Knaster-Tarski fixed point theorem

ℕaproche Formalization: Andrei Paskevich (2007), Steffen Frerix (2018), Erik Sturzenhecker (2020), and Peter Koepke (2021)

The Knaster-Tarski theorem is a result from lattice theory about fixed points of monotone functions. Bronisław Knaster and Alfred Tarski established it in 1928 for the special case of power set lattices (Knaster, B. with Tarski, A.: Un théorème sur les fonctions d'ensembles, Ann. Soc. Polon. Math. vol. 6 (1928), 133-134). This more general result was stated by Tarski in 1955 (Tarski, A.: A lattice-theoretical fixpoint theorem and its applications, Pacific Journal of Mathematics vol. 5 (1955), no. 2, 285-309).

Our formalization checks in about 45 seconds on a modest laptop.

1 Preliminaries

We first import a small ℕaproche formalization of an ontology with sets, functions, and objects:

[read examples/preliminaries.ftl.tex]

2 Partial Orders

We work with a "global" order relation $\leq$.

Let x, y, z, u, v, w denote elements. Let S, T denote sets.

Signature 1. $x \leq y$ is a relation.

Axiom 2. $x \leq x$.

Axiom 3. $x \leq y \leq x \implies x = y$.

Axiom 4. $x \leq y \leq z \implies x \leq z$.

Definition 5. Let S be a subset of T. A lower bound of S in T is an element u of T such that $u \leq x$ for every $x \in S$.

Definition 6. Let S be a subset of T. An upper bound of S in T is an element u of T such that $x \leq u$ for every $x \in S$.

Definition 7. Let S be a subset of T. An infimum of S in T is an element

u of T such that u is a lower bound of S in T and for every lower bound v of S in T we have $v \leq u$.

Definition 8. Let S be a subset of T. A supremum of S in T is an element u of T such that u is a upper bound of S in T and for every upper bound v of S in T we have $u \leq v$.

Lemma 9. Let S be a subset of T. Let u, v be suprema of S in T. Then $u = v$.

Lemma 10. Let S be a subset of T. Let u, v be infima of S in T. Then $u = v$.

Definition 11. A complete lattice is a set S such that every subset of S has an infimum in S and a supremum in S.

3 Functions

Let f stand for a function.

Definition 12. A fixed point of f is an element x of $\mathrm{dom}(f)$ such that $f(x) = x$.

Definition 13. f is monotone iff for all $x, y \in \mathrm{dom}(f)$ $x \leq y \implies f(x) \leq f(y)$.

4 Knaster-Tarski theorem

Let f stand for a function.

Theorem 14 (KnasterTarski). Let U be a complete lattice. Let $f : U \to U$ and f be monotone. Let S be the class of fixed points of f. Then S is a complete lattice.

Proof. Let T be a subset of S. Let us show that T has a supremum in S. Define

$$P = \{x \in U \mid f(x) \leq x \text{ and } x \text{ is an upper bound of } T \text{ in } U\}.$$

Take an infimum p of P in U. $f(p)$ is a lower bound of P in U and an upper bound of T in U. Hence p is a fixed point of f and a supremum of T in S. qed.

Let us show that T has an infimum in S. Define

$$Q = \{x \in U \mid x \leq f(x) \text{ and } x \text{ is a lower bound of } T \text{ in } U\}.$$

Take a supremum q of Q in U. $f(q)$ is an upper bound of Q in U and a lower bound of T in U. Hence q is a fixed point of f. q is an infimum of T in S. qed. $\square$