The mutilated checkerboard

Naproche formalization and commentary:
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1 Introduction

Max Black proposed the mutilated checkerboard problem in his book *Critical Thinking* (1946). It was later discussed by Martin Gardner in his *Scientific American* column, *Mathematical Games*. John McCarthy, one of the founders of Artificial Intelligence described it as a *Tough Nut for Proof Procedures* and discussed fully automatic or interactive proofs of the solution.

There have been several formalization of the Checkerboard problem before. A survey article by Manfred Kerber and Martin Pollet called *A Tough Nut for Mathematical Knowledge Management* lists a couple of formalizations. Our formalization is based on a set-theoretical formalization by John McCarthy. Checking takes 2 or 3 minutes on a modest laptop.

[read examples/preliminaries.ftl.tex] [synonym coordinate/-s] [synonym square/-s] [synonym subset/-s]

Let $f : X \to Y$ stand for $f : X \to Y$.

2 Preliminaries about cardinality

Definition 1. Let B, C be sets. B is equinumerous with C iff there exists a map F such that $F : B \leftrightarrow C$.

Lemma 2. Let B, C be sets. Assume that B is equinumerous with C. Then C is equinumerous with B.

Lemma 3. Let A, B, C be sets. Assume A is equinumerous with B. Assume B is equinumerous with C. Then A is equinumerous with C.

Proof. Take a map F such that $F : A \leftrightarrow B$. Take a map G such that $G : B \to A$ and (for all elements x of A we have $G(F(x)) = x$) and (for all elements y of B we have $F(G(y)) = y$). Take a map H such that $H : B \leftrightarrow C$. Take a map I such that $I : C \to B$ and (for all elements x of

B we have $I(H(x)) = x$) and (for all elements y of C we have $H(I(y)) = y$). For every element x of A $H(F(x))$ is an object. Define $J(x) = H(F(x))$ for x in A. For every element y of C $G(I(y))$ is an object. [timelimit 30] $J : A \leftrightarrow C$. Indeed define $K(y) = G(I(y))$ for y in C. [/timelimit] $\square$

Definition 4. Let X be a set. X is Dedekind finite iff every proper subset of X is not equinumerous with X.

3 Setting up the checkerboard

We introduce integer constants to model checkerboards as a Cartesian product of the integers $0, 1, 2, \ldots, 7$.

Signature 5. An integer is an object.

Signature 6. 0 is a integer.

Signature 7. 1 is a integer.

Signature 8. 2 is a integer.

Signature 9. 3 is a integer.

Signature 10. 4 is a integer.

Signature 11. 5 is a integer.

Signature 12. 6 is a integer.

Signature 13. 7 is a integer.

Definition 14. A coordinate is an integer x such that $x = 0$ or $x = 1$ or $x = 2$ or $x = 3$ or $x = 4$ or $x = 5$ or $x = 6$ or $x = 7$.

Let m, n, i, j, k, l denote coordinates.

Signature 15. A square is an object.

Axiom 16. (m, n) is a square.

Axiom 17. Let x be a square. Then $x = (m, n)$ for some coordinate m and some coordinate n.

Let x, y, z denote squares.

The checkerboard is the Dedekind finite set containing all squares.

Definition 18. The checkerboard is the class of all squares.

Axiom 19. The checkerboard is a set.

Let **C** stand for the checkerboard.

Axiom 20. Every subset of the checkerboard is Dedekind finite.

4 The Mutilated Checkerboard

Defining the mutilated checkerboard is straightforward: we simply remove the two corners.

Definition 21. $\mathbf{C}' = \{(0,0),(7,7)\}$.

Definition 22. $\mathbf{M} = \mathbf{C} \setminus \mathbf{C}'$.

Let the mutilated checkerboard stand for $\mathbf{M}$.

5 Dominoes

To define dominoes, we introduce concepts of adjacency by first declaring new relations and then axiomatizing them. As usual, chaining of relation symbols indicates a conjunction.

Let m,n,i,j,k,l denote coordinates. Let x,y,z denote squares.

Signature 23. Let m be an integer. A neighbour of m is an integer.

Let $m \sim n$ stand for m is a neighbour of n.

Axiom 24. If $m \sim n$ then $n \sim m$.

Axiom 25. $0 \sim 1 \sim 2 \sim 3 \sim 4 \sim 5 \sim 6 \sim 7$.

Definition 26. x is adjacent to y iff there exist integers b,c,d,e such that $x = (b,c)$ and $y = (d,e)$ and $((b = d$ and $c \sim e)$ or $(b \sim d$ and $c = e))$.

Definition 27. A domino is a set D such that $D = \{x,y\}$ for some adjacent squares x,y.

6 Domino Tilings

Definition 28. A domino tiling is a disjoint family T such that every element of T is a domino.

Definition 29. Let A be a subset of the checkerboard. A domino tiling of A is a domino tiling T such that for every square x x is an element of A iff x is an element of some element of T.

We shall prove that the mutilated checkerboard has no domino tiling.

7 Colours

We shall solve the mutilated checkerboard problem by a cardinality argument. Squares on an actual checkerboard are coloured black and white and we can count colours on dominoes and on the mutilated checkerboard $\mathbf{M}$.

The introduction of colours can be viewed as a creative move typical of mathematics: changing perspectives and introducing aspects that are not part of the original problem. The mutilated checkerboard was first discussed under a cognition-theoretic perspective: can one solve the problem *without* inventing new concepts and completely stay within the realm of squares, subsets of the checkerboard and dominoes.

Let x, y, z denote squares.

Signature 30. x is black is a relation.

Let x is white stand for x is not black.

Axiom 31. If x is adjacent to y then x is black iff y is white.

Axiom 32. $(0, 0)$ is black.

Axiom 33. $(7, 7)$ is black.

Definition 34. $\mathbf{B}$ is the class of black elements of $\mathbf{C}$.

Definition 35. $\mathbf{W}$ is the class of white elements of $\mathbf{C}$.

Lemma 36. $\mathbf{B}$ is a set.

Lemma 37. $\mathbf{W}$ is a set.

8 Counting Colours on Checkerboards

The original checkerboard has an equal number of black and white squares. Since our setup does not include numbers for counting, we rather work with equinumerosity. The following argument formalizes that we can invert the colours of a checkerboard by swapping the files 0 and 1, 2 and 3, etc.. We formalize swapping by a first-order function symbol Swap.

Let m, n, i, j, k, l denote coordinates.

Signature 38. Let x be an element of $\mathbf{C}$. $\mathrm{Swap}\, x$ is an element of $\mathbf{C}$.

Axiom 39. $\mathrm{Swap}(0, n) = (1, n)$ and $\mathrm{Swap}(1, n) = (0, n)$.

Axiom 40. $\mathrm{Swap}(2, n) = (3, n)$ and $\mathrm{Swap}(3, n) = (2, n)$.

Axiom 41. $\mathrm{Swap}(4, n) = (5, n)$ and $\mathrm{Swap}(5, n) = (4, n)$.

Axiom 42. $\mathrm{Swap}(6, n) = (7, n)$ and $\mathrm{Swap}(7, n) = (6, n)$.

Lemma 43. Let x be an element of $\mathbf{C}$. $\mathrm{Swap}\, x$ is adjacent to x.

Proof. Take integers i, j such that $x = (i, j)$. Case $i = 0$. End. Case $i = 1$. End. Case $i = 2$. End. Case $i = 3$. End. Case $i = 4$. End. Case $i = 5$. End. Case $i = 6$. End. Case $i = 7$. End. □

Swap is an involution.

Lemma 44. Let x be an element of $\mathbf{C}$. $\mathrm{Swap}(\mathrm{Swap}\,x) = x$.

Proof. Take integers i, j such that $x = (i, j)$. Case $i = 0$. End. Case $i = 1$. End. Case $i = 2$. End. Case $i = 3$. End. Case $i = 4$. End. Case $i = 5$. End. Case $i = 6$. End. Case $i = 7$. End. ☐

Lemma 45. Let x be an element of $\mathbf{C}$. x is black iff $\mathrm{Swap}\,x$ is white.

Using Swap we can define a witness of $\mathbf{B} \leftrightarrow \mathbf{W}$.

Lemma 46. $\mathbf{B}$ is equinumerous with $\mathbf{W}$.

Proof. Define $F(x) = \mathrm{Swap}\,x$ for x in $\mathbf{B}$. Define $G(x) = \mathrm{Swap}\,x$ for x in $\mathbf{W}$. $F(x)$ is white for all elements x of $\mathrm{dom}(F)$. $G(y)$ is black for all elements y of $\mathrm{dom}(G)$. Then $F : \mathbf{B} \to \mathbf{W}$ and $G : \mathbf{W} \to \mathbf{B}$. For all elements x of $\mathbf{B}$ we have $G(F(x)) = x$. For all elements x of $\mathbf{W}$ we have $F(G(x)) = x$. $F : \mathbf{B} \leftrightarrow \mathbf{W}$. ☐

Given a domino tiling one can also swap the squares of each domino, leading to similar properties.

Signature 47. Let A be a subset of the checkerboard. Let T be a domino tiling of A. Let x be an element of A. $\mathrm{Swap}^A_T(x)$ is a square y such that there is an element D of T such that $D = \{x, y\}$.

Lemma 48. Let A be a subset of the checkerboard. Assume that T is a domino tiling of A. Let x be an element of A. Then $\mathrm{Swap}^A_T(x)$ is an element of A.

Proof. Let $y = \mathrm{Swap}^A_T(x)$. Take an element D of T such that $D = \{x, y\}$. Then y is an element of A. ☐

Swapping dominoes is also an involution.

Lemma 49. Let A be a subset of the checkerboard. Assume that T is a domino tiling of A. Let x be an element of A. Then $\mathrm{Swap}^A_T(\mathrm{Swap}^A_T(x)) = x$.

Proof. Let $y = \mathrm{Swap}^A_T(x)$. Take an element Y of T such that $Y = \{x, y\}$. Let $z = \mathrm{Swap}^A_T(y)$. Take an element Z of T such that $Z = \{y, z\}$. Then $x = z$. ☐

Lemma 50. Let A be a subset of the checkerboard. Assume that T is a domino tiling of A. Let x be a white element of A. Then $\mathrm{Swap}^A_T(x)$ is black.

Proof. Let $y = \mathrm{Swap}^A_T(x)$. Take an element Y of T such that $Y = \{x, y\}$. [timelimit 10] Then x is adjacent to y. [/timelimit] Thus y is black. ☐

Lemma 51. Let A be a subset of the checkerboard. Assume that T is a domino tiling of A. Let x be a black element of A. Then $\mathrm{Swap}_T^A(x)$ is white.

Proof. Let $y = \mathrm{Swap}_T^A(x)$. Take an element Y of T such that $Y = \{x, y\}$. [timelimit 10] Then x is adjacent to y. [/timelimit] Thus y is white. □

9 The Theorem

We can easily show that a domino tiling involves as many black as white squares.

Lemma 52. Let A be a subset of the checkerboard. Let T be a domino tiling of A. Then $A \cap \mathbf{B}$ is equinumerous with $A \cap \mathbf{W}$.

Proof. Define $F(x) = \mathrm{Swap}_T^A(x)$ for x in $A \cap \mathbf{B}$. Define $G(x) = \mathrm{Swap}_T^A(x)$ for x in $A \cap \mathbf{W}$. $F : A \cap \mathbf{B} \to A \cap \mathbf{W}$. Indeed $F : A \cap \mathbf{B} \to \mathbf{W}$. $G : A \cap \mathbf{W} \to A \cap \mathbf{B}$. Indeed $G : A \cap \mathbf{W} \to \mathbf{B}$. For all elements x of $A \cap \mathbf{B}$ we have $G(F(x)) = x$. For all elements x of $A \cap \mathbf{W}$ we have $F(G(x)) = x$. $F : A \cap \mathbf{B} \leftrightarrow A \cap \mathbf{W}$. □

In mutilating the checkerboard, one only removes black squares

Lemma 53. $\mathbf{M} \cap \mathbf{W} = \mathbf{W}$.

Lemma 54. $\mathbf{M} \cap \mathbf{B}$ is a proper subset of $\mathbf{B}$.

Proof. $(0, 0)$ is an element of $\mathbf{B}$. $(0, 0)$ is not an element of $\mathbf{M}$. Thus $(0, 0)$ is not an element of $\mathbf{M} \cap \mathbf{B}$. □

Now the theorem follows by putting together the previous cardinality properties. Note that the phrasing *[...] has no domino tiling* in the theorem is automatically derived from the definition of *a domino tiling of [...]*.

Theorem 55. The mutilated checkerboard has no domino tiling.

Proof by contradiction. Assume T is a domino tiling of $\mathbf{M}$. $\mathbf{M} \cap \mathbf{B}$ is equinumerous with $\mathbf{M} \cap \mathbf{W}$. $\mathbf{M} \cap \mathbf{B}$ is equinumerous with $\mathbf{W}$. $\mathbf{M} \cap \mathbf{B}$ is equinumerous with $\mathbf{B}$. Contradiction. □