_well_ord_converse)
done




lemma LEMMA_imp_WO1: "LEMMA ⟹ WO1"
  unfolding WO1_def LEMMA_def Finite_def eqpoll_def
apply (blast intro!: well_ord_rvimage [OF bij_is_inj nat_implies_well_ord])
done
```

```
lemma converse_Memrel_not_wf_on:
    "⟦Ord(a); ¬Finite(a)⟧ ⟹ ¬wf[a](converse(Memrel(a)))"
  unfolding wf_on_def wf_def
apply (drule nat_le_infinite_Ord [THEN le_imp_subset], assumption)
apply (rule notI)
apply (erule_tac x = nat in allE, blast)
done

lemma converse_Memrel_not_well_ord:
    "⟦Ord(a); ¬Finite(a)⟧ ⟹ ¬well_ord(a,converse(Memrel(a)))"
  unfolding well_ord_def
apply (blast dest: converse_Memrel_not_wf_on)
done

lemma well_ord_rvimage_ordertype:
     "well_ord(A,r) ⟹
       rvimage (ordertype(A,r), converse(ordermap(A,r)),r) =
       Memrel(ordertype(A,r))"
by (blast intro: ordertype_ord_iso [THEN ord_iso_sym] ord_iso_rvimage_eq
            Memrel_type [THEN subset_Int_iff [THEN iffD1]] trans)

lemma well_ord_converse_Memrel:
     "⟦well_ord(A,r); well_ord(A,converse(r))⟧
      ⟹ well_ord(ordertype(A,r), converse(Memrel(ordertype(A,r))))"

apply (subst well_ord_rvimage_ordertype [symmetric], assumption)
apply (rule rvimage_converse [THEN subst])
apply (blast intro: ordertype_ord_iso ord_iso_sym ord_iso_is_bij
                    bij_is_inj well_ord_rvimage)
done

lemma WO1_imp_LEMMA: "WO1 ⟹ LEMMA"
apply (unfold WO1_def LEMMA_def, clarify)
apply (blast dest: well_ord_converse_Memrel
                   Ord_ordertype [THEN converse_Memrel_not_well_ord]
             intro: ordertype_ord_iso ord_iso_is_bij bij_is_inj lepoll_Finite
```

```
                    lepoll_def [THEN def_imp_iff, THEN iffD2] )
done

lemma WO1_iff_WO7: "WO1 ⟷ WO7"
apply (simp add: WO7_iff_LEMMA)
apply (blast intro: LEMMA_imp_WO1 WO1_imp_LEMMA)
done




lemma WO1_WO8: "WO1 ⟹ WO8"
by (unfold WO1_def WO8_def, fast)



lemma WO8_WO1: "WO8 ⟹ WO1"
  unfolding WO1_def WO8_def
apply (rule allI)
apply (erule_tac x = "{{x}. x ∈ A}" in allE)
apply (erule impE)
 apply (rule_tac x = "λa ∈ {{x}. x ∈ A}. THE x. a={x}" in exI)
 apply (force intro!: lam_type simp add: singleton_eq_iff the_equality)
apply (blast intro: lam_sing_bij bij_is_inj well_ord_rvimage)
done

end


theory AC7_AC9
imports AC_Equiv
begin




lemma Sigma_fun_space_not0: "⟦0∉A; B ∈ A⟧ ⟹ (nat->⋃(A)) * B ≠ 0"
by (blast dest!: Sigma_empty_iff [THEN iffD1] Union_empty_iff [THEN iffD1])

lemma inj_lemma:
        "C ∈ A ⟹ (λg ∈ (nat->⋃(A))*C.
                (λn ∈ nat. if(n=0, snd(g), fst(g)`(n #- 1))))
                ∈ inj((nat->⋃(A))*C, (nat->⋃(A)) ) "
```

**unfolding** *inj_def*
**apply** *(rule CollectI)*
**apply** *(fast intro!: lam_type if_type apply_type fst_type snd_type, auto)*

**apply** *(rule fun_extension, assumption+)*
**apply** *(drule lam_eqE [OF _ nat_succI], assumption, simp)*
**apply** *(drule lam_eqE [OF _ nat_0I], simp)*
**done**

**lemma** *Sigma_fun_space_eqpoll:*
"⟦*C ∈ A; 0∉A*⟧ ⟹ *(nat->⋃(A)) * C ≈ (nat->⋃(A))*"
**apply** *(rule eqpollI)*
**apply** *(simp add: lepoll_def)*
**apply** *(fast intro!: inj_lemma)*
**apply** *(fast elim!: prod_lepoll_self not_sym [THEN not_emptyE] subst_elem*

*elim: swap)*
**done**

**lemma** *AC6_AC7: "AC6 ⟹ AC7"*
**by** *(unfold AC6_def AC7_def, blast)*

**lemma** *lemma1_1: "y ∈ (∏B ∈ A. Y*B) ⟹ (λB ∈ A. snd(y‘B)) ∈ (∏B ∈ A. B)"*
**by** *(fast intro!: lam_type snd_type apply_type)*

**lemma** *lemma1_2:*
"*y ∈ (∏B ∈ {Y*C. C ∈ A}. B) ⟹ (λB ∈ A. y‘(Y*B)) ∈ (∏B ∈ A. Y*B)*"
**apply** *(fast intro!: lam_type apply_type)*
**done**

**lemma** *AC7_AC6_lemma1:*
"*(∏B ∈ {(nat->⋃(A))*C. C ∈ A}. B) ≠ 0 ⟹ (∏B ∈ A. B) ≠ 0*"
**by** *(fast intro!: equals0I lemma1_1 lemma1_2)*

**lemma** *AC7_AC6_lemma2: "0 ∉ A ⟹ 0 ∉ {(nat -> ⋃(A)) * C. C ∈ A}"*
**by** *(blast dest: Sigma_fun_space_not0)*

```
lemma AC7_AC6: "AC7 ⟹ AC6"
  unfolding AC6_def AC7_def
apply (rule allI)
apply (rule impI)
apply (case_tac "A=0", simp)
apply (rule AC7_AC6_lemma1)
apply (erule allE)
apply (blast del: notI
             intro!: AC7_AC6_lemma2 intro: eqpoll_sym eqpoll_trans
                    Sigma_fun_space_eqpoll)
done
```

```
lemma AC1_AC8_lemma1:
        "∀B ∈ A. ∃B1 B2. B=⟨B1,B2⟩ ∧ B1 ≈ B2
        ⟹ 0 ∉ { bij(fst(B),snd(B)). B ∈ A }"
apply (unfold eqpoll_def, auto)
done

lemma AC1_AC8_lemma2:
     "⟦f ∈ (∏X ∈ RepFun(A,p). X); D ∈ A⟧ ⟹ (λx ∈ A. f‘p(x))‘D ∈
p(D)"
apply (simp, fast elim!: apply_type)
done

lemma AC1_AC8: "AC1 ⟹ AC8"
  unfolding AC1_def AC8_def
apply (fast dest: AC1_AC8_lemma1 AC1_AC8_lemma2)
done
```

```
lemma AC8_AC9_lemma:
     "∀B1 ∈ A. ∀B2 ∈ A. B1 ≈ B2
      ⟹ ∀B ∈ A*A. ∃B1 B2. B=⟨B1,B2⟩ ∧ B1 ≈ B2"
by fast

lemma AC8_AC9: "AC8 ⟹ AC9"
  unfolding AC8_def AC9_def
```

```
apply (intro allI impI)
apply (erule allE)
apply (erule impE, erule AC8_AC9_lemma, force)
done
```

```
lemma snd_lepoll_SigmaI: "b ∈ B ⟹ X ≲ B × X"
by (blast intro: lepoll_trans prod_lepoll_self eqpoll_imp_lepoll
                 prod_commute_eqpoll)

lemma nat_lepoll_lemma:
     "⟦0 ∉ A; B ∈ A⟧ ⟹ nat ≲ ((nat → ⋃(A)) × B) × nat"
by (blast dest: Sigma_fun_space_not0 intro: snd_lepoll_SigmaI)

lemma AC9_AC1_lemma1:
     "⟦0∉A;  A≠0;
        C = {((nat->⋃(A))*B)*nat. B ∈ A}  ∪
            {cons(0,((nat->⋃(A))*B)*nat). B ∈ A};
        B1 ∈ C;  B2 ∈ C⟧
      ⟹ B1 ≈ B2"
by (blast intro!: nat_lepoll_lemma Sigma_fun_space_eqpoll
                    nat_cons_eqpoll [THEN eqpoll_trans]
                    prod_eqpoll_cong [OF _ eqpoll_refl]
             intro: eqpoll_trans eqpoll_sym )

lemma AC9_AC1_lemma2:
     "∀B1 ∈ {(F*B)*N. B ∈ A} ∪ {cons(0,(F*B)*N). B ∈ A}.
      ∀B2 ∈ {(F*B)*N. B ∈ A} ∪ {cons(0,(F*B)*N). B ∈ A}.
        f‘⟨B1,B2⟩ ∈ bij(B1, B2)
      ⟹ (λB ∈ A. snd(fst((f‘<cons(0,(F*B)*N),(F*B)*N>)‘0))) ∈ (∏ X
∈ A. X)"
apply (intro lam_type snd_type fst_type)
apply (rule apply_type [OF _ consI1])
apply (fast intro!: fun_weaken_type bij_is_fun)
done

lemma AC9_AC1: "AC9 ⟹ AC1"
  unfolding AC1_def AC9_def
apply (intro allI impI)
apply (erule allE)
apply (case_tac "A≠0")
```

```
apply (blast dest: AC9_AC1_lemma1 AC9_AC1_lemma2, force)
done

end


theory WO1_AC
imports AC_Equiv
begin




theorem WO1_AC1: "WO1 ⟹ AC1"
by (unfold AC1_def WO1_def, fast elim!: ex_choice_fun)




lemma lemma1: "⟦WO1; ∀B ∈ A. ∃C ∈ D(B). P(C,B)⟧ ⟹ ∃f. ∀B ∈ A. P(f‘B,B)"
  unfolding WO1_def
apply (erule_tac x = "⋃({{C ∈ D (B) . P (C,B) }. B ∈ A}) " in allE)
apply (erule exE, drule ex_choice_fun, fast)
apply (erule exE)
apply (rule_tac x = "λx ∈ A. f‘{C ∈ D (x) . P (C,x) }" in exI)
apply (simp, blast dest!: apply_type [OF _ RepFunI])
done

lemma lemma2_1: "⟦¬Finite(B); WO1⟧ ⟹ |B| + |B| ≈  B"
  unfolding WO1_def
apply (rule eqpoll_trans)
prefer 2 apply (fast elim!: well_ord_cardinal_eqpoll)
apply (rule eqpoll_sym [THEN eqpoll_trans])
apply (fast elim!: well_ord_cardinal_eqpoll)
apply (drule spec [of _ B])
apply (clarify dest!: eqpoll_imp_Finite_iff [OF well_ord_cardinal_eqpoll])

apply (simp add: cadd_def [symmetric]
            eqpoll_refl InfCard_cdouble_eq Card_cardinal Inf_Card_is_InfCard)

done

lemma lemma2_2:
     "f ∈ bij(D+D, B) ⟹ {{f‘Inl(i), f‘Inr(i)}. i ∈ D} ∈ Pow(Pow(B))"
by (fast elim!: bij_is_fun [THEN apply_type])
```

```
lemma lemma2_3:
        "f ∈ bij(D+D, B) ⟹ pairwise_disjoint({{f`Inl(i), f`Inr(i)}.
i ∈ D})"
  unfolding pairwise_disjoint_def
apply (blast dest: bij_is_inj [THEN inj_apply_equality])
done

lemma lemma2_4:
     "⟦f ∈ bij(D+D, B); 1≤n⟧
      ⟹ sets_of_size_between({{f`Inl(i), f`Inr(i)}. i ∈ D}, 2, succ(n))"
apply (simp (no_asm_simp) add: sets_of_size_between_def succ_def)
apply (blast intro!: cons_lepoll_cong
            intro: singleton_eqpoll_1 [THEN eqpoll_imp_lepoll]
                   le_imp_subset [THEN subset_imp_lepoll]  lepoll_trans

            dest: bij_is_inj [THEN inj_apply_equality] elim!: mem_irrefl)
done

lemma lemma2_5:
     "f ∈ bij(D+D, B) ⟹ ⋃({{f`Inl(i), f`Inr(i)}. i ∈ D})=B"
  unfolding bij_def surj_def
apply (fast elim!: inj_is_fun [THEN apply_type])
done

lemma lemma2:
     "⟦WO1; ¬Finite(B); 1≤n⟧
      ⟹ ∃C ∈ Pow(Pow(B)). pairwise_disjoint(C) ∧
                sets_of_size_between(C, 2, succ(n)) ∧
                ⋃(C)=B"
apply (drule lemma2_1 [THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]],

       assumption)
apply (blast intro!: lemma2_2 lemma2_3 lemma2_4 lemma2_5)
done

theorem WO1_AC10: "⟦WO1; 1≤n⟧ ⟹ AC10(n)"
  unfolding AC10_def
apply (fast intro!: lemma1 elim!: lemma2)
done

end


theory Hartog
imports AC_Equiv
begin

definition
  Hartog :: "i ⇒ i"  where
```

```
   "Hartog(X) ≡ μ i. ¬ i ≲ X"

lemma Ords_in_set: "∀a. Ord(a) ⟶ a ∈ X ⟹ P"
apply (rule_tac X = "{y ∈ X. Ord (y) }" in ON_class [elim_format])
apply fast
done

lemma Ord_lepoll_imp_ex_well_ord:
     "⟦Ord(a); a ≲ X⟧
      ⟹ ∃Y. Y ⊆ X ∧ (∃R. well_ord(Y,R) ∧ ordertype(Y,R)=a)"
  unfolding lepoll_def
apply (erule exE)
apply (intro exI conjI)
  apply (erule inj_is_fun [THEN fun_is_rel, THEN image_subset])
 apply (rule well_ord_rvimage [OF bij_is_inj well_ord_Memrel])
  apply (erule restrict_bij [THEN bij_converse_bij])
apply (rule subset_refl, assumption)
apply (rule trans)
apply (rule bij_ordertype_vimage)
apply (erule restrict_bij [THEN bij_converse_bij])
apply (rule subset_refl)
apply (erule well_ord_Memrel)
apply (erule ordertype_Memrel)
done

lemma Ord_lepoll_imp_eq_ordertype:
     "⟦Ord(a); a ≲ X⟧ ⟹ ∃Y. Y ⊆ X ∧ (∃R. R ⊆ X*X ∧ ordertype(Y,R)=a)"
apply (drule Ord_lepoll_imp_ex_well_ord, assumption, clarify)
apply (intro exI conjI)
apply (erule_tac [3] ordertype_Int, auto)
done

lemma Ords_lepoll_set_lemma:
     "(∀a. Ord(a) ⟶ a ≲ X) ⟹
       ∀a. Ord(a) ⟶
        a ∈ {b. Z ∈ Pow(X)*Pow(X*X), ∃Y R. Z=⟨Y,R⟩ ∧ ordertype(Y,R)=b}"
apply (intro allI impI)
apply (elim allE impE, assumption)
apply (blast dest!: Ord_lepoll_imp_eq_ordertype intro: sym)
done

lemma Ords_lepoll_set: "∀a. Ord(a) ⟶ a ≲ X ⟹ P"
by (erule Ords_lepoll_set_lemma [THEN Ords_in_set])

lemma ex_Ord_not_lepoll: "∃a. Ord(a) ∧ ¬a ≲ X"
apply (rule ccontr)
apply (best intro: Ords_lepoll_set)
done
```

```
lemma not_Hartog_lepoll_self: "¬ Hartog(A) ≲ A"
  unfolding Hartog_def
apply (rule ex_Ord_not_lepoll [THEN exE])
apply (rule LeastI, auto)
done

lemmas Hartog_lepoll_selfE = not_Hartog_lepoll_self [THEN notE]

lemma Ord_Hartog: "Ord(Hartog(A))"
by (unfold Hartog_def, rule Ord_Least)

lemma less_HartogE1: "⟦i < Hartog(A); ¬ i ≲ A⟧ ⟹ P"
by (unfold Hartog_def, fast elim: less_LeastE)

lemma less_HartogE: "⟦i < Hartog(A); i ≈ Hartog(A)⟧ ⟹ P"
by (blast intro: less_HartogE1 eqpoll_sym eqpoll_imp_lepoll
                 lepoll_trans [THEN Hartog_lepoll_selfE])

lemma Card_Hartog: "Card(Hartog(A))"
by (fast intro!: CardI Ord_Hartog elim: less_HartogE)

end


theory HH
imports AC_Equiv Hartog
begin

definition
  HH :: "[i, i, i] ⇒ i"  where
    "HH(f,x,a) ≡ transrec(a, λb r. let z = x - (⋃c ∈ b. r`c)
                                   in  if f`z ∈ Pow(z)-{0} then f`z else
{x})"
```

## 0.1 Lemmas useful in each of the three proofs

```
lemma HH_def_satisfies_eq:
     "HH(f,x,a) = (let z = x - (⋃b ∈ a. HH(f,x,b))
                   in  if f`z ∈ Pow(z)-{0} then f`z else {x})"
by (rule HH_def [THEN def_transrec, THEN trans], simp)

lemma HH_values: "HH(f,x,a) ∈ Pow(x)-{0} | HH(f,x,a)={x}"
apply (rule HH_def_satisfies_eq [THEN ssubst])
apply (simp add: Let_def Diff_subset [THEN PowI], fast)
done

lemma subset_imp_Diff_eq:
     "B ⊆ A ⟹ X-(⋃a ∈ A. P(a)) = X-(⋃a ∈ A-B. P(a))-(⋃b ∈ B. P(b))"
by fast
```

```
lemma Ord_DiffE: "⟦c ∈ a-b; b<a⟧ ⟹ c=b | b<c ∧ c<a"
apply (erule ltE)
apply (drule Ord_linear [of _ c])
apply (fast elim: Ord_in_Ord)
apply (fast intro!: ltI intro: Ord_in_Ord)
done

lemma Diff_UN_eq_self: "(⋀y. y∈A ⟹ P(y) = {x}) ⟹ x - (⋃y ∈ A.
P(y)) = x"
by (simp, fast elim!: mem_irrefl)

lemma HH_eq: "x - (⋃b ∈ a. HH(f,x,b)) = x - (⋃b ∈ a1. HH(f,x,b))
              ⟹ HH(f,x,a) = HH(f,x,a1)"
apply (subst HH_def_satisfies_eq [of _ _ a1])
apply (rule HH_def_satisfies_eq [THEN trans], simp)
done

lemma HH_is_x_gt_too: "⟦HH(f,x,b)={x}; b<a⟧ ⟹ HH(f,x,a)={x}"
apply (rule_tac P = "b<a" in impE)
prefer 2 apply assumption+
apply (erule lt_Ord2 [THEN trans_induct])
apply (rule impI)
apply (rule HH_eq [THEN trans])
prefer 2 apply assumption+
apply (rule leI [THEN le_imp_subset, THEN subset_imp_Diff_eq, THEN ssubst],

       assumption)
apply (rule_tac t = "λz. z-X" for X in subst_context)
apply (rule Diff_UN_eq_self)
apply (drule Ord_DiffE, assumption)
apply (fast elim: ltE, auto)
done

lemma HH_subset_x_lt_too:
     "⟦HH(f,x,a) ∈ Pow(x)-{0}; b<a⟧ ⟹ HH(f,x,b) ∈ Pow(x)-{0}"
apply (rule HH_values [THEN disjE], assumption)
apply (drule HH_is_x_gt_too, assumption)
apply (drule subst, assumption)
apply (fast elim!: mem_irrefl)
done

lemma HH_subset_x_imp_subset_Diff_UN:
    "HH(f,x,a) ∈ Pow(x)-{0} ⟹ HH(f,x,a) ∈ Pow(x - (⋃b ∈ a. HH(f,x,b)))-{0}"
apply (drule HH_def_satisfies_eq [THEN subst])
apply (rule HH_def_satisfies_eq [THEN ssubst])
apply (simp add: Let_def Diff_subset [THEN PowI])
apply (drule split_if [THEN iffD1])
apply (fast elim!: mem_irrefl)
```

**done**

**lemma** *HH_eq_arg_lt:*
"⟦*HH(f,x,v)=HH(f,x,w); HH(f,x,v)* ∈ *Pow(x)-{0}; v* ∈ *w*⟧ ⟹ *P*"
**apply** *(frule_tac P = "λy. y* ∈ *Pow (x) -{0}"* **in** *subst, assumption)*
**apply** *(drule_tac a = w* **in** *HH_subset_x_imp_subset_Diff_UN)*
**apply** *(drule subst_elem, assumption)*
**apply** *(fast intro!: singleton_iff [THEN iffD2] equals0I)*
**done**

**lemma** *HH_eq_imp_arg_eq:*
"⟦*HH(f,x,v)=HH(f,x,w); HH(f,x,w)* ∈ *Pow(x)-{0}; Ord(v); Ord(w)*⟧ ⟹ *v=w*"
**apply** *(rule_tac j = w* **in** *Ord_linear_lt)*
**apply** *(simp_all (no_asm_simp))*
**apply** *(drule subst_elem, assumption)*
**apply** *(blast dest: ltD HH_eq_arg_lt)*
**apply** *(blast dest: HH_eq_arg_lt [OF sym] ltD)*
**done**

**lemma** *HH_subset_x_imp_lepoll:*
"⟦*HH(f, x, i)* ∈ *Pow(x)-{0}; Ord(i)*⟧ ⟹ *i* ≲ *Pow(x)-{0}*"
**unfolding** *lepoll_def inj_def*
**apply** *(rule_tac x = "λj* ∈ *i. HH (f, x, j) "* **in** *exI)*
**apply** *(simp (no_asm_simp))*
**apply** *(fast del: DiffE*
*elim!: HH_eq_imp_arg_eq Ord_in_Ord HH_subset_x_lt_too*
*intro!: lam_type ballI ltI intro: bexI)*
**done**

**lemma** *HH_Hartog_is_x: "HH(f, x, Hartog(Pow(x)-{0})) = {x}"*
**apply** *(rule HH_values [THEN disjE])*
**prefer** *2* **apply** *assumption*
**apply** *(fast del: DiffE*
*intro!: Ord_Hartog*
*dest!: HH_subset_x_imp_lepoll*
*elim!: Hartog_lepoll_selfE)*
**done**

**lemma** *HH_Least_eq_x: "HH(f, x,* μ *i. HH(f, x, i) = {x}) = {x}"*
**by** *(fast intro!: Ord_Hartog HH_Hartog_is_x LeastI)*

**lemma** *less_Least_subset_x:*
"a ∈ (μ *i. HH(f,x,i)={x}*) ⟹ *HH(f,x,*a*)* ∈ *Pow(x)-{0}*"
**apply** *(rule HH_values [THEN disjE], assumption)*
**apply** *(rule less_LeastE)*
**apply** *(erule_tac [2] ltI [OF _ Ord_Least], assumption)*
**done**

## 0.2 Lemmas used in the proofs of *AC1* $\Longrightarrow$ *WO2* and *AC17* $\Longrightarrow$ *AC1*

**lemma** *lam_Least_HH_inj_Pow:*
*"(λa ∈ (μ i. HH(f,x,i)={x}). HH(f,x,a))*
*∈ inj(μ i. HH(f,x,i)={x}, Pow(x)-{0})"*
**apply** *(unfold inj_def, simp)*
**apply** *(fast intro!: lam_type dest: less_Least_subset_x*
*elim!: HH_eq_imp_arg_eq Ord_Least [THEN Ord_in_Ord])*
**done**

**lemma** *lam_Least_HH_inj:*
*"∀a ∈ (μ i. HH(f,x,i)={x}). ∃z ∈ x. HH(f,x,a) = {z}*
*⟹ (λa ∈ (μ i. HH(f,x,i)={x}). HH(f,x,a))*
*∈ inj(μ i. HH(f,x,i)={x}, {{y}. y ∈ x})"*
**by** *(rule lam_Least_HH_inj_Pow [THEN inj_strengthen_type], simp)*

**lemma** *lam_surj_sing:*
*"⟦x - (⋃a ∈ A. F(a)) = 0; ∀a ∈ A. ∃z ∈ x. F(a) = {z}⟧*
*⟹ (λa ∈ A. F(a)) ∈ surj(A, {{y}. y ∈ x})"*
**apply** *(simp add: surj_def lam_type Diff_eq_0_iff)*
**apply** *(blast elim: equalityE)*
**done**

**lemma** *not_emptyI2: "y ∈ Pow(x)-{0} ⟹ x ≠ 0"*
**by** *auto*

**lemma** *f_subset_imp_HH_subset:*
*"f`(x - (⋃j ∈ i. HH(f,x,j))) ∈ Pow(x - (⋃j ∈ i. HH(f,x,j)))-{0}*

*⟹ HH(f, x, i) ∈ Pow(x) - {0}"*
**apply** *(rule HH_def_satisfies_eq [THEN ssubst])*
**apply** *(simp add: Let_def Diff_subset [THEN PowI] not_emptyI2 [THEN if_P],*
*fast)*
**done**

**lemma** *f_subsets_imp_UN_HH_eq_x:*
*"∀z ∈ Pow(x)-{0}. f`z ∈ Pow(z)-{0}*
*⟹ x - (⋃j ∈ (μ i. HH(f,x,i)={x}). HH(f,x,j)) = 0"*
**apply** *(case_tac "P ∈ {0}"* **for** *P, fast)*
**apply** *(drule Diff_subset [THEN PowI, THEN DiffI])*
**apply** *(drule bspec, assumption)*
**apply** *(drule f_subset_imp_HH_subset)*
**apply** *(blast dest!: subst_elem [OF _ HH_Least_eq_x [symmetric]]*
*elim!: mem_irrefl)*
**done**

**lemma** *HH_values2: "HH(f,x,i) = f`(x - (⋃j ∈ i. HH(f,x,j))) | HH(f,x,i)={x}"*
**apply** *(rule HH_def_satisfies_eq [THEN ssubst])*
**apply** *(simp add: Let_def Diff_subset [THEN PowI])*
**done**

```
lemma HH_subset_imp_eq:
     "HH(f,x,i): Pow(x)-{0} ⟹ HH(f,x,i)=f`(x - (⋃j ∈ i. HH(f,x,j)))"
apply (rule HH_values2 [THEN disjE], assumption)
apply (fast elim!: equalityE mem_irrefl dest!: singleton_subsetD)
done

lemma f_sing_imp_HH_sing:
     "⟦f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x};
        a ∈ (μ i. HH(f,x,i)={x})⟧ ⟹ ∃z ∈ x. HH(f,x,a) = {z}"
apply (drule less_Least_subset_x)
apply (frule HH_subset_imp_eq)
apply (drule apply_type)
apply (rule Diff_subset [THEN PowI, THEN DiffI])
apply (fast dest!: HH_subset_x_imp_subset_Diff_UN [THEN not_emptyI2],
force)
done

lemma f_sing_lam_bij:
     "⟦x - (⋃j ∈ (μ i. HH(f,x,i)={x}). HH(f,x,j)) = 0;
        f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x}⟧
      ⟹ (λa ∈ (μ i. HH(f,x,i)={x}). HH(f,x,a))
          ∈ bij(μ i. HH(f,x,i)={x}, {{y}. y ∈ x})"
  unfolding bij_def
apply (fast intro!: lam_Least_HH_inj lam_surj_sing f_sing_imp_HH_sing)
done

lemma lam_singI:
     "f ∈ (∏X ∈ Pow(x)-{0}. F(X))
      ⟹ (λX ∈ Pow(x)-{0}. {f`X}) ∈ (∏X ∈ Pow(x)-{0}. {{z}. z ∈ F(X)})"
by (fast del: DiffI DiffE
           intro!: lam_type singleton_eq_iff [THEN iffD2] dest: apply_type)


lemmas bij_Least_HH_x =
    comp_bij [OF f_sing_lam_bij [OF _ lam_singI]
              lam_sing_bij [THEN bij_converse_bij]]
```

## 0.3 The proof of *AC1* ⟹ *WO2*

```
lemma bijection:
     "f ∈ (∏X ∈ Pow(x) - {0}. X)
      ⟹ ∃g. g ∈ bij(x, μ i. HH(λX ∈ Pow(x)-{0}. {f`X}, x, i) = {x})"
apply (rule exI)
apply (rule bij_Least_HH_x [THEN bij_converse_bij])
apply (rule f_subsets_imp_UN_HH_eq_x)
apply (intro ballI apply_type)
apply (fast intro: lam_type apply_type del: DiffE, assumption)
apply (fast intro: Pi_weaken_type)
```

**done**

**lemma** *AC1_WO2: "AC1 ⟹ WO2"*
  **unfolding** *AC1_def WO2_def eqpoll_def*
  **apply** *(intro allI)*
  **apply** *(drule_tac x = "Pow(A) - {0}"* **in** *spec)*
  **apply** *(blast dest: bijection)*
  **done**

**end**

**theory** *AC15_WO6*
**imports** *HH Cardinal_aux*
**begin**

**lemma** *lepoll_Sigma: "A≠0 ⟹ B ≲ A*B"*
  **unfolding** *lepoll_def*
**apply** *(erule not_emptyE)*
**apply** *(rule_tac x = "λz ∈ B. ⟨x,z⟩"* **in** *exI)*
**apply** *(fast intro!: snd_conv lam_injective)*
**done**

**lemma** *cons_times_nat_not_Finite:*
     *"0∉A ⟹ ∀B ∈ {cons(0,x*nat). x ∈ A}. ¬Finite(B)"*
**apply** *clarify*
**apply** *(rule nat_not_Finite [THEN notE] )*
**apply** *(subgoal_tac "x ≠ 0")*
 **apply** *(blast intro: lepoll_Sigma [THEN lepoll_Finite])+*
**done**

**lemma** *lemma1: "⟦⋃(C)=A; a ∈ A⟧ ⟹ ∃B ∈ C. a ∈ B ∧ B ⊆ A"*
**by** *fast*

**lemma** *lemma2:*
        *"⟦pairwise_disjoint(A); B ∈ A; C ∈ A; a ∈ B; a ∈ C⟧ ⟹ B=C"*
**by** *(unfold pairwise_disjoint_def, blast)*

**lemma** *lemma3:*
     *"∀B ∈ {cons(0, x*nat). x ∈ A}. pairwise_disjoint(f‘B) ∧*
            *sets_of_size_between(f‘B, 2, n) ∧ ⋃(f‘B)=B*

```
     ⟹ ∀B ∈ A. ∃! u. u ∈ f‘cons(0, B*nat) ∧ u ⊆ cons(0, B*nat) ∧

                0 ∈ u ∧ 2 ≲ u ∧ u ≲ n"
  unfolding sets_of_size_between_def
apply (rule ballI)
apply (erule_tac x="cons(0, B*nat)" in ballE)
 apply (blast dest: lemma1 intro!: lemma2, blast)
done

lemma lemma4: "⟦A ≲ i; Ord(i)⟧ ⟹ {P(a). a ∈ A} ≲ i"
  unfolding lepoll_def
apply (erule exE)
apply (rule_tac x = "λx ∈ RepFun(A,P). μ j. ∃a∈A. x=P(a) ∧ f‘a=j"
       in exI)
apply (rule_tac d = "λy. P (converse (f) ‘y) " in lam_injective)
apply (erule RepFunE)
apply (frule inj_is_fun [THEN apply_type], assumption)
apply (fast intro: LeastI2 elim!: Ord_in_Ord inj_is_fun [THEN apply_type])
apply (erule RepFunE)
apply (rule LeastI2)
  apply fast
 apply (fast elim!: Ord_in_Ord inj_is_fun [THEN apply_type])
apply (fast elim: sym left_inverse [THEN ssubst])
done

lemma lemma5_1:
     "⟦B ∈ A; 2 ≲ u(B)⟧ ⟹ (λx ∈ A. {fst(x). x ∈ u(x)-{0}})‘B ≠ 0"
apply simp
apply (fast dest: lepoll_Diff_sing
            elim: lepoll_trans [THEN succ_lepoll_natE] ssubst
            intro!: lepoll_refl)
done

lemma lemma5_2:
     "⟦B ∈ A; u(B) ⊆ cons(0, B*nat)⟧
      ⟹ (λx ∈ A. {fst(x). x ∈ u(x)-{0}})‘B ⊆ B"
apply auto
done

lemma lemma5_3:
     "⟦n ∈ nat; B ∈ A; 0 ∈ u(B); u(B) ≲ succ(n)⟧
      ⟹ (λx ∈ A. {fst(x). x ∈ u(x)-{0}})‘B ≲ n"
apply simp
apply (fast elim!: Diff_lepoll [THEN lemma4 [OF _ nat_into_Ord]])
done

lemma ex_fun_AC13_AC15:
     "⟦∀B ∈ {cons(0, x*nat). x ∈ A}.
               pairwise_disjoint(f‘B) ∧
```

```
            sets_of_size_between(f`B, 2, succ(n)) ∧ ⋃(f`B)=B;
        n ∈ nat⟧
      ⟹ ∃f. ∀B ∈ A. f`B ≠ 0 ∧ f`B ⊆ B ∧ f`B ≲ n"
by (fast del: subsetI notI
        dest!: lemma3 theI intro!: lemma5_1 lemma5_2 lemma5_3)
```

```
theorem AC10_AC11: "⟦n ∈ nat; 1≤n; AC10(n)⟧ ⟹ AC11"
by (unfold AC10_def AC11_def, blast)
```

```
theorem AC11_AC12: "AC11 ⟹ AC12"
by (unfold AC10_def AC11_def AC11_def AC12_def, blast)
```

```
theorem AC12_AC15: "AC12 ⟹ AC15"
  unfolding AC12_def AC15_def
apply (blast del: ballI
            intro!: cons_times_nat_not_Finite ex_fun_AC13_AC15)
done
```

```
lemma OUN_eq_UN: "Ord(x) ⟹ (⋃a<x. F(a)) = (⋃a ∈ x. F(a))"
by (fast intro!: ltI dest!: ltD)

lemma AC15_WO6_aux1:
     "∀x ∈ Pow(A)-{0}. f`x≠0 ∧ f`x ⊆ x ∧ f`x ≲ m
      ⟹ (⋃i<μ x. HH(f,A,x)={A}. HH(f,A,i)) = A"
apply (simp add: Ord_Least [THEN OUN_eq_UN])
apply (rule equalityI)
apply (fast dest!: less_Least_subset_x)
apply (blast del: subsetI
```

```
          intro!: f_subsets_imp_UN_HH_eq_x [THEN Diff_eq_0_iff [THEN
iffD1]])
done

lemma AC15_WO6_aux2:
     "∀x ∈ Pow(A)-{0}. f‘x≠0 ∧ f‘x ⊆ x ∧ f‘x ≲ m
      ⟹ ∀x < (μ x. HH(f,A,x)={A}). HH(f,A,x) ≲ m"
apply (rule oallI)
apply (drule ltD [THEN less_Least_subset_x])
apply (frule HH_subset_imp_eq)
apply (erule ssubst)
apply (blast dest!: HH_subset_x_imp_subset_Diff_UN [THEN not_emptyI2])

done

theorem AC15_WO6: "AC15 ⟹ WO6"
  unfolding AC15_def WO6_def
apply (rule allI)
apply (erule_tac x = "Pow (A) -{0}" in allE)
apply (erule impE, fast)
apply (elim bexE conjE exE)
apply (rule bexI)
 apply (rule conjI, assumption)
 apply (rule_tac x = "μ i. HH (f,A,i) ={A}" in exI)
 apply (rule_tac x = "λj ∈ (μ i. HH (f,A,i) ={A}) . HH (f,A,j) " in exI)
 apply (simp_all add: ltD)
apply (fast intro!: Ord_Least lam_type [THEN domain_of_fun]
            elim!: less_Least_subset_x AC15_WO6_aux1 AC15_WO6_aux2)
done
```

```
theorem AC10_AC13: "⟦n ∈ nat; 1≤n; AC10(n)⟧ ⟹ AC13(n)"
apply (unfold AC10_def AC13_def, safe)
apply (erule allE)
apply (erule impE [OF _ cons_times_nat_not_Finite], assumption)
apply (fast elim!: impE [OF _ cons_times_nat_not_Finite]
            dest!: ex_fun_AC13_AC15)
```

**done**

**lemma** *AC1_AC13: "AC1 ⟹ AC13(1)"*
  **unfolding** *AC1_def AC13_def*
**apply** *(rule allI)*
**apply** *(erule allE)*
**apply** *(rule impI)*
**apply** *(drule mp, assumption)*
**apply** *(elim exE)*
**apply** *(rule_tac x = "λx ∈ A. {f`x}" in exI)*
**apply** *(simp add: singleton_eqpoll_1 [THEN eqpoll_imp_lepoll])*
**done**

**lemma** *AC13_mono: "⟦m≤n; AC13(m)⟧ ⟹ AC13(n)"*
  **unfolding** *AC13_def*
**apply** *(drule le_imp_lepoll)*
**apply** *(fast elim!: lepoll_trans)*
**done**

**theorem** *AC13_AC14: "⟦n ∈ nat; 1≤n; AC13(n)⟧ ⟹ AC14"*
**by** *(unfold AC13_def AC14_def, auto)*

**theorem** *AC14_AC15: "AC14 ⟹ AC15"*
**by** *(unfold AC13_def AC14_def AC15_def, fast)*

```
lemma lemma_aux: "⟦A≠0; A ≲ 1⟧ ⟹ ∃a. A={a}"
by (fast elim!: not_emptyE lepoll_1_is_sing)

lemma AC13_AC1_lemma:
     "∀B ∈ A. f(B)≠0 ∧ f(B)<=B ∧ f(B) ≲ 1
      ⟹ (λx ∈ A. THE y. f(x)={y}) ∈ (∏X ∈ A. X)"
apply (rule lam_type)
apply (drule bspec, assumption)
apply (elim conjE)
apply (erule lemma_aux [THEN exE], assumption)
apply (simp add: the_equality)
done

theorem AC13_AC1: "AC13(1) ⟹ AC1"
  unfolding AC13_def AC1_def
apply (fast elim!: AC13_AC1_lemma)
done




theorem AC11_AC14: "AC11 ⟹ AC14"
  unfolding AC11_def AC14_def
apply (fast intro!: AC10_AC13)
done

end


theory AC16_lemmas
imports AC_Equiv Hartog Cardinal_aux
begin

lemma cons_Diff_eq: "a∉A ⟹ cons(a,A)-{a}=A"
by fast

lemma nat_1_lepoll_iff: "1≲X ⟷ (∃x. x ∈ X)"
  unfolding lepoll_def
apply (rule iffI)
apply (fast intro: inj_is_fun [THEN apply_type])
```

```
apply (erule exE)
apply (rule_tac x = "λa ∈ 1. x" in exI)
apply (fast intro!: lam_injective)
done

lemma eqpoll_1_iff_singleton: "X≈1 ⟷ (∃x. X={x})"
apply (rule iffI)
apply (erule eqpollE)
apply (drule nat_1_lepoll_iff [THEN iffD1])
apply (fast intro!: lepoll_1_is_sing)
apply (fast intro!: singleton_eqpoll_1)
done

lemma cons_eqpoll_succ: "⟦x≈n; y∉x⟧ ⟹ cons(y,x)≈succ(n)"
  unfolding succ_def
apply (fast elim!: cons_eqpoll_cong mem_irrefl)
done

lemma subsets_eqpoll_1_eq: "{Y ∈ Pow(X). Y≈1} = {{x}. x ∈ X}"
apply (rule equalityI)
apply (rule subsetI)
apply (erule CollectE)
apply (drule eqpoll_1_iff_singleton [THEN iffD1])
apply (fast intro!: RepFunI)
apply (rule subsetI)
apply (erule RepFunE)
apply (rule CollectI, fast)
apply (fast intro!: singleton_eqpoll_1)
done

lemma eqpoll_RepFun_sing: "X≈{{x}. x ∈ X}"
  unfolding eqpoll_def bij_def
apply (rule_tac x = "λx ∈ X. {x}" in exI)
apply (rule IntI)
apply (unfold inj_def surj_def, simp)
apply (fast intro!: lam_type RepFunI intro: singleton_eq_iff [THEN iffD1],
simp)
apply (fast intro!: lam_type)
done

lemma subsets_eqpoll_1_eqpoll: "{Y ∈ Pow(X). Y≈1}≈X"
apply (rule subsets_eqpoll_1_eq [THEN ssubst])
apply (rule eqpoll_RepFun_sing [THEN eqpoll_sym])
done

lemma InfCard_Least_in:
     "⟦InfCard(x); y ⊆ x; y ≈ succ(z)⟧ ⟹ (μ i. i ∈ y) ∈ y"
apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll,
                         THEN succ_lepoll_imp_not_empty, THEN not_emptyE])
```

```
apply (fast intro: LeastI
            dest!: InfCard_is_Card [THEN Card_is_Ord]
            elim: Ord_in_Ord)
done

lemma subsets_lepoll_lemma1:
     "⟦InfCard(x); n ∈ nat⟧
      ⟹ {y ∈ Pow(x). y≈succ(succ(n))} ≲ x*{y ∈ Pow(x). y≈succ(n)}"
  unfolding lepoll_def
apply (rule_tac x = "λy ∈ {y ∈ Pow(x) . y≈succ (succ (n))}.
                     <μ i. i ∈ y, y-{μ i. i ∈ y}>" in exI)
apply (rule_tac d = "λz. cons (fst(z), snd(z))" in lam_injective)
 apply (blast intro!: Diff_sing_eqpoll intro: InfCard_Least_in)
apply (simp, blast intro: InfCard_Least_in)
done

lemma set_of_Ord_succ_Union: "(∀y ∈ z. Ord(y)) ⟹ z ⊆ succ(⋃(z))"
apply (rule subsetI)
apply (case_tac "∀y ∈ z. y ⊆ x", blast )
apply (simp, erule bexE)
apply (rule_tac i=y and j=x in Ord_linear_le)
apply (blast dest: le_imp_subset elim: leE ltE)+
done

lemma subset_not_mem: "j ⊆ i ⟹ i ∉ j"
by (fast elim!: mem_irrefl)

lemma succ_Union_not_mem:
     "(⋀y. y ∈ z ⟹ Ord(y)) ⟹ succ(⋃(z)) ∉ z"
apply (rule set_of_Ord_succ_Union [THEN subset_not_mem], blast)
done

lemma Union_cons_eq_succ_Union:
     "⋃(cons(succ(⋃(z)),z)) = succ(⋃(z))"
by fast

lemma Un_Ord_disj: "⟦Ord(i); Ord(j)⟧ ⟹ i ∪ j = i | i ∪ j = j"
by (fast dest!: le_imp_subset elim: Ord_linear_le)

lemma Union_eq_Un: "x ∈ X ⟹ ⋃(X) = x ∪ ⋃(X-{x})"
by fast

lemma Union_in_lemma [rule_format]:
     "n ∈ nat ⟹ ∀z. (∀y ∈ z. Ord(y)) ∧ z≈n ∧ z≠0 ⟶ ⋃(z) ∈ z"
apply (induct_tac "n")
apply (fast dest!: eqpoll_imp_lepoll [THEN lepoll_0_is_0])
apply (intro allI impI)
apply (erule natE)
apply (fast dest!: eqpoll_1_iff_singleton [THEN iffD1]
```

```
           intro!: Union_singleton, clarify)
apply (elim not_emptyE)
apply (erule_tac x = "z-{xb}" in allE)
apply (erule impE)
apply (fast elim!: Diff_sing_eqpoll
                   Diff_sing_eqpoll [THEN eqpoll_succ_imp_not_empty])
apply (subgoal_tac "xb ∪ ⋃(z - {xb}) ∈ z")
apply (simp add: Union_eq_Un [symmetric])
apply (frule bspec, assumption)
apply (drule bspec)
apply (erule Diff_subset [THEN subsetD])
apply (drule Un_Ord_disj, assumption, auto)
done

lemma Union_in: "⟦∀x ∈ z. Ord(x); z≈n; z≠0; n ∈ nat⟧ ⟹ ⋃(z) ∈ z"
by (blast intro: Union_in_lemma)

lemma succ_Union_in_x:
     "⟦InfCard(x); z ∈ Pow(x); z≈n; n ∈ nat⟧ ⟹ succ(⋃(z)) ∈ x"
apply (rule Limit_has_succ [THEN ltE])
prefer 3 apply assumption
apply (erule InfCard_is_Limit)
apply (case_tac "z=0")
apply (simp, fast intro!: InfCard_is_Limit [THEN Limit_has_0])
apply (rule ltI [OF PowD [THEN subsetD] InfCard_is_Card [THEN Card_is_Ord]],
assumption)
apply (blast intro: Union_in
                    InfCard_is_Card [THEN Card_is_Ord, THEN Ord_in_Ord])+
done

lemma succ_lepoll_succ_succ:
     "⟦InfCard(x); n ∈ nat⟧
      ⟹ {y ∈ Pow(x). y≈succ(n)} ≲ {y ∈ Pow(x). y≈succ(succ(n))}"
  unfolding lepoll_def
apply (rule_tac x = "λz ∈ {y∈Pow(x). y≈succ(n)}. cons(succ(⋃(z)), z)"

       in exI)
apply (rule_tac d = "λz. z-{⋃(z) }" in lam_injective)
apply (blast intro!: succ_Union_in_x succ_Union_not_mem
             intro: cons_eqpoll_succ Ord_in_Ord
             dest!: InfCard_is_Card [THEN Card_is_Ord])
apply (simp only: Union_cons_eq_succ_Union)
apply (rule cons_Diff_eq)
apply (fast dest!: InfCard_is_Card [THEN Card_is_Ord]
            elim: Ord_in_Ord
            intro!: succ_Union_not_mem)
done

lemma subsets_eqpoll_X:
```

```
    "⟦InfCard(X); n ∈ nat⟧ ⟹ {Y ∈ Pow(X). Y≈succ(n)} ≈ X"
apply (induct_tac "n")
apply (rule subsets_eqpoll_1_eqpoll)
apply (rule eqpollI)
apply (rule subsets_lepoll_lemma1 [THEN lepoll_trans], assumption+)
apply (rule eqpoll_trans [THEN eqpoll_imp_lepoll])
 apply (erule eqpoll_refl [THEN prod_eqpoll_cong])
apply (erule InfCard_square_eqpoll)
apply (fast elim: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_trans]

            intro!: succ_lepoll_succ_succ)
done

lemma image_vimage_eq:
    "⟦f ∈ surj(A,B); y ⊆ B⟧ ⟹ f``(converse(f)``y) = y"
  unfolding surj_def
apply (fast dest: apply_equality2 elim: apply_iff [THEN iffD2])
done

lemma vimage_image_eq: "⟦f ∈ inj(A,B); y ⊆ A⟧ ⟹ converse(f)``(f``y)
= y"
by (fast elim!: inj_is_fun [THEN apply_Pair] dest: inj_equality)

lemma subsets_eqpoll:
    "A≈B ⟹ {Y ∈ Pow(A). Y≈n}≈{Y ∈ Pow(B). Y≈n}"
  unfolding eqpoll_def
apply (erule exE)
apply (rule_tac x = "λX ∈ {Y ∈ Pow (A) . ∃f. f ∈ bij (Y, n) }. f``X"
in exI)
apply (rule_tac d = "λZ. converse (f) ``Z" in lam_bijective)
apply (fast intro!: bij_is_inj [THEN restrict_bij, THEN bij_converse_bij,

                               THEN comp_bij]
            elim!: bij_is_fun [THEN fun_is_rel, THEN image_subset])
apply (blast intro!:  bij_is_inj [THEN restrict_bij]
                      comp_bij bij_converse_bij
                      bij_is_fun [THEN fun_is_rel, THEN image_subset])
apply (fast elim!: bij_is_inj [THEN vimage_image_eq])
apply (fast elim!: bij_is_surj [THEN image_vimage_eq])
done

lemma WO2_imp_ex_Card: "WO2 ⟹ ∃a. Card(a) ∧ X≈a"
  unfolding WO2_def
apply (drule spec [of _ X])
apply (blast intro: Card_cardinal eqpoll_trans
          well_ord_Memrel [THEN well_ord_cardinal_eqpoll, THEN eqpoll_sym])
done

lemma lepoll_infinite: "⟦X≲Y; ¬Finite(X)⟧ ⟹ ¬Finite(Y)"
```

**by** *(blast intro: lepoll_Finite)*

**lemma** *infinite_Card_is_InfCard: "⟦¬Finite(X); Card(X)⟧ ⟹ InfCard(X)"*
  **unfolding** *InfCard_def*
**apply** *(fast elim!: Card_is_Ord [THEN nat_le_infinite_Ord])*
**done**

**lemma** *WO2_infinite_subsets_eqpoll_X: "⟦WO2; n ∈ nat; ¬Finite(X)⟧*
*⟹ {Y ∈ Pow(X). Y≈succ(n)}≈X"*
**apply** *(drule WO2_imp_ex_Card)*
**apply** *(elim allE exE conjE)*
**apply** *(frule eqpoll_imp_lepoll [THEN lepoll_infinite], assumption)*
**apply** *(drule infinite_Card_is_InfCard, assumption)*
**apply** *(blast intro: subsets_eqpoll subsets_eqpoll_X eqpoll_sym eqpoll_trans)*

**done**

**lemma** *well_ord_imp_ex_Card: "well_ord(X,R) ⟹ ∃a. Card(a) ∧ X≈a"*
**by** *(fast elim!: well_ord_cardinal_eqpoll [THEN eqpoll_sym]*
*intro!: Card_cardinal)*

**lemma** *well_ord_infinite_subsets_eqpoll_X:*
*"⟦well_ord(X,R); n ∈ nat; ¬Finite(X)⟧ ⟹ {Y ∈ Pow(X). Y≈succ(n)}≈X"*
**apply** *(drule well_ord_imp_ex_Card)*
**apply** *(elim allE exE conjE)*
**apply** *(frule eqpoll_imp_lepoll [THEN lepoll_infinite], assumption)*
**apply** *(drule infinite_Card_is_InfCard, assumption)*
**apply** *(blast intro: subsets_eqpoll subsets_eqpoll_X eqpoll_sym eqpoll_trans)*

**done**

**end**

**theory** *WO2_AC16* **imports** *AC_Equiv AC16_lemmas Cardinal_aux* **begin**

**definition**
*recfunAC16 :: "[i,i,i,i] ⇒ i"* **where**
*"recfunAC16(f,h,i,a) ≡*
*transrec2(i, 0,*
*λg r. if (∃y ∈ r. h‘g ⊆ y) then r*
*else r ∪ {f‘(μ i. h‘g ⊆ f‘i ∧*
*(∀b<a. (h‘b ⊆ f‘i ⟶ (∀t ∈ r. ¬ h‘b ⊆ t))))})"

```
lemma recfunAC16_0: "recfunAC16(f,h,0,a) = 0"
by (simp add: recfunAC16_def)

lemma recfunAC16_succ:
     "recfunAC16(f,h,succ(i),a) =
      (if (∃y ∈ recfunAC16(f,h,i,a). h ‘ i ⊆ y) then recfunAC16(f,h,i,a)

       else recfunAC16(f,h,i,a) ∪
            {f ‘ (μ j. h ‘ i ⊆ f ‘ j ∧
             (∀b<a. (h‘b ⊆ f‘j
              ⟶ (∀t ∈ recfunAC16(f,h,i,a). ¬ h‘b ⊆ t)))))})"
apply (simp add: recfunAC16_def)
done

lemma recfunAC16_Limit: "Limit(i)
       ⟹ recfunAC16(f,h,i,a) = (⋃j<i. recfunAC16(f,h,j,a))"
by (simp add: recfunAC16_def transrec2_Limit)




lemma transrec2_mono_lemma [rule_format]:
     "⟦⋀g r. r ⊆ B(g,r);  Ord(i)⟧
      ⟹ j<i ⟶ transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
apply (erule trans_induct)
apply (rule Ord_cases, assumption+, fast)
apply (simp (no_asm_simp))
apply (blast elim!: leE)
apply (simp add: transrec2_Limit)
apply (blast intro: OUN_I ltI Ord_in_Ord [THEN le_refl]
             elim!: Limit_has_succ [THEN ltE])
done

lemma transrec2_mono:
     "⟦⋀g r. r ⊆ B(g,r); j≤i⟧
      ⟹ transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
apply (erule leE)
apply (rule transrec2_mono_lemma)
apply (auto intro: lt_Ord2 )
done




lemma recfunAC16_mono:
       "i≤j ⟹ recfunAC16(f, g, i, a) ⊆ recfunAC16(f, g, j, a)"
```

**unfolding** *recfunAC16_def*
**apply** *(rule transrec2_mono, auto)*
**done**

**lemma** *lemma3_1:*
"⟦∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) ⟶ (∃! Y. Y ∈ F(y) ∧ f(z)<=Y);
∀i j. i≤j ⟶ F(i) ⊆ F(j); j≤i; i<x; z<a;
V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W⟧
⟹ V = W"
**apply** *(erule asm_rl allE impE)+*
**apply** *(drule subsetD, assumption, blast)*
**done**

**lemma** *lemma3:*
"⟦∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) ⟶ (∃! Y. Y ∈ F(y) ∧ f(z)<=Y);
∀i j. i≤j ⟶ F(i) ⊆ F(j); i<x; j<x; z<a;
V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W⟧
⟹ V = W"
**apply** *(rule_tac j=j* **in** *Ord_linear_le [OF lt_Ord lt_Ord], assumption+)*
**apply** *(erule lemma3_1 [symmetric], assumption+)*
**apply** *(erule lemma3_1, assumption+)*
**done**

**lemma** *lemma4:*
"⟦∀y<x. F(y) ⊆ X ∧
(∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) ⟶
(∃! Y. Y ∈ F(y) ∧ h(x) ⊆ Y));
x < a⟧
⟹ ∀y<x. ∀z<a. z < y | (∃Y ∈ F(y). h(z) ⊆ Y) ⟶
(∃! Y. Y ∈ F(y) ∧ h(z) ⊆ Y)"
**apply** *(intro oallI impI)*
**apply** *(drule ospec, assumption, clarify)*
**apply** *(blast elim!: oallE )*
**done**

**lemma** *lemma5:*
"⟦∀y<x. F(y) ⊆ X ∧
(∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) ⟶
(∃! Y. Y ∈ F(y) ∧ h(x) ⊆ Y));
x < a; Limit(x); ∀i j. i≤j ⟶ F(i) ⊆ F(j)⟧
⟹ (⋃x<x. F(x)) ⊆ X ∧

```
          (∀xa<a. xa < x | (∃x ∈ ⋃x<x. F(x). h(xa) ⊆ x)
               ⟶ (∃! Y. Y ∈ (⋃x<x. F(x)) ∧ h(xa) ⊆ Y))"
apply (rule conjI)
apply (rule subsetI)
apply (erule OUN_E)
apply (drule ospec, assumption, fast)
apply (drule lemma4, assumption)
apply (rule oallI)
apply (rule impI)
apply (erule disjE)
apply (frule ospec, erule Limit_has_succ, assumption)
apply (drule_tac A = a and x = xa in ospec, assumption)
apply (erule impE, rule le_refl [THEN disjI1], erule lt_Ord)
apply (blast intro: lemma3 Limit_has_succ)
apply (blast intro: lemma3)
done
```

```
lemma dbl_Diff_eqpoll_Card:
      "⟦A≈a; Card(a); ¬Finite(a); B≺a; C≺a⟧ ⟹ A - B - C≈a"
by (blast intro: Diff_lesspoll_eqpoll_Card)
```

```
lemma Finite_lesspoll_infinite_Ord:
      "⟦Finite(X); ¬Finite(a); Ord(a)⟧ ⟹ X≺a"
  unfolding lesspoll_def
apply (rule conjI)
apply (drule nat_le_infinite_Ord [THEN le_imp_lepoll], assumption)
  unfolding Finite_def
apply (blast intro: leI [THEN le_imp_subset, THEN subset_imp_lepoll]
                    ltI eqpoll_imp_lepoll lepoll_trans)
apply (blast intro: eqpoll_sym [THEN eqpoll_trans])
done

lemma Union_lesspoll:
```

```
    "⟦∀x ∈ X. x ≲ n ∧ x ⊆ T; well_ord(T, R); X ≲ b;
       b<a; ¬Finite(a); Card(a); n ∈ nat⟧
     ⟹ ⋃(X)≺a"
apply (case_tac "Finite (X)")
apply (blast intro: Card_is_Ord Finite_lesspoll_infinite_Ord
                    lepoll_nat_imp_Finite Finite_Union)
apply (drule lepoll_imp_ex_le_eqpoll)
apply (erule lt_Ord)
apply (elim exE conjE)
apply (frule eqpoll_imp_lepoll [THEN lepoll_infinite], assumption)
apply (erule eqpoll_sym [THEN eqpoll_def [THEN def_imp_iff, THEN iffD1],
                         THEN exE])
apply (frule bij_is_surj [THEN surj_image_eq])
apply (drule image_fun [OF bij_is_fun subset_refl])
apply (drule sym [THEN trans], assumption)
apply (blast intro: lt_Ord UN_lepoll lt_Card_imp_lesspoll
                    lt_trans1 lesspoll_trans1)
done
```

```
lemma Un_sing_eq_cons: "A ∪ {a} = cons(a, A)"
by fast

lemma Un_lepoll_succ: "A ≲ B ⟹ A ∪ {a} ≲ succ(B)"
apply (simp add: Un_sing_eq_cons succ_def)
apply (blast elim!: mem_irrefl intro: cons_lepoll_cong)
done

lemma Diff_UN_succ_empty: "Ord(a) ⟹ F(a) - (⋃b<succ(a). F(b)) = 0"
by (fast intro!: le_refl)

lemma Diff_UN_succ_subset: "Ord(a) ⟹ F(a) ∪ X - (⋃b<succ(a). F(b))
⊆ X"
by blast

lemma recfunAC16_Diff_lepoll_1:
     "Ord(x)
      ⟹ recfunAC16(f, g, x, a) - (⋃i<x. recfunAC16(f, g, i, a)) ≲ 1"
apply (erule Ord_cases)
  apply (simp add: recfunAC16_0 empty_subsetI [THEN subset_imp_lepoll])

prefer 2 apply (simp add: recfunAC16_Limit Diff_cancel
                          empty_subsetI [THEN subset_imp_lepoll])

apply (simp add: recfunAC16_succ
                 Diff_UN_succ_empty [of _ "λj. recfunAC16(f,g,j,a)"]
```

```
                empty_subsetI [THEN subset_imp_lepoll])
apply (best intro: Diff_UN_succ_subset [THEN subset_imp_lepoll]
                   singleton_eqpoll_1 [THEN eqpoll_imp_lepoll] lepoll_trans)
done

lemma in_Least_Diff:
     "⟦z ∈ F(x); Ord(x)⟧
      ⟹ z ∈ F(μ i. z ∈ F(i)) - (⋃j<(μ i. z ∈ F(i)). F(j))"
by (fast elim: less_LeastE elim!: LeastI)

lemma Least_eq_imp_ex:
     "⟦(μ i. w ∈ F(i)) = (μ i. z ∈ F(i));
        w ∈ (⋃i<a. F(i)); z ∈ (⋃i<a. F(i))⟧
      ⟹ ∃b<a. w ∈ (F(b) - (⋃c<b. F(c))) ∧ z ∈ (F(b) - (⋃c<b. F(c)))"
apply (elim OUN_E)
apply (drule in_Least_Diff, erule lt_Ord)
apply (frule in_Least_Diff, erule lt_Ord)
apply (rule oexI, force)
apply (blast intro: lt_Ord Least_le [THEN lt_trans1])
done


lemma two_in_lepoll_1: "⟦A ≲ 1; a ∈ A; b ∈ A⟧ ⟹ a=b"
by (fast dest!: lepoll_1_is_sing)


lemma UN_lepoll_index:
     "⟦∀i<a. F(i)-(⋃j<i. F(j)) ≲ 1; Limit(a)⟧
      ⟹ (⋃x<a. F(x)) ≲ a"
apply (rule lepoll_def [THEN def_imp_iff [THEN iffD2]])
apply (rule_tac x = "λz ∈ (⋃x<a. F (x)). μ i. z ∈ F (i) " in exI)
  unfolding inj_def
apply (rule CollectI)
apply (rule lam_type)
apply (erule OUN_E)
apply (erule Least_in_Ord)
apply (erule ltD)
apply (erule lt_Ord2)
apply (intro ballI)
apply (simp (no_asm_simp))
apply (rule impI)
apply (drule Least_eq_imp_ex, assumption+)
apply (fast elim!: two_in_lepoll_1)
done


lemma recfunAC16_lepoll_index: "Ord(y) ⟹ recfunAC16(f, h, y, a) ≲
y"
apply (erule trans_induct3)
```

```
apply (simp (no_asm_simp) add: recfunAC16_0 lepoll_refl)

apply (simp (no_asm_simp) add: recfunAC16_succ)
apply (blast dest!: succI1 [THEN rev_bspec]
             intro: subset_succI [THEN subset_imp_lepoll] Un_lepoll_succ

                    lepoll_trans)
apply (simp (no_asm_simp) add: recfunAC16_Limit)
apply (blast intro: lt_Ord [THEN recfunAC16_Diff_lepoll_1] UN_lepoll_index)
done


lemma Union_recfunAC16_lesspoll:
     "⟦recfunAC16(f,g,y,a) ⊆ {X ∈ Pow(A). X≈n};
        A≈a;  y<a;  ¬Finite(a);  Card(a);  n ∈ nat⟧
      ⟹ ⋃(recfunAC16(f,g,y,a))≺a"
apply (erule eqpoll_def [THEN def_imp_iff, THEN iffD1, THEN exE])
apply (rule_tac T=A in Union_lesspoll, simp_all)
apply (blast intro!: eqpoll_imp_lepoll)
apply (blast intro: bij_is_inj Card_is_Ord [THEN well_ord_Memrel]
                    well_ord_rvimage)
apply (erule lt_Ord [THEN recfunAC16_lepoll_index])
done

lemma dbl_Diff_eqpoll:
     "⟦recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
        Card(a); ¬ Finite(a); A≈a;
        k ∈ nat;  y<a;
        h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)})⟧
      ⟹ A - ⋃(recfunAC16(f, h, y, a)) - h‘y≈a"
apply (rule dbl_Diff_eqpoll_Card, simp_all)
apply (simp add: Union_recfunAC16_lesspoll)
apply (rule Finite_lesspoll_infinite_Ord)
apply (rule Finite_def [THEN def_imp_iff, THEN iffD2])
apply (blast dest: ltD bij_is_fun [THEN apply_type], assumption)
apply (blast intro: Card_is_Ord)
done



lemmas disj_Un_eqpoll_nat_sum =
    eqpoll_trans [THEN eqpoll_trans,
                  OF disj_Un_eqpoll_sum sum_eqpoll_cong nat_sum_eqpoll_sum]


lemma Un_in_Collect: "⟦x ∈ Pow(A - B - h‘i); x≈m;
        h ∈ bij(a, {x ∈ Pow(A) . x≈k}); i<a; k ∈ nat; m ∈ nat⟧
        ⟹ h ‘ i ∪ x ∈ {x ∈ Pow(A) . x≈k #+ m}"
```

**by** *(blast intro: disj_Un_eqpoll_nat_sum*
*dest: ltD bij_is_fun [THEN apply_type])*

**lemma** *lemma6:*
*"⟦∀y<succ(j). F(y)<=X ∧ (∀x<a. x<y | P(x,y) ⟶ Q(x,y)); succ(j)<a⟧*
*⟹ F(j)<=X ∧ (∀x<a. x<j | P(x,j) ⟶ Q(x,j))"*
**by** *(blast intro!: lt_Ord succI1 [THEN ltI, THEN lt_Ord, THEN le_refl])*

**lemma** *lemma7:*
*"⟦∀x<a. x<j | P(x,j) ⟶ Q(x,j); succ(j)<a⟧*
*⟹ P(j,j) ⟶ (∀x<a. x≤j | P(x,j) ⟶ Q(x,j))"*
**by** *(fast elim!: leE)*

**lemma** *ex_subset_eqpoll:*
*"⟦A≈a; ¬ Finite(a); Ord(a); m ∈ nat⟧ ⟹ ∃X ∈ Pow(A). X≈m"*
**apply** *(rule lepoll_imp_eqpoll_subset [of m A, THEN exE])*
**apply** *(rule lepoll_trans, rule leI [THEN le_imp_lepoll])*
**apply** *(blast intro: lt_trans2 [OF ltI nat_le_infinite_Ord] Ord_nat)*
**apply** *(erule eqpoll_sym [THEN eqpoll_imp_lepoll])*
**apply** *(fast elim!: eqpoll_sym)*
**done**

**lemma** *subset_Un_disjoint: "⟦A ⊆ B ∪ C; A ∩ C = 0⟧ ⟹ A ⊆ B"*
**by** *blast*

**lemma** *Int_empty:*
*"⟦X ∈ Pow(A - ⋃(B) -C); T ∈ B; F ⊆ T⟧ ⟹ F ∩ X = 0"*
**by** *blast*

**lemma** *subset_imp_eq_lemma:*

```
    "m ∈ nat ⟹ ∀A B. A ⊆ B ∧ m ≲ A ∧ B ≲ m ⟶ A=B"
apply (induct_tac "m")
apply (fast dest!: lepoll_0_is_0)
apply (intro allI impI)
apply (elim conjE)
apply (rule succ_lepoll_imp_not_empty [THEN not_emptyE], assumption)
apply (frule subsetD [THEN Diff_sing_lepoll], assumption+)
apply (frule lepoll_Diff_sing)
apply (erule allE impE)+
apply (rule conjI)
prefer 2 apply fast
apply fast
apply (blast elim: equalityE)
done


lemma subset_imp_eq: "⟦A ⊆ B; m ≲ A; B ≲ m; m ∈ nat⟧ ⟹ A=B"
by (blast dest!: subset_imp_eq_lemma)


lemma bij_imp_arg_eq:
    "⟦f ∈ bij(a, {Y ∈ X. Y≈succ(k)}); k ∈ nat; f‘b ⊆ f‘y; b<a; y<a⟧

     ⟹ b=y"
apply (drule subset_imp_eq)
apply (erule_tac [3] nat_succI)
  unfolding bij_def inj_def
apply (blast intro: eqpoll_sym eqpoll_imp_lepoll
            dest:  ltD apply_type)+
done


lemma ex_next_set:
    "⟦recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
       Card(a); ¬ Finite(a); A≈a;
       k ∈ nat; m ∈ nat; y<a;
       h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
       ¬ (∃Y ∈ recfunAC16(f, h, y, a). h‘y ⊆ Y)⟧
     ⟹ ∃X ∈ {Y ∈ Pow(A). Y≈succ(k #+ m)}. h‘y ⊆ X ∧
              (∀b<a. h‘b ⊆ X ⟶
              (∀T ∈ recfunAC16(f, h, y, a). ¬ h‘b ⊆ T))"
apply (erule_tac m1=m in dbl_Diff_eqpoll [THEN ex_subset_eqpoll, THEN
bexE],
       assumption+)
apply (erule Card_is_Ord, assumption)
apply (frule Un_in_Collect, (erule asm_rl nat_succI)+)
apply (erule CollectE)
apply (rule rev_bexI, simp)
apply (rule conjI, blast)
```

```
apply (intro ballI impI oallI notI)
apply (drule subset_Un_disjoint, rule Int_empty, assumption+)
apply (blast dest: bij_imp_arg_eq)
done
```

```
lemma ex_next_Ord:
     "⟦recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
        Card(a); ¬ Finite(a); A≈a;
        k ∈ nat; m ∈ nat; y<a;
        h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
        f ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k #+ m)});
        ¬ (∃Y ∈ recfunAC16(f, h, y, a). h‘y ⊆ Y)⟧
      ⟹ ∃c<a. h‘y ⊆ f‘c ∧
               (∀b<a. h‘b ⊆ f‘c ⟶
               (∀T ∈ recfunAC16(f, h, y, a). ¬ h‘b ⊆ T))"
apply (drule ex_next_set, assumption+)
apply (erule bexE)
apply (rule_tac x="converse(f)‘X" in oexI)
apply (simp add: right_inverse_bij)
apply (blast intro: bij_converse_bij bij_is_fun [THEN apply_type] ltI
                    Card_is_Ord)
done
```

```
lemma lemma8:
     "⟦∀x<a. x<j | (∃xa ∈ F(j). P(x, xa))
        ⟶ (∃! Y. Y ∈ F(j) ∧ P(x, Y)); F(j) ⊆ X;
        L ∈ X; P(j, L) ∧ (∀x<a. P(x, L) ⟶ (∀xa ∈ F(j). ¬P(x, xa)))⟧

      ⟹ F(j) ∪ {L} ⊆ X ∧
          (∀x<a. x≤j | (∃xa ∈ (F(j) ∪ {L}). P(x, xa)) ⟶
                (∃! Y. Y ∈ (F(j) ∪ {L}) ∧ P(x, Y)))"
apply (rule conjI)
apply (fast intro!: singleton_subsetI)
apply (rule oallI)
apply (blast elim!: leE oallE)
done
```

```
lemma main_induct:
     "⟦b < a; f ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k #+ m)});
        h ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k)});
        ¬Finite(a); Card(a); A≈a; k ∈ nat; m ∈ nat⟧
      ⟹ recfunAC16(f, h, b, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)} ∧
          (∀x<a. x < b | (∃Y ∈ recfunAC16(f, h, b, a). h ‘ x ⊆ Y) ⟶

          (∃! Y. Y ∈ recfunAC16(f, h, b, a) ∧ h ‘ x ⊆ Y))"
apply (erule lt_induct)
apply (frule lt_Ord)
apply (erule Ord_cases)

apply (simp add: recfunAC16_0)

prefer 2  apply (simp add: recfunAC16_Limit)
          apply (rule lemma5, assumption+)
          apply (blast dest!: recfunAC16_mono)

apply clarify
apply (erule lemma6 [THEN conjE], assumption)
apply (simp (no_asm_simp) split del: split_if add: recfunAC16_succ)
apply (rule conjI [THEN split_if [THEN iffD2]])
 apply (simp, erule lemma7, assumption)
apply (rule impI)
apply (rule ex_next_Ord [THEN oexE],
       assumption+, rule le_refl [THEN lt_trans], assumption+)
apply (erule lemma8, assumption)
 apply (rule bij_is_fun [THEN apply_type], assumption)
 apply (erule Least_le [THEN lt_trans2, THEN ltD])
  apply (erule lt_Ord)
 apply (erule succ_leI)
apply (erule LeastI)
apply (erule lt_Ord)
done
```

```
lemma lemma_simp_induct:
     "⟦∀b. b<a ⟶ F(b) ⊆ S ∧ (∀x<a. (x<b | (∃Y ∈ F(b). f‘x ⊆ Y))
                                  ⟶ (∃! Y. Y ∈ F(b) ∧ f‘x ⊆ Y));
        f ∈ a->f‘‘(a); Limit(a);
        ∀i j. i≤j ⟶ F(i) ⊆ F(j)⟧
       ⟹ (⋃j<a. F(j)) ⊆ S ∧
```

```
            (∀x ∈ f``a. ∃! Y. Y ∈ (⋃j<a. F(j)) ∧ x ⊆ Y)"
apply (rule conjI)
apply (rule subsetI)
apply (erule OUN_E, blast)
apply (rule ballI)
apply (erule imageE)
apply (drule ltI, erule Limit_is_Ord)
apply (drule Limit_has_succ, assumption)
apply (frule_tac x1="succ(xa)" in spec [THEN mp], assumption)
apply (erule conjE)
apply (drule ospec)

apply (erule leI [THEN succ_leE])
apply (erule impE)
apply (fast elim!: leI [THEN succ_leE, THEN lt_Ord, THEN le_refl])
apply (drule apply_equality, assumption)
apply (elim conjE ex1E)

apply (rule ex1I, blast)
apply (elim conjE OUN_E)
apply (erule_tac i="succ(xa)" and j=aa
       in Ord_linear_le [OF lt_Ord lt_Ord], assumption)
prefer 2
apply (drule spec [THEN spec, THEN mp, THEN subsetD], assumption+, blast)


apply (drule_tac x1=aa in spec [THEN mp], assumption)
apply (frule succ_leE)
apply (drule spec [THEN spec, THEN mp, THEN subsetD], assumption+, blast)

done
```

```
theorem WO2_AC16: "⟦WO2; 0<m; k ∈ nat; m ∈ nat⟧ ⟹ AC16(k #+ m,k)"
  unfolding AC16_def
apply (rule allI)
apply (rule impI)
apply (frule WO2_infinite_subsets_eqpoll_X, assumption+)
apply (frule_tac n="k #+ m" in WO2_infinite_subsets_eqpoll_X, simp, simp)

apply (frule WO2_imp_ex_Card)
apply (elim exE conjE)
apply (drule eqpoll_trans [THEN eqpoll_sym,
                           THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]],
       assumption)
apply (drule eqpoll_trans [THEN eqpoll_sym,
```

```
                        THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]],

       assumption+)
apply (elim exE)
apply (rename_tac h)
apply (rule_tac x = "⋃j<a. recfunAC16 (h,f,j,a) " in exI)
apply (rule_tac P="λz. Y ∧ (∀x ∈ z. Z(x))"  for Y Z
       in bij_is_surj [THEN surj_image_eq, THEN subst], assumption)
apply (rule lemma_simp_induct)
apply (blast del: conjI notI
             intro!: main_induct eqpoll_imp_lepoll [THEN lepoll_infinite]
)
apply (blast intro: bij_is_fun [THEN surj_image, THEN surj_is_fun])
apply (erule eqpoll_imp_lepoll [THEN lepoll_infinite,
                                THEN infinite_Card_is_InfCard,
                                THEN InfCard_is_Limit],
       assumption+)
apply (blast dest!: recfunAC16_mono)
done

end


theory AC16_WO4
imports AC16_lemmas
begin




lemma lemma1:
     "⟦Finite(A); 0<m; m ∈ nat⟧
      ⟹ ∃a f. Ord(a) ∧ domain(f) = a ∧
                (⋃b<a. f`b) = A ∧ (∀b<a. f`b ≲ m)"
  unfolding Finite_def
apply (erule bexE)
apply (drule eqpoll_sym [THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]])
apply (erule exE)
apply (rule_tac x = n in exI)
apply (rule_tac x = "λi ∈ n. {f`i}" in exI)
apply (simp add: ltD bij_def surj_def)
apply (fast intro!: ltI nat_into_Ord lam_funtype [THEN domain_of_fun]

              singleton_eqpoll_1 [THEN eqpoll_imp_lepoll, THEN lepoll_trans]

                   nat_1_lepoll_iff [THEN iffD2]
         elim!: apply_type ltE)
done
```

**lemmas** *well_ord_paired = paired_bij [THEN bij_is_inj, THEN well_ord_rvimage]*

**lemma** *lepoll_trans1: "⟦A ≲ B; ¬ A ≲ C⟧ ⟹ ¬ B ≲ C"*
**by** *(blast intro: lepoll_trans)*

**lemmas** *lepoll_paired = paired_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll]*

**lemma** *lemma2: "∃y R. well_ord(y,R) ∧ x ∩ y = 0 ∧ ¬y ≲ z ∧ ¬Finite(y)"*
**apply** *(rule_tac x = "{{a,x}. a ∈ nat ∪ Hartog (z) }"* **in** *exI)*
**apply** *(rule well_ord_Un [OF Ord_nat [THEN well_ord_Memrel]*
*Ord_Hartog [THEN well_ord_Memrel], THEN exE])*
**apply** *(blast intro!: Ord_Hartog well_ord_Memrel well_ord_paired*
*lepoll_trans1 [OF _ not_Hartog_lepoll_self]*
*lepoll_trans [OF subset_imp_lepoll lepoll_paired]*
*elim!: nat_not_Finite [THEN notE]*
*elim: mem_asym*
*dest!: Un_upper1 [THEN subset_imp_lepoll, THEN lepoll_Finite]*
*lepoll_paired [THEN lepoll_Finite])*
**done**

**lemma** *infinite_Un: "¬Finite(B) ⟹ ¬Finite(A ∪ B)"*
**by** *(blast intro: subset_Finite)*

**lemma** *succ_not_lepoll_lemma:*
*"⟦¬(∃x ∈ A. f`x=y); f ∈ inj(A, B); y ∈ B⟧*
*⟹ (λa ∈ succ(A). if(a=A, y, f`a)) ∈ inj(succ(A), B)"*

```
apply (rule_tac d = "λz. if (z=y, A, converse (f) `z) " in lam_injective)
apply (force simp add: inj_is_fun [THEN apply_type])

apply (simp (no_asm_simp))
apply force
done

lemma succ_not_lepoll_imp_eqpoll: "⟦¬A ≈ B; A ≲ B⟧ ⟹ succ(A) ≲ B"
  unfolding lepoll_def eqpoll_def bij_def surj_def
apply (fast elim!: succ_not_lepoll_lemma inj_is_fun)
done
```

```
lemmas ordertype_eqpoll =
       ordermap_bij [THEN exI [THEN eqpoll_def [THEN def_imp_iff, THEN
iffD2]]]

lemma cons_cons_subset:
     "⟦a ⊆ y; b ∈ y-a; u ∈ x⟧ ⟹ cons(b, cons(u, a)) ∈ Pow(x ∪ y)"
by fast

lemma cons_cons_eqpoll:
     "⟦a ≈ k; a ⊆ y; b ∈ y-a; u ∈ x; x ∩ y = 0⟧
      ⟹ cons(b, cons(u, a)) ≈ succ(succ(k))"
by (fast intro!: cons_eqpoll_succ)

lemma set_eq_cons:
     "⟦succ(k) ≈ A; k ≈ B; B ⊆ A; a ∈ A-B; k ∈ nat⟧ ⟹ A = cons(a,
B)"
apply (rule equalityI)
prefer 2 apply fast
apply (rule Diff_eq_0_iff [THEN iffD1])
apply (rule equals0I)
apply (drule eqpoll_sym [THEN eqpoll_imp_lepoll])
apply (drule eqpoll_sym [THEN cons_eqpoll_succ], fast)
apply (drule cons_eqpoll_succ, fast)
apply (fast elim!: lepoll_trans [THEN lepoll_trans, THEN succ_lepoll_natE,
         OF eqpoll_sym [THEN eqpoll_imp_lepoll] subset_imp_lepoll])
done

lemma cons_eqE: "⟦cons(x,a) = cons(y,a); x ∉ a⟧ ⟹ x = y "
by (fast elim!: equalityE)

lemma eq_imp_Int_eq: "A = B ⟹ A ∩ C = B ∩ C"
by blast
```

```
lemma eqpoll_sum_imp_Diff_lepoll_lemma [rule_format]:
     "⟦k ∈ nat; m ∈ nat⟧
      ⟹ ∀A B. A ≈ k #+ m ∧ k ≲ B ∧ B ⊆ A ⟶ A-B ≲ m"
apply (induct_tac "k")
apply (simp add: add_0)
apply (blast intro: eqpoll_imp_lepoll lepoll_trans
                    Diff_subset [THEN subset_imp_lepoll])
apply (intro allI impI)
apply (rule succ_lepoll_imp_not_empty [THEN not_emptyE], fast)
apply (erule_tac x = "A - {xa}" in allE)
apply (erule_tac x = "B - {xa}" in allE)
apply (erule impE)
apply (simp add: add_succ)
apply (fast intro!: Diff_sing_eqpoll lepoll_Diff_sing)
apply (subgoal_tac "A - {xa} - (B - {xa}) = A - B", simp)
apply blast
done

lemma eqpoll_sum_imp_Diff_lepoll:
     "⟦A ≈ succ(k #+ m); B ⊆ A; succ(k) ≲ B;  k ∈ nat; m ∈ nat⟧
      ⟹ A-B ≲ m"
apply (simp only: add_succ [symmetric])
apply (blast intro: eqpoll_sum_imp_Diff_lepoll_lemma)
done
```

```
lemma eqpoll_sum_imp_Diff_eqpoll_lemma [rule_format]:
     "⟦k ∈ nat; m ∈ nat⟧
      ⟹ ∀A B. A ≈ k #+ m ∧ k ≈ B ∧ B ⊆ A ⟶ A-B ≈ m"
apply (induct_tac "k")
apply (force dest!: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_0_is_0])
apply (intro allI impI)
apply (rule succ_lepoll_imp_not_empty [THEN not_emptyE])
apply (fast elim!: eqpoll_imp_lepoll)
apply (erule_tac x = "A - {xa}" in allE)
apply (erule_tac x = "B - {xa}" in allE)
apply (erule impE)
apply (force intro: eqpoll_sym intro!: Diff_sing_eqpoll)
apply (subgoal_tac "A - {xa} - (B - {xa}) = A - B", simp)
apply blast
done
```

**lemma** *eqpoll_sum_imp_Diff_eqpoll:*
"⟦A ≈ succ(k #+ m); B ⊆ A; succ(k) ≈ B; k ∈ nat; m ∈ nat⟧
⟹ A-B ≈ m"
**apply** *(simp only: add_succ [symmetric])*
**apply** *(blast intro: eqpoll_sum_imp_Diff_eqpoll_lemma)*
**done**

**lemma** *subsets_lepoll_0_eq_unit: "{x ∈ Pow(X). x ≲ 0} = {0}"*
**by** *(fast dest!: lepoll_0_is_0 intro!: lepoll_refl)*

**lemma** *subsets_lepoll_succ:*
"n ∈ nat ⟹ {z ∈ Pow(y). z ≲ succ(n)} =
{z ∈ Pow(y). z ≲ n} ∪ {z ∈ Pow(y). z ≈ succ(n)}"
**by** *(blast intro: leI le_imp_lepoll nat_into_Ord*
*lepoll_trans eqpoll_imp_lepoll*
*dest!: lepoll_succ_disj)*

**lemma** *Int_empty:*
"n ∈ nat ⟹ {z ∈ Pow(y). z ≲ n} ∩ {z ∈ Pow(y). z ≈ succ(n)} =
0"
**by** *(blast intro: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_trans]*

*succ_lepoll_natE)*

**locale** *AC16* =
**fixes** *x* **and** *y* **and** *k* **and** *l* **and** *m* **and** *t_n* **and** *R* **and** *MM* **and** *LL* **and** *GG* **and** *s*
**defines** *k_def:* "k ≡ succ(l)"
**and** *MM_def:* "MM ≡ {v ∈ t_n. succ(k) ≲ v ∩ y}"
**and** *LL_def:* "LL ≡ {v ∩ y. v ∈ MM}"
**and** *GG_def:* "GG ≡ λv ∈ LL. (THE w. w ∈ MM ∧ v ⊆ w) - v"
**and** *s_def:* "s(u) ≡ {v ∈ t_n. u ∈ v ∧ k ≲ v ∩ y}"
**assumes** *all_ex:* "∀z ∈ {z ∈ Pow(x ∪ y) . z ≈ succ(k)}.
∃! w. w ∈ t_n ∧ z ⊆ w "
**and** *disjoint[iff]:* "x ∩ y = 0"
**and** *"includes":* "t_n ⊆ {v ∈ Pow(x ∪ y). v ≈ succ(k #+ m)}"
**and** *WO_R[iff]:* "well_ord(y,R)"
**and** *lnat[iff]:* "l ∈ nat"
**and** *mnat[iff]:* "m ∈ nat"
**and** *mpos[iff]:* "0<m"
**and** *Infinite[iff]:* "¬ Finite(y)"
**and** *noLepoll:* "¬ y ≲ {v ∈ Pow(x). v ≈ m}"
**begin**

```
lemma knat [iff]: "k ∈ nat"
by (simp add: k_def)
```

```
lemma Diff_Finite_eqpoll: "⟦l ≈ a; a ⊆ y⟧ ⟹ y - a ≈ y"
apply (insert WO_R Infinite lnat)
apply (rule eqpoll_trans)
apply (rule Diff_lesspoll_eqpoll_Card)
apply (erule well_ord_cardinal_eqpoll [THEN eqpoll_sym])
apply (blast intro: lesspoll_trans1
            intro!: Card_cardinal
                    Card_cardinal [THEN Card_is_Ord, THEN nat_le_infinite_Ord,
                                   THEN le_imp_lepoll]
            dest: well_ord_cardinal_eqpoll
                   eqpoll_sym  eqpoll_imp_lepoll
                   n_lesspoll_nat [THEN lesspoll_trans2]
                   well_ord_cardinal_eqpoll [THEN eqpoll_sym,
                          THEN eqpoll_imp_lepoll, THEN lepoll_infinite])+
done
```

```
lemma s_subset: "s(u) ⊆ t_n"
by (unfold s_def, blast)

lemma sI:
      "⟦w ∈ t_n; cons(b,cons(u,a)) ⊆ w; a ⊆ y; b ∈ y-a; l ≈ a⟧
       ⟹ w ∈ s(u)"
  unfolding s_def succ_def k_def
apply (blast intro!: eqpoll_imp_lepoll [THEN cons_lepoll_cong]
             intro: subset_imp_lepoll lepoll_trans)
done

lemma in_s_imp_u_in: "v ∈ s(u) ⟹ u ∈ v"
by (unfold s_def, blast)
```

```
lemma ex1_superset_a:
     "⟦l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x⟧
      ⟹ ∃! c. c ∈ s(u) ∧ a ⊆ c ∧ b ∈ c"
apply (rule all_ex [simplified k_def, THEN ballE])
 apply (erule ex1E)
 apply (rule_tac a = w in ex1I, blast intro: sI)
 apply (blast dest: s_subset [THEN subsetD] in_s_imp_u_in)
```

```
apply (blast del: PowI
            intro!: cons_cons_subset eqpoll_sym [THEN cons_cons_eqpoll])
done

lemma the_eq_cons:
     "⟦∀v ∈ s(u). succ(l) ≈ v ∩ y;
        l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x⟧
      ⟹ (THE c. c ∈ s(u) ∧ a ⊆ c ∧ b ∈ c) ∩ y = cons(b, a)"
apply (frule ex1_superset_a [THEN theI], assumption+)
apply (rule set_eq_cons)
apply (fast+)
done

lemma y_lepoll_subset_s:
     "⟦∀v ∈ s(u). succ(l) ≈ v ∩ y;
        l ≈ a;  a ⊆ y;  u ∈ x⟧
      ⟹ y ≲ {v ∈ s(u). a ⊆ v}"
apply (rule Diff_Finite_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll,

                                THEN lepoll_trans],  fast+)
apply (rule_tac  f3 = "λb ∈ y-a. THE c. c ∈ s (u) ∧ a ⊆ c ∧ b ∈ c"

        in exI [THEN lepoll_def [THEN def_imp_iff, THEN iffD2]])
apply (simp add: inj_def)
apply (rule conjI)
apply (rule lam_type)
apply (frule ex1_superset_a [THEN theI], fast+, clarify)
apply (rule cons_eqE [of _ a])
apply (drule_tac A = "THE c. P (c)" and C = y for P in eq_imp_Int_eq)
apply (simp_all add: the_eq_cons)
done

lemma x_imp_not_y [dest]: "a ∈ x ⟹ a ∉ y"
by (blast dest:  disjoint [THEN equalityD1, THEN subsetD, OF IntI])

lemma w_Int_eq_w_Diff:
     "w ⊆ x ∪ y ⟹ w ∩ (x - {u}) = w - cons(u, w ∩ y)"
by blast

lemma w_Int_eqpoll_m:
     "⟦w ∈ {v ∈ s(u). a ⊆ v};
        l ≈ a;  u ∈ x;
```

```
        ∀v ∈ s(u). succ(l) ≈ v ∩ y⟧
      ⟹ w ∩ (x - {u}) ≈ m"
apply (erule CollectE)
apply (subst w_Int_eq_w_Diff)
apply (fast dest!: s_subset [THEN subsetD]
                   "includes" [simplified k_def, THEN subsetD])
apply (blast dest: s_subset [THEN subsetD]
                   "includes" [simplified k_def, THEN subsetD]
             dest: eqpoll_sym [THEN cons_eqpoll_succ, THEN eqpoll_sym]

                   in_s_imp_u_in
            intro!: eqpoll_sum_imp_Diff_eqpoll)
done
```

```
lemma eqpoll_m_not_empty: "a ≈ m ⟹ a ≠ 0"
apply (insert mpos)
apply (fast elim!: zero_lt_natE dest!: eqpoll_succ_imp_not_empty)
done

lemma cons_cons_in:
     "⟦z ∈ xa ∩ (x - {u}); l ≈ a; a ⊆ y; u ∈ x⟧
      ⟹ ∃! w. w ∈ t_n ∧ cons(z, cons(u, a)) ⊆ w"
apply (rule all_ex [THEN bspec])
  unfolding k_def
apply (fast intro!: cons_eqpoll_succ elim: eqpoll_sym)
done

lemma subset_s_lepoll_w:
     "⟦∀v ∈ s(u). succ(l) ≈ v ∩ y; a ⊆ y; l ≈ a; u ∈ x⟧
      ⟹ {v ∈ s(u). a ⊆ v} ≲ {v ∈ Pow(x). v ≈ m}"
apply (rule_tac f3 = "λw ∈ {v ∈ s (u) . a ⊆ v}. w ∩ (x - {u})"
       in exI [THEN lepoll_def [THEN def_imp_iff, THEN iffD2]])
apply (simp add: inj_def)
apply (intro conjI lam_type CollectI)
  apply fast
 apply (blast intro: w_Int_eqpoll_m)
apply (intro ballI impI)

apply (rule w_Int_eqpoll_m [THEN eqpoll_m_not_empty, THEN not_emptyE])
apply (blast, assumption+)
apply (drule equalityD1 [THEN subsetD], assumption)
apply (frule cons_cons_in, assumption+)
apply (blast dest: ex1_two_eq intro: s_subset [THEN subsetD] in_s_imp_u_in)+
```

**done**

**lemma** *well_ord_subsets_eqpoll_n:*
"n ∈ nat ⟹ ∃S. well_ord({z ∈ Pow(y) . z ≈ succ(n)}, S)"
**apply** *(rule WO_R [THEN well_ord_infinite_subsets_eqpoll_X,*
*THEN eqpoll_def [THEN def_imp_iff, THEN iffD1], THEN*
*exE])*
**apply** *(fast intro: bij_is_inj [THEN well_ord_rvimage])+*
**done**

**lemma** *well_ord_subsets_lepoll_n:*
"n ∈ nat ⟹ ∃R. well_ord({z ∈ Pow(y). z ≲ n}, R)"
**apply** *(induct_tac "n")*
**apply** *(force intro!: well_ord_unit simp add: subsets_lepoll_0_eq_unit)*
**apply** *(erule exE)*
**apply** *(rule well_ord_subsets_eqpoll_n [THEN exE], assumption)*
**apply** *(simp add: subsets_lepoll_succ)*
**apply** *(drule well_ord_radd, assumption)*
**apply** *(erule Int_empty [THEN disj_Un_eqpoll_sum,*
*THEN eqpoll_def [THEN def_imp_iff, THEN iffD1], THEN*
*exE])*
**apply** *(fast elim!: bij_is_inj [THEN well_ord_rvimage])*
**done**

**lemma** *LL_subset: "LL ⊆ {z ∈ Pow(y). z ≲ succ(k #+ m)}"*
**unfolding** *LL_def MM_def*
**apply** *(insert "includes")*
**apply** *(blast intro: subset_imp_lepoll eqpoll_imp_lepoll lepoll_trans)*
**done**

**lemma** *well_ord_LL: "∃S. well_ord(LL,S)"*
**apply** *(rule well_ord_subsets_lepoll_n [THEN exE, of "succ(k#+m)"])*
**apply** *simp*
**apply** *(blast intro: well_ord_subset [OF _ LL_subset])*
**done**

**lemma** *unique_superset_in_MM:*
"v ∈ LL ⟹ ∃! w. w ∈ MM ∧ v ⊆ w"
**apply** *(unfold MM_def LL_def, safe, fast)*

```
apply (rule lepoll_imp_eqpoll_subset [THEN exE], assumption)
apply (rule_tac x = x in all_ex [THEN ballE])
apply (blast intro: eqpoll_sym)+
done
```

```
lemma Int_in_LL: "w ∈ MM ⟹ w ∩ y ∈ LL"
by (unfold LL_def, fast)

lemma in_LL_eq_Int:
     "v ∈ LL ⟹ v = (THE x. x ∈ MM ∧ v ⊆ x) ∩ y"
apply (unfold LL_def, clarify)
apply (subst unique_superset_in_MM [THEN the_equality2])
apply (auto simp add: Int_in_LL)
done

lemma unique_superset1: "a ∈ LL ⟹ (THE x. x ∈ MM ∧ a ⊆ x) ∈ MM"
by (erule unique_superset_in_MM [THEN theI, THEN conjunct1])

lemma the_in_MM_subset:
     "v ∈ LL ⟹ (THE x. x ∈ MM ∧ v ⊆ x) ⊆ x ∪ y"
apply (drule unique_superset1)
  unfolding MM_def
apply (fast dest!: unique_superset1 "includes" [THEN subsetD])
done

lemma GG_subset: "v ∈ LL ⟹ GG ‘ v ⊆ x"
  unfolding GG_def
apply (frule the_in_MM_subset)
apply (frule in_LL_eq_Int)
apply (force elim: equalityE)
done

lemma nat_lepoll_ordertype: "nat ≲ ordertype(y, R)"
apply (rule nat_le_infinite_Ord [THEN le_imp_lepoll])
 apply (rule Ord_ordertype [OF WO_R])
apply (rule ordertype_eqpoll [THEN eqpoll_imp_lepoll, THEN lepoll_infinite])

 apply (rule WO_R)
apply (rule Infinite)
done
```

```
lemma ex_subset_eqpoll_n: "n ∈ nat ⟹ ∃z. z ⊆ y ∧ n ≈ z"
apply (erule nat_lepoll_imp_ex_eqpoll_n)
apply (rule lepoll_trans [OF nat_lepoll_ordertype])
apply (rule ordertype_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll])

apply (rule WO_R)
done


lemma exists_proper_in_s: "u ∈ x ⟹ ∃v ∈ s(u). succ(k) ≲ v ∩ y"
apply (rule ccontr)
apply (subgoal_tac "∀v ∈ s (u) . k ≈ v ∩ y")
prefer 2 apply (simp add: s_def, blast intro: succ_not_lepoll_imp_eqpoll)
  unfolding k_def
apply (insert all_ex "includes" lnat)
apply (rule ex_subset_eqpoll_n [THEN exE], assumption)
apply (rule noLepoll [THEN notE])
apply (blast intro: lepoll_trans [OF y_lepoll_subset_s subset_s_lepoll_w])
done

lemma exists_in_MM: "u ∈ x ⟹ ∃w ∈ MM. u ∈ w"
apply (erule exists_proper_in_s [THEN bexE])
apply (unfold MM_def s_def, fast)
done

lemma exists_in_LL: "u ∈ x ⟹ ∃w ∈ LL. u ∈ GG`w"
apply (rule exists_in_MM [THEN bexE], assumption)
apply (rule bexI)
apply (erule_tac [2] Int_in_LL)
  unfolding GG_def
apply (simp add: Int_in_LL)
apply (subst unique_superset_in_MM [THEN the_equality2])
apply (fast elim!: Int_in_LL)+
done

lemma OUN_eq_x: "well_ord(LL,S) ⟹
      (⋃b<ordertype(LL,S). GG ` (converse(ordermap(LL,S)) ` b)) = x"
apply (rule equalityI)
apply (rule subsetI)
apply (erule OUN_E)
apply (rule GG_subset [THEN subsetD])
prefer 2 apply assumption
apply (blast intro: ltD  ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun,
                                       THEN apply_type])
apply (rule subsetI)
apply (erule exists_in_LL [THEN bexE])
apply (force intro: ltI [OF _ Ord_ordertype]
                    ordermap_type [THEN apply_type]
```

```
            simp add: ordermap_bij [THEN bij_is_inj, THEN left_inverse])
done
```

```
lemma in_MM_eqpoll_n: "w ∈ MM ⟹ w ≈ succ(k #+ m)"
  unfolding MM_def
apply (fast dest: "includes" [THEN subsetD])
done

lemma in_LL_eqpoll_n: "w ∈ LL ⟹ succ(k) ≲ w"
by (unfold LL_def MM_def, fast)

lemma in_LL: "w ∈ LL ⟹ w ⊆ (THE x. x ∈ MM ∧ w ⊆ x)"
by (erule subset_trans [OF in_LL_eq_Int [THEN equalityD1] Int_lower1])

lemma all_in_lepoll_m:
      "well_ord(LL,S) ⟹
       ∀b<ordertype(LL,S). GG ‘ (converse(ordermap(LL,S)) ‘ b) ≲ m"
  unfolding GG_def
apply (rule oallI)
apply (simp add: ltD ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun,
THEN apply_type])
apply (insert "includes")
apply (rule eqpoll_sum_imp_Diff_lepoll)
apply (blast del: subsetI
             dest!: ltD
             intro!: eqpoll_sum_imp_Diff_lepoll in_LL_eqpoll_n
             intro: in_LL   unique_superset1 [THEN in_MM_eqpoll_n]
                    ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun,

                                  THEN apply_type])+
done

lemma "conclusion":
     "∃a f. Ord(a) ∧ domain(f) = a ∧ (⋃b<a. f ‘ b) = x ∧ (∀b<a. f ‘
b ≲ m)"
apply (rule well_ord_LL [THEN exE])
apply (rename_tac S)
apply (rule_tac x = "ordertype (LL,S)" in exI)
apply (rule_tac x = "λb ∈ ordertype(LL,S).
                     GG ‘ (converse (ordermap (LL,S)) ‘ b)"  in exI)
apply (simp add: ltD)
apply (blast intro: lam_funtype [THEN domain_of_fun]
                    Ord_ordertype  OUN_eq_x  all_in_lepoll_m [THEN ospec])
done
```

**end**

**theorem** *AC16_WO4:*
"⟦*AC_Equiv.AC16(k #+ m, k); 0 < k; 0 < m; k ∈ nat; m ∈ nat*⟧ ⟹ *WO4(m)*"
**unfolding** *AC_Equiv.AC16_def WO4_def*
**apply** *(rule allI)*
**apply** *(case_tac "Finite (A)")*
**apply** *(rule lemma1, assumption+)*
**apply** *(cut_tac lemma2)*
**apply** *(elim exE conjE)*
**apply** *(erule_tac x = "A ∪ y"* **in** *allE)*
**apply** *(frule infinite_Un, drule mp, assumption)*
**apply** *(erule zero_lt_natE, assumption, clarify)*
**apply** *(blast intro: AC16.conclusion [OF AC16.intro])*
**done**

**end**

**theory** *AC17_AC1*
**imports** *HH*
**begin**

**lemma** *AC0_AC1_lemma:* "⟦*f:(∏X ∈ A. X); D ⊆ A*⟧ ⟹ ∃*g. g:(∏X ∈ D. X)*"
**by** *(fast intro!: lam_type apply_type)*

**lemma** *AC0_AC1:* "*AC0* ⟹ *AC1*"
**unfolding** *AC0_def AC1_def*
**apply** *(blast intro: AC0_AC1_lemma)*
**done**

**lemma** *AC1_AC0:* "*AC1* ⟹ *AC0*"
**by** *(unfold AC0_def AC1_def, blast)*

**lemma** *AC1_AC17_lemma:* "*f* ∈ (∏*X* ∈ *Pow(A) - {0}. X)* ⟹ *f* ∈ *(Pow(A) - {0} -> A)*"

```
apply (rule Pi_type, assumption)
apply (drule apply_type, assumption, fast)
done

lemma AC1_AC17: "AC1 ⟹ AC17"
  unfolding AC1_def AC17_def
apply (rule allI)
apply (rule ballI)
apply (erule_tac x = "Pow (A) -{0}" in allE)
apply (erule impE, fast)
apply (erule exE)
apply (rule bexI)
apply (erule_tac [2] AC1_AC17_lemma)
apply (rule apply_type, assumption)
apply (fast dest!: AC1_AC17_lemma elim!: apply_type)
done

lemma UN_eq_imp_well_ord:
     "⟦x - (⋃j ∈ μ i. HH(λX ∈ Pow(x)-{0}. {f`X}, x, i) = {x}.
        HH(λX ∈ Pow(x)-{0}. {f`X}, x, j)) = 0;
        f ∈ Pow(x)-{0} -> x⟧
        ⟹ ∃r. well_ord(x,r)"
apply (rule exI)
apply (erule well_ord_rvimage
        [OF bij_Least_HH_x [THEN bij_converse_bij, THEN bij_is_inj]
            Ord_Least [THEN well_ord_Memrel]], assumption)
done

lemma not_AC1_imp_ex:
     "¬AC1 ⟹ ∃A. ∀f ∈ Pow(A)-{0} -> A. ∃u ∈ Pow(A)-{0}. f`u ∉ u"
  unfolding AC1_def
apply (erule swap)
apply (rule allI)
apply (erule swap)
apply (rule_tac x = "⋃(A)" in exI)
apply (blast intro: lam_type)
done
```

```
lemma AC17_AC1_aux1:
     "⟦∀f ∈ Pow(x) - {0} -> x. ∃u ∈ Pow(x) - {0}. f`u∉u;
        ∃f ∈ Pow(x)-{0}->x.
           x - (⋃a ∈ (μ i. HH(λX ∈ Pow(x)-{0}. {f`X},x,i)={x}).
           HH(λX ∈ Pow(x)-{0}. {f`X},x,a)) = 0⟧
       ⟹ P"
apply (erule bexE)
apply (erule UN_eq_imp_well_ord [THEN exE], assumption)
apply (erule ex_choice_fun_Pow [THEN exE])
apply (erule ballE)
apply (fast intro: apply_type del: DiffE)
apply (erule notE)
apply (rule Pi_type, assumption)
apply (blast dest: apply_type)
done

lemma AC17_AC1_aux2:
      "¬ (∃f ∈ Pow(x)-{0}->x. x - F(f) = 0)
       ⟹ (λf ∈ Pow(x)-{0}->x . x - F(f))
           ∈ (Pow(x) -{0} -> x) -> Pow(x) - {0}"
by (fast intro!: lam_type dest!: Diff_eq_0_iff [THEN iffD1])

lemma AC17_AC1_aux3:
     "⟦f`Z ∈ Z; Z ∈ Pow(x)-{0}⟧
      ⟹ (λX ∈ Pow(x)-{0}. {f`X})`Z ∈ Pow(Z)-{0}"
by auto

lemma AC17_AC1_aux4:
     "∃f ∈ F. f`((λf ∈ F. Q(f))`f) ∈ (λf ∈ F. Q(f))`f
      ⟹ ∃f ∈ F. f`Q(f) ∈ Q(f)"
by simp

lemma AC17_AC1: "AC17 ⟹ AC1"
  unfolding AC17_def
apply (rule classical)
apply (erule not_AC1_imp_ex [THEN exE])
apply (case_tac
       "∃f ∈ Pow(x)-{0} -> x.
        x - (⋃a ∈ (μ i. HH (λX ∈ Pow (x) -{0}. {f`X},x,i) ={x}) . HH
(λX ∈ Pow (x) -{0}. {f`X},x,a)) = 0")
apply (erule AC17_AC1_aux1, assumption)
apply (drule AC17_AC1_aux2)
apply (erule allE)
apply (drule bspec, assumption)
apply (drule AC17_AC1_aux4)
apply (erule bexE)
apply (drule apply_type, assumption)
apply (simp add: HH_Least_eq_x del: Diff_iff )
apply (drule AC17_AC1_aux3, assumption)
```

**apply** *(fast dest!: subst_elem [OF _ HH_Least_eq_x [symmetric]]*
*f_subset_imp_HH_subset elim!: mem_irrefl)*
**done**

**lemma** *AC1_AC2_aux1:*
*"⟦f:(∏ X ∈ A. X); B ∈ A; 0∉A⟧ ⟹ {f‘B} ⊆ B ∩ {f‘C. C ∈ A}"*
**by** *(fast elim!: apply_type)*

**lemma** *AC1_AC2_aux2:*
*"⟦pairwise_disjoint(A); B ∈ A; C ∈ A; D ∈ B; D ∈ C⟧ ⟹ f‘B = f‘C"*
**by** *(unfold pairwise_disjoint_def, fast)*

**lemma** *AC1_AC2: "AC1 ⟹ AC2"*
**unfolding** *AC1_def AC2_def*
**apply** *(rule allI)*
**apply** *(rule impI)*
**apply** *(elim asm_rl conjE allE exE impE, assumption)*
**apply** *(intro exI ballI equalityI)*
**prefer** *2* **apply** *(rule AC1_AC2_aux1, assumption+)*
**apply** *(fast elim!: AC1_AC2_aux2 elim: apply_type)*
**done**

**lemma** *AC2_AC1_aux1: "0∉A ⟹ 0 ∉ {B*{B}. B ∈ A}"*
**by** *(fast dest!: sym [THEN Sigma_empty_iff [THEN iffD1]])*

**lemma** *AC2_AC1_aux2: "⟦X*{X} ∩ C = {y}; X ∈ A⟧*
*⟹ (THE y. X*{X} ∩ C = {y}): X*A"*
**apply** *(rule subst_elem [of y])*
**apply** *(blast elim!: equalityE)*
**apply** *(auto simp add: singleton_eq_iff)*
**done**

**lemma** *AC2_AC1_aux3:*
*"∀D ∈ {E*{E}. E ∈ A}. ∃y. D ∩ C = {y}*
*⟹ (λx ∈ A. fst(THE z. (x*{x} ∩ C = {z}))) ∈ (∏ X ∈ A. X)"*
**apply** *(rule lam_type)*

```
apply (drule bspec, blast)
apply (blast intro: AC2_AC1_aux2 fst_type)
done

lemma AC2_AC1: "AC2 ⟹ AC1"
  unfolding AC1_def AC2_def pairwise_disjoint_def
apply (intro allI impI)
apply (elim allE impE)
prefer 2 apply (fast elim!: AC2_AC1_aux3)
apply (blast intro!: AC2_AC1_aux1)
done

lemma empty_notin_images: "0 ∉ {R``{x}. x ∈ domain(R)}"
by blast

lemma AC1_AC4: "AC1 ⟹ AC4"
  unfolding AC1_def AC4_def
apply (intro allI impI)
apply (drule spec, drule mp [OF _ empty_notin_images])
apply (best intro!: lam_type elim!: apply_type)
done

lemma AC4_AC3_aux1: "f ∈ A->B ⟹ (⋃z ∈ A. {z}*f`z) ⊆ A*⋃(B)"
by (fast dest!: apply_type)

lemma AC4_AC3_aux2: "domain(⋃z ∈ A. {z}*f(z)) = {a ∈ A. f(a)≠0}"
by blast

lemma AC4_AC3_aux3: "x ∈ A ⟹ (⋃z ∈ A. {z}*f(z))``{x} = f(x)"
by fast

lemma AC4_AC3: "AC4 ⟹ AC3"
  unfolding AC3_def AC4_def
apply (intro allI ballI)
apply (elim allE impE)
apply (erule AC4_AC3_aux1)
apply (simp add: AC4_AC3_aux2 AC4_AC3_aux3 cong add: Pi_cong)
done
```

```
lemma AC3_AC1_lemma:
     "b∉A ⟹ (∏ x ∈ {a ∈ A. id(A)`a≠b}. id(A)`x) = (∏ x ∈ A. x)"
apply (simp add: id_def cong add: Pi_cong)
apply (rule_tac b = A in subst_context, fast)
done

lemma AC3_AC1: "AC3 ⟹ AC1"
  unfolding AC1_def AC3_def
apply (fast intro!: id_type elim: AC3_AC1_lemma [THEN subst])
done

lemma AC4_AC5: "AC4 ⟹ AC5"
  unfolding range_def AC4_def AC5_def
apply (intro allI ballI)
apply (elim allE impE)
apply (erule fun_is_rel [THEN converse_type])
apply (erule exE)
apply (rename_tac g)
apply (rule_tac x=g in bexI)
apply (blast dest: apply_equality range_type)
apply (blast intro: Pi_type dest: apply_type fun_is_rel)
done

lemma AC5_AC4_aux1: "R ⊆ A*B ⟹ (λx ∈ R. fst(x)) ∈ R -> A"
by (fast intro!: lam_type fst_type)

lemma AC5_AC4_aux2: "R ⊆ A*B ⟹ range(λx ∈ R. fst(x)) = domain(R)"
by (unfold lam_def, force)

lemma AC5_AC4_aux3: "⟦∃f ∈ A->C. P(f,domain(f)); A=B⟧ ⟹  ∃f ∈ B->C.
P(f,B)"
apply (erule bexE)
apply (frule domain_of_fun, fast)
done

lemma AC5_AC4_aux4: "⟦R ⊆ A*B; g ∈ C->R; ∀x ∈ C. (λz ∈ R. fst(z))`
```

```
(g`x) = x⟧
                ⟹ (λx ∈ C. snd(g`x)): (∏x ∈ C. R``{x})"
apply (rule lam_type)
apply (force dest: apply_type)
done

lemma AC5_AC4: "AC5 ⟹ AC4"
apply (unfold AC4_def AC5_def, clarify)
apply (elim allE ballE)
apply (drule AC5_AC4_aux3 [OF _ AC5_AC4_aux2], assumption)
apply (fast elim!: AC5_AC4_aux4)
apply (blast intro: AC5_AC4_aux1)
done

lemma AC1_iff_AC6: "AC1 ⟷ AC6"
by (unfold AC1_def AC6_def, blast)

end


theory AC18_AC19
imports AC_Equiv
begin

definition
  uu    :: "i ⇒ i" where
    "uu(a) ≡ {c ∪ {0}. c ∈ a}"

lemma PROD_subsets:
     "⟦f ∈ (∏b ∈ {P(a). a ∈ A}. b);  ∀a ∈ A. P(a)<=Q(a)⟧
      ⟹ (λa ∈ A. f`P(a)) ∈ (∏a ∈ A. Q(a))"
by (rule lam_type, drule apply_type, auto)

lemma lemma_AC18:
     "⟦∀A. 0 ∉ A ⟶ (∃f. f ∈ (∏X ∈ A. X)); A ≠ 0⟧
      ⟹ (⋂a ∈ A. ⋃b ∈ B(a). X(a, b)) ⊆
          (⋃f ∈ ∏a ∈ A. B(a). ⋂a ∈ A. X(a, f`a))"
apply (rule subsetI)
apply (erule_tac x = "{{b ∈ B (a) . x ∈ X (a,b) }. a ∈ A}" in allE)
```

```
apply (erule impE, fast)
apply (erule exE)
apply (rule UN_I)
 apply (fast elim!: PROD_subsets)
apply (simp, fast elim!: not_emptyE dest: apply_type [OF _ RepFunI])
done

lemma AC1_AC18: "AC1 ⟹ PROP AC18"
  unfolding AC1_def
apply (rule AC18.intro)
apply (fast elim!: lemma_AC18 apply_type intro!: equalityI INT_I UN_I)
done




theorem (in AC18) AC19
  unfolding AC19_def
apply (intro allI impI)
apply (rule AC18 [of _ "λx. x", THEN mp], blast)
done




lemma RepRep_conj:
        "⟦A ≠ 0; 0 ∉ A⟧ ⟹ {uu(a). a ∈ A} ≠ 0 ∧ 0 ∉ {uu(a). a ∈ A}"
apply (unfold uu_def, auto)
apply (blast dest!: sym [THEN RepFun_eq_0_iff [THEN iffD1]])
done

lemma lemma1_1: "⟦c ∈ a; x = c ∪ {0}; x ∉ a⟧ ⟹ x - {0} ∈ a"
apply clarify
apply (rule subst_elem, assumption)
apply (fast elim: notE subst_elem)
done

lemma lemma1_2:
        "⟦f`(uu(a)) ∉ a; f ∈ (∏ B ∈ {uu(a). a ∈ A}. B); a ∈ A⟧
                ⟹ f`(uu(a))-{0} ∈ a"
apply (unfold uu_def, fast elim!: lemma1_1 dest!: apply_type)
done

lemma lemma1: "∃f. f ∈ (∏ B ∈ {uu(a). a ∈ A}. B) ⟹ ∃f. f ∈ (∏ B
∈ A. B)"
apply (erule exE)
apply (rule_tac x = "λa∈A. if (f` (uu(a)) ∈ a, f` (uu(a)), f` (uu(a))-{0})"
```

```
      in exI)
apply (rule lam_type)
apply (simp add: lemma1_2)
done

lemma lemma2_1: "a≠0 ⟹ 0 ∈ (⋃b ∈ uu(a). b)"
by (unfold uu_def, auto)

lemma lemma2: "⟦A≠0; 0∉A⟧ ⟹ (⋂x ∈ {uu(a). a ∈ A}. ⋃b ∈ x. b) ≠
0"
apply (erule not_emptyE)
apply (rule_tac a = 0 in not_emptyI)
apply (fast intro!: lemma2_1)
done

lemma AC19_AC1: "AC19 ⟹ AC1"
apply (unfold AC19_def AC1_def, clarify)
apply (case_tac "A=0", force)
apply (erule_tac x = "{uu (a) . a ∈ A}" in allE)
apply (erule impE)
apply (erule RepRep_conj, assumption)
apply (rule lemma1)
apply (drule lemma2, assumption, auto)
done

end


theory DC
imports AC_Equiv Hartog Cardinal_aux
begin

lemma RepFun_lepoll: "Ord(a) ⟹ {P(b). b ∈ a} ≲ a"
  unfolding lepoll_def
apply (rule_tac x = "λz ∈ RepFun (a,P) . μ i. z=P (i) " in exI)
apply (rule_tac d="λz. P (z)" in lam_injective)
 apply (fast intro!: Least_in_Ord)
apply (rule sym)
apply (fast intro: LeastI Ord_in_Ord)
done
```

Trivial in the presence of AC, but here we need a wellordering of X

```
lemma image_Ord_lepoll: "⟦f ∈ X->Y; Ord(X)⟧ ⟹ f``X ≲ X"
  unfolding lepoll_def
apply (rule_tac x = "λx ∈ f``X. μ y. f`y = x" in exI)
apply (rule_tac d = "λz. f`z" in lam_injective)
apply (fast intro!: Least_in_Ord apply_equality, clarify)
apply (rule LeastI)
```

```
 apply (erule apply_equality, assumption+)
apply (blast intro: Ord_in_Ord)
done

lemma range_subset_domain:
      "⟦R ⊆ X*X;   ⋀g. g ∈ X ⟹ ∃u. ⟨g,u⟩ ∈ R⟧
       ⟹ range(R) ⊆ domain(R)"
by blast

lemma cons_fun_type: "g ∈ n->X ⟹ cons(⟨n,x⟩, g) ∈ succ(n) -> cons(x,
X)"
  unfolding succ_def
apply (fast intro!: fun_extend elim!: mem_irrefl)
done

lemma cons_fun_type2:
     "⟦g ∈ n->X; x ∈ X⟧ ⟹ cons(⟨n,x⟩, g) ∈ succ(n) -> X"
by (erule cons_absorb [THEN subst], erule cons_fun_type)

lemma cons_image_n: "n ∈ nat ⟹ cons(⟨n,x⟩, g)``n = g``n"
by (fast elim!: mem_irrefl)

lemma cons_val_n: "g ∈ n->X ⟹ cons(⟨n,x⟩, g)`n = x"
by (fast intro!: apply_equality elim!: cons_fun_type)

lemma cons_image_k: "k ∈ n ⟹ cons(⟨n,x⟩, g)``k = g``k"
by (fast elim: mem_asym)

lemma cons_val_k: "⟦k ∈ n; g ∈ n->X⟧ ⟹ cons(⟨n,x⟩, g)`k = g`k"
by (fast intro!: apply_equality consI2 elim!: cons_fun_type apply_Pair)

lemma domain_cons_eq_succ: "domain(f)=x ⟹ domain(cons(⟨x,y⟩, f)) =
succ(x)"
by (simp add: domain_cons succ_def)

lemma restrict_cons_eq: "g ∈ n->X ⟹ restrict(cons(⟨n,x⟩, g), n) =
g"
apply (simp add: restrict_def Pi_iff)
apply (blast intro: elim: mem_irrefl)
done

lemma succ_in_succ: "⟦Ord(k); i ∈ k⟧ ⟹ succ(i) ∈ succ(k)"
apply (rule Ord_linear [of "succ(i)" "succ(k)", THEN disjE])
apply (fast elim: Ord_in_Ord mem_irrefl mem_asym)+
done

lemma restrict_eq_imp_val_eq:
      "⟦restrict(f, domain(g)) = g; x ∈ domain(g)⟧
       ⟹ f`x = g`x"
```

**by** *(erule subst, simp add: restrict)*

**lemma** *domain_eq_imp_fun_type:* "⟦*domain(f)=A; f ∈ B->C*⟧ ⟹ *f ∈ A->C*"
**by** *(frule domain_of_fun, fast)*

**lemma** *ex_in_domain:* "⟦*R ⊆ A * B; R ≠ 0*⟧ ⟹ ∃*x. x ∈ domain(R)*"
**by** *(fast elim!: not_emptyE)*

**definition**
  *DC :: "i ⇒ o"* **where**
    "DC(a) ≡ ∀*X R. R ⊆ Pow(X)*X ∧
                  (∀Y ∈ Pow(X). Y ≺ a ⟶ (∃x ∈ X. ⟨Y,x⟩ ∈ R))
                  ⟶ (∃f ∈ a->X. ∀b<a. <f‘‘b,f‘b> ∈ R)"

**definition**
  *DC0 :: o* **where**
    "DC0 ≡ ∀*A B R. R ⊆ A*B ∧ R≠0 ∧ range(R) ⊆ domain(R)*
                  ⟶ (∃*f ∈ nat->domain(R). ∀n ∈ nat. <f‘n,f‘succ(n)>:R)"

**definition**
  *ff :: "[i, i, i, i] ⇒ i"* **where**
    "*ff(b, X, Q, R) ≡*
          *transrec(b, λc r. THE x. first(x, {x ∈ X. <r‘‘c, x> ∈ R},*
*Q))"*

**locale** *DC0_imp* =
  **fixes** *XX* **and** *RR* **and** *X* **and** *R*

  **assumes** *all_ex:* "∀*Y ∈ Pow(X). Y ≺ nat ⟶ (∃x ∈ X. ⟨Y, x⟩ ∈ R)*"

  **defines** *XX_def:* "*XX ≡ (⋃n ∈ nat. {f ∈ n->X. ∀k ∈ n. <f‘‘k, f‘k> ∈*
*R})*"
    **and** *RR_def:* "*RR ≡ {⟨z1,z2⟩:XX*XX. domain(z2)=succ(domain(z1))*
                                  *∧ restrict(z2, domain(z1)) = z1}*"
**begin**

```
lemma lemma1_1: "RR ⊆ XX*XX"
by (unfold RR_def, fast)

lemma lemma1_2: "RR ≠ 0"
  unfolding RR_def XX_def
apply (rule all_ex [THEN ballE])
apply (erule_tac [2] notE [OF _ empty_subsetI [THEN PowI]])
apply (erule_tac impE [OF _ nat_0I [THEN n_lesspoll_nat]])
apply (erule bexE)
apply (rule_tac a = "<0, {⟨0, x⟩}>" in not_emptyI)
apply (rule CollectI)
apply (rule SigmaI)
apply (rule nat_0I [THEN UN_I])
apply (simp (no_asm_simp) add: nat_0I [THEN UN_I])
apply (rule nat_1I [THEN UN_I])
apply (force intro!: singleton_fun [THEN Pi_type]
             simp add: singleton_0 [symmetric])
apply (simp add: singleton_0)
done

lemma lemma1_3: "range(RR) ⊆ domain(RR)"
  unfolding RR_def XX_def
apply (rule range_subset_domain, blast, clarify)
apply (frule fun_is_rel [THEN image_subset, THEN PowI,
                         THEN all_ex [THEN bspec]])
apply (erule impE[OF _ lesspoll_trans1[OF image_Ord_lepoll
                                          [OF _ nat_into_Ord] n_lesspoll_nat]],
       assumption+)
apply (erule bexE)
apply (rule_tac x = "cons (⟨n,x⟩, g) " in exI)
apply (rule CollectI)
apply (force elim!: cons_fun_type2
             simp add: cons_image_n cons_val_n cons_image_k cons_val_k)
apply (simp add: domain_of_fun succ_def restrict_cons_eq)
```

**done**

**lemma** *lemma2:*
```
     "⟦∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  n ∈ nat⟧
      ⟹ ∃k ∈ nat. f‘succ(n) ∈ k -> X ∧ n ∈ k
                  ∧ <f‘succ(n)‘‘n, f‘succ(n)‘n> ∈ R"
```
**apply** *(induct_tac "n")*
**apply** *(drule apply_type [OF _ nat_1I])*
**apply** *(drule bspec [OF _ nat_0I])*
**apply** *(simp add: XX_def, safe)*
**apply** *(rule rev_bexI, assumption)*
**apply** *(subgoal_tac "0 ∈ y", force)*
**apply** *(force simp add: RR_def*
*intro: ltD elim!: nat_0_le [THEN leE])*

**apply** *(drule bspec [OF _ nat_succI], assumption)*
**apply** *(subgoal_tac "f ‘ succ (succ (x)) ∈ succ (k) ->X")*
**apply** *(drule apply_type [OF _ nat_succI [THEN nat_succI]], assumption)*
**apply** *(simp (no_asm_use) add: XX_def RR_def)*
**apply** *safe*
**apply** *(frule_tac a="succ(k)"* **in** *domain_of_fun [symmetric, THEN trans],*

*assumption)*
**apply** *(frule_tac a=y* **in** *domain_of_fun [symmetric, THEN trans],*
*assumption)*
**apply** *(fast elim!: nat_into_Ord [THEN succ_in_succ]*
*dest!: bspec [OF _ nat_into_Ord [THEN succ_in_succ]])*
**apply** *(drule domain_of_fun)*
**apply** *(simp add: XX_def RR_def, clarify)*
**apply** *(blast dest: domain_of_fun [symmetric, THEN trans] )*
**done**

**lemma** *lemma3_1:*
```
     "⟦∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  m ∈ nat⟧
      ⟹  {f‘succ(x)‘x. x ∈ m} = {f‘succ(m)‘x. x ∈ m}"
```
**apply** *(subgoal_tac "∀x ∈ m. f‘succ (m) ‘x = f‘succ (x) ‘x")*
**apply** *simp*
**apply** *(induct_tac "m", blast)*
**apply** *(rule ballI)*
**apply** *(erule succE)*
**apply** *(rule restrict_eq_imp_val_eq)*
**apply** *(drule bspec [OF _ nat_succI], assumption)*
**apply** *(simp add: RR_def)*
**apply** *(drule lemma2, assumption+)*
**apply** *(fast dest!: domain_of_fun)*
**apply** *(drule_tac x = xa* **in** *bspec, assumption)*
**apply** *(erule sym [THEN trans, symmetric])*
**apply** *(rule restrict_eq_imp_val_eq [symmetric])*
**apply** *(drule bspec [OF _ nat_succI], assumption)*

```
 apply (simp add: RR_def)
apply (drule lemma2, assumption+)
apply (blast dest!: domain_of_fun
             intro: nat_into_Ord OrdmemD [THEN subsetD])
done

lemma lemma3:
     "⟦∀n ∈ nat. <f`n, f`succ(n)> ∈ RR;  f ∈ nat -> XX;  m ∈ nat⟧
      ⟹ (λx ∈ nat. f`succ(x)`x) `` m = f`succ(m)``m"
apply (erule natE, simp)
apply (subst image_lam)
 apply (fast elim!: OrdmemD [OF nat_succI Ord_nat])
apply (subst lemma3_1, assumption+)
 apply fast
apply (fast dest!: lemma2
            elim!: image_fun [symmetric, OF _ OrdmemD [OF _ nat_into_Ord]])
done

end

theorem DC0_imp_DC_nat: "DC0 ⟹ DC(nat)"
apply (unfold DC_def DC0_def, clarify)
apply (elim allE)
apply (erule impE)

apply (blast intro!: DC0_imp.lemma1_1 [OF DC0_imp.intro] DC0_imp.lemma1_2
[OF DC0_imp.intro] DC0_imp.lemma1_3 [OF DC0_imp.intro])
apply (erule bexE)
apply (rule_tac x = "λn ∈ nat. f`succ (n) `n" in rev_bexI)
 apply (rule lam_type, blast dest!: DC0_imp.lemma2 [OF DC0_imp.intro]
intro: fun_weaken_type)
apply (rule oallI)
apply (frule DC0_imp.lemma2 [OF DC0_imp.intro], assumption)
  apply (blast intro: fun_weaken_type)
 apply (erule ltD)

apply (subst DC0_imp.lemma3 [OF DC0_imp.intro], assumption+)
  apply (fast elim!: fun_weaken_type)
 apply (erule ltD)
apply (force simp add: lt_def)
done




lemma singleton_in_funs:
 "x ∈ X ⟹ {⟨0,x⟩} ∈
           (⋃n ∈ nat. {f ∈ succ(n)->X. ∀k ∈ n. <f`k, f`succ(k)> ∈
R})"
```

```
apply (rule nat_0I [THEN UN_I])
apply (force simp add: singleton_0 [symmetric]
            intro!: singleton_fun [THEN Pi_type])
done


locale imp_DC0 =
  fixes XX and RR and x and R and f and allRR
  defines XX_def: "XX ≡ (⋃n ∈ nat.
                     {f ∈ succ(n)->domain(R). ∀k ∈ n. <f‘k, f‘succ(k)>
∈ R})"
      and RR_def:
         "RR ≡ {⟨z1,z2⟩:Fin(XX)*XX.
                 (domain(z2)=succ(⋃f ∈ z1. domain(f))
                   ∧ (∀f ∈ z1. restrict(z2, domain(f)) = f))
                 | (¬ (∃g ∈ XX. domain(g)=succ(⋃f ∈ z1. domain(f))

                    ∧ (∀f ∈ z1. restrict(g, domain(f)) = f)) ∧ z2={⟨0,x⟩})}"
      and allRR_def:
        "allRR ≡ ∀b<nat.
                  <f‘‘b, f‘b> ∈
                   {⟨z1,z2⟩∈Fin(XX)*XX. (domain(z2)=succ(⋃f ∈ z1. domain(f))
                                 ∧ (⋃f ∈ z1. domain(f)) = b
                                 ∧ (∀f ∈ z1. restrict(z2,domain(f))
= f))}"
begin

lemma lemma4:
     "⟦range(R) ⊆ domain(R);  x ∈ domain(R)⟧
      ⟹ RR ⊆ Pow(XX)*XX ∧
            (∀Y ∈ Pow(XX). Y ≺ nat ⟶ (∃x ∈ XX. ⟨Y,x⟩:RR))"
apply (rule conjI)
apply (force dest!: FinD [THEN PowI] simp add: RR_def)
apply (rule impI [THEN ballI])
apply (drule Finite_Fin [OF lesspoll_nat_is_Finite PowD], assumption)
apply (case_tac
       "∃g ∈ XX. domain (g) =
            succ(⋃f ∈ Y. domain(f)) ∧ (∀f∈Y. restrict(g, domain(f))
= f)")
apply (simp add: RR_def, blast)
apply (safe del: domainE)
  unfolding XX_def RR_def
apply (rule rev_bexI, erule singleton_in_funs)
apply (simp add: nat_0I [THEN rev_bexI] cons_fun_type2)
done

lemma UN_image_succ_eq:
     "⟦f ∈ nat->X; n ∈ nat⟧
      ⟹ (⋃x ∈ f‘‘succ(n). P(x)) =  P(f‘n) ∪ (⋃x ∈ f‘‘n. P(x))"
```

```
by (simp add: image_fun OrdmemD)

lemma UN_image_succ_eq_succ:
     "⟦(⋃x ∈ f``n. P(x)) = y; P(f`n) = succ(y);
        f ∈ nat -> X; n ∈ nat⟧ ⟹ (⋃x ∈ f``succ(n). P(x)) = succ(y)"
by (simp add: UN_image_succ_eq, blast)

lemma apply_domain_type:
     "⟦h ∈ succ(n) -> D;  n ∈ nat; domain(h)=succ(y)⟧ ⟹ h`y ∈ D"
by (fast elim: apply_type dest!: trans [OF sym domain_of_fun])

lemma image_fun_succ:
     "⟦h ∈ nat -> X; n ∈ nat⟧ ⟹ h``succ(n) = cons(h`n, h``n)"
by (simp add: image_fun OrdmemD)

lemma f_n_type:
     "⟦domain(f`n) = succ(k); f ∈ nat -> XX;  n ∈ nat⟧
      ⟹ f`n ∈ succ(k) -> domain(R)"
  unfolding XX_def
apply (drule apply_type, assumption)
apply (fast elim: domain_eq_imp_fun_type)
done

lemma f_n_pairs_in_R [rule_format]:
     "⟦h ∈ nat -> XX;  domain(h`n) = succ(k);  n ∈ nat⟧
      ⟹ ∀i ∈ k. <h`n`i, h`n`succ(i)> ∈ R"
  unfolding XX_def
apply (drule apply_type, assumption)
apply (elim UN_E CollectE)
apply (drule domain_of_fun [symmetric, THEN trans], assumption, simp)
done

lemma restrict_cons_eq_restrict:
     "⟦restrict(h, domain(u))=u;  h ∈ n->X;  domain(u) ⊆ n⟧
      ⟹ restrict(cons(⟨n, y⟩, h), domain(u)) = u"
  unfolding restrict_def
apply (simp add: restrict_def Pi_iff)
apply (erule sym [THEN trans, symmetric])
apply (blast elim: mem_irrefl)
done

lemma all_in_image_restrict_eq:
     "⟦∀x ∈ f``n. restrict(f`n, domain(x))=x;
        f ∈ nat -> XX;
        n ∈ nat;  domain(f`n) = succ(n);
        (⋃x ∈ f``n. domain(x)) ⊆ n⟧
      ⟹ ∀x ∈ f``succ(n). restrict(cons(<succ(n),y>, f`n), domain(x))
= x"
apply (rule ballI)
```

```
apply (simp add: image_fun_succ)
apply (drule f_n_type, assumption+)
apply (erule disjE)
 apply (simp add: domain_of_fun restrict_cons_eq)
apply (blast intro!: restrict_cons_eq_restrict)
done

lemma simplify_recursion:
     "⟦∀ b<nat. <f‘‘b, f‘b> ∈ RR;
        f ∈ nat -> XX; range(R) ⊆ domain(R); x ∈ domain(R)⟧
      ⟹ allRR"
  unfolding RR_def allRR_def
apply (rule oallI, drule ltD)
apply (erule nat_induct)
apply (drule_tac x=0 in ospec, blast intro: Limit_has_0)
apply (force simp add: singleton_fun [THEN domain_of_fun] singleton_in_funs)



apply (simp only: separation split)
apply (drule_tac x="succ(xa)" in ospec, blast intro: ltI)
apply (elim conjE disjE)
apply (force elim!: trans subst_context
            intro!: UN_image_succ_eq_succ)
apply (erule notE)
apply (simp add: XX_def UN_image_succ_eq_succ)
apply (elim conjE bexE)
apply (drule apply_domain_type, assumption+)
apply (erule domainE)+
apply (frule f_n_type)
apply (simp add: XX_def, assumption+)
apply (rule rev_bexI, erule nat_succI)
apply (rename_tac m i j y z)
apply (rule_tac x = "cons(<succ(m), z>, f‘m)" in bexI)
prefer 2 apply (blast intro: cons_fun_type2)
apply (rule conjI)
prefer 2 apply (fast del: ballI subsetI
                elim: trans [OF _ subst_context, THEN domain_cons_eq_succ]
                      subst_context
                      all_in_image_restrict_eq [simplified XX_def]
                      trans equalityD1)

apply (rule ballI)
apply (erule succE)

 apply (simp add: cons_val_n cons_val_k)

apply (drule f_n_pairs_in_R [simplified XX_def, OF _ domain_of_fun],
       assumption, assumption, assumption)
```

```
apply (simp add: nat_into_Ord [THEN succ_in_succ] succI2 cons_val_k)
done


lemma lemma2:
     "⟦allRR; f ∈ nat->XX; range(R) ⊆ domain(R); x ∈ domain(R); n ∈
nat⟧
      ⟹ f`n ∈ succ(n) -> domain(R) ∧ (∀i ∈ n. <f`n`i, f`n`succ(i)>:R)"
  unfolding allRR_def
apply (drule ospec)
apply (erule ltI [OF _ Ord_nat])
apply (erule CollectE, simp)
apply (rule conjI)
prefer 2 apply (fast elim!: f_n_pairs_in_R trans subst_context)
  unfolding XX_def
apply (fast elim!: trans [THEN domain_eq_imp_fun_type] subst_context)
done

lemma lemma3:
     "⟦allRR; f ∈ nat->XX; n∈nat; range(R) ⊆ domain(R);  x ∈ domain(R)⟧
      ⟹ f`n`n = f`succ(n)`n"
apply (frule lemma2 [THEN conjunct1, THEN domain_of_fun], assumption+)
  unfolding allRR_def
apply (drule ospec)
apply (drule ltI [OF nat_succI Ord_nat], assumption, simp)
apply (elim conjE ballE)
apply (erule restrict_eq_imp_val_eq [symmetric], force)
apply (simp add: image_fun OrdmemD)
done

end


theorem DC_nat_imp_DC0: "DC(nat) ⟹ DC0"
  unfolding DC_def DC0_def
apply (intro allI impI)
apply (erule asm_rl conjE ex_in_domain [THEN exE] allE)+
apply (erule impE [OF _ imp_DC0.lemma4], assumption+)
apply (erule bexE)
apply (drule imp_DC0.simplify_recursion, assumption+)
apply (rule_tac x = "λn ∈ nat. f`n`n" in bexI)
apply (rule_tac [2] lam_type)
apply (erule_tac [2] apply_type [OF imp_DC0.lemma2 [THEN conjunct1] succI1])
apply (rule ballI)
apply (frule_tac n="succ(n)" in imp_DC0.lemma2,
       (assumption|erule nat_succI)+)
apply (drule imp_DC0.lemma3, auto)
done
```

```
lemma fun_Ord_inj:
      "⟦f ∈ a->X;  Ord(a);
         ⋀b c. ⟦b<c; c ∈ a⟧ ⟹ f‘b≠f‘c⟧
       ⟹ f ∈ inj(a, X)"
apply (unfold inj_def, simp)
apply (intro ballI impI)
apply (rule_tac j=x in Ord_in_Ord [THEN Ord_linear_lt], assumption+)
apply (blast intro: Ord_in_Ord, auto)
apply (atomize, blast dest: not_sym)
done

lemma value_in_image: "⟦f ∈ X->Y; A ⊆ X; a ∈ A⟧ ⟹ f‘a ∈ f‘‘A"
by (fast elim!: image_fun [THEN ssubst])

lemma lesspoll_lemma: "⟦¬ A ≺ B; C ≺ B⟧ ⟹ A - C ≠ 0"
  unfolding lesspoll_def
apply (fast dest!: Diff_eq_0_iff [THEN iffD1, THEN subset_imp_lepoll]
            intro!: eqpollI elim: notE
            elim!: eqpollE lepoll_trans)
done

theorem DC_WO3: "(∀K. Card(K) ⟶ DC(K)) ⟹ WO3"
  unfolding DC_def WO3_def
apply (rule allI)
apply (case_tac "A ≺ Hartog (A)")
apply (fast dest!: lesspoll_imp_ex_lt_eqpoll
            intro!: Ord_Hartog leI [THEN le_imp_subset])
apply (erule allE impE)+
apply (rule Card_Hartog)
apply (erule_tac x = A in allE)
apply (erule_tac x = "{⟨z1,z2⟩ ∈ Pow (A) *A . z1 ≺ Hartog (A) ∧ z2 ∉
z1}"
                in allE)
apply simp
apply (erule impE, fast elim: lesspoll_lemma [THEN not_emptyE])
apply (erule bexE)
apply (rule Hartog_lepoll_selfE)
apply (rule lepoll_def [THEN def_imp_iff, THEN iffD2])
apply (rule exI, rule fun_Ord_inj, assumption, rule Ord_Hartog)
apply (drule value_in_image)
apply (drule OrdmemD, rule Ord_Hartog, assumption+, erule ltD)
apply (drule ospec)
apply (blast intro: ltI Ord_Hartog, force)
done
```

```
lemma images_eq:
     "⟦∀x ∈ A. f`x=g`x; f ∈ Df->Cf; g ∈ Dg->Cg; A ⊆ Df; A ⊆ Dg⟧
      ⟹ f``A = g``A"
apply (simp (no_asm_simp) add: image_fun)
done

lemma lam_images_eq:
     "⟦Ord(a); b ∈ a⟧ ⟹ (λx ∈ a. h(x))``b = (λx ∈ b. h(x))``b"
apply (rule images_eq)
    apply (rule ballI)
    apply (drule OrdmemD [THEN subsetD], assumption+)
    apply simp
   apply (fast elim!: RepFunI OrdmemD intro!: lam_type)+
done

lemma lam_type_RepFun: "(λb ∈ a. h(b)) ∈ a -> {h(b). b ∈ a}"
by (fast intro!: lam_type RepFunI)

lemma lemmaX:
     "⟦∀Y ∈ Pow(X). Y ≺ K ⟶ (∃x ∈ X. ⟨Y, x⟩ ∈ R);
        b ∈ K; Z ∈ Pow(X); Z ≺ K⟧
      ⟹ {x ∈ X. ⟨Z,x⟩ ∈ R} ≠ 0"
by blast


lemma WO1_DC_lemma:
     "⟦Card(K); well_ord(X,Q);
        ∀Y ∈ Pow(X). Y ≺ K ⟶ (∃x ∈ X. ⟨Y, x⟩ ∈ R); b ∈ K⟧
      ⟹ ff(b, X, Q, R) ∈ {x ∈ X. <(λc ∈ b. ff(c, X, Q, R))``b, x> ∈
R}"
apply (rule_tac P = "b ∈ K" in impE, (erule_tac [2] asm_rl)+)
apply (rule_tac i=b in Card_is_Ord [THEN Ord_in_Ord, THEN trans_induct],

       assumption+)
apply (rule impI)
apply (rule ff_def [THEN def_transrec, THEN ssubst])
apply (erule the_first_in, fast)
apply (simp add: image_fun [OF lam_type_RepFun subset_refl])
apply (erule lemmaX, assumption)
 apply (blast intro: Card_is_Ord OrdmemD [THEN subsetD])
apply (blast intro: lesspoll_trans1 in_Card_imp_lesspoll RepFun_lepoll)
done

theorem WO1_DC_Card: "WO1 ⟹ ∀K. Card(K) ⟶ DC(K)"
  unfolding DC_def WO1_def
```

```
apply (rule allI impI)+
apply (erule allE exE conjE)+
apply (rule_tac x = "λb ∈ K. ff (b, X, Ra, R) " in bexI)
 apply (simp add: lam_images_eq [OF Card_is_Ord ltD])
 apply (fast elim!: ltE WO1_DC_lemma [THEN CollectD2])
apply (rule_tac lam_type)
apply (rule WO1_DC_lemma [THEN CollectD1], assumption+)
done

end
```

# References


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[2] Herman Rubin and Jean E. Rubin. *Equivalents of the Axiom of Choice, II.* North-Holland, 1985.