Definition 1. Let α be an ordinal. $\mathrm{succ}(\alpha) := \alpha \cup \{\alpha\}$.

Proposition 2. Let α be an ordinal. Then $\mathrm{succ}(\alpha)$ is an ordinal.

Proof. (1) $\mathrm{succ}(\alpha)$ is transitive.

Proof. Let $x \in \mathrm{succ}(\alpha)$ and $y \in x$. Then $x \in \alpha$ or $x = \alpha$. Hence $y \in \alpha$. Thus $y \in \mathrm{succ}(\alpha)$. □

(2) Every element of $\mathrm{succ}(\alpha)$ is transitive.

Proof. Let $x \in \mathrm{succ}(\alpha)$. Then $x \in \alpha$ or $x = \alpha$. Hence x is transitive. Indeed α is transitive and every element of α is transitive. □ ■

Definition 3. A *successor ordinal* is an ordinal α such that $\alpha = \mathrm{succ}(\beta)$ for some ordinal β.

Proposition 4. Let α, β be ordinals. If $\mathrm{succ}(\alpha) = \mathrm{succ}(\beta)$ then $\alpha = \beta$.

Proof. Assume $\mathrm{succ}(\alpha) = \mathrm{succ}(\beta)$.

(1) $\alpha \subset \beta$.

Proof. Let $\gamma \in \alpha$. Then $\gamma \in \alpha \cup \{\alpha\} = \mathrm{succ}(\alpha) = \mathrm{succ}(\beta) = \beta \cup \{\beta\}$. Hence $\gamma \in \beta$ or $\gamma = \beta$. Assume $\gamma = \beta$. Then $\beta \in \alpha$. Hence $\beta = (\beta \cup \{\beta\}) \setminus \{\gamma\} = (\alpha \cup \{\alpha\}) \setminus \{\gamma\} = (\alpha \setminus \{\gamma\}) \cup$
Therefore $\alpha \in \beta$. Consequently $\alpha \in \beta \in \alpha$. Contradiction. □

(2) $\beta \subset \alpha$.

Proof. Let $\gamma \in \beta$. Then $\gamma \in \beta \cup \{\beta\} = \mathrm{succ}(\beta) = \mathrm{succ}(\alpha) = \alpha \cup \{\alpha\}$. Hence $\gamma \in \alpha$ or $\gamma = \alpha$. Assume $\gamma = \alpha$. Then $\alpha \in \beta$. Hence $\alpha = (\alpha \cup \{\alpha\}) \setminus \{\gamma\} = (\beta \cup \{\beta\}) \setminus \{\gamma\} = (\beta \setminus \{\gamma\}) \cup$
Therefore $\beta \in \alpha$. Consequently $\beta \in \alpha \in \beta$. Contradiction. □ ■

Definition 5. Let α be a successor ordinal. $\mathrm{pred}(\alpha)$ is the ordinal β such that $\alpha = \mathrm{succ}(\beta)$.