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## Beginnings of Tarskian geometry in Naproche (draft)

### Abstract

We present a literate formalization of the beginnings of Tarskian geometry. We follow *Metamathematische Methoden in der Geometrie* by Schwabhäuser, Szmielew, and Tarski (SST), covering most of the material up to Satz 6.7, including Gupta’s result that outer Pasch follows from inner Pasch. Throughout, figures help the human reader keep up with the automated theorem prover.

### 1. Introduction

Tarski’s axiomatization of geometry is characterized by its logical elegance, relying only on two basic relations between points and a few simple axioms. These properties make Tarskian geometry attractive for formalization: in particular for research in automated deduction, see for instance Narboux (2006), Beeson and Wos (2017). For more detailed accounts of Tarski’s axioms and their history see SST, Beeson (2015), and Narboux (2006).

### 2. The language of Tarskian geometry

The only objects under consideration are *points*. They are subject to two primitive relations: quaternary *congruence*  $(-)(-)\equiv(-)(-)$  and ternary *betweenness*  $(-)(-)(-)$ . Congruence (also called *equidistance*) expresses that the distance between the first two points is equal to the distance of the last two points, and betweenness expresses that the second point lies between the other two on a shared line. Informally we will also talk about segments and lines, indicating them by concatenation  $(-)(-)$  of points.

**2.1. Signature.** A point is an object.

**2.2. Convention.** Let  $a, b, c, d, e, f$  denote points.

**2.3. Signature (Congruence).**  $ab \equiv cd$  is a relation.

**2.4. Convention.** Let  $ab \not\equiv cd$  stand for it is wrong that  $ab \equiv cd$ .

**2.5. Signature (Betweenness).**  $abc$  is a relation.

Points are *collinear* when they lie on a single line. We will later see that betweenness is symmetric ( $abc$  implies  $cba$ ), so we only need to consider three of the six permutations

of three points in the definition of collinearity.

**2.6. Definition** (Collinearity).  $a$  is collinear with  $b$  and  $c$  iff  $abc$  or  $bca$  or  $cab$ .

**2.7. Axiom** (Reflexivity of congruence). We have  $ab \equiv ba$ .

**2.8. Axiom** (Pseudotransitivity of congruence). If  $cd \equiv ab$  and  $cd \equiv ef$  then  $ab \equiv ef$ .

**2.9. Axiom** (Identity of congruence). If  $ab \equiv cc$  then  $a = b$ .

Segment construction allows us to extend a segment  $ab$  by a length specified by another segment  $de$ .

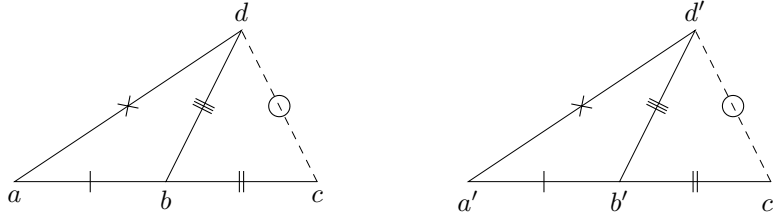
**2.10. Axiom** (Segment construction). There exists a point  $c$  such that  $abc$  and  $bc \equiv de$ .

We say that the points  $x, y, z, r, u, v, w, p$  are in an *outer five segment configuration* whenever  $\text{OFS}(\begin{smallmatrix} x & y & z & r \\ u & v & w & p \end{smallmatrix})$ .

**2.11. Convention.** Let  $a', b', c', d'$  denote points. Let  $x, y, z, u, v, w, p, q, r$  denote points.

**2.12. Definition.**  $\text{OFS}(\begin{smallmatrix} x & y & z & r \\ u & v & w & p \end{smallmatrix})$  if and only if  $xyz \wedge uvw$  and we have  $xy \equiv uv \wedge yz \equiv vw \wedge xr \equiv up \wedge yr \equiv vp$ .

Using the concept of an outer five segment configuration, we can state the five segment axiom in a concise form.



**2.13. Axiom** (Five segment axiom). If  $\text{OFS}(\begin{smallmatrix} a & b & c & d \\ a' & b' & c' & d' \end{smallmatrix})$  and  $a \neq b$  then  $cd \equiv c'd'$ .

**2.14. Axiom** (Identity of betweenness). If  $aba$  then  $a = b$ .

Tarski splits the classical axiom of Pasch into two axioms by making an inner/outer distinction, leading to logically simpler statements. We will later see that outer Pasch follows from inner Pasch, which was first demonstrated by Gupta (1965).

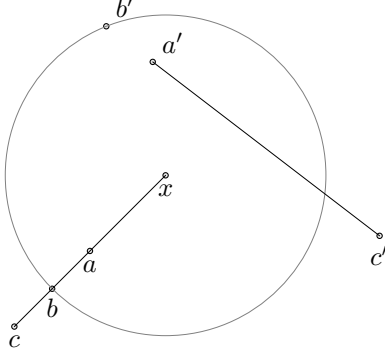
**2.15. Axiom** (Inner Pasch). If  $xuz$  and  $yvz$  then there exists a point  $w$  such that  $uwv$  and  $vwx$ .

**2.16. Axiom** (Lower dimension). There exist points  $\alpha, \beta, \gamma$  such that  $\alpha$  is not collinear with  $\beta$  and  $\gamma$ .

**2.17. Axiom** (Upper dimension). If  $xu \equiv xv$  and  $yu \equiv yv$  and  $zu \equiv zv$  and  $u \neq v$  then  $x$  is collinear with  $y$  and  $z$ .

**2.18. Axiom** (Euclid). Assume  $x \neq r$ . If  $xrv$  and  $yrz$  then there exist points  $s, t$  such that  $xy s$  and  $xzt$  and  $svt$ .

Circle continuity is equivalent to the the statement *a line that has a point within a circle intersects that circle*.



**2.19. Axiom** (Circle continuity). Assume  $xab$  and  $xbc$ . Assume  $xa' \equiv xa$  and  $xc' \equiv xc$ . Then there exists a point  $b'$  such that  $xb' \equiv xb$  and  $a'b'c'$ .

**2.20. Lemma** (Reflexivity of congruence). For all points  $x, y$  we have  $xy \equiv xy$ .

**2.21. Lemma** (Symmetry of congruence). If  $xy \equiv vw$  then  $vw \equiv xy$ .

**2.22. Lemma** (Transitivity of congruence). If  $xy \equiv vw$  and  $vw \equiv pq$  then  $xy \equiv pq$ .

**2.23. Lemma** (Congruence is independent of the order of the pairs). If  $xy \equiv vw$  then  $yx \equiv vw$ .

**2.24. Lemma**. If  $xy \equiv vw$  then  $xy \equiv wv$ .

**2.25. Lemma** (Zero segments are congruent). For all point  $x, y$  we have  $xx \equiv yy$ .

**2.26. Lemma** (Concatenation of segments). Assume  $xyz$  and  $rvw$ . Assume  $xy \equiv rv$  and  $yz \equiv vw$ . Then  $xz \equiv rw$ .

*Proof.* We have OFS  $\left( \begin{smallmatrix} x & y & z \\ r & v & w \end{smallmatrix} \begin{smallmatrix} x \\ r \end{smallmatrix} \right)$ . If  $x = y$  then  $r = v$ . If  $x \neq y$  then  $xz \equiv rw$ .  $\square$

**2.27. Lemma** (Uniqueness of segment construction). Assume  $a \neq b$ . Suppose  $abc$  and  $bc \equiv de$ . Suppose  $abc'$  and  $bc' \equiv de$ . Then  $c = c'$ .

*Proof.* We have  $ac \equiv ac'$ . Thus  $bc \equiv bc'$ . Thus OFS  $\left( \begin{smallmatrix} a & b & c \\ a & b & c \end{smallmatrix} \begin{smallmatrix} c \\ c \end{smallmatrix} \right)$ . Therefore  $cc \equiv cc'$ .  $\square$

**2.28. Lemma** (Right betweenness). For all points  $x, y$  we have  $xyy$ .

**2.29. Lemma** (Symmetry of betweenness). Assume  $xyz$ . Then  $zyx$ .

Left betweenness follows directly from right betweenness and symmetry of betweenness.

**2.30. Lemma** (Left betweenness). For all points  $x, y$  we have  $xxxy$ .

**2.31. Lemma.** Assume  $xyz$  and  $yxz$ . Then  $x = y$ .

*Proof.* Take a point  $w$  such that  $ywy$  and  $xwx$ . Then  $x = w = y$ .  $\square$

**2.32. Lemma.** Assume  $xyv$  and  $yzv$ . Then  $xyz$ .

*Proof.* Take a point  $w$  such that  $ywy$  and  $zwx$ .  $\square$

**2.33. Lemma.** Assume  $xyz$  and  $yzr$  and  $y \neq z$ . Then  $xzr$ .

*Proof.* Take  $v$  such that  $xzv$  and  $zv \equiv zr$ . Then  $yzv$  and  $zv \equiv zr$ . Hence  $v = r$ .  $\square$

**2.34. Lemma.** Assume  $xyv$  and  $yzv$ . Then  $xzv$ .

*Proof.* If  $y = z$  then  $xzv$ . Assume  $y \neq z$ . We have  $xyz$ .  $\square$

**2.35. Lemma.** Assume  $xyz$  and  $xzr$ . Then  $yzr$ .

**2.36. Lemma.** Assume  $xyz$  and  $xzr$ . Then  $xyr$ .

*Proof.* We have  $rxz$ . We have  $zyx$ . Thus  $ryx$ . Thus  $xyz$ .  $\square$

**2.37. Lemma.** Assume  $y \neq z$ . If  $xyz$  and  $yzr$  then  $xyr$ .

Existence of at least two points follows from the lower dimension axiom. All other axioms also hold in a one-point space.

**2.38. Lemma.** We have  $x \neq y$  for some  $x, y$ .

**2.39. Lemma.** There exist  $z$  such that  $xyz$  and  $y \neq z$ .

The following follows from invoking inner Pasch twice.

**2.40. Lemma.** Assume  $xyz$  and  $uvz$  and  $xpu$ . Then there exist  $q$  such that  $pqz$  and  $yqv$ .

*Proof.* We have  $xpu$  and  $zvu$ . Take  $r$  such that  $vrz$  and  $prz$ . [prover vampire] Take  $q$  such that  $rqz$  and  $vqy$ .  $\square$

We say that the points  $a, b, c, d, a', b', c', d'$  are in an inner five segment configuration whenever  $\text{IFS} \left( \begin{smallmatrix} a & b & c & d \\ a' & b' & c' & d' \end{smallmatrix} \right)$ .

**2.41. Definition.**  $\text{IFS} \left( \begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix} \right)$  iff  $xyz$  and  $vwp$  and  $xz \equiv vp$  and  $yz \equiv wp$  and  $xr \equiv vq$  and  $zr \equiv pq$ .

We can swap  $x, y$  with  $v, w$ .

[timelimit 20]

**2.42. Lemma.** Assume  $\text{IFS} \left( \begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix} \right)$ . Then  $yr \equiv wq$ .

*Proof.* Case  $x \neq z$ . Take points  $g, h$  such that  $g \neq z$  and  $xzg$  and  $vph$  and  $ph \equiv zg$ . Then OFS  $\begin{pmatrix} x & z & g & r \\ v & p & h & q \end{pmatrix}$ . Thus  $gr \equiv hq$ . Thus OFS  $\begin{pmatrix} g & z & y & r \\ h & p & w & q \end{pmatrix}$ . Thus  $yr \equiv wq$ . End.  $\square$

**2.43. Lemma** (Overlapping segments). Assume  $xyz$  and  $rvw$  and  $xz \equiv rw$  and  $yz \equiv vw$ . Then  $xy \equiv rv$ .

*Proof.* We have IFS  $\begin{pmatrix} x & y & z & x \\ r & v & w & r \end{pmatrix}$ .  $\square$

**2.44. Definition.**  $xyz \equiv uvw$  iff  $xy \equiv uv$  and  $xz \equiv uw$  and  $yz \equiv vw$ .

**2.45. Lemma.**  $xyz \equiv uvw$  iff  $yxz \equiv vuw$ .

**2.46. Lemma.**  $xyz \equiv uvw$  iff  $zyx \equiv wvu$ .

**2.47. Lemma.**  $xyz \equiv uvw$  iff  $xzy \equiv uwx$ .

If we have two congruent segments, then an inner point of one segment can be transferred congruently onto the other segment.

**2.48. Lemma.** Assume  $xyz$  and  $xz \equiv rw$ . Then there exists  $v$  such that  $rvw$  and  $xyz \equiv rvw$ .

*Proof.* Take  $u$  such that  $wru$  and  $r \neq u$ . Then take  $v$  such that  $urv$  and  $rv \equiv xy$ . Take a point  $g$  such that  $uv g$  and  $vg \equiv yz$ . Then  $xz \equiv rw$ . Therefore  $g = w$ .  $\square$

**2.49. Lemma.** Assume  $xyz$  and  $xyz \equiv rvw$ . Then  $rvw$ .

*Proof.* Take  $u$  such that  $ruw$  and  $xyz \equiv ruw$ . Then  $ruw \equiv rvw$  and IFS  $\begin{pmatrix} r & u & w & u \\ r & u & w & v \end{pmatrix}$ . Then  $uu \equiv uv$ . Hence  $u = v$ . Hence  $rvw$ .  $\square$

### 3. Collinearity

Until now we have only used the concept of collinearity to abbreviate some axioms. We first make the straightforward observation that collinearity is invariant under permutation of the arguments.

**3.1. Convention.** Let  $a, a', b, b', c, c', d, d', e, f, x, y, z, u, v, w, p, q, r$  denote points.

**3.2. Lemma.** Assume that  $a$  is collinear with  $b$  and  $c$ . Then  $b$  is collinear with  $c$  and  $a$ .

**3.3. Lemma.** Assume that  $a$  is collinear with  $b$  and  $c$ . Then  $c$  is collinear with  $a$  and  $b$ .

**3.4. Lemma.** Assume that  $a$  is collinear with  $b$  and  $c$ . Then  $c$  is collinear with  $b$  and  $a$ .

**3.5. Lemma.** Assume that  $a$  is collinear with  $b$  and  $c$ . Then  $b$  is collinear with  $a$  and  $c$ .

**3.6. Lemma.** Assume that  $a$  is collinear with  $b$  and  $c$ . Then  $a$  is collinear with  $c$  and  $b$ .

Similarly, it is easy to find a common line between just two points instead of three.

**3.7. Lemma.**  $a$  is collinear with  $a$  and  $b$  for all points  $a, b$ .

**3.8. Lemma.** Assume  $a$  is collinear with  $b$  and  $c$ . Assume  $ab \equiv a'b'$ . Then there exists  $c'$  such that  $abc \equiv a'b'c'$ .

*Proof.* Case  $abc$ . Take  $c'$  such that  $a'b'c'$  and  $b'c' \equiv bc$ . End. Case  $bac$ . Take  $c'$  such that  $b'a'c'$  and  $a'c' \equiv ac$ . Then  $bc \equiv b'c'$ . End. Then  $acb$ . Take  $c'$  such that  $a'c'b'$  and  $acb \equiv a'c'b'$ .  $\square$

#### 4. Five segment configuration

**4.1. Convention.** Let  $a, a', b, b', c, c', d, d', e, f, x, y, z, u, v, w, p, q, r$  denote points.

**4.2. Definition.**  $\text{FS} \left( \begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix} \right)$  iff  $x$  is collinear with  $y$  and  $z$  and  $xyz \equiv vwp$  and  $xr \equiv vq$  and  $yr \equiv wq$ .

The following lemma summarizes previous statements about outer/inner five segment configurations.

**4.3. Lemma.** Assume  $\text{FS} \left( \begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix} \right)$  and  $x \neq y$ . Then  $zr \equiv pq$ .

*Proof.* Case  $xyz$ . We have  $xyz \equiv vwp$ . Thus  $vwp$ . Thus  $\text{OFS} \left( \begin{smallmatrix} x & y & z & r \\ v & w & p & q \end{smallmatrix} \right)$ . End. Case  $zxy$ . We have  $zxy \equiv pvw$ . Thus  $pvw$ . Then  $\text{OFS} \left( \begin{smallmatrix} y & x & z & r \\ w & v & p & q \end{smallmatrix} \right)$ . End. Then  $yzx$ . We have  $yzx \equiv wpv$ . Thus  $wpv$ . Then  $\text{IFS} \left( \begin{smallmatrix} y & z & x & r \\ w & p & v & q \end{smallmatrix} \right)$ .  $\square$

**4.4. Lemma.** Assume  $x \neq y$ . Assume  $x$  is collinear with  $y$  and  $z$ . Assume  $xp \equiv xq$  and  $yp \equiv yq$ . Then  $zp \equiv zq$ .

*Proof.* We have  $\text{FS} \left( \begin{smallmatrix} x & y & z & p \\ x & y & z & q \end{smallmatrix} \right)$ .  $\square$

**4.5. Lemma.** Assume  $a \neq b$ . Assume  $a$  is collinear with  $b$  and  $c$ . Assume  $ac \equiv ac'$  and  $bc \equiv bc'$ . Then  $c' = c$ .

**4.6. Lemma.** Assume  $xzy$  and  $xz \equiv xp$  and  $yz \equiv yp$ . Then  $z = p$ .

*Proof.* Assume  $x = y$ . Then  $x = z$  and  $x = p$ . Hence  $z = p$ . Assume  $x \neq y$ .  $\square$

#### 5. Connexity of betweenness

**5.1. Convention.** Let  $a, a', b, b', c, c', d, d', e, f, x, y, z, u, v, w, p, q, r$  denote points.

Gupta (1965) proved that outer Pasch follows from inner Pasch. To prove Gupta's theorem, we need a few preparatory lemmas.

**5.2. Definition.**  $abcd$  iff  $abc$  and  $abd$  and  $acd$  and  $bcd$ .

**5.3. Definition.**  $abcde$  iff  $abc$  and  $abd$  and  $abe$  and  $acd$  and  $ace$  and  $ade$  and  $bcd$  and  $bce$  and  $bde$  and  $cde$ .

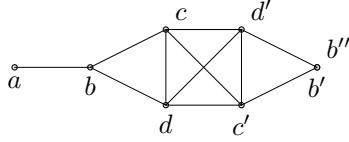
**5.4. Lemma** (Extension to quaternary betweenness). If  $abc$  and  $acd$  then  $abcd$ .

**5.5. Lemma** (Extension to quinary betweenness). If  $abcd$  and  $ade$  then  $abcde$ .

**5.6. Lemma.** Assume  $x \neq y$  and  $xyz$  and  $xyr$ . Then there exist points  $\alpha, \beta$  such that  $xr\alpha$  and  $r\alpha \equiv zr$  and  $xz\beta$  and  $z\beta \equiv zr$ .

*Proof.* Take point  $a$  such that  $xra$  and  $ra \equiv zr$  (by 2.10). Take point  $b$  such that  $xzb$  and  $zb \equiv zr$  (by 2.10).  $\square$

**5.7. Lemma.** Assume  $x \neq y$  and  $xyz$  and  $xyr$  and  $xrp$  and  $rp \equiv zr$  and  $xzq$  and  $zq \equiv zr$ . Then there exist points  $s, t$  such that  $zqt$  and  $rps$ .



**5.8. Theorem** (Outer Pasch). Assume  $a \neq b$ . Assume  $abc$  and  $abd$ . Then  $acd$  or  $adc$ .

*Proof.* Take a point  $c'$  such that  $adc'$  and  $dc' \equiv cd$ . Take a point  $d'$  such that  $acd'$  and  $cd' \equiv cd$ . Then  $c = c'$  or  $d = d'$ .

*Proof.* We have  $abcd'$  (by 5.4). We have  $abdc'$  (by 5.4). Take a point  $b'$  such that  $ac'b'$  and  $c'b' \equiv cb$  (by 2.10). Take a point  $b''$  such that  $ad'b''$  and  $d'b'' \equiv bd$  (by 2.10). Then  $abcd'b''$ . Then  $abdc'b'$ .

Thus  $bc' \equiv b''c$ .

Thus  $bb' \equiv b''b$ .

We have  $abb'$  and  $abb''$ . Thus  $b'' = b'$ .

We have OFS  $\begin{pmatrix} b & c & d' & c' \\ b' & c' & d & c \end{pmatrix}$ . Thus  $c'd' \equiv cd$ . [timelimit 65] Take a point  $e$  such that  $cec'$  and  $ded'$ . Then IFS  $\begin{pmatrix} d & e & d' & c' \\ d & e & d' & c' \end{pmatrix}$  and IFS  $\begin{pmatrix} c & e & c' & d \\ c & e & c' & d' \end{pmatrix}$ . Thus  $ec \equiv ec'$  and  $ed \equiv ed'$ .

Case  $c \neq c'$ . We have  $c \neq d'$ .

Take a point  $p$  such that  $c'cp$  and  $cp \equiv cd'$ . Take a point  $r$  such that  $d'cr$  and  $cr \equiv ce$ . Take a point  $q$  such that  $prq$  and  $rq \equiv rp$ .

Then OFS  $\begin{pmatrix} d' & c & r & p \\ p & c & e & d' \end{pmatrix}$ . Thus  $rp \equiv ed'$ . Thus  $rq \equiv ed$ .

Then OFS  $\begin{pmatrix} d' & e & d & c \\ p & r & q & c \end{pmatrix}$ . Thus  $d'd \equiv pq$ . Thus  $cq \equiv cd$ . Thus  $cp \equiv cq$ .

We have  $rp \equiv rq$ . We have  $r \neq c$ .

Then  $r$  is collinear with  $c$  and  $d'$ . Thus  $d'p \equiv d'q$ .

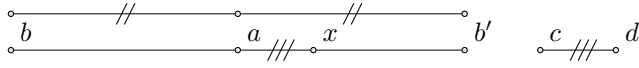
We have  $c \neq d'$ . Then  $c$  is collinear with  $d'$  and  $b$ . Then  $c$  is collinear with  $d'$  and  $b'$ . Thus  $bp \equiv bq$  and  $b'p \equiv b'q$ .

Thus  $b \neq b'$ . Then  $b$  is collinear with  $c'$  and  $b'$ . Thus  $c'p \equiv c'q$ .

$c'$  is collinear with  $c$  and  $p$ . Thus  $pp \equiv pq$ . Thus  $p = q$ . Thus  $d = d'$ . End. End. Therefore  $acd$  or  $adc$ .  $\square$

**5.9. Lemma.** Assume  $a \neq b$ . If  $abc$  and  $abd$  then  $bcd$  or  $bdc$ .

**5.10. Theorem.** If  $xyw$  and  $xzw$  then  $xyz$  or  $xzy$ .



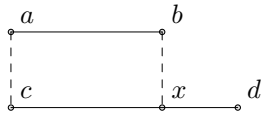
**5.11. Lemma.** Assume  $a \neq b$ . Then we have  $ax \equiv cd$  for some point  $x$  such that  $abx$  or  $axb$ .

*Proof.* Take  $b'$  such that  $bab'$  and  $ab' \equiv ab$ . Take  $x$  such that  $b'ax$  and  $ax \equiv cd$ .  $\square$

## 6. Comparing segments

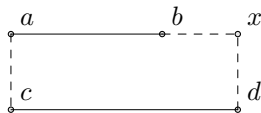
**6.1. Convention.** Let  $a, a', b, b', c, c', d, d', e, f, x, y, z, u, v, w, p, q, r$  denote points.

Informally, a segment  $ab$  is smaller than a segment  $cd$  whenever we can find a subsegment  $cx$  of  $cd$  of the same length as  $ab$ .



**6.2. Definition.**  $ab \leq cd$  iff there exists  $x$  such that  $cx \equiv ab$  and  $x$  is on  $cd$ .

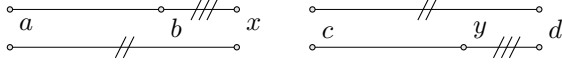
Alternatively, we can say that a segment  $ab$  is smaller than  $cd$  whenever we can extend  $ab$  to a segment  $ax$  of length  $cd$ .



**6.3. Lemma.** Assume  $ab \leq cd$ . Then there exists  $x$  such that  $abx$  and  $ax \equiv cd$ .



*Proof.* Take  $y$  such that  $cyd$  and  $ab \equiv cy$ . Take  $x$  such that  $abx$  and  $bx \equiv yd$ . Then  $ax \equiv cd$ .  $\square$



**6.4. Lemma.** Let  $x$  be a point such that  $abx$  and  $ax \equiv cd$ . Then  $ab \leq cd$ .

*Proof.* Take  $b'$  such that  $cb'd$  and  $abx \equiv cb'd$ .  $\square$

**6.5. Lemma** (Transitivity of congruence and comparison). Assume  $ab \equiv cd$  and  $cd \leq ef$ . Then  $ab \leq ef$ .

*Proof.* Take  $x$  such that  $exf$  and  $cd \equiv ex$ .  $\square$

**6.6. Lemma** (Transitivity of comparison and congruence). Assume  $ab \leq cd$  and  $cd \equiv ef$ . Then  $ab \leq ef$ .

*Proof.* Take  $x$  such that  $abx$  and  $ax \equiv cd$ .  $\square$

**6.7. Lemma** (Reflexivity of comparison). For all points  $a, b$  we have  $ab \leq ab$ .

**6.8. Lemma** (Transitivity of comparison). Assume  $ab \leq cd$  and  $cd \leq ef$ . Then  $ab \leq ef$ .

*Proof.* Take  $x$  such that  $abx$  and  $ax \equiv cd$ . Take  $y$  such that  $cdy$  and  $cy \equiv ef$ .  $\square$

**6.9. Lemma** (Antisymmetry of comparison). Assume  $ab \leq cd$  and  $cd \leq ab$ . Then  $ab \equiv cd$ .

*Proof.* Take  $x$  such that  $cx d$  and  $ab \equiv cx$ . Take  $y$  such that  $cdy$  and  $ab \equiv cy$ . Then  $cx \equiv cy$ . Thus  $x = d = y$ .  $\square$

**6.10. Lemma** (Connexity of comparison). Let  $a, b, c, d$  be points. Then  $ab \leq cd$  or  $cd \leq ab$ .

*Proof.* Case  $a \neq b$ . Take  $x$  such that  $(bax$  or  $bxa)$  and  $bx \equiv cd$ . End.  $\square$

**6.11. Lemma.** For all points  $a, b, c$  we have  $aa \leq bc$ .

**6.12. Lemma.** Assume  $abc$ . Then  $ab \leq ac$ .

**6.13. Lemma.** Assume  $abc$ . Then  $bc \leq ac$ .

*Proof.*  $a$  is a point such that  $cba$  and  $ca \equiv ac$ .  $\square$

**6.14. Lemma.** Assume that  $a$  is collinear with  $b$  and  $c$ . Assume  $ab \leq ac$  and  $bc \leq ac$ . Then  $abc$ .

**6.15.** *Definition.*  $pq < xy$  iff  $pq \leq xy$  and  $pq \neq xy$ .

**6.16.** *Definition.*  $pq > xy$  iff  $xy < pq$ .

## 7. Rays and lines

**7.1.** *Convention.* Let  $a, a', b, b', c, c', d, d', e, f, x, y, z, u, v, w, p, q, r$  denote points.

**7.2.** *Definition.*  $a$  and  $b$  lie on opposite sides of  $u$  iff  $a, b, u$  are pairwise nonequal and  $aub$ .

**7.3.** *Definition.*  $a$  and  $b$  are equivalent with respect to  $u$  iff  $a, b \neq u$  and  $(uab \text{ or } uba)$ .

**7.4.** *Convention.* Let  $a \approx_u b$  stand for  $a$  and  $b$  are equivalent with respect to  $u$ .

We will see that two points are equivalent with respect to a point  $u$  iff they determine the same ray with origin  $u$ .

**7.5.** *Lemma.* Suppose  $a, b, c \neq u$ . Suppose  $auc$ . Then  $buc$  iff  $a \approx_u b$ .

**7.6.** *Lemma.* Suppose  $a$  and  $b$  are equivalent with respect to  $u$ . Then  $a, b \neq u$  and there exists a point  $c$  such that  $c \neq u$  and  $auc$  and  $buc$ .

**7.7.** *Lemma.* Suppose  $a, b \neq u$ . Suppose there exists a point  $c$  such that  $c \neq u$  and  $auc$  and  $buc$ . Then  $a$  and  $b$  are equivalent with respect to  $u$ .

**7.8.** *Lemma.*  $a$  and  $b$  are equivalent with respect to  $u$  iff  $a$  is collinear with  $u$  and  $b$  and  $\neg aub$ .

**7.9.** *Lemma* (Reflexivity of relative equivalence). Suppose  $a \neq u$ . Then  $a \approx_u a$ .

**7.10.** *Lemma* (Symmetry of relative equivalence). If  $a \approx_u b$  then  $b \approx_u a$ .

**7.11.** *Lemma* (Transitivity of relative equivalence). Assume  $a \approx_u b$  and  $b \approx_u c$ . Then  $a \approx_u c$ .

*Proof.* Case  $uab$ . End. □

**7.12.** *Lemma.* Suppose  $r \neq a$  and  $b \neq c$ . Then there exists a point  $x$  such that  $x \approx_a r$  and  $ax \equiv bc$ .

[timelimit 120]

**7.13.** *Lemma.* Suppose  $r \neq a$  and  $b \neq c$ . Let  $x$  be a point such that  $x \approx_a r$  and  $ax \equiv bc$ . Let  $x'$  be a point such that  $x' \approx_a r$  and  $ax' \equiv bc$ . Then  $x' = x$ .

**7.14.** *Lemma.* Suppose  $a \approx_u b$  and  $ua \leq ub$ . Then  $uab$ .

**7.15.** *Lemma.* Suppose  $a \approx_u b$  and  $uab$ . Then  $ua \leq ub$ .

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