Regularity of successor cardinals

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1 Preliminaries

[read `examples/preliminaries.ftl.tex`]

Let x, y, X, Y denote sets. Let f denote a function.

Let f is surjective from X onto Y stand for $\mathrm{dom}(f) = X$ and $f[X] = Y$.

Let $f : X \twoheadrightarrow Y$ stand for f is surjective from X onto Y.

Let X is nonempty stand for X has an element.

2 Ordinals

Let x, y denote sets.

Signature 1. An ordinal is a set.

Let α, β denote ordinals.

Axiom 2. Every element of α is an ordinal.

Axiom 3. Let $x \in y \in \alpha$. Then $x \in \alpha$.

Signature 4. Let α, β be ordinals. $\alpha < \beta$ is a relation.

Axiom 5. $\alpha < \beta$ or $\beta < \alpha$ or $\beta = \alpha$.

Axiom 6. If $\alpha < \beta$ then α is an element of β.

Let $a \leq b$ stand for $a = b$ or $a < b$.

3 Cardinals

Let X, Y denote sets.

Signature 7. A cardinal is an ordinal.

Let μ, ν denote cardinals.

Signature 8. μ is infinite is an atom.

Let κ denote an infinite cardinal.

Signature 9. $|X|$ is a cardinal.

Axiom 10 (existence of surjection). Assume X has an element. $|X| \leq |Y|$ iff there exists a function that is surjective from Y onto X.

Axiom 11 (Hessenberg). $|\kappa \times \kappa| = \kappa$.

Axiom 12. $|\kappa| = \kappa$.

Axiom 13. Let Y be a subset of X. $|Y| \leq |X|$.

Signature 14. Let κ be an infinite cardinal. κ^+ is an infinite cardinal.

Axiom 15. $\kappa < \kappa^+$.

Axiom 16. $|\alpha| \leq \kappa$ for every element α of κ^+.

Axiom 17. For no cardinals μ, ν we have $\mu < \nu$ and $\nu < \mu$.

Axiom 18. There is no cardinal ν such that $\kappa < \nu < \kappa^+$.

Axiom 19. The empty set is a cardinal η such that η is an element of every nonempty ordinal.

Definition 20. The constant zero on X is the function f such that $\mathrm{dom}(f) = X$ and $f(x)$ is the empty set for every element x of X.

Let 0^X stand for the constant zero on X.

4 Cofinality and regular cardinals

Let κ denote an infinite cardinal.

Definition 21 (Cofinality). Let Y be a subset of κ. Y is cofinal in κ iff for every element x of κ there exists an element y of Y such that $x < y$.

Let a cofinal subset of κ stand for a subset of κ that is cofinal in κ.

Definition 22. κ is regular iff $|x| = \kappa$ for every cofinal subset x of κ.

5 Hausdorff's theorem

The following result appears in [1, p. 443], where Hausdorff mentions that the proof is *"ganz einfach"* (*"very simple"*) and can be skipped.

Theorem 23 (Hausdorff). Let κ be an infinite cardinal. Then κ^+ is regular.

Proof by contradiction. Assume the contrary. Take a cofinal subset x of κ^+ such that $|x| \neq \kappa^+$. Then $|x| \leq \kappa$. Take a function f that is surjective from κ onto x (by existence of surjection). Indeed x has an element and

$|\kappa| = \kappa$.

Define

$$g(z) = \begin{cases} \text{choose a function } h \text{ such that } h : \kappa \twoheadrightarrow z \text{ in } h & : z \text{ has an element} \\ 0^\kappa & : z \text{ has no element} \end{cases}$$

for z in κ^+.

For all $\xi, \zeta \in \kappa$ $g(f(\xi))$ is a map such that $\zeta \in \mathrm{dom}(g(f(\xi)))$. Define $h(\xi, \zeta) = g(f(\xi))(\zeta)$ for (ξ, ζ) in $\kappa \times \kappa$.

Let us show that h is surjective from $\kappa \times \kappa$ onto κ^+.

Every element of κ^+ is an element of $h[\kappa \times \kappa]$.
Proof. Let n be an element of κ^+. Take an element ξ of κ such that $n < f(\xi)$. Take an element ζ of κ such that $g(f(\xi))(\zeta) = n$. Indeed $g(f(\xi))$ is a function that is surjective from κ onto $f(\xi)$. Then $n = h(\xi, \zeta)$. Therefore the thesis. Indeed (ξ, ζ) is an element of $\kappa \times \kappa$. End.

Every element of $h[\kappa \times \kappa]$ is an element of κ^+.
Proof. Let n be an element of $h[\kappa \times \kappa]$. We can take elements a, b of κ such that $n = h(a, b)$. Then $n = g(f(a))(b)$. $f(a)$ is an element of κ^+. Every element of $f(a)$ is an element of κ^+.

Case $f(a)$ has an element. Then $g(f(a))$ is a function that is surjective from κ onto $f(a)$. Hence $n \in f(a) \in \kappa^+$. Thus $n \in \kappa^+$. End.

Case $f(a)$ has no element. Then $g(f(a)) = 0^\kappa$. Hence n is the empty set. Thus $n \in \kappa^+$. End. End. End.

Therefore $\kappa^+ \leq |\kappa \times \kappa| = \kappa$. Contradiction. $\square$

References

[1] Felix Hausdorff. “Grundzüge einer Theorie der geordneten Mengen”. In: *Mathematische Annalen* 65 (1908), pp. 435–505.