

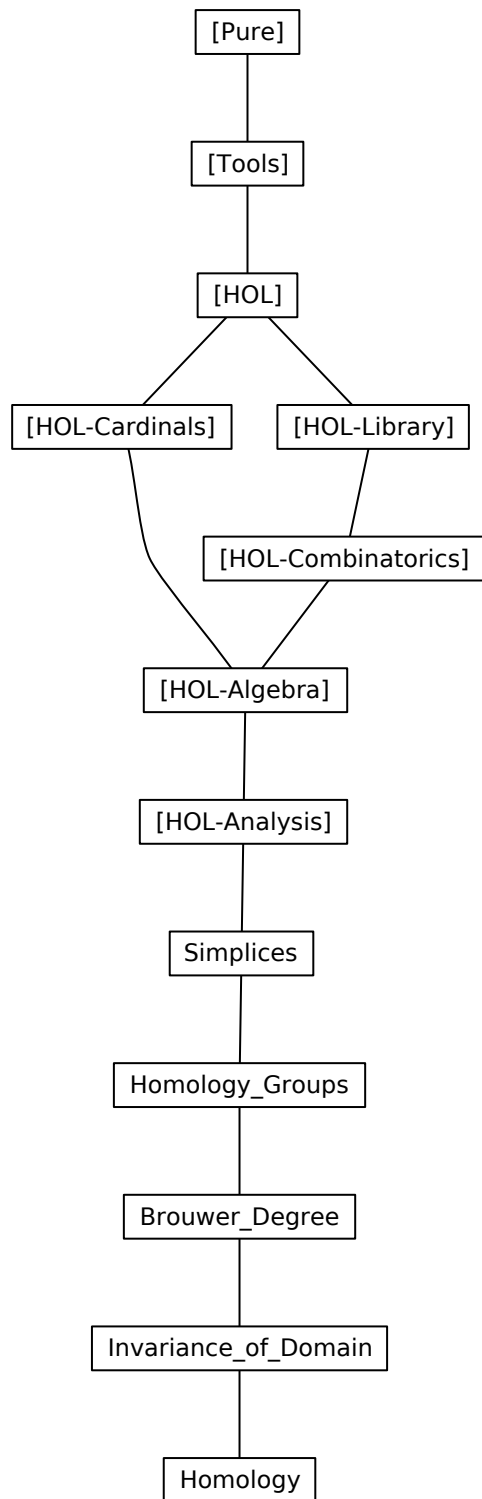
Homology

March 13, 2025

Contents

0.1	Homology, I: Simplices	4
0.1.1	Standard simplices, all of which are topological subspaces of \mathbb{R}^n	4
0.1.2	Face map	4
0.1.3	Singular simplices, forcing canonicity outside the intended domain	4
0.1.4	Singular chains	4
0.1.5	Boundary homomorphism for singular chains	4
0.1.6	Factoring out chains in a subtopology for relative homology	4
0.1.7	Relative cycles $Z_p X(S)$ where X is a topology and S a subset	4
0.1.8	Relative boundaries $B_p X(S)$, where X is a topology and S a subset.	4
0.1.9	The (relative) homology relation	4
0.1.10	Show that all boundaries are cycles, the key "chain complex" property.	4
0.1.11	Operations induced by a continuous map g between topological spaces	4
0.1.12	Homology of one-point spaces degenerates except for $p = 0$	4
0.1.13	Simplicial chains	5
0.1.14	The cone construction on simplicial simplices.	5
0.1.15	Barycentric subdivision of a linear ("simplicial") simplex's image	5
0.1.16	Singular subdivision	5
0.1.17	Excision argument that we keep doing singular subdivision	5
0.1.18	Homotopy invariance	5
0.2	Homology, II: Homology Groups	6
0.2.1	Homology Groups	6
0.2.2	Towards the Eilenberg-Steenrod axioms	6
0.2.3	Additivity axiom	6
0.2.4	Special properties of singular homology	7

	3
0.2.5 More basic properties of homology groups, deduced from the E-S axioms	8
0.2.6 Generalize exact homology sequence to triples	8
0.3 Homology, III: Brouwer Degree	9
0.3.1 Reduced Homology	9
0.3.2 More homology properties of deformations, retracts, contractible spaces	9
0.3.3 Homology groups of spheres	10
0.3.4 Brouwer degree of a Map	10
0.4 Invariance of Domain	11
0.4.1 Degree invariance mod 2 for map between pairs . . .	11
0.4.2 General Jordan-Brouwer separation theorem and invariance of dimension	11
0.4.3 Relating two variants of Euclidean space, one within product topology.	13
0.4.4 Invariance of dimension and domain	13



0.1 Homology, I: Simplices

theory *Simplices*

imports

HOL-Analysis.Function_Metric

HOL-Analysis.Abstract_Euclidean_Space

HOL-Algebra.Free_Abelian_Groups

begin

0.1.1 Standard simplices, all of which are topological subspaces of $R^{\wedge n}$.

0.1.2 Face map

0.1.3 Singular simplices, forcing canonicity outside the intended domain

0.1.4 Singular chains

0.1.5 Boundary homomorphism for singular chains

0.1.6 Factoring out chains in a subtopology for relative homology

0.1.7 Relative cycles $Z_p X(S)$ where X is a topology and S a subset

0.1.8 Relative boundaries $B_p X S$, where X is a topology and S a subset.

0.1.9 The (relative) homology relation

0.1.10 Show that all boundaries are cycles, the key "chain complex" property.

0.1.11 Operations induced by a continuous map g between topological spaces

0.1.12 Homology of one-point spaces degenerates except for $p = 0$.

0.1.13 Simplicial chains

0.1.14 The cone construction on simplicial simplices.

0.1.15 Barycentric subdivision of a linear ("simplicial") simplex's image

0.1.16 Singular subdivision

0.1.17 Excision argument that we keep doing singular subdivision

proposition *sufficient_iterated_singular_subdivision_exists:*

assumes $\mathcal{C}: \bigwedge U. U \in \mathcal{C} \implies \text{openin } X \ U$

and $X: \text{topspace } X \subseteq \bigcup \mathcal{C}$

and $p: \text{singular_chain } p \ X \ c$

obtains n **where** $\bigwedge m f. \llbracket n \leq m; f \in \text{Poly_Mapping.keys } ((\text{singular_subdivision } p \ \frown m) \ c) \rrbracket$

$\implies \exists V \in \mathcal{C}. f \text{ ' } (\text{standard_simplex } p) \subseteq V$

0.1.18 Homotopy invariance

theorem *homotopic_imp_homologous_rel_chain_maps:*

assumes $\text{hom}: \text{homotopic_with } (\lambda h. h \text{ ' } T \subseteq V) \ S \ U \ f \ g$ **and** $c: \text{singular_recycle } p \ S \ T \ c$

shows $\text{homologous_rel } p \ U \ V \ (\text{chain_map } p \ f \ c) \ (\text{chain_map } p \ g \ c)$

end

0.2 Homology, II: Homology Groups

theory *Homology_Groups*
imports *Simplices HOL-Algebra.Exact_Sequence*

begin

0.2.1 Homology Groups

0.2.2 Towards the Eilenberg-Steenrod axioms

proposition *homology_homotopy_axiom:*
assumes *homotopic_with* $(\lambda h. h \cdot S \subseteq T) \ X \ Y \ f \ g$
shows $\text{hom_induced } p \ X \ S \ Y \ T \ f = \text{hom_induced } p \ X \ S \ Y \ T \ g$

proposition *homology_excision_axiom:*
assumes $X \text{ closure_of } U \subseteq X \text{ interior_of } T \ T \subseteq S$
shows
 $\text{hom_induced } p \ (\text{subtopology } X \ (S - U)) \ (T - U) \ (\text{subtopology } X \ S) \ T \ id$
 $\in \text{iso} \ (\text{relative_homology_group } p \ (\text{subtopology } X \ (S - U)) \ (T - U))$
 $(\text{relative_homology_group } p \ (\text{subtopology } X \ S) \ T)$

0.2.3 Additivity axiom

proposition *iso_cycle_group_sum:*
assumes *disj: pairwise disjoint \mathcal{U} and $UU: \bigcup \mathcal{U} = \text{topspace } X$*

and *subs*: $\bigwedge C T. \llbracket \text{compactin } X C; \text{path_connectedin } X C; T \in \mathcal{U}; \neg \text{disjnt } C \rrbracket \implies C \subseteq T$
shows $(\lambda f. \text{sum}' f \mathcal{U}) \in \text{iso } (\text{sum_group } \mathcal{U} (\lambda T. \text{relcycle_group } p (\text{subtopology } X T) \{\}))$
 $(\text{relcycle_group } p X \{\})$

proposition *homology_additivity_axiom_gen*:

assumes *disj*: *pairwise disjnt* \mathcal{U} **and** *UU*: $\bigcup \mathcal{U} = \text{topspace } X$
and *subs*: $\bigwedge C T. \llbracket \text{compactin } X C; \text{path_connectedin } X C; T \in \mathcal{U}; \neg \text{disjnt } C \rrbracket \implies C \subseteq T$
shows $(\lambda x. \text{gfinprod } (\text{homology_group } p X)$
 $(\lambda V. \text{hom_induced } p (\text{subtopology } X V) \{\} X \{\} \text{id } (x V)) \mathcal{U})$
 $\in \text{iso } (\text{sum_group } \mathcal{U} (\lambda S. \text{homology_group } p (\text{subtopology } X S))) (\text{homology_group } p X)$
(is $?h \in \text{iso } ?SG ?HG)$

corollary *homology_additivity_axiom*:

assumes *disj*: *pairwise disjnt* \mathcal{U} **and** *UU*: $\bigcup \mathcal{U} = \text{topspace } X$
and *ope*: $\bigwedge v. v \in \mathcal{U} \implies \text{openin } X v$
shows $(\lambda x. \text{gfinprod } (\text{homology_group } p X)$
 $(\lambda v. \text{hom_induced } p (\text{subtopology } X v) \{\} X \{\} \text{id } (x v)) \mathcal{U})$
 $\in \text{iso } (\text{sum_group } \mathcal{U} (\lambda S. \text{homology_group } p (\text{subtopology } X S))) (\text{homology_group } p X)$

0.2.4 Special properties of singular homology

proposition *iso_integer_zeroth_homology_group*:

assumes *X*: *path_connected_space* X **and** *f*: *singular_simplex* $0 X f$
shows $\text{pow } (\text{homology_group } 0 X) (\text{homologous_rel_set } 0 X \{\} (\text{frag_of } f))$
 $\in \text{iso_integer_group } (\text{homology_group } 0 X) (\text{is_pow } ?H ?q \in \text{iso_} ?H)$

corollary *isomorphic_integer_zeroth_homology_group*:

assumes *X*: *path_connected_space* X *topspace* $X \neq \{\}$
shows $\text{homology_group } 0 X \cong \text{integer_group}$

corollary *homology_coefficients*:

$\text{topspace } X = \{a\} \implies \text{homology_group } 0 X \cong \text{integer_group}$

proposition *zeroth_homology_group*:

$\text{homology_group } 0 X \cong \text{free_Abelian_group } (\text{path_components_of } X)$

0.2.5 More basic properties of homology groups, deduced from the E-S axioms

corollary *mon_hom_induced_section_map:*

assumes *section_map* $X\ Y\ f$
shows $(\text{hom_induced } p\ X\ \{\}\ Y\ \{\}\ f) \in \text{mon } (\text{homology_group } p\ X) (\text{homology_group } p\ Y)$

corollary *epi_hom_induced_retraction_map:*

assumes *retraction_map* $X\ Y\ f$
shows $(\text{hom_induced } p\ X\ \{\}\ Y\ \{\}\ f) \in \text{epi } (\text{homology_group } p\ X) (\text{homology_group } p\ Y)$

0.2.6 Generalize exact homology sequence to triples

proposition *homology_exactness_triple_1:*

assumes $T \subseteq S$
shows $\text{exact_seq } ([\text{relative_homology_group}(p-1) (\text{subtopology } X\ S)\ T,$
 $\text{relative_homology_group } p\ X\ S,$
 $\text{relative_homology_group } p\ X\ T],$
 $[\text{hom_relboundary } p\ X\ S\ T, \text{hom_induced } p\ X\ T\ X\ S\ \text{id}])$
(is exact_seq ([?G1,?G2,?G3], [?h1,?h2]))

proposition *homology_exactness_triple_2:*

assumes $T \subseteq S$
shows $\text{exact_seq } ([\text{relative_homology_group}(p-1) X\ T,$
 $\text{relative_homology_group}(p-1) (\text{subtopology } X\ S)\ T,$
 $\text{relative_homology_group } p\ X\ S],$
 $[\text{hom_induced } (p-1) (\text{subtopology } X\ S)\ T\ X\ T\ \text{id}, \text{hom_relboundary}$
 $p\ X\ S\ T])$
(is exact_seq ([?G1,?G2,?G3], [?h1,?h2]))

proposition *homology_exactness_triple_3:*

assumes $T \subseteq S$

```

shows exact_seq ([relative_homology_group  $p\ X\ S$ ,
                    relative_homology_group  $p\ X\ T$ ,
                    relative_homology_group  $p\ (\text{subtopology } X\ S)\ T$ ],
                  [hom_induced  $p\ X\ T\ X\ S\ id$ , hom_induced  $p\ (\text{subtopology } X\ S)\ T$ 
                   $X\ T\ id$ ])
  (is exact_seq ([?G1, ?G2, ?G3], [?h1, ?h2]))
end

```

0.3 Homology, III: Brouwer Degree

```

theory Brouwer_Degree
  imports Homology_Groups HOL-Algebra.Multiplicative_Group

begin

```

0.3.1 Reduced Homology

0.3.2 More homology properties of deformations, retracts, contractible spaces

```

corollary isomorphic_relative_homology_groups_relativization_contractible:
  assumes contractible_space(subtopology  $X\ S$ ) contractible_space(subtopology  $X\ T$ )
   $T \subseteq S\ \text{topspace } X \cap T \neq \{\}$ 
  shows relative_homology_group  $p\ X\ T \cong \text{relative\_homology\_group } p\ X\ S$ 

corollary isomorphic_relative_homology_groups_inclusion_contractible:
  assumes contractible_space  $X$  contractible_space(subtopology  $X\ S$ )  $T \subseteq S\ \text{topspace } X \cap S \neq \{\}$ 

```

shows $\text{relative_homology_group } p \text{ (subtopology } X \text{ } S) \text{ } T \cong \text{relative_homology_group } p \text{ } X \text{ } T$

corollary $\text{isomorphic_relative_homology_groups_relboundary_contractible}$:

assumes $\text{contractible_space } X \text{ contractible_space (subtopology } X \text{ } T) \text{ } T \subseteq S \text{ topspace } X \cap T \neq \{\}$

shows $\text{relative_homology_group } p \text{ } X \text{ } S \cong \text{relative_homology_group } (p - 1) \text{ (subtopology } X \text{ } S) \text{ } T$

0.3.3 Homology groups of spheres

proposition $\text{iso_relative_homology_group_upper_hemisphere}$:

$(\text{hom_induced } p \text{ (subtopology (nsphere } n) \{x. x \text{ } k \geq 0\}) \{x. x \text{ } k = 0\} \text{ (nsphere } n) \{x. x \text{ } k \leq 0\} \text{ id})$

$\in \text{iso (relative_homology_group } p \text{ (subtopology (nsphere } n) \{x. x \text{ } k \geq 0\}) \{x. x \text{ } k = 0\})$

$(\text{relative_homology_group } p \text{ (nsphere } n) \{x. x \text{ } k \leq 0\}) \text{ (is ?h } \in \text{iso ?G ?H)}$

corollary $\text{iso_upper_hemisphere_reduced_homology_group}$:

$(\text{hom_boundary } (1 + p) \text{ (subtopology (nsphere (Suc } n) \{x. x(\text{Suc } n) \geq 0\}) \{x. x(\text{Suc } n) = 0\})$

$\in \text{iso (relative_homology_group } (1 + p) \text{ (subtopology (nsphere (Suc } n) \{x. x(\text{Suc } n) \geq 0\}) \{x. x(\text{Suc } n) = 0\})$

$(\text{reduced_homology_group } p \text{ (nsphere } n))$

corollary $\text{iso_reduced_homology_group_upper_hemisphere}$:

assumes $k \leq n$

shows $\text{hom_induced } p \text{ (nsphere } n) \{\} \text{ (nsphere } n) \{x. x \text{ } k \geq 0\} \text{ id}$

$\in \text{iso (reduced_homology_group } p \text{ (nsphere } n) \text{ (relative_homology_group } p \text{ (nsphere } n) \{x. x \text{ } k \geq 0\})$

0.3.4 Brouwer degree of a Map

corollary *Brouwer_degree2_nonsurjective:*

$\llbracket \text{continuous_map}(\text{nsphere } p) (\text{nsphere } p) f; f \text{ ' } \text{topspace } (\text{nsphere } p) \neq \text{topspace } (\text{nsphere } p) \rrbracket$
 $\implies \text{Brouwer_degree2 } p \text{ } f = 0$

proposition *Brouwer_degree2_reflection:*

$\text{Brouwer_degree2 } p (\lambda x \text{ i. if } i = 0 \text{ then } -x \text{ i else } x \text{ i}) = -1$ (**is** $\text{Brouwer_degree2 } _ \text{ ?r} = -1$)

end

0.4 Invariance of Domain

theory *Invariance_of_Domain*

imports *Brouwer_Degree HOL-Analysis.Continuous_Extension HOL-Analysis.Homeomorphism*

begin

0.4.1 Degree invariance mod 2 for map between pairs

theorem *Borsuk_odd_mapping_degree_step:*

assumes *cmf*: $\text{continuous_map } (\text{nsphere } n) (\text{nsphere } n) f$
and *f*: $\bigwedge x. x \in \text{topspace}(\text{nsphere } n) \implies (f \circ (\lambda x \text{ i. } -x \text{ i})) x = ((\lambda x \text{ i. } -x \text{ i}) \circ f) x$
and *fm*: $f \text{ ' } (\text{topspace}(\text{nsphere}(n - \text{Suc } 0))) \subseteq \text{topspace}(\text{nsphere}(n - \text{Suc } 0))$
shows *even* $(\text{Brouwer_degree2 } n \text{ } f - \text{Brouwer_degree2 } (n - \text{Suc } 0) \text{ } f)$

0.4.2 General Jordan-Brouwer separation theorem and invariance of dimension

proposition *relative_homology_group_Euclidean_complement_step:*

assumes *closedin* $(\text{Euclidean_space } n) S$
shows $\text{relative_homology_group } p (\text{Euclidean_space } n) (\text{topspace}(\text{Euclidean_space } n) - S)$
 $\cong \text{relative_homology_group } (p + k) (\text{Euclidean_space } (n+k)) (\text{topspace}(\text{Euclidean_space } (n+k)) - S)$

proposition *isomorphic_relative_homology_groups_Euclidean_complements:*

assumes *S*: $\text{closedin } (\text{Euclidean_space } n) S$ **and** *T*: $\text{closedin } (\text{Euclidean_space } n) T$
and *hom*: $(\text{subtopology } (\text{Euclidean_space } n) S) \text{ homeomorphic_space } (\text{subtopology } (\text{Euclidean_space } n) T)$
shows $\text{relative_homology_group } p (\text{Euclidean_space } n) (\text{topspace}(\text{Euclidean_space } n) - S)$

$$\cong \text{relative_homology_group } p \text{ (Euclidean_space } n \text{) (topspace(Euclidean_space } n \text{) - } T)$$

theorem *invariance_of_dimension_Euclidean_space:*

Euclidean_space m homeomorphic_space Euclidean_space n \longleftrightarrow m = n

theorem *invariance_of_domain_Euclidean_space:*

assumes *U: openin (Euclidean_space n) U*

and *cmf: continuous_map (subtopology (Euclidean_space n) U) (Euclidean_space n) f*

and *inj_on f U*

shows *openin (Euclidean_space n) (f ' U) (is openin ?E (f ' U))*

corollary *invariance_of_domain_Euclidean_space_embedding_map:*

assumes *openin (Euclidean_space n) U*

and *cmf: continuous_map (subtopology (Euclidean_space n) U) (Euclidean_space n) f*

and *inj_on f U*

shows *embedding_map (subtopology (Euclidean_space n) U) (Euclidean_space n) f*

corollary *invariance_of_domain_Euclidean_space_gen:*

assumes *n ≤ m and U: openin (Euclidean_space m) U*

and *cmf: continuous_map (subtopology (Euclidean_space m) U) (Euclidean_space n) f*

and *inj_on f U*

shows *openin (Euclidean_space n) (f ' U)*

corollary *invariance_of_domain_Euclidean_space_embedding_map_gen:*

assumes *n ≤ m and U: openin (Euclidean_space m) U*

and *cmf: continuous_map (subtopology (Euclidean_space m) U) (Euclidean_space n) f*

and *inj_on f U*

shows *embedding_map (subtopology (Euclidean_space m) U) (Euclidean_space n) f*

0.4.3 Relating two variants of Euclidean space, one within product topology.

proposition *homeomorphic_maps_Euclidean_space_euclidean_gen_OLD*:
fixes $B :: 'n::\text{euclidean_space set}$
assumes *finite B independent B* **and** *orth: pairwise orthogonal B* **and** $n: \text{card } B = n$
obtains $f\ g$ **where** *homeomorphic_maps (Euclidean_space n) (top_of_set (span B)) f g*

proposition *homeomorphic_maps_Euclidean_space_euclidean_gen*:
fixes $B :: 'n::\text{euclidean_space set}$
assumes *independent B* **and** *orth: pairwise orthogonal B* **and** $n: \text{card } B = n$
and $1: \bigwedge u. u \in B \implies \text{norm } u = 1$
obtains $f\ g$ **where** *homeomorphic_maps (Euclidean_space n) (top_of_set (span B)) f g*
and $\bigwedge x. x \in \text{topspace (Euclidean_space } n) \implies (\text{norm } (f\ x))^2 = (\sum_{i < n. (x\ i)^2})$

corollary *homeomorphic_maps_Euclidean_space_euclidean*:
obtains $f :: (\text{nat} \Rightarrow \text{real}) \Rightarrow 'n::\text{euclidean_space}$ **and** g
where *homeomorphic_maps (Euclidean_space (DIM('n))) euclidean f g*

0.4.4 Invariance of dimension and domain

corollary *invariance_of_domain_subspaces*:
fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes *ope: openin (top_of_set U) S*
and *subspace U subspace V* **and** $VU: \text{dim } V \leq \text{dim } U$
and *contf: continuous_on S f* **and** *fim: f ' S \subseteq V*
and *injf: inj_on f S*
shows *openin (top_of_set V) (f ' S)*

corollary *invariance_of_dimension_subspaces*:
fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes *ope: openin (top_of_set U) S*
and *subspace U subspace V*
and *contf: continuous_on S f* **and** *fim: f ' S \subseteq V*
and *injf: inj_on f S* **and** $S \neq \{\}$
shows $\text{dim } U \leq \text{dim } V$

corollary *invariance_of_domain_affine_sets*:
fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes *ope: openin (top_of_set U) S*
and *aff: affine U affine V* $\text{aff_dim } V \leq \text{aff_dim } U$
and *contf: continuous_on S f* **and** *fim: f ' S \subseteq V*
and *injf: inj_on f S*

shows $\text{openin } (\text{top_of_set } V) (f \text{ ` } S)$

corollary *invariance_of_dimension_affine_sets*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{ope}: \text{openin } (\text{top_of_set } U) S$
and $\text{aff}: \text{affine } U \text{ affine } V$
and $\text{contf}: \text{continuous_on } S f$ **and** $\text{fim}: f \text{ ` } S \subseteq V$
and $\text{injf}: \text{inj_on } f S$ **and** $S \neq \{\}$
shows $\text{aff_dim } U \leq \text{aff_dim } V$

corollary *invariance_of_dimension*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{contf}: \text{continuous_on } S f$ **and** $\text{open } S$
and $\text{injf}: \text{inj_on } f S$ **and** $S \neq \{\}$
shows $\text{DIM}('a) \leq \text{DIM}('b)$

corollary *continuous_injective_image_subspace_dim_le*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{subspace } S \text{ subspace } T$
and $\text{contf}: \text{continuous_on } S f$ **and** $\text{fim}: f \text{ ` } S \subseteq T$
and $\text{injf}: \text{inj_on } f S$
shows $\text{dim } S \leq \text{dim } T$

corollary *invariance_of_domain_homeomorphic*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{open } S \text{ continuous_on } S f \text{ DIM}('b) \leq \text{DIM}('a) \text{ inj_on } f S$
shows $S \text{ homeomorphic } (f \text{ ` } S)$

proposition *homeomorphic_interiors*:

fixes $S :: 'a::\text{euclidean_space set}$ **and** $T :: 'b::\text{euclidean_space set}$
assumes $S \text{ homeomorphic } T \text{ interior } S = \{\} \longleftrightarrow \text{interior } T = \{\}$
shows $(\text{interior } S) \text{ homeomorphic } (\text{interior } T)$

proposition *uniformly_continuous_homeomorphism_UNIV_trivial*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'a$
assumes $\text{contf}: \text{uniformly_continuous_on } S f$ **and** $\text{hom}: \text{homeomorphism } S$
 $\text{UNIV } f g$
shows $S = \text{UNIV}$

proposition *invariance_of_domain_sphere_affine_set_gen*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{contf}: \text{continuous_on } S f$ **and** $\text{injf}: \text{inj_on } f S$ **and** $\text{fim}: f \text{ ` } S \subseteq T$
and $U: \text{bounded } U \text{ convex } U$
and $\text{affine } T$ **and** $\text{affTU}: \text{aff_dim } T < \text{aff_dim } U$
and $\text{ope}: \text{openin } (\text{top_of_set } (\text{rel_frontier } U)) S$
shows $\text{openin } (\text{top_of_set } T) (f \text{ ` } S)$

end

```
theory Homology  
  imports Invariance_of_Domain  
begin  
  
end
```