Definition 1. $\mathbb{C}$ is the collection of all infinite cardinals.

Proposition 2. Let α be an infinite ordinal. Then $|\mathrm{succ}(\alpha)| = |\alpha|$.

Proof. For any $\beta \in \mathrm{succ}(\alpha)$ we have $\beta < \omega$ or $\omega \leq \beta < \alpha$ or $\beta = \alpha$. Define

$$f(\beta) := \begin{cases} \mathrm{succ}(\beta) & : \beta < \omega \\ \beta & : \omega \leq \beta < \alpha \\ 0 & : \beta = \alpha \end{cases}$$

for $\beta \in \mathrm{succ}(\alpha)$.

Then f is a map from $\mathrm{succ}(\alpha)$ to α. Indeed we can show that $f(\beta) \in \alpha$ for all $\beta \in \mathrm{succ}(\alpha)$.

Proof. Let $\beta \in \mathrm{succ}(\alpha)$.

Case $\beta < \omega$. Then $f(\beta) = \mathrm{succ}(\beta) < \omega \leq \alpha$. $\square$

Case $\omega \leq \beta < \alpha$. Then $f(\beta) = \beta < \alpha$. $\square$

Case $\beta = \alpha$. Then $f(\beta) = 0 < \alpha$. $\square$

$\square$

f is surjective onto α. Indeed we can show that for any $\beta \in \alpha$ there exists a $\gamma \in \mathrm{succ}(\alpha)$ such that $\beta = f(\gamma)$.

Proof. Let $\beta \in \alpha$. Then $\beta = 0$ or $0 < \beta < \omega$ or $\beta \geq \omega$.

Case $\beta = 0$. Then $\beta = f(\alpha)$. $\square$

Case $0 < \beta < \omega$. Take an ordinal β' such that $\beta = \mathrm{succ}(\beta')$. Then $\beta' < \omega$. Hence $\beta = f(\beta')$. $\square$

Case $\beta \geq \omega$. Then $\beta = f(\beta)$. $\square$

$\square$

f is injective. Indeed we can show that for all $\beta, \gamma \in \mathrm{succ}(\alpha)$ if $\beta \neq \gamma$ then $f(\beta) \neq f(\gamma)$.

Proof. Let $\beta, \gamma \in \mathrm{succ}(\alpha)$. Assume $\beta \neq \gamma$.

Case $\beta < \omega$. If $\gamma = \alpha$ then $f(\beta) = \mathrm{succ}(\beta) \neq 0 = f(\gamma)$. If $\omega \leq \gamma < \alpha$ then $f(\beta) = \mathrm{succ}(\beta) < \omega \leq \gamma = f(\gamma)$. $\square$

Case $\omega \leq \beta < \alpha$. If $\gamma = \alpha$ then $f(\beta) = \beta \geq \omega > 0 = f(\gamma)$. If $\gamma < \omega$ then $f(\beta) = \beta \geq \omega > \mathrm{succ}(\gamma) = f(\gamma)$. $\square$

Case $\beta = \alpha$. If $\gamma < \omega$ then $f(\beta) = 0 \neq \mathrm{succ}(\gamma) = f(\gamma)$. If $\omega \leq \gamma < \alpha$ then $f(\beta) = 0 < \omega \leq \gamma = f(\gamma)$. □

□

Hence f is a bijection between $\mathrm{succ}(\alpha)$ and α. Therefore $\mathrm{succ}(\alpha)$ and α are equinumerous. Consequently $|\mathrm{succ}(\alpha)| = |\alpha|$. ■

Proposition 3. Every infinite cardinal is a limit ordinal.

Proof. Let κ be an infinite cardinal. Suppose that κ is not a limit ordinal. $\kappa \neq 0$. Hence κ is a successor ordinal. Indeed κ is an ordinal. Thus we can take an ordinal α such that $\kappa = \mathrm{succ}(\alpha)$. We have $\alpha > \kappa \geq \omega$. Hence $|\mathrm{succ}(\alpha)| = |\alpha|$. Thus $\alpha < |\kappa|$ and κ is equinumerous to κ. Contradiction. ■