Theorem 1. Let $n \in \omega$. Then $|n| = n$.

Proof. Define $\Phi := \{n' \in \omega \mid |n'| = n'\}$.

(1) $0 \in \Phi$. Indeed $|0| = |\emptyset| = 0$.

(2) For all $n' \in \Phi$ we have $\mathrm{succ}(n') \in \Phi$.

Proof. Let $n' \in \Phi$. Then $|n'| = n'$. We have $|\mathrm{succ}(n')| \leq \mathrm{succ}(n')$.

Let us show that $\mathrm{succ}(n') \leq |\mathrm{succ}(n')|$. Assume the contrary. Then $|\mathrm{succ}(n')| < \mathrm{succ}(n')$. Take a bijection f between $|\mathrm{succ}(n')|$ and $\mathrm{succ}(n')$. $|\mathrm{succ}(n')|$ is nonzero. Hence we can take a $m \in \omega$ such that $|\mathrm{succ}(n')| = \mathrm{succ}(m)$. Indeed $|\mathrm{succ}(n')| \in \mathrm{succ}(\mathrm{succ}(n')) \in \omega$. Then $f^{-1}(n') \leq m$.

We can show that $f^{-1}(n') < m$. Assume the contrary. Then $f^{-1}(n') = m$. $f \upharpoonright m$ is a bijection between m and $f[m]$ (by bijectivity of restriction of injection). Indeed f is an injective map from $|\mathrm{succ}(n')|$ to $\mathrm{succ}(n')$ and $m \subset |\mathrm{succ}(n')|$. We have $f[m] \subset n'$ and $n' \subset f[m]$. Hence $f[m] = n'$. Thus $f \upharpoonright m$ is a bijection between m and n'. Therefore $n' = |n'| \leq m < |\mathrm{succ}(n')| < \mathrm{succ}(n')$. Consequently $m = n'$. Then we have $\mathrm{succ}(n') = |\mathrm{succ}(n')| < \mathrm{succ}(n')$. Contradiction. End.

Define

$$g(i) := \begin{cases} f(i) & : i \neq f^{-1}(n') \\ f(m) & : i = f^{-1}(n') \end{cases}$$

for $i \in m$.

g is a map from m to n'. Indeed we can show that $g(i) \in n'$ for each $i \in m$.

Proof. Let $i \in m$. We have $g(i) \in \mathrm{succ}(n')$. Indeed $f(i), f(m) \in \mathrm{succ}(n')$ and $(g(i) = f(i)$ or $g(i) = f(m))$. Hence if $g(i) \neq n'$ then $g(i) \in n'$.

Case $i \neq f^{-1}(n')$. Then $g(i) = f(i) \in \mathrm{succ}(n')$. If $g(i) = n'$ then $f(i) = n' = f(f^{-1}(n'))$. Hence if $g(i) = n'$ then $i = f^{-1}(n')$. Thus $g(i) \neq n'$. Therefore $g(i) \in n'$. □

Case $i = f^{-1}(n')$. Then $g(i) = f(m) \neq f(f^{-1}(n')) = n'$. Hence $g(i) \in n'$. □

□

g is surjective onto n'. Indeed we can show that for all $k \in n'$ there exists a $l \in m$ such that $k = g(l)$.

Proof. Let $k \in n'$. Then $f^{-1}(k) \neq f^{-1}(n')$.

Case $f^{-1}(k) = m$. Then $k = f(f^{-1}(k)) = f(m) = g(f^{-1}(n'))$. □

Case $f^{-1}(k) \neq m$. Then $f^{-1}(k) \in m$. Indeed $f^{-1}(k) \in |\mathrm{succ}(n')| = \mathrm{succ}(m) = m \cup \{m\}$. Hence $k = f(f^{-1}(k)) = g(f^{-1}(k))$. □

□

g is injective. Indeed we can show that for all $i, j \in m$ if $i \neq j$ then $g(i) \neq g(j)$.

Proof. Let $i, j \in m$. Assume $i \neq j$.

Case $i, j \neq f^{-1}(n')$. Then $g(i) = f(i) \neq f(j) = g(j)$. □

Case $i = f^{-1}(n')$. Then $j \neq f^{-1}(n')$. Hence $g(i) = g(f^{-1}(n')) = f(m) \neq f(j) = g(j)$. Indeed $m \neq j$. □

Case $j = f^{-1}(n')$. Then $i \neq f^{-1}(n')$. Hence $g(i) = f(i) \neq f(m) = g(f^{-1}(n')) = g(j)$. Indeed $i \neq m$. □

□ End. □

Thus $\omega \subset \Phi$ (by transfinite induction III). Consequently $n \in \Phi$. Therefore $|n| = n$. ■

Corollary 2. Every natural number is a cardinal.