Theorem 1 (Transfinite Induction I). Let Φ be a class. Assume that for all ordinals α if Φ contains all $\beta \in \alpha$ then Φ contains α. Then Φ contains every ordinal.

Proof. Define $B := \{x \mid x \text{ is a set and if } x \in \mathbb{O} \text{ then } x \in \Phi\}$.

Let us show that for all sets x if B contains every element of x that is a set then B contains x. Let x be a set. Assume that every element of x that is a set is contained in B.

Case $x \notin \mathbb{O}$. $\square$

Case $x \in \mathbb{O}$. Then Φ contains all ordinals less than x. Hence Φ contains x. Thus $x \in B$. $\square$

End.

[prover vampire] Hence B contains every set (by $\in$-induction). [prover eprover] Thus Φ contains every ordinal. ∎

Theorem 2 (Transfinite Induction II). Let Φ be a class. (Initial case) Assume that Φ contains 0. (Successor step) Assume that for all ordinals α if $\alpha \in \Phi$ then $\mathrm{succ}(\alpha) \in \Phi$. (Limit step) Assume that for all limit ordinals λ if every $\alpha \in \lambda$ is contained in Φ then $\lambda \in \Phi$. Then Φ contains every ordinal.

Proof. Let us show that for all ordinals α if Φ contains all ordinals less than α then Φ contains α. Let α be an ordinal. Then $\alpha = 0$ or α is a successor ordinal or α is a limit ordinal. Assume that Φ contains all $\beta \in \alpha$.

Case $\alpha = 0$. $\square$

Case α is a successor ordinal. Take an ordinal β such that $\alpha = \mathrm{succ}(\beta)$. Then $\beta \in \Phi$. Hence $\alpha \in \Phi$ (by successor step). $\square$

Case α is a limit ordinal. Then $\beta \in \Phi$ for all ordinals β less than α. Hence $\alpha \in \Phi$ (by limit step). $\square$

End.

[prover vampire] Thus Φ contains every ordinal (by transfinite induction I). [prover eprover] ∎