Proposition 1. $|\omega| = \omega$.

Proof. We have $|\omega| \leq \omega$.

Let us show that $|\omega|$ is not less than ω. Assume the contrary. Then $|\omega| \in \omega$. Take $n = |\omega|$ and a bijection f between n and ω.

Define

$$g(k) := \begin{cases} \mathrm{succ}(f(k)) & : k < n \\ 0 & : k = n \end{cases}$$

for $k \in \mathrm{succ}(n)$. Then g is a map from $\mathrm{succ}(n)$ to ω. Indeed we can show that $g(k) \in \omega$ for all $k \in \mathrm{succ}(n)$. Let $k \in \mathrm{succ}(n)$.

Case $k < n$. □

Case $k = n$. □

End.

g is injective. Indeed we can show that for all $k, k' \in \mathrm{succ}(n)$ if $k \neq k'$ then $g(k) \neq g(k')$.

Proof. Let $k, k' \in \mathrm{succ}(n)$. Assume $k \neq k'$.

Case $k, k' < n$. Then $f(k) \neq f(k')$. Hence $\mathrm{succ}(f(k)) \neq \mathrm{succ}(f(k'))$. Thus $g(k) \neq g(k')$. □

Case $k < n$ and $k' = n$. We have $\mathrm{succ}(f(k)) \neq 0$. Hence $g(k) \neq g(k')$. □

Case $k = n$ and $k' < n$. We have $\mathrm{succ}(f(k')) \neq 0$. Hence $g(k) \neq g(k')$. □

□

g is surjective onto ω. Indeed we can show that for any $m \in \omega$ there exists a $k \in \mathrm{succ}(n)$ such that $m = g(k)$.

Proof. Let $m \in \omega$. Then $f^{-1}(m) \in n$.

Case $m = 0$. Then $m = g(n)$. □

Case $m \neq 0$. Take $m' \in \omega$ such that $m = \mathrm{succ}(m')$. Then $m = \mathrm{succ}(m') = \mathrm{succ}(f(f^{-1}(m'))) = g(f^{-1}(m'))$. Indeed $f(f^{-1}(m')) = m'$ and $f^{-1}(m') < n$. □

□

Hence g is a bijection between $\mathrm{succ}(n)$ and ω. Then we have $n = |n| = |\mathrm{succ}(n)| = \mathrm{succ}(n)$. Contradiction. End. ■

Corollary 2. ω is a cardinal.