Cantor's Theorem

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In this document we give a proof of Cantor's Theorem:

Theorem. There is no surjection from a set onto its powerset.

Some basic notions and set-theoretic axioms used to formulate and prove it are taken from:

[read `examples/preliminaries.ftl.tex`]

Moreover, we need to provide certain definitions concerning surjective functions and the notion of powerset.

Definition. Let X be a set. A function of X is a function f such that $\mathrm{dom}(f) = X$.

Definition. Let f be a function and Y be a set. f surjects onto Y iff $Y = \{f(x) \mid x \in \mathrm{dom}(f)\}$.

Let a surjective function from X to Y stand for a function of X that surjects onto Y.

Definition. Let X be a set. The powerset of X is the collection of subsets of X.

Axiom. The powerset of any set is a set.

On this basis Cantor's theorem and its proof can be formalized as follows.

Theorem (Cantor). Let M be a set. No function of M surjects onto the powerset of M.

Proof. Assume the contrary. Take a surjective function f from M to the powerset of M. The value of f at any element of M is a set. Define

$$N = \{x \in M \mid x \text{ is not an element of } f(x)\}.$$

N is a subset of M. Consider an element z of M such that $f(z) = N$. Then

$$z \in N \iff z \notin f(z) = N.$$

Contradiction. $\square$