Definition 1. Let α, β be ordinals. $\alpha < \beta$ iff $\alpha \in \beta$. Let α is *less than* β stand for $\alpha < \beta$. Let $\alpha \nless \beta$ stand for $\neg\, \alpha < \beta$. Let $\alpha > \beta$ stand for $\beta < \alpha$. Let α is *greater than* β stand for $\beta < \alpha$. Let $\alpha \ngtr \beta$ stand for $\neg\, \alpha > \beta$.

Definition 2. Let α, β be ordinals. $\alpha \leq \beta$ iff $\alpha < \beta$ or $\alpha = \beta$. Let α is *less than or equal to* β stand for $\alpha \leq \beta$. Let $\alpha \nleq \beta$ stand for $\neg\, \alpha \leq \beta$. Let $\alpha \geq \beta$ stand for $\beta \leq \alpha$. Let α is *greater than or equal to* β stand for $\beta \leq \alpha$. Let $\alpha \ngeq \beta$ stand for $\neg\, \alpha \geq \beta$.

Proposition 3. Let α, β be ordinals. If $\alpha \leq \beta$ then $\alpha \subset \beta$.

Proof. Assume $\alpha \leq \beta$. Then $\alpha < \beta$ or $\alpha = \beta$. Let $x \in \alpha$. If $\alpha < \beta$ then $x \in \alpha \in \beta$. Hence if $\alpha < \beta$ then $x \in \beta$. If $\alpha = \beta$ then $x \in \beta$. Thus $x \in \beta$. ■

Proposition 4. Let α be an ordinal. Then $\alpha \nless \alpha$.

Proof. Assume $\alpha < \alpha$. Then $\alpha \in \alpha$. Contradiction. ■

Proposition 5. Let α, β, γ be ordinals. If $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$.

Proof. Assume $\alpha < \beta$ and $\beta < \gamma$. Then $\alpha \in \beta \in \gamma$. Hence $\alpha \in \gamma$. Thus $\alpha < \gamma$. ■

Proposition 6. Let α, β be ordinals. Then $\alpha < \beta$ or $\alpha = \beta$ or $\alpha > \beta$.

Proof. Assume the contrary.

Define $A := \{\alpha' \in \mathbb{O} \mid$ there exists an ordinal β' such that neither $\alpha' < \beta'$ nor $\alpha' = \beta'$ nor $\alpha' > \beta'\}$.

A is nonempty. Hence we can take a $\alpha' \in A$ such that for no $\gamma \in A$ we have $\gamma < \alpha'$.

Define $B := \{\beta' \in \mathbb{O} \mid \text{ neither } \alpha' < \beta' \text{ nor } \alpha' = \beta' \text{ nor } \alpha' > \beta'\}$.

B is nonempty. Hence we can take a $\beta' \in B$ such that for no $\gamma \in B$ we have $\gamma < \beta'$.

Let us show that $\alpha' \subset \beta'$. Let $a \in \alpha'$. Then $a < \beta'$ or $a = \beta'$ or $a > \beta'$. Indeed if neither $a < \beta'$ nor $a = \beta'$ nor $a > \beta'$ then $a \in A$. If $a = \beta'$ then $\beta' < \alpha'$. If $a > \beta'$ then $\beta' < \alpha'$. Hence $a < \beta'$. Thus $a \in \beta'$. End.

Let us show that $\beta' \subset \alpha'$. Let $b \in \beta'$. Then $b < \alpha'$ or $b = \alpha'$ or $b > \alpha'$. If $b = \alpha'$ then $\alpha' < \beta'$. If $b > \alpha'$ then $\alpha' < \beta'$. Hence $b < \alpha'$. Thus $b \in \alpha'$. End.

Hence $\alpha' = \beta'$. Contradiction. ■

Proposition 7. Let α, β be ordinals. If $\alpha \subset \beta$ then $\alpha \leq \beta$.

Proof. Assume $\alpha \subset \beta$.

Case $\alpha = \beta$. □

Case $\alpha \neq \beta$. Then $\alpha < \beta$ or $\alpha > \beta$. Assume $\alpha > \beta$. Then $\beta \in \alpha$. Hence $\beta \in \beta$. Contradiction. □

■

Proposition 8. Let α be an ordinal. Then $\alpha < \mathrm{succ}(\alpha)$.

Proposition 9. Let α, β be ordinals. If $\beta < \mathrm{succ}(\alpha)$ then $\beta \leq \alpha$.

Proof. Assume $\beta < \mathrm{succ}(\alpha)$. Then $\beta \in \mathrm{succ}(\alpha) = \alpha \cup \{\alpha\}$. Hence $\beta \in \alpha$ or $\beta \in \{\alpha\}$. Thus $\beta < \alpha$ or $\beta = \alpha$. Therefore $\beta \leq \alpha$. ■

Proposition 10. Let α be an ordinal. There exists no ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$.

Proof. Assume the contrary. Consider an ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$. Then $\beta < \alpha$ or $\beta = \alpha$. Hence $\alpha < \alpha$. Contradiction. ■

Proposition 11. Let α be an ordinal. There exists no ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$.

Proof. Assume the contrary. Choose an ordinal β such that $\alpha < \beta < \mathrm{succ}(\alpha)$. Then $\alpha \in \beta \in \alpha \cup \{\alpha\}$. Hence $\beta \in \alpha$ or $\beta = \alpha$. Then $\alpha \in \alpha$. Contradiction. ■

Proposition 12. Let λ be a limit ordinal and $\alpha \in \lambda$. Then λ contains $\mathrm{succ}(\alpha)$.

Proof. If $\mathrm{succ}(\alpha) \notin \lambda$ then $\alpha < \lambda < \mathrm{succ}(\alpha)$. ■